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Oscillation of Noncanonical Second-Order Functional Differential Equations via Canonical Transformation

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Abstract

The oscillatory properties of solutions to the second order functional differential equation

$$\mathcal{L}x(t) + f(t)x^{\beta}(\sigma(t)) = 0, \qquad t \geq t_0 > 0$$

where $\mathcal{L}x(t) = (\eta(t)x'(t))'$ is a noncanonical operator, are studied. The main idea is to transform the noncanonical equation into canonical form which simplifies the investigation of oscillation of the equation. The obtained criteria are new and complement to the existing results reported in the literature. Examples illustrating the main results are presented.

Keywords Second-order · Non-canonical · Oscillation · Monotonic properties

Mathematics Subject Classification 34C10 · 34K11

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1 Introduction

This paper is concerned with oscillation of a second-order nonlinear functional differential equation

$$\mathcal{L}x(t) + f(t)x^{\beta}(\sigma(t)) = 0, \qquad t \geq t_0 > 0 \tag{1.1}$$

where $\mathcal{L}x(t) = (\eta(t)x'(t))'$, subject to the following conditions:

(H_1) $\eta, f \in C([t_0, \infty))$, $\eta(t) > 0$ and $f(t) > 0$ for all $t \geq t_0$;
(H_2) β is the ratio of two positive odd integers;
(H_3) $\sigma \in C([t_0, \infty))$, $\sigma(t) \leq t$, $\sigma'(t) > 0$ and $\lim_{t\to\infty} \sigma(t) = \infty$;
(H_4) the operator $\mathcal{L}$ is in noncanonical form, that is,

$$\Pi(t_0) = \int_{t_0}^{\infty} \frac{1}{\eta(t)} dt < \infty.$$

By a solution of (1.1), we mean a function $x(t) \in C^1([T, \infty),)$, $T \geq t_0$, which has the property $\eta(t)x'(t) \in C^1([T, \infty))$ and satisfies (1.1) on $[T, \infty)$. We consider only those solutions $x(t)$ of (1.1) which exit on some half-line $[T, \infty)$ and the condition satisfy $\sup\{|x(t)| : t \geq T_1\} > 0$ for all $T_1 \geq T$. It is assumed that (1.1) possesses such a solution. A solution of (1.1) is called *oscillatory* if it has arbitrary large zeros on $[T, \infty)$; otherwise it is called to be *nonoscillatory*. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

Determining oscillation and nonoscillation criteria for various types of differential equations with deviating arguments has been a very active and interesting area of research over the past several decades. Many references and reviews of known results are found in the monographs [1–3, 12, 17, 21], research papers [4–11, 13–16, 18, 19] and the references cited therein.

Usually, the oscillatory properties of solutions of (1.1) are studied in so-called canonical type equations, that is, when

$$\int_{t_0}^{\infty} \frac{1}{\eta(t)} dt = \infty$$

is satisfied. However in [4, 5, 7, 14–16], the authors studied the Eq. (1.1) when (H_4) holds. In this case, it is well-known that the first derivative of any positive solution of (1.1) is of one sign eventually, that is, this solution is either increasing eventually or decreasing eventually. Thus, to obtain oscillation criteria for noncanonical equation these two possible cases have to be eliminated. Generally, the authors use two condition criteria to investigate the oscillatory behavior of noncanonical equations. Particularly in [4, 5, 15, 16], the authors combine two conditions into single-condition and obtained criteria for the oscillation of noncanonical equations.

Therefore in this paper, first the noncanonical Eq. (1.1) is converted into canonical form which simplifies the investigation of oscillatory behavior of (1.1) as in this case any positive solution of (1.1) is only increasing. The obtained criteria are in new form and complement to the existing results reported in the literature. Some particular examples are given to illustrate the main results.

2 Preliminary Results

For the sake of brevity and clarity , we use the following notations:
$w(t) = \eta(t)\Pi^2(t)$, $\quad \xi(t) = \frac{x(t)}{\Pi(t)}$,
$F(t) = \Pi(t)\Pi^\beta(\sigma(t))f(t)$, and $\Omega(t) = \int_{t_1}^t \frac{1}{w(s)}ds$ for every $t_1 \geq t_0$.

From the form of (1.1), while considering nonoscillatory solutions of (1.1), our attention is restricted to positive ones. From the well-known results in [12, 21], it follows that the set of positive solutions of (1.1) has the following structure:

Lemma 2.1 *Assume that $x(t)$ is an eventually positive solution of* (1.1). *Then $x(t)$ satisfies one of the following cases:*

(I) $x(t) > 0, \eta(t)x'(t) > 0, (\eta(t)x'(t))' < 0$;
(II) $x(t) > 0, \eta(t)x'(t) < 0, (\eta(t)x'(t))' < 0$

eventually.

To establish oscillation criteria for the noncanonical Eq. (1.1), the above mentioned two cases have to be eliminated. However, if (1.1) is transformed into canonical form, then the number of cases will be reduced to one. Therefore, this essentially simplifies the examination of (1.1).

Theorem 2.2 *The noncanonical operator $\mathcal{L}x$ has the following unique canonical representation*

$$\mathcal{L}x = \frac{1}{\Pi(t)}\left(\eta(t)\Pi^2(t)\left(\frac{x(t)}{\Pi(t)}\right)'\right)'. \tag{2.1}$$

Proof A straightforward computation reveals that

$$\eta(t)\Pi^2(t)\left(\frac{x(t)}{\Pi(t)}\right)' = \Pi(t)\eta(t)x'(t) + x(t).$$

Therefore

$$\frac{1}{\Pi(t)}\left(\eta(t)\Pi^2(t)\left(\frac{x(t)}{\Pi(t)}\right)'\right)' = (\eta(t)x'(t))'.$$

Now, we shall show that (2.1) is in canonical form, that is,

$$\int_{t_0}^{\infty} \frac{1}{\eta(t)\Pi^2(t)}dt = \lim_{t\to\infty} \frac{1}{\Pi(t)} - \frac{1}{\Pi(t_0)} = \infty.$$

Trench proved in [23] that there exists the only one canonical representation of $\mathcal{L}$ (up to multiplicative constants with product 1) and so our canonical form is unique. The theorem is proved. $\square$

Now from Theorem 2.2, the Eq. (1.1) can be written in the equivalent canonical form as

$$(w(t)\xi'(t))' + F(t)\xi^{\beta}(\sigma(t)) = 0 \tag{2.2}$$

has the solution $\xi(t) = \frac{x(t)}{\Pi(t)}$, and the following result is immediate.

Theorem 2.3 *The noncanonical differential Eq. (1.1) has a solution $x(t)$ if and only if the canonical equation (2.2) has a solution $\xi(t)$.*

Corollary 2.4 *Noncanonical differential Eq. (1.1) has an eventually positive solution if and only if the canonical Eq. (2.2) has an eventually positive solution.*

Clearly Corollary 2.4 simplifies examination of (1.1) since for (2.2), we deal with only one class of an eventually positive solution, namely,

$$\xi(t) > 0,\, w(t)\xi'(t) > 0 \text{ and } (w(t)\xi'(t))' < 0. \tag{2.3}$$

Lemma 2.5 *Assume that $\xi(t)$ is an eventually positive solution of (2.2). Then $\frac{\xi(t)}{\Omega(t)}$ is decreasing for all $t \geq t_1 \geq t_0$. Further, if*

$$\int_{t_0}^{\infty} \Omega^{\beta}(\sigma(t))F(t)dt = \infty, \tag{2.4}$$

then

$$\lim_{t\to\infty} \frac{\xi(t)}{\Omega(t)} = 0.$$

Proof Let $\xi(t)$ be a positive solution of (2.2). Then $\xi(t)$ satisfies (2.3) and so $w(t)\xi'(t) > 0$ and decreasing for all $t \geq t_0 \geq 0$. Now

$$\xi(t) \geq \int_{t_1}^{t} \frac{w(s)\xi'(s)}{w(s)} ds \geq w(t)\xi'(t)\Omega(t),$$

which implies that $\left(\frac{\xi(t)}{\Omega(t)}\right)' < 0$. On the other hand, since $\frac{\xi(t)}{\Omega(t)}$ is positive and decreasing there exists

$$\lim_{t\to\infty} \frac{\xi(t)}{\Omega(t)} = m \geq 0.$$

We claim that $m = 0$. If not, then $\frac{\xi(t)}{\Omega(t)} \geq m > 0, t \geq t_1$.

Integrating (2.2) from t_1 to t, we obtain

$$w(t_1)\xi(t_1) \geq m^{\beta} \int_{t_1}^{t} F(s)\Omega^{\beta}(\sigma(s))ds,$$

which for $t \to \infty$ contradicts with (2.4). Hence $\lim_{t\to\infty} \frac{\xi(t)}{\Omega(t)} = 0$. The lemma is proved. □

Theorem 2.6 *If (2.4) holds, then every nonoscillatory solution $x(t)$ of (1.1) satisfies*

$$\lim_{t\to\infty} \frac{x(t)}{\Pi(t)\Omega(t)} = 0.$$

Proof Assume that $x(t)$ is an eventually positive solution of (1.1). By Corollary 2.4, the corresponding function $\frac{x(t)}{\Pi(t)}$ is a positive solution of (2.2) and the result now follows from Lemma 2.5. The proof is completed. □

Since $\Omega(t)$ is increasing, there exists a $\lambda \geq 1$ such that

$$\frac{\Omega(t)}{\Omega(\sigma(t))} \geq \lambda. \tag{2.5}$$

Lemma 2.7 *Let $\beta = 1$ and (2.4) hold. If there exists a positive constant α such that*

$$\Omega(\sigma(t))w(t)\Omega(t)F(t) \geq \alpha \text{ for } t \geq t_0 \tag{2.6}$$

then for every positive solution $x(t)$ of (1.1), we have

$$\frac{x(t)}{\Pi(t)\Omega^{1-\alpha}(t)} \text{ is decreasing for } t \geq t_1,$$

and

$$\frac{x(t)}{\Pi(t)\Omega^{\alpha_0}(t)} \text{ is increasing for } t \geq t_1 \geq t_0,$$

where $\alpha_0 = \alpha\lambda^\alpha$.

Proof The proof is similar to that of Theorem 2.3 of [6] and therefore the details are omitted. □

3 Oscillation Criteria

In this section, we apply the results from the previous section to get new oscillation criteria.

Theorem 3.1 *Let $\beta > 1$. If*

$$\int_{t_1}^{\infty} \Omega^{1-\beta}(t)\Omega^{\beta}(\sigma(t))F(t)dt = \infty \tag{3.1}$$

for all $t_1 \geq t_0$, then (1.1) is oscillatory.

Proof Let $x(t)$ be an eventually positive solution of (1.1). It follows from Corollary 2.4 that $\xi(t) = \frac{x(t)}{\Pi(t)}$ is also a positive solution of (2.2) and satisfies (2.3). Define

$$R(t) = \Omega(t)\frac{w(t)\xi'(t)}{\xi^{\beta}(t)}, \quad t \geq t_0 \geq 0.$$

Then $R(t) > 0$, and using (2.2) we see that

$$\begin{aligned} R'(t) &= -\Omega(t)F(t)\frac{\xi^{\beta}(\sigma(t))}{\xi^{\beta}(t)} + \frac{\xi'(t)}{\xi^{\beta}(t)} - \frac{\beta\Omega(t)w(t)(\xi'(t))^2}{\xi^{\beta+1}(t)} \\ &\leq -\Omega^{1-\beta}(t)F(t)\Omega^{\beta}(\sigma(t)) + \frac{\xi'(t)}{\xi^{\beta}(t)}, \end{aligned} \tag{3.2}$$

where we have used $\frac{\xi(t)}{\Omega(t)}$ is deceasing for all $t \geq t_1$.

Integrating (3.2) from t_1 to t, we obtain

$$\begin{aligned} \int_{t_1}^{t} \Omega^{1-\beta}(s)\Omega^{\beta}(\sigma(s))F(s)ds &\leq R(t_1) + \int_{t_1}^{\infty} \frac{\xi'(s)}{\xi^{\beta}(s)}ds \\ &\leq R(t_1) + \frac{1}{(\beta-1)\xi^{\beta-1}(t_1)} < \infty, \end{aligned}$$

which contradicts (3.1) as $t \to \infty$. The theorem is proved. □

Theorem 3.2 *Let $\beta = 1$, (2.4) and (2.6) hold. If*

$$\int_{t_0}^{\infty} F(t)\Omega^{\alpha_0}(\sigma(t))dt = \infty \tag{3.3}$$

where α_0 is as defined in Lemma 2.7, then (1.1) is oscillatory.

Proof Let $x(t)$ be an eventually positive solution of (1.1). By Corollary 2.4, $\xi(t)$ is also a positive solution of (2.2) satisfying (2.3) for all $t \geq t_1$. Since $\frac{\xi(t)}{\Omega^{\alpha_0}(t)}$ is increasing and positive there exists $M > 0$ such that $\xi(t) \geq M\Omega^{\alpha_0}(t)$ for all $t \geq t_1$. Using this in (2.2), we obtain

$$(w(t)\xi'(t))' + MF(t)\Omega^{\alpha_0}(\sigma(t)) \leq 0, t \geq t_1.$$

Integrating the last inequality from t_1 to t, we have

$$M\int_{t_1}^{t} F(s)\Omega^{\alpha_0}(\sigma(s))ds \leq w(t_1)\xi'(t_1) < \infty,$$

which contradicts (3.3). The theorem is proved. □

Theorem 3.3 *Let* $\beta = 1$, (2.4) *and* (2.6) *hold. If there exists a positive function* $\rho(t) \in C([t_0, \infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\rho(s)\Omega^{1-\alpha}(\sigma(s))\Omega^{\alpha-1}(s)F(s) - \frac{w(s)(\rho'(s))^2}{4\rho(s)} \right] ds = \infty \tag{3.4}$$

then (1.1) *is oscillatory.*

Proof Let $x(t)$ be an eventually positive solutions of (1.1). It follows from Corollary 2.4, $\xi(t) = \frac{x(t)}{\Pi(t)}$ is also positive solution of (2.2) and satisfies (2.4). Define

$$R(t) = \rho(t)\frac{w(t)\xi'(t)}{\xi(t)}, t \geq t_1.$$

Then $R(t) > 0$ and using(2.2), we have

$$\begin{aligned} R'(t) &= -\rho(t)F(t)\frac{\xi(\sigma(t))}{\xi(t)} + \frac{\rho'(t)}{\rho(t)}R(t) - \frac{R^2(t)}{\rho(t)w(t)} \\ &\leq -\rho(t)F(t)\Omega^{1-\alpha}(\sigma(t))\Omega^{\alpha-1}(t) + \frac{w(t)\rho'(t)^2 w(t)}{4\rho}, \end{aligned}$$

where we have used $\frac{\xi(t)}{\Omega^{1-\alpha}(t)}$ is decreasing for all $t \geq t_1$. Integrating the last inequality from t_1 to t, we obtain

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\rho(s)\Omega^{1-\alpha}(\sigma(s))\Omega^{\alpha-1}(s)F(s) - \frac{w(s)(\rho'(s))^2}{4\rho(s)} \right] ds \leq R(t_1) < \infty$$

which contradicts (3.4) as $t \to \infty$. The theorem is proved. □

Theorem 3.4 *Let* $\beta > 1$. *If*

$$\begin{aligned} \limsup_{t\to\infty} \Bigg[&\frac{1}{\Omega^{\beta}(\sigma(t))} \int_{t_1}^{\sigma(t)} \Omega(t)F(s)\Omega^{\beta}(\sigma(s))ds + \Omega^{1-\beta}(\sigma(t)) \int_{\sigma(t)}^{t} F(s)\Omega^{\beta}(\sigma(s))ds \\ &+ \Omega(\sigma(t)) \int_{t}^{\infty} F(s)ds \Bigg] = \infty. \end{aligned} \tag{3.5}$$

then (1.1) *is oscillatory.*

Proof Let $x(t)$ be an eventually positive solution of (1.1). By Corollary 2.4, the function $\xi(t) = \frac{x(t)}{\Pi(t)}$ is also a positive solution of (2.2) and satisfies (2.3). An integration of (2.2) yields

$$\xi'(t) \geq \frac{1}{w(t)} \int_{t}^{\infty} F(s)\xi^{\beta}(\sigma(s))ds.$$

Integrating again, we get

$$\begin{aligned}\xi(t) &\geq \int_{t_1}^{t} \frac{1}{w(s)} \int_{s}^{\infty} F(u)\xi^{\beta}(\sigma(u))du\,ds.\\ &= \int_{t_1}^{t} \frac{1}{w(s)} \int_{s}^{t} F(u)\xi^{\beta}(\sigma(u))du\,ds + \int_{t_1}^{t} \frac{1}{w(s)} \int_{t}^{\infty} F(u)\xi^{\beta}(\sigma(u))du\,ds.\end{aligned}$$

Applying integrating by parts formula, we obtain

$$\xi(t) \geq \int_{t_1}^{t} \Omega(s)F(s)\xi^{\beta}(\sigma(s))ds + \Omega(t)\int_{t}^{\infty} F(s)\xi^{\beta}(\sigma(s))ds.$$

Hence

$$\begin{aligned}\xi(\sigma(t)) \geq \int_{t_1}^{\sigma(t)} \Omega(s)F(s)\xi^{\beta}(\sigma(s))ds &+ \Omega(\sigma(t))\int_{\sigma(t)}^{t} F(s)\xi^{\beta}(\sigma(s))ds\\ &+ \Omega(\sigma(t))\int_{t}^{\infty} F(s)\xi^{\beta}(\sigma(s))ds.\end{aligned}$$

Using the fact that $\xi(t)$ is increasing and $\frac{\xi(t)}{\Omega(t)}$ is decreasing the previous inequality implies

$$\begin{aligned}\xi(\sigma(t)) \geq &\left(\frac{\xi(\sigma(t))}{\Omega(\sigma(t))}\right)^{\beta} \int_{t_1}^{\sigma(t)} \Omega(s)F(s)\Omega^{\beta}(\sigma(s))ds\\ &+ \Omega(\sigma(t))\left(\frac{\xi(\sigma(t))}{\Omega(\sigma(t))}\right)^{\beta} \int_{\sigma(t)}^{t} F(s)\Omega^{\beta}(\sigma(s))ds\\ &+ \Omega(\sigma(t))\xi^{\beta}(\sigma(t))\int_{t}^{\infty} F(s)ds,\end{aligned}$$

or

$$\begin{aligned}\xi^{1-\beta}(\sigma(t)) \geq &\frac{1}{\Omega^{\beta}(\sigma(t))} \int_{t_1}^{\sigma(t)} \Omega(s)F(s)\Omega^{\beta}(\sigma(s))ds\\ &+ \Omega^{1-\beta}(\sigma(t))\int_{\sigma(t)}^{t} F(s)\Omega^{\beta}(\sigma(s))ds\\ &+ \Omega(\sigma(t))\int_{t}^{\infty} F(s)ds.\end{aligned} \tag{3.6}$$

Since $\xi(t)$ is increasing and $\beta > 1$, there exists a constant $M > 0$ such that $\xi^{1-\beta}(\sigma(t)) \leq M^{1-\beta}$ for all $t \geq t_1$. Using this in (3.6) and letting $t \to \infty$, we obtain a contradiction with (3.5). The theorem is proved. □

Theorem 3.5 *Let* $0 < \beta < 1$. *If*

$$\limsup_{t\to\infty}\left[\frac{1}{\Omega(\sigma(t))}\int_{t_1}^{\sigma(t)}\Omega(t)F(s)\Omega^{\beta}(\sigma(s))ds + \int_{\sigma(t)}^{t}F(s)\Omega^{\beta}(\sigma(s))ds \right. \\ \left. + \Omega^{\beta}(\sigma(t))\int_{t}^{\infty}F(s)ds\right] = \infty. \tag{3.7}$$

then (1.1) *is oscillatory.*

Proof Proceeding as in Theorem 3.4, we are led to (3.6). Dividing (3.6) by $\Omega^{1-\beta}(\sigma(t))$, we have

$$\left(\frac{\xi(\sigma(t))}{\Omega(\sigma(t))}\right)^{1-\beta} \geq \frac{1}{\Omega(\sigma(t))}\int_{t_1}^{\sigma(t)}\Omega(s)F(s)\Omega^{\beta}(\sigma(s))ds + \int_{\sigma(t)}^{t}F(s)\Omega^{\beta}(\sigma(s))ds \\ + \Omega^{\beta}(\sigma(s))\int_{s}^{\infty}F(s)ds. \tag{3.8}$$

Since $0 < \beta < 1$ and $\frac{\xi(t)}{\Omega(t)}$ is decreasing, there exists a constant $M_1 > 0$ such that$\left(\frac{\xi(\sigma(t))}{\Omega(\sigma(t))}\right)^{1-\beta} \leq M_1^{1-\beta}$ for all $t \geq t_1$. Using this in (3.8) and letting $t \to \infty$, we obtain a contradiction with (3.7). The theorem is proved. □

In our final result, we make use comparison method to get oscillation of (1.1).

Theorem 3.6 *If the first-order delay differential equation*

$$R'(t) + F(t)\Omega^{\beta}(\sigma(t))R^{\beta}(\sigma(t)) = 0 \tag{3.9}$$

is oscillatory, then (1.1) *is oscillatory.*

Proof Let $x(t)$ be an eventually positive solution of (1.1). By Corollary 2.4, the function $\xi(t) = \frac{x(t)}{\Pi(t)}$ is also a positive solution of (2.2) and satisfies (2.3). From (2) we have

$$\xi^{\beta}(\sigma(t)) \geq (w(\sigma(t))\xi'(\sigma(t)))^{\beta}\Omega^{\beta}(\sigma(t)). \tag{3.10}$$

Set $R(t) = w(t)\xi'(t)$. Then $R(t) > 0$ and from (2.2) and (3.10). We see that $R(t)$ obeys the differential inequality

$$R'(t) + F(t)\Omega^{\beta}(\sigma(t))R^{\beta}(\sigma(t)) \leq 0.$$

However, by Corollary 1 in [20] ensures that the corresponding differential Eq. (3.9) has a positive solution. This is a contradiction and the theorem is proved. □

Employing criteria for oscillation of (3.9), we immediately obtain criteria for oscillation of (1.1).

Corollary 3.7 *Let $\beta = 1$. If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} F(s)\Omega(\sigma(s))ds > \frac{1}{e} \tag{3.11}$$

then (1.1) is oscillatory.

Proof The proof follows from Theorem 2.1.1 of [17] and Theorem 3.6. □

Corollary 3.8 *Let $0 < \beta < 1$. If*

$$\int_{t_0}^{\infty} F(t)\Omega^{\beta}(\sigma(t))dt = \infty \tag{3.12}$$

then (1.1) is oscillatory.

Proof The proof follows from Theorem 3.9.3 of [17] and Theorem 3.6. □

Corollary 3.9 *Assume that $\beta > 1$ and there exists $\theta \in (0, 1)$ such that $\sigma(t) = \theta t$. If there exists*

$$\mu > -\ln(\beta)/\ln\theta$$

such that

$$\liminf_{t\to\infty} \left[F(t)\Omega^{\beta}(\sigma(t))\exp(-t^{\mu})\right] > 0 \tag{3.13}$$

holds, then (1.1) is oscillatory.

Corollary 3.10 *Assume that $\beta > 1$ and there exists $\theta \in (0, 1)$ such that $\sigma(t) = t^{\theta}$. If there exists*

$$\mu > -\ln(\beta)/\ln\theta$$

such that

$$\liminf_{t\to\infty} \left[F(t)\Omega^{\beta}(\sigma(t))\exp(-(\ln t)^{\mu})\right] > 0 \tag{3.14}$$

holds then (1.1) is oscillatory.

The proof of Corollaries 3.9 and 3.10 follow from Theorem 4 and Theorem 5 of [22] respectively.

4 Examples

In this section, we present some examples to show the applicability and the importance of the obtained results.

Example 1 Consider the second-order nonlinear delay differential equation

$$(t^3x'(t))' + \frac{t^3}{8}x^3(t/2) = 0, \;\; t \geq 1. \tag{4.1}$$

Here $\eta(t) = t^3$, $f(t) = t^3/8, \sigma(t) = t/2$ and $\beta = 3$. It is easy to verify that $\Pi(t) = \frac{1}{2t^2}$, $w(t) = \frac{1}{4t}$, $F(t) = \frac{1}{4t^5}$. The transformed equation

$$\left(\frac{1}{t}\xi'(t)\right)' + \frac{1}{t^5}\xi^3(t/2) = 0, t \geq 1$$

is canonical since $\Omega(t) = t^2/2 \to \infty$ as $t \to \infty$. The condition (2.4) becomes

$$\int_1^\infty t \; dt = \infty,$$

that is, (2.4) is satisfied. Hence by Theorem 2.6, any nonoscillatory solution $x(t)$ of (4.1) satisfies

$$\lim_{t\to\infty} \frac{x(t)}{\Pi(t)\Omega(t)} = 0.$$

In fact, one such nonoscillatory of (4.1) is $x(t) = 1/t$. Further note that the condition (3.1) of Theorem 3.1 is not satisfied.

Example 2 Consider the second-order nonlinear differential equation

$$(t^2x'(t))' + t^2x^3(t/2) = 0, \;\; t \geq 1. \tag{4.2}$$

Here $\eta(t) = t^2$, $f(t) = t^3$, $\sigma(t) = t/2$ and $\beta = 3$. A simple computation shows that $\Pi(t) = \frac{1}{t}$, $w(t) = 1$, $F(t) = \frac{8}{t^2}$ and $\Omega(t) = t$. Clearly the transformed equation

$$\xi''(t) + \frac{8}{t^2}\xi^3(t/2) = 0$$

is canonical. The condition (3.1) becomes

$$\int_1^\infty \frac{1}{t} \; dt = \infty,$$

that is, (3.1) holds. Hence Theorem 3.1 implies that (4.2) is oscillatory.

Example 3 Consider the second-order nonlinear differential equation

$$(t^2x'(t))' + t^{1/3}x^{1/3}(t/2) = 0, \quad t \geq 1. \tag{4.3}$$

Here $\eta(t) = t^2$, $f(t) = t^{1/3}$, $\sigma(t) = t/2$ and $\beta = 1/3$. It is easy to verify that $\Pi(t) = \frac{1}{t}$, $w(t) = 1$, $\Omega(t) = t$, $F(t) = \frac{2^{1/3}}{t}$. Clearly the transformed equation

$$\xi''(t) + \frac{2^{1/3}}{t}\xi^{1/3}(t/2) = 0$$

is canonical. The condition (3.12) becomes

$$\int_1^\infty \frac{1}{t^{2/3}}\,dt = \infty,$$

that is, (3.12) is satisfied. Hence Corollary 3.8 implies that Eq. (4.3) is oscillatory.

Example 4 Consider the second-order linear differential equation

$$(t^2x'(t))' + q_0x(t/2) = 0, \quad t \geq 1, \tag{4.4}$$

where $q_0 > 0$. Here $\eta(t) = t^2$, $f(t) = q_0$, $\sigma(t) = t/2$ and $\beta = 1$. It is easy to verify that $\Pi(t) = \frac{1}{t}$, $w(t) = 1$, $F(t) = 2q_0/t^2$ and $\Omega(t) = t$. The transformed equation

$$\xi''(t) + \frac{2q_0}{t^2}\xi(t/2) = 0$$

is in canonical form . Clearly the condition (2.5) holds and the condition (2.8) holds if $q_0 \geq \alpha \in (0, 1)$. By taking $\rho(t) = t$, we see that condition (3.4) is satisfied if $q_0 > \frac{1}{2^{\alpha+2}}$. Hence by Theorem 3.3, the Eq. (4.4) is oscillatory if $q_0 > max\{\alpha, \frac{1}{2^{\alpha+2}}\}$.

Let $\alpha = 0.15$. Equation (4.4) is oscillatory by Theorem 3.3 if $q_0 > 0.2253$. This equation is oscillatory by Theorem 1 of [5] if $q_0 > 0.37$, by Theorem 3 of [15] if $q_0 > 1$, by Theorem 3.2 of [4] if $q_0 > 0.5$ and by Theorem 2.7 of [16] if $q_0 > 0.25$. Hence our Theorem 3.3 improved the above mentioned theorems in [4, 5, 15, 16].

5 Conclusion

The results obtained in [4, 5, 7, 14–16] cannot be applied to Eqs. (4.1) to (4.3) since the equations are not linear or half-linear. Further we have obtained the oscillation criteria of (1.1) using canonical transformation technique, and therefore the results presented in this paper are different from those reported in [4, 5, 7, 14–16]. Moreover, we will study the case $\sigma(t) \geq t$ in the future work using this method.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author declares that they have no conflict of interest.

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