



Higher order computational method for a singularly perturbed nonlinear system of differential equations

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Abstract

In this article, a boundary value problem for a nonlinear system of singularly perturbed two second order differential equations in which only the first equation is multiplied by a small positive parameter is considered. The first component of the solution exhibits boundary layers whereas the second component exhibits less-severe layers. A numerical method composed of a classical finite difference scheme applied on a piecewise uniform Shishkin mesh is suggested to solve the system. The method is proved to be essentially second order convergent in the maximum norm uniformly with respect to the perturbation parameter. Numerical illustration presented supports the proved theoretical results.

Keywords Singular perturbation problems · Boundary layers · System of nonlinear differential equations · Finite difference scheme · Shishkin mesh · Parameter-uniform convergence

Mathematics Subject Classification 65L11 · 65L12 · 65L20 · 65L70

1 Introduction

Singular perturbation problems are widespread in nature. For instance, these problems arise in various fields of applied mathematics such as fluid dynamics [1] and control systems [14]. Systems of singularly perturbed nonlinear reaction–diffusion equations

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arise, for example, in catalytic reaction theory [2]. Classical numerical methods are not suitable for these problems due to the multiscale behaviour of the solutions [1].

Various numerical methods for several scalar singularly perturbed nonlinear differential equations are reported in [3–8]. Different systems of singularly perturbed semilinear reaction–diffusion equations are solved asymptotically by Jeffries [9] and numerically by Shishkina and Shishkin [10]. In [11], a third order uniformly convergent numerical method consisting of a finite difference scheme of Hermite type with standard central difference on a piecewise-uniform Shishkin mesh is developed for a semilinear system of singularly perturbed second order differential equations with same perturbation parameters. Article [12] deals with the construction of a first order convergent numerical method for a semilinear system of singularly perturbed differential equations with different perturbation parameters.

Recently, Manikandan and Tamilselvan [13] have developed a second order parameter-uniform convergent numerical method for a semilinear system of singularly perturbed second order differential equations of reaction–diffusion type with different perturbation parameters.

All the singular perturbation parameters associated with a system of differential equations need not have small magnitudes. For example, a two-time scale system in control systems can be modeled by a system of two differential equations [14] in which the coefficient of the highest derivative term in one equation is a very small positive parameter whereas the coefficient of the highest derivative term in the other equation is one. In [15], a first order convergent numerical method is constructed for a system of singularly perturbed differential equations of reaction diffusion in which only the first equation is multiplied by a small parameter ε . For this same problem a second order convergent numerical method is developed in [16]. Induced by the works reported in [15,16], a second order convergent numerical method based on classical finite difference operator is designed in [17] for a general system of $n(n \geq 2)$ singularly perturbed differential equations of reaction diffusion type in which only the first $m(m < n)$ equations are perturbed. Motivated by [13,15–17], in the present article an essentially second order parameter-uniform convergent computational method is developed for a nonlinear system of singularly perturbed second order differential equations of reaction–diffusion type.

The following system of partially singularly perturbed nonlinear differential equations is considered in this article.

$$\mathbf{T}\mathbf{u}(x) := -E \mathbf{u}''(x) + \mathbf{f}(x, \mathbf{u}) = \mathbf{0} \text{ on } \Omega = (0, 1) \quad (1)$$

$$\text{with } \mathbf{u}(0) = \boldsymbol{\beta} \text{ and } \mathbf{u}(1) = \boldsymbol{\mu} \quad (2)$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$ and $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$ are constant vectors. For all $x \in \overline{\Omega} = [0, 1]$, $\mathbf{u}(x) = (u_1(x), u_2(x))^T$, $\mathbf{f}(x, \mathbf{u}) = (f_1(x, \mathbf{u}), f_2(x, \mathbf{u}))^T \in C^4(\overline{\Omega} \times \mathbb{R}^2)$. E is a 2×2 diagonal matrix with diagonal elements $\varepsilon_1, \varepsilon_2$ such that $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$. It is assumed that for all $(x, \mathbf{u}) \in \overline{\Omega} \times \mathbb{R}^2$, the nonlinear terms satisfy

$$\frac{\partial f_k(x, \mathbf{u})}{\partial u_j} \leq 0, \quad k, j = 1, 2 \text{ and } k \neq j, \quad (3)$$

$$\min_{\substack{x \in \bar{\Omega} \\ i=1,2}} \left(\sum_{j=1}^2 \frac{\partial f_i(x, \mathbf{u})}{\partial u_j} \right) \geq \alpha > 0, \text{ for a constant } \alpha. \quad (4)$$

The implicit function theorem together with conditions (3) and (4) ensures the existence of a unique solution $\mathbf{u}$ to the problem (1)–(2) such that $\mathbf{u} \in (C^4(\bar{\Omega}))^2$.

Based on the magnitudes of the parameters ε_1 and ε_2 the following four cases arise.

$$\begin{aligned} \text{Case (i):} & \quad 0 < \varepsilon_1 < \varepsilon_2 < 1 \\ \text{Case (ii):} & \quad 0 < \varepsilon_1 < \varepsilon_2 = 1 \\ \text{Case (iii):} & \quad 0 < \varepsilon_1 = \varepsilon_2 = 1 \\ \text{Case (iv):} & \quad 0 < \varepsilon_1 = \varepsilon_2 < 1. \end{aligned} \quad (5)$$

Case (i) is explained in [13] in detail. The problem is unperturbed with respect to Case (iii). The first component u_1 alone exhibits boundary layers with respect to Case (ii). Both the components exhibit boundary layers of same width with respect to Case (iv). Hence the problem with respect to Case (iv) can be solved by following similar technique which is used to solve the problem with respect to Case (ii). Thus Case (ii) is discussed elaborately in this article. With respect to Case (ii), the problem (1)–(2) is rewritten as follows

$$\mathbf{T}\mathbf{u}(x) := -E \mathbf{u}''(x) + \mathbf{f}(x, \mathbf{u}) = \mathbf{0} \text{ on } \Omega \quad (6)$$

$$\text{with } \mathbf{u}(0) = \boldsymbol{\beta} \text{ and } \mathbf{u}(1) = \boldsymbol{\mu} \quad (7)$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$ and $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$ are constant vectors. E is a 2×2 diagonal matrix with diagonal elements $\varepsilon, 1$ such that $0 < \varepsilon < 1$.

The reduced problem (obtained by putting $\varepsilon = 0$) corresponding to (6)–(7) is defined by

$$f_1(x, \mathbf{r}) = 0, \quad (8)$$

$$-r_2''(x) + f_2(x, \mathbf{r}) = 0, \text{ on } \Omega, \quad r_2(0) = \beta_2, r_2(1) = \mu_2 \quad (9)$$

where $\mathbf{r} = (r_1, r_2)^T$. Conditions (3) and (4) together with the implicit function theorem ensure the existence of a unique solution for (8)–(9). Moreover, $\mathbf{r}$ and its derivatives are bounded independently of ε . Hence,

$$|r_l^{(k)}(x)| \leq C \text{ for } l = 1, 2, \quad k = 0, 1, 2 \text{ and } x \in \bar{\Omega}. \quad (10)$$

Throughout the article C, C_1 and C_2 indicate positive constants, which are free from x, ε and the discretization parameter(N).

2 Analytical results

Shishkin decomposition of the solution $\mathbf{u}$ of (6)–(7) is considered in the following form

$$\mathbf{u}(x) = \mathbf{v}(x) + \mathbf{w}(x)$$

where $\mathbf{v}(x)$ is the solution of

$$-E \mathbf{v}''(x) + \mathbf{f}(x, \mathbf{v}) = \mathbf{0} \text{ on } \Omega, \quad \mathbf{v}(0) = \mathbf{r}(0), \quad \mathbf{v}(1) = \mathbf{r}(1) \quad (11)$$

and $\mathbf{w}(x)$ is the solution of

$$-E \mathbf{w}''(x) + \mathbf{f}(x, \mathbf{v} + \mathbf{w}) - \mathbf{f}(x, \mathbf{v}) = \mathbf{0} \text{ on } \Omega, \quad (12)$$

$$\mathbf{w}(0) = \mathbf{u}(0) - \mathbf{v}(0), \quad \mathbf{w}(1) = \mathbf{u}(1) - \mathbf{v}(1). \quad (13)$$

In the following lemma the bounds of $\mathbf{v}(x)$ and its derivatives are established.

Lemma 1 For all $x \in \overline{\Omega}$,

$$|v_i^{(j)}(x)| \leq C, \text{ for } i = 1, 2 \text{ and } j = 0, 1, 2,$$

$$|v_1^{(k)}(x)| \leq C \left(1 + \varepsilon^{1-\frac{k}{2}}\right) \text{ and } |v_2^{(k)}(x)| \leq C, \text{ for } k = 3, 4.$$

Proof For convenience, $\mathbf{v}$ is further decomposed as follows

$$\mathbf{v} = \tilde{\mathbf{q}} + \hat{\mathbf{q}}$$

where $\hat{\mathbf{q}}$ is the solution of

$$f_1(x, \hat{\mathbf{q}}) = 0, \quad (14)$$

$$-\frac{d^2 \hat{q}_2}{dx^2} + f_2(x, \hat{\mathbf{q}}) = 0, \quad x \in \Omega, \quad (15)$$

$$\hat{q}_2(0) = \hat{q}_2(1) = 0 \quad (16)$$

and $\tilde{\mathbf{q}}$ is the solution of

$$-\varepsilon \frac{d^2 \tilde{q}_1}{dx^2} + f_1(x, \tilde{\mathbf{q}} + \hat{\mathbf{q}}) - f_1(x, \hat{\mathbf{q}}) = \varepsilon \frac{d^2 \hat{q}_1}{dx^2}, \quad (17)$$

$$-\frac{d^2 \tilde{q}_2}{dx^2} + f_2(x, \tilde{\mathbf{q}} + \hat{\mathbf{q}}) - f_2(x, \hat{\mathbf{q}}) = 0, \quad x \in \Omega, \quad (18)$$

$$\tilde{q}_1(0) = v_1(0), \quad \tilde{q}_1(1) = v_1(1), \quad \tilde{q}_2(0) = v_2(0), \quad \tilde{q}_2(1) = v_2(1). \quad (19)$$

Let $x \in \overline{\Omega}$. Using (8) and (14),

$$\theta_{11}(x)(\hat{q}_1 - r_1) + \theta_{12}(x)(\hat{q}_2 - r_2) = 0 \quad (20)$$

and using (9) and (15),

$$-\frac{d^2}{dx^2}(\hat{q}_2 - r_2) + \theta_{21}(x)(\hat{q}_1 - r_1) + \theta_{22}(x)(\hat{q}_2 - r_2) = 0 \quad (21)$$

where $\theta_{ij}(x) = \frac{\partial f_i}{\partial u_j}(x, \chi_{f_i}(x))$, $i, j = 1, 2$, are intermediate values.

Using (20) in (21),

$$-\frac{d^2}{dx^2}(\hat{q}_2 - r_2) + \left(\theta_{22}(x) - \frac{\theta_{12}(x)\theta_{21}(x)}{\theta_{11}(x)} \right) (\hat{q}_2 - r_2) = 0.$$

Consider the linear operator,

$$L\psi(x) := -\psi''(x) + \left(\theta_{22}(x) - \frac{\theta_{12}(x)\theta_{21}(x)}{\theta_{11}(x)} \right) \psi(x) = 0 \quad (22)$$

where $\psi = \hat{q}_2 - r_2$. The operator L satisfies the maximum principle in Chapter 6 of [18]. Thus, $\|\hat{q}_2 - r_2\| \leq C$ and $\left\| \frac{d^2(\hat{q}_2 - r_2)}{dx^2} \right\| \leq C$. Using the mean value theorem, $|(\hat{q}_2 - r_2)'(x)| = |\psi'(x)| \leq C$. Differentiating (22) with respect to x once and twice and using the bounds of ψ , ψ' and ψ'' , we get $|\psi'''(x)| \leq C$ and $|\psi^{(iv)}(x)| \leq C$. Using the bound of ψ in (20), $\|\hat{q}_1 - r_1\| \leq C$. Hence

$$\|\hat{q}_2^{(k)}\| \leq C, \text{ for } k = 0, 1, 2, 3, 4 \text{ and } \|\hat{q}_1\| \leq C.$$

Differentiating (20) with respect to x once, twice, thrice and four times and using the estimates of $\|\hat{q}_2^{(k)}\| \leq C$, $k = 1, 2, 3, 4$, we get

$$\|\hat{q}_1^{(k)}\| \leq C, \text{ for } k = 1, 2, 3, 4.$$

From (17)–(19),

$$\begin{aligned} -\varepsilon \frac{d^2 \tilde{q}_1}{dx^2} + \theta_{11}^*(x) \tilde{q}_1 + \theta_{12}^*(x) \tilde{q}_2 &= \varepsilon \frac{d^2 \hat{q}_1}{dx^2}, \\ -\frac{d^2 \tilde{q}_2}{dx^2} + \theta_{21}^*(x) \tilde{q}_1 + \theta_{22}^*(x) \tilde{q}_2 &= 0, \\ \tilde{q}_1(0) = v_1(0), \tilde{q}_1(1) = v_1(1), \tilde{q}_2(0) = v_2(0), \tilde{q}_2(1) = v_2(1) \end{aligned} \quad (23)$$

where $\theta_{ij}^*(x) = \frac{\partial f_i}{\partial u_j}(x, \eta_{f_i}(x))$, $i, j = 1, 2$, are intermediate values. From (23), for $i = 1, 2$,

$$\left\| \frac{d^k \tilde{q}_i}{dx^k} \right\| \leq C, \quad k = 0, 1, 2, \quad \left\| \frac{d^k \tilde{q}_2}{dx^k} \right\| \leq C \quad \text{and}$$

$$\left\| \frac{d^k \tilde{q}_1}{dx^k} \right\| \leq C \left(1 + \varepsilon^{1-\frac{k}{2}} \right), \quad k = 3, 4.$$

Hence from the bounds for $\tilde{\mathbf{q}}$ and $\hat{\mathbf{q}}$, the required bounds of $\mathbf{v}$ follow. $\square$

From (12),

$$-\varepsilon w_1''(x) + \lambda_{11}(x)w_1(x) + \lambda_{12}(x)w_2(x) = 0, \quad (24)$$

$$-w_2''(x) + \lambda_{21}(x)w_1(x) + \lambda_{22}(x)w_2(x) = 0, \quad x \in \Omega \quad (25)$$

where $\lambda_{ij}(x) = \frac{\partial f_i}{\partial u_j}(x, \phi_{f_i}(x))$, $i, j = 1, 2$, are intermediate values.

The component $\mathbf{w}$ is given a further decomposition

$$\mathbf{w}(x) = \mathbf{w}^l(x) + \mathbf{w}^r(x) \quad (26)$$

$$\text{satisfying } -E(\mathbf{w}^l)''(x) + \Lambda(x)\mathbf{w}^l(x) = \mathbf{0}, \quad x \in \Omega \quad (27)$$

with $\mathbf{w}^l(0) = \mathbf{w}(0)$, $\mathbf{w}^l(1) = \mathbf{0}$ and

$$-E(\mathbf{w}^r)''(x) + \Lambda(x)\mathbf{w}^r(x) = \mathbf{0}, \quad x \in \Omega \quad (28)$$

with $\mathbf{w}^r(0) = \mathbf{0}$, $\mathbf{w}^r(1) = \mathbf{w}(1)$ where $\Lambda(x) = \begin{bmatrix} \lambda_{11}(x) & \lambda_{12}(x) \\ \lambda_{21}(x) & \lambda_{22}(x) \end{bmatrix}$.

The layer functions B^l , B^r and B related with the solution $\mathbf{u}$ of (6)–(7) are defined by

$$\begin{aligned} B^l(x) &= e^{-x\sqrt{\alpha}/\sqrt{\varepsilon}}, \quad B^r(x) = e^{-(1-x)\sqrt{\alpha}/\sqrt{\varepsilon}}, \\ B(x) &= B^l(x) + B^r(x) \text{ on } \overline{\Omega}. \end{aligned}$$

In the following lemma the bounds of $\mathbf{w}^l$ and its derivatives are established. Analogous results hold for $\mathbf{w}^r$ and its derivatives by following similar procedure with $1-x$ instead of x .

Lemma 2 For any $x \in \overline{\Omega}$,

$$\begin{aligned} |w_1^l(x)| &\leq C_1 B^l(x) + C_2 \varepsilon (1 - B^l(x)), \\ |(w_1^l)^{(k)}(x)| &\leq C \varepsilon^{-k/2} B^l(x), \quad k = 1, 2, \\ |(w_1^l)^{(3)}(x)| &\leq C \varepsilon^{-3/2} B^l(x), \quad |\varepsilon (w_1^l)^{(4)}(x)| \leq C \varepsilon^{-1} B^l(x) \end{aligned}$$

and

$$\begin{aligned} |w_2^l(x)| &\leq C_2 \varepsilon (1 - B^l(x)), \\ |(w_2^l)^{(k)}(x)| &\leq C_1 B^l(x) + C_2 \varepsilon (1 - B^l(x)), \quad k = 1, 2, \\ |(w_2^l)^{(k)}(x)| &\leq C \varepsilon^{-(k-2)/2} B^l(x), \quad k = 3, 4. \end{aligned}$$

Proof From (27) we find that the defining equations for $\mathbf{w}^l$ are same as those in [17]. Hence the bounds of $\mathbf{w}^l$ and its derivatives can be derived as in [17]. $\square$

3 The Shishkin mesh and the discrete problem

On the interval $\overline{\Omega}$ a piecewise uniform Shishkin mesh with N mesh-intervals is now constructed as follows. Let $\Omega^N = \{x_j\}_{j=1}^{N-1}$ then $\overline{\Omega}^N = \{x_j\}_{j=0}^N$. The interval $\overline{\Omega}$ is subdivided into 3 sub-intervals as follows: $[0, \tau] \cup (\tau, 1 - \tau] \cup (1 - \tau, 1]$. The transition parameter τ , separating the uniform meshes, is defined by

$$\tau = \min \left\{ \frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\sqrt{\alpha}} \ln N \right\}. \quad (29)$$

From the total N mesh points, $\frac{N}{2}$ mesh points are placed on the outer domain $(\tau, 1 - \tau]$ uniformly and on each of the inner domains $[0, \tau]$ and $(1 - \tau, 1]$ a uniform fine mesh of $\frac{N}{4}$ mesh points is placed.

The discrete problem corresponding to (6)–(7) is defined to be

$$\mathbf{T}^N \mathbf{U}(x_j) = -E \delta^2 \mathbf{U}(x_j) + \mathbf{f}(x_j, \mathbf{U}(x_j)) = \mathbf{0}, \quad \text{for } x_j \in \Omega^N, \quad (30)$$

$$\mathbf{U}(x_0) = \mathbf{u}(x_0) \quad \text{and} \quad \mathbf{U}(x_N) = \mathbf{u}(x_N). \quad (31)$$

$$\text{Here } \delta^2 V(x_j) = \frac{(D^+ - D^-)V(x_j)}{\bar{h}_j}, \quad D^+ V(x_j) = \frac{V(x_{j+1}) - V(x_j)}{h_{j+1}}, \quad D^- V(x_j) = \frac{V(x_j) - V(x_{j-1}))}{h_j},$$

$$h_j = x_j - x_{j-1}, \quad \bar{h}_j = \frac{h_{j+1} + h_j}{2}, \quad \bar{h}_0 = \frac{h_1}{2} \quad \text{and} \quad \bar{h}_N = \frac{h_N}{2}.$$

4 Error analysis

Let Ψ and Θ be any two mesh functions defined on $\overline{\Omega}^N$. Then for any $x_j \in \Omega^N$, we have

$$\begin{aligned} & (\mathbf{T}^N \Psi - \mathbf{T}^N \Theta)(x_j) \\ &= -E \delta^2 (\Psi - \Theta)(x_j) + \mathbf{f}(x_j, \Psi(x_j)) - \mathbf{f}(x_j, \Theta(x_j)) \\ &= -E \delta^2 (\Psi - \Theta)(x_j) + \frac{\partial \mathbf{f}}{\partial u_1}(x_j, \zeta(x_j))(\Psi_1 - \Theta_1)(x_j) \\ &\quad + \frac{\partial \mathbf{f}}{\partial u_2}(x_j, \zeta(x_j))(\Psi_2 - \Theta_2)(x_j) \\ &= (\mathbf{T}^N)'(\Psi - \Theta)(x_j) \end{aligned} \quad (32)$$

where $\frac{\partial \mathbf{f}}{\partial u_i}(x_j, \zeta(x_j))$, $i = 1, 2$, are intermediate values and $(\mathbf{T}^N)'$ is the Frechet derivative of the nonlinear operator $\mathbf{T}^N$. Since $(\mathbf{T}^N)'$ is a linear operator, it satisfies the discrete maximum principle presented in [17]. Hence,

$$\|\Psi - \Theta\| \leq C \|(\mathbf{T}^N)'(\Psi - \Theta)\| = C \|\mathbf{T}^N \Psi - \mathbf{T}^N \Theta\| \quad \text{on } \Omega^N. \quad (33)$$

Lemma 3 Let $\mathbf{u}$ be the solution of (6)–(7) and $\mathbf{U}$ be the solution of (30)–(31). Then,

$$\|\mathbf{U} - \mathbf{u}\| \leq C \left(N^{-1} \ln N \right)^2. \quad (34)$$

Proof From (33), we have

$$\|\mathbf{U} - \mathbf{u}\| \leq C \|\mathbf{T}^N \mathbf{U} - \mathbf{T}^N \mathbf{u}\|.$$

Consider,

$$\|\mathbf{T}^N \mathbf{u}\| = \|\mathbf{T}^N \mathbf{u} - \mathbf{T}^N \mathbf{U}\|.$$

Hence,

$$\begin{aligned} \|\mathbf{T}^N \mathbf{u} - \mathbf{T}^N \mathbf{U}\| &= \|\mathbf{T}^N \mathbf{u}\| \\ &= \|\mathbf{T}^N \mathbf{u} - \mathbf{T} \mathbf{u}\| \\ &= E \|\delta^2 \mathbf{u} - \mathbf{u}''\|(x_j) \\ &\leq E (\|\delta^2 \mathbf{v} - \mathbf{v}''\|(x_j) + \|\delta^2 \mathbf{w} - \mathbf{w}''\|(x_j)). \end{aligned}$$

We note that the bounds for $\mathbf{v}$ and $\mathbf{w}$ are the same as those in [17]. Hence for $x_j \in \overline{\Omega}^N$, the required result (34) follows by using the same arguments in [17] to the linear operator $(\mathbf{T}^N)'$. $\square$

5 The continuation method

System of nonlinear differential equations in (6)–(7) is modified to an artificial system of nonlinear partial differential equations as follows.

$$\begin{aligned} \frac{\partial \mathbf{u}(x, t)}{\partial t} - E \frac{\partial^2 \mathbf{u}(x, t)}{\partial x^2} + \mathbf{f}(x, \mathbf{u}(x, t)) &= \mathbf{0}, \quad (x, t) \in (0, 1) \times (0, T], \\ \mathbf{u}(0, t) = \mathbf{u}(0), \quad \mathbf{u}(1, t) &= \mathbf{u}(1), \quad t \geq 0 \quad \text{and} \quad \mathbf{u}(x, 0) = \mathbf{u}_{init}(x), \quad 0 < x < 1. \end{aligned} \quad (35)$$

The continuation method mentioned for a scalar nonlinear differential equation in [1] is modified appropriately for a system of nonlinear differential equations as given below which is used to solve (35). For $j = 1, \dots, N$ and $k = 1, \dots, K$,

$$D_t^- \mathbf{U}(x_j, t_k) - E \delta_x^2 \mathbf{U}(x_j, t_k) + \mathbf{f}(x_j, \mathbf{U}(x_j, t_{k-1})) = \mathbf{0}, \quad (36)$$

$$\begin{aligned} \mathbf{U}(0, t_k) &= \mathbf{u}(0), \quad \mathbf{U}(1, t_k) = \mathbf{u}(1) \quad \text{for all } k \quad \text{and} \\ \mathbf{U}(x_j, 0) &= \mathbf{u}_{init}(x_j) \quad \text{for all } x_j \in \overline{\Omega}^N \end{aligned} \quad (37)$$

$$\begin{aligned} \text{where } \delta_x^2 V(x_j, t_k) &= \frac{(D_x^+ - D_x^-)V(x_j, t_k)}{h_j}, \quad D_x^+ V(x_j, t_k) = \frac{V(x_{j+1}, t_k) - V(x_j, t_k)}{h_{j+1}}, \\ D_x^- V(x_j, t_k) &= \frac{V(x_j, t_k) - V(x_{j-1}, t_k)}{h_j}, \quad D_t^- V(x_j, t_k) = \frac{V(x_j, t_k) - V(x_j, t_{k-1})}{h_t}. \end{aligned}$$

The initial guess $\mathbf{u}_{init}(x)$ is taken to be $\mathbf{u}(0) + x(\mathbf{u}(1) - \mathbf{u}(0))$. The choices of the step size $h_t = t_k - t_{k-1}$ and the number of iterations K are determined as follows. Define, for each $i = 1, 2$,

$$\begin{aligned} err_i(k) &= \max_{1 \leq j \leq N} \left(\frac{|U_i(x_j, t_k) - U_i(x_j, t_{k-1})|}{h_t} \right) \text{ for } k = 1, \dots, K, \\ err(k) &= \max_{i=1,2} err_i(k). \end{aligned} \quad (38)$$

The step size h_t is chosen sufficiently small so that the error decreases with the increasing k . Precisely, we choose h_t such that

$$err(k) \leq err(k-1) \text{ for all } k, 1 < k \leq K. \quad (39)$$

The number of iterations K is based on the condition that

$$err(K) \leq tol \quad (40)$$

where tol is a prescribed small tolerance. Methodology given below is used to evaluate the numerical solution for (35).

Algorithm:

- Step 1: Start from t_0 with initial step size $h_t = 1.0$.
- Step 2: Suppose at some value of k , condition (39) is not satisfied, then leave the present step and start from the previous step t_{k-1} with h_t as $h_t/2$ and then continue halving the step size h_t until finding a h_t for which condition (39) is satisfied.
- Step 3: If condition (39) is satisfied at each step h_t , then continue the process until either condition (40) is satisfied or $K = 100$.
- Step 4: If condition (40) is not satisfied, then it is assumed that the stepping process is stalled because of the choice of a large initial step. In such a case, the entire process is repeated from t_0 by halving the initial step size h_t to $h_t/2$.
- Step 5: If condition (40) is satisfied, then the final values of $\mathbf{U}(x_j, t_K)$ are taken as the numerical approximations to the solution of the corresponding continuous problem.

6 Numerical illustration

In order to show the efficiency and accuracy of the proposed method for the system of partially singularly perturbed nonlinear second order differential equations, a numerical example is presented in this section.

Example 1 Consider the BV

$$\begin{aligned} -E \mathbf{u}''(x) + \mathbf{f}(x, \mathbf{u}) &= \mathbf{0}, \text{ for } x \in (0, 1), \\ \mathbf{u}(0) &= (0.5, 0.1)^T \text{ and } \mathbf{u}(1) = (0.5, 0.1)^T \end{aligned}$$

Table 1 Values of D^N , p^N & C_p^N for $\alpha = 0.9$.

ϵ	Number of mesh points N					
	64	128	256	512	1024	2048
2^{-1}	5.9067e-04	3.0205e-04	1.5272e-04	7.6787e-05	3.8500e-05	1.9277e-05
2^{-2}	1.4046e-04	7.6773e-05	4.0060e-05	2.0453e-05	1.9277e-05	5.1930e-06
2^{-3}	2.1591e-04	1.0771e-04	5.3798e-05	2.6882e-05	1.3436e-05	6.7171e-06
2^{-4}	3.3474e-04	1.6711e-04	8.3488e-05	4.1729e-05	2.0860e-05	1.0429e-05
2^{-5}	4.2400e-04	2.1180e-04	1.0584e-04	5.2903e-05	2.6447e-05	1.3222e-05
2^{-6}	4.8974e-04	2.4462e-04	1.2224e-04	6.1104e-05	3.0548e-05	1.5273e-05
2^{-7}	7.3474e-04	2.6857e-04	1.3425e-04	6.7106e-05	3.3548e-05	1.6773e-05
2^{-8}	1.4441e-03	3.7334e-04	1.4299e-04	7.1475e-05	3.5734e-05	1.7866e-05
2^{-9}	2.6552e-03	7.3476e-04	1.8773e-04	7.4641e-05	3.7315e-05	1.8657e-05
2^{-10}	2.2417e-03	7.1854e-04	3.7334e-04	9.4215e-05	3.8454e-05	1.9226e-05
2^{-11}	2.2417e-03	1.1972e-03	5.1086e-04	2.1862e-04	7.9713e-05	1.9634e-05
2^{-12}	2.2417e-03	1.1972e-03	5.1086e-04	2.1862e-04	8.2674e-05	2.8328e-05
2^{-13}	2.2417e-03	1.1972e-03	5.1086e-04	2.1862e-04	8.2674e-05	2.8328e-05
D^N	2.6552e-03	1.1972e-03	5.1086e-04	2.1862e-04	8.2674e-05	2.8328e-05
p^N	1.1492	1.2287	1.2245	1.4029	1.5452	
C_p^N	3.1128e-01	2.7580e-01	2.3126e-01	1.9448e-01	1.4452e-01	9.7306e-02

where $E = \text{diag}(\epsilon, 1)$ and $\mathbf{f}(x, \mathbf{u}) = ((u_1(x))^5 + 2u_1(x) - \sin(0.1)u_2(x), (u_2(x))^3 + 2u_2(x) - u_1(x))^T$.

Problem in Example 1 is solved by the continuation method constructed in Sect. 5 for a system of partially singularly perturbed nonlinear differential equations. The tolerance 'tol' is taken to be 0.00001.

The ϵ -uniform order of convergence and the ϵ -uniform error constant are computed using the general methodology from [1]. The notations D^N , p^N and C_p^N bear the same meaning as in [1].

The maximum pointwise errors and the rate of convergence for the above BVP are presented in Table 1. The solution of this problem for $N = 128$ and $\epsilon = 2^{-3}, 2^{-6}, 2^{-9}, 2^{-14}$ is portrayed in Fig. 1.

7 Conclusion

From the table, it is evident that the maximum pointwise error (D^N) decreases when the number of mesh points (N) increases and the maximum pointwise error stabilizes for each N as ϵ approaches zero. Further, from the table, we also observe that the proposed method is almost second order parameter-uniform convergent. This is in agreement with Lemma 3.

From Fig. 1 we observe that the component u_1 of the solution exhibits boundary layers at both the boundaries $x = 0$ and $x = 1$ whereas the component u_2 remains

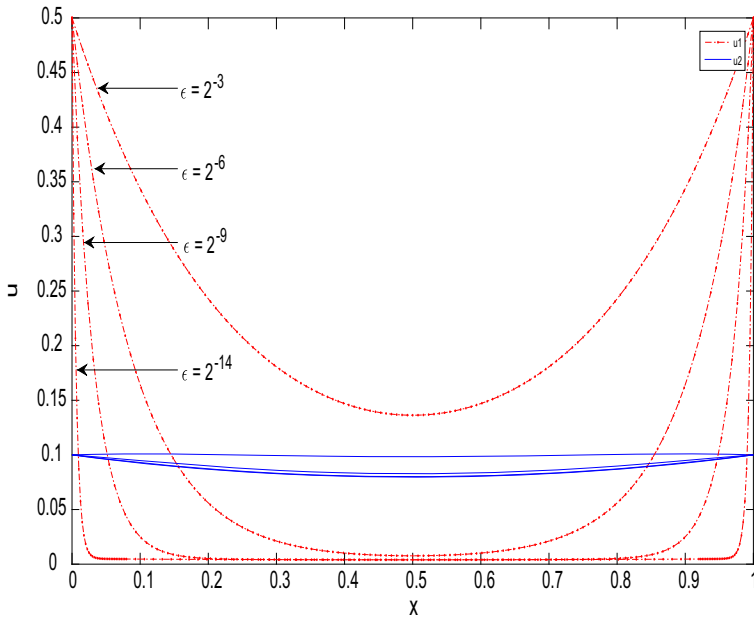


Fig. 1 Solution profile

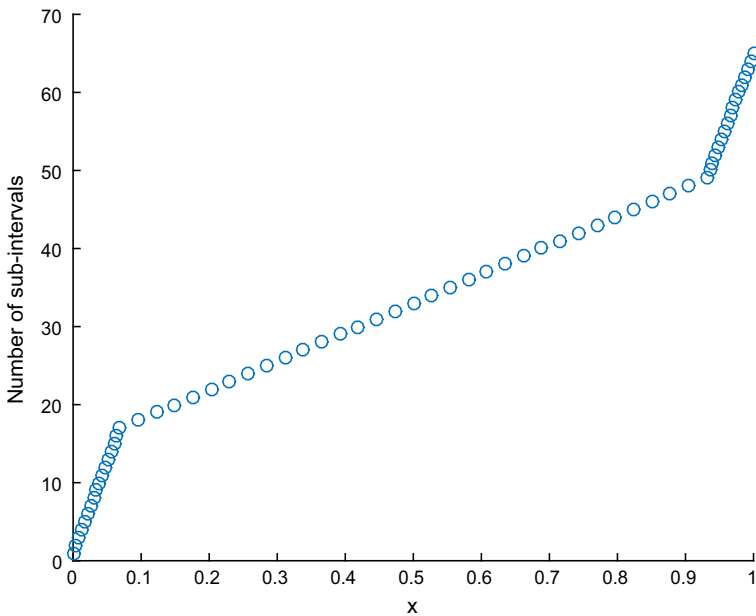


Fig. 2 Decomposition of $[0, 1]$ for $N = 64$ and $\epsilon = 2^{-14}$

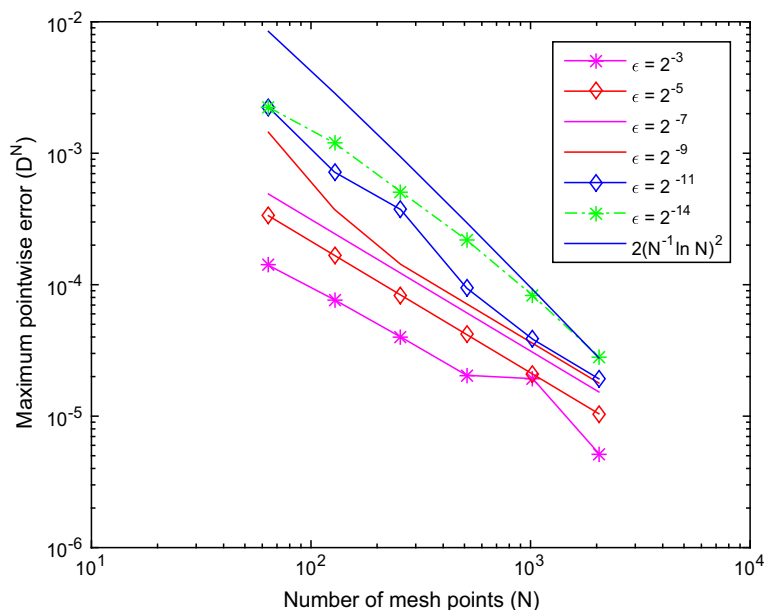


Fig. 3 *log-log plot for the error*

smooth throughout the domain. Moreover, we perceive that the component u_1 in Fig. 1 changes very rapidly near the boundaries as ε approaches zero.

The decomposition of the domain $[0, 1]$ for $N = 64$ and $\varepsilon = 2^{-14}$ is portrayed in Fig. 2. It is evident from Fig. 2 that, for very small values of the parameter ε , the domain $[0, 1]$ is decomposed into sub-intervals of different lengths and the mesh is dense near the boundaries and coarse away from the boundaries.

The *Log-log plot* for the error in the suggested numerical method for the above problem is presented in Fig. 3. From Fig. 3 we perceive that the maximum pointwise errors are bounded by $2(N^{-1} \ln N)^2$ which is proved in Lemma 3.

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