Journal of Applied Mathematics and Computing (2022) 68:1069–1086
https://doi.org/10.1007/s12190-021-01559-0

ORIGINAL RESEARCH

An efficient numerical method for a nonlinear system of singularly perturbed differential equations arising in a two-time scale system

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Received: 7 January 2021 / Revised: 1 March 2021 / Accepted: 17 May 2021 / Published online: 20 May 2021

Abstract
In this article, a nonlinear system of singularly perturbed differential equations arising in a two-time scale system is considered. A computational method consists of the standard backward difference operator and a piecewise uniform Shishkin mesh is constructed to solve the system. The computational method is proved to be first order convergent uniformly with respect to the perturbation parameter. Numerical experiments support the theoretical results.

Keywords Singular perturbation problems · Boundary layers · Nonlinear differential equations · Finite difference scheme · Shishkin mesh · Parameter-uniform convergence · Two-time scale systems

Mathematics Subject Classification 65L11, 65L12, 65L20, 65L70

1 Introduction

Singular perturbation problems are widespread in nature. For instance, these problems arise in various fields of applied mathematics such as fluid dynamics and control systems [1,14]. Classical computational methods are not suitable for these problems due to the multiscale behaviour of the solutions [1].

The first author has been supported with Dr. D. S. Kothari Postdoctoral Fellowship by the University Grants Commission, Government of India.

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A good number of non-classical numerical methods are available in literature for different scalar singularly perturbed linear and nonlinear differential equations. In [2], a numerical method based on a finite difference operator and a Shishkin mesh is constructed for a linear system of Singularly Perturbed Initial Value Problems (SPIVPs) with a parameter ε. The same method for a linear system of SPIVPs with different parameters is reported in [3]. Li-Bin Liu et. al. [4] have developed an adaptive moving grid method for a linear system of SPIVPs. Numerical methods based on the backward Euler finite difference scheme with a Shishkin mesh and a Bakhvalov mesh are reported in [5] for a linear system of SPIVPs. Chandra Sekhara Rao et. al. [6] have developed a second order numerical method based on high order difference approximation with identity expansions technique for a linear system of two SPIVPs.

O'Malley [7] investigated the behaviour of a nonlinear system of $m + n$ SPIVPs in which the first m equations are unperturbed and the remaining n equations are perturbed by a parameter ε. Balser–Kostov method is used to study the nature of a nonlinear system of SPIVPs with a parameter ε in [8]. And in [9], a numerical scheme based on a finite difference operator and a Shishkin mesh is developed for a nonlinear system of two SPIVPs with different parameters. Zhongdi Cen et. al. [10] have designed a second order hybrid finite difference scheme for a nonlinear system of two SPIVPs with two parameters ε_1 and ε_2. In this work the conditions $\varepsilon_1 \leq CN^{-1}$ and $\varepsilon_2 \leq CN^{-1}$ are imposed on ε_1 and ε_2. A second order numerical method based on a weighted monotone hybrid scheme for a nonlinear system of two SPIVPs with two parameters is reported in [11].

Based on a mathematical model, a study of the Michaelis-Menten kinetic equations is reported in [12]. With the aid of dimensionless quantities introduced in [13], the equations mentioned in [12] are converted into a system of two SPIVPs in [9]. In this system the derivative term in the first equation alone multiplied by a parameter ε ($\varepsilon << 1$) and the nonlinear terms are free from ε. A mathematical model consists of a nonlinear system of $m + n$ SPIVPs for a two-time scale system is reported in [14]. In this model the derivative terms in n equations are multiplied by a parameter ε ($\varepsilon << 1$) and the parameter ε also occurs in the nonlinear terms. It is to be noted that the model reported in [14] is more general than the model mentioned in [9].

The importance of singular perturbation problems in power systems are investigated in [15,16]. A survey on singular perturbation and time-scale methods in control theory is reported in [17,18]. Applications of singular perturbation techniques to control problems are listed in [19,20]. Modeling of generators and their controls in power system simulations using singular perturbations are presented in [14].

A nonlinear two-time scale system can be represented by [14]

$$\varepsilon \frac{d\vec{z}}{dt} = \vec{f}(t, \varepsilon, \vec{z}, \vec{x}),$$
$$\frac{d\vec{x}}{dt} = \vec{g}(t, \varepsilon, \vec{z}, \vec{x}), \quad t \in (t_0, T] \tag{1}$$
$$\text{with} \quad \vec{z}(t_0) = \vec{z}_0, \quad \vec{x}(t_0) = \vec{x}_0 \ \text{ and } \ 0 < \varepsilon << 1 \tag{2}$$

where $\vec{z}$ and $\vec{x}$ represent vectors of fast and slow states, respectively. The small positive parameter, ε, determines the time scale separation of the states. By setting $\varepsilon = 0$ in the above system, the reduced or degenerate system is given by

$$\begin{aligned} 0 &= \vec{f}(t,\, 0,\, \vec{z},\, \vec{x}), \\ \frac{d\vec{x}}{dt} &= \vec{g}(t,\, 0,\, \vec{z},\, \vec{x}), \quad t \in (t_0, T\,],\ \vec{x}(t_0) = \vec{x}_0. \end{aligned} \tag{3}$$

In (3), the fast states are removed from the differential equations. Note that the fast states follow the slow states and can be obtained by solving the algebraic equation $\vec{z} = \vec{h}(t,\, \vec{x})$.

In power system simulators it is a common practice to neglect the fast network transients. But the negligence of the fast network transients may lead to inaccuracies if the system frequency changes substantially. Many other fast states are involved in a power system dynamic model, for example, the states related with the generator amortisseur windings and the states related with fast Automatic Voltage Regulators (AVR). In many cases it is not acceptable to neglect the dynamics of the fast states. For example, due to the phase shift they may introduce, ignoring the dynamics of the AVR fast states can have a considerable effect on the stability of controls; and neglecting the rates of change of the amortisseur windings states is equivalent to assuming zero amortisseur currents or no amortisseurs, an approximation which implies a reduced system damping and is clearly unsatisfactory. This fact lead to the occurrence of nonlinear system of singular perturbation problems in two-time scale systems. Motivated by the mathematical model reported in [14], in the present article a nonlinear system of n SPIVPs arising in a two-time scale system is considered in which the derivative terms in the first m $(m < n)$ equations are multiplied by a parameter ε and the parameter ε also occurs in the nonlinear terms.

Some novelties of the present article are as follows: no artificial condition is imposed on the parameter ε and the nonlinear terms are taken together with the parameter ε. The nonlinear terms in one of the three mathematical model problems presented in Sect. 6 of this article are free from the parameter ε. Such a problem is presented to demonstrate that the numerical method developed in the present article is also applicable to the problems considered in [9] in which the nonlinear terms are assumed to be free from the parameter ε. Furthermore, no mathematical technique is available in the literature to investigate two-time scale systems. Thus the numerical method designed in this article creates a new way to examine two-time scale systems.

For convenience, the two-time scale system represented in (1)–(2) is considered in the following form

$$\begin{aligned} \varepsilon\, u_i'(t) + f_i(t,\, \varepsilon,\, \vec{u}\,) &= 0, \ \text{for } i = 1, ..., m,\ m < n, n \geq 2, \\ u_j'(t) + f_j(t,\, \varepsilon,\, \vec{u}\,) &= 0, \ \text{for } j = m+1, ..., n, \quad \text{on } \ \Omega = (0, 1] \qquad (4) \\ \text{with } \ u_k(0) &= a_k, \quad \text{for } k = 1, ..., n,\ 0 < \varepsilon << 1. \qquad (5) \end{aligned}$$

For all $(t, \vec{y}) \in \overline{\Omega} \times \mathbb{R}^n$, the following conditions are assumed

$$\frac{\partial f_k(t, \varepsilon, \vec{y})}{\partial u_l} \leq 0, \quad \text{for } k, l = 1, ..., n \text{ and } k \neq l, \tag{6}$$

$$\min_{t,k} \left(\sum_{l=1}^{n} \frac{\partial f_k(t, \varepsilon, \vec{y})}{\partial u_l} \right) \geq \alpha > 0, \text{ for some constant } \alpha \tag{7}$$

where $\overline{\Omega} = [0, 1]$. Assumptions (6) and (7) together with the implicit function theorem ensure that $\vec{u} \in (C^2(\overline{\Omega}))^n$. The problem (4)-(5) can be written as

$$\vec{T}\vec{u}(t) = E\, \vec{u}\,'(t) + \vec{f}(t, \varepsilon, \vec{u}) = \vec{0} \text{ on } \Omega \text{ with } \vec{u}(0) = \vec{a} \tag{8}$$

where $\vec{a} = (a_1, ..., a_n)^T$. For all $t \in \overline{\Omega}$, $\vec{u}(t) = (u_1(t), ..., u_n(t))^T$, $\vec{f}(t, \varepsilon, \vec{u}) = (f_1(t, \varepsilon, \vec{u}), ..., f_n(t, \varepsilon, \vec{u}))^T \in C^2(\overline{\Omega} \times \mathbb{R}^n)$ and E is an $n \times n$ matrix such that $E = \text{diag}(\varepsilon, ..., \varepsilon, 1, ..., 1)$ where ε occurs m times.

Throughout the article C and C_1 denote positive constants, which are independent of t, ε and N.

2 Analytical results

The reduced problem (obtained by putting $\varepsilon = 0$) corresponding to (4)–(5) is defined by

$$f_i(t, 0, \vec{r}) = 0, \text{ for } i = 1, ..., m, \tag{9}$$

$$r_j'(t) + f_j(t, 0, \vec{r}) = 0, \text{ for } j = m+1, ..., n, \text{ on } \Omega, \; r_j(0) = a_j. \tag{10}$$

Note that $r_i(0) = f_i(0, 0, \vec{u}(0))$ for $i = 1, ..., m$. Conditions (6) and (7) together with the implicit function theorem ensure the existence of a unique solution for (9). Further, the function $\vec{r}$ and its derivatives are bounded independently of ε. Hence,

$$|r_l^{(k)}(t)| \leq C \quad \text{for } l = 1, ..., n, \quad k = 0, 1, 2 \text{ and } t \in \overline{\Omega}. \tag{11}$$

Decompose $\vec{u}(t)$ into two components $\vec{v}(t)$ and $\vec{w}(t)$ such that $\vec{u}(t) = \vec{v}(t) + \vec{w}(t)$ where

$$E\, \vec{v}\,'(t) + \vec{f}(t, \varepsilon, \vec{v}) = \vec{0} \text{ on } \Omega, \; \vec{v}(0) = \vec{r}(0) \tag{12}$$

and

$$E\, \vec{w}\,'(t) + \vec{f}(t, \varepsilon, \vec{v} + \vec{w}) - \vec{f}(t, \varepsilon, \vec{v}) = \vec{0} \text{ on } \Omega, \; \vec{w}(0) = \vec{u}(0) - \vec{v}(0). \tag{13}$$

Before estimating the bounds of the components $\vec{v}$, $\vec{w}$ and their derivatives, we establish the following results. Let $P(t)$ be an $n \times n$ matrix with entries $p_{kl}(t)$ such that for all

$t \in \overline{\Omega}$,

$$p_{kl}(t) \leq 0 \text{ for } k, l = 1, ..., n, \ k \neq l \quad \text{and } \min_{t,k} \left(\sum_{l=1}^{n} p_{kl}(t) \right) \geq \alpha > 0. \tag{14}$$

And let $\vec{L}$ be a linear operator such that

$$\vec{L}\vec{\phi}(t) = E\,\vec{\phi}\,'(t) + P(t)\vec{\phi}(t), \ t \in \Omega \quad \text{with } \vec{\phi}(0) = \vec{\gamma} \tag{15}$$

where $\vec{\gamma}$ is a vector constant.

Lemma 1 *Let $P(t)$ satisfy the conditions in (14). If $\vec{\phi}(0) \geq \vec{0}$ and $\vec{L}\vec{\phi} \geq \vec{0}$ on Ω then $\vec{\phi} \geq \vec{0}$ on $\overline{\Omega}$.*

Proof Choose k^*, t^* such that $\phi_{k^*}(t^*) = \min_{t,k} \phi_k(t)$. Suppose $\phi_{k^*}(t^*) < 0$ then $t^* \neq 0$ and $\phi'_{k^*}(t^*) = 0$. Let $t^* \in \Omega$. Consider,

$$(\vec{L}\vec{\phi})_{k^*}(t^*) = \begin{cases} \varepsilon\,\phi'_{k^*}(t^*) + \sum_{l=1}^{n} p_{k^*l}(t^*)\phi_l(t^*), & \text{if } k^* = 1, ..., m \\ \phi'_{k^*}(t^*) + \sum_{l=1}^{n} p_{k^*l}(t^*)\phi_l(t^*), & \text{if } k^* = m+1, ..., n. \end{cases}$$

Using (14), $(\vec{L}\vec{\phi})_{k^*}(t^*) < 0$, a contradiction. Hence $\phi_{k^*}(t^*) \geq 0$ which gives $\vec{\phi} \geq \vec{0}$ on $\overline{\Omega}$. □

Lemma 2 *Let $P(t)$ satisfy the conditions in (14). Then for all $t \in \overline{\Omega}$,*

$$\| \vec{\phi}(t) \| \leq \max \left\{ \| \vec{\phi}(0) \|, \frac{1}{\alpha} \| \vec{L}\vec{\phi} \| \right\}.$$

Proof Let $M = \max\{\| \vec{\phi}(0) \|, \frac{1}{\alpha} \| \vec{L}\vec{\phi} \|\}$ and $\vec{\Theta}^{\pm}(t) = M\vec{e} \pm \vec{\phi}(t)$ where $\vec{e} = (1, ..., 1)^T$. Then $\vec{\Theta}^{\pm}(0) \geq \vec{0}$ and $\vec{L}\vec{\Theta}^{\pm}(t) = MP(t)\vec{e} \pm \vec{L}\vec{\phi}(t)$. Using (14), $\vec{L}\vec{\Theta}^{\pm} \geq \vec{0}$ on Ω. Hence from Lemma 1, $\vec{\Theta}^{\pm} \geq \vec{0}$ on $\overline{\Omega}$, which proves the result. □

Lemma 3 *Let conditions (6) and (7) hold. Then for all $t \in \overline{\Omega}$, for $l = 1, ..., n$, $s = 1, ..., m$ and $\ell = m+1, ..., n$,*

$$|v_l^{(k)}(t)| \leq C, \ \textit{for } k = 0, 1, \ |v''_s(t)| \leq C\,\varepsilon^{-1} \ \textit{ and } \ |v''_\ell(t)| \leq C.$$

Proof Decompose $\vec{v}(t)$ further as $\vec{v}(t) = \vec{\hat{q}}(t) + \vec{\tilde{q}}(t)$ where

$$f_i(t, \ \varepsilon, \ \vec{\hat{q}}\,) = 0, \ \text{for } i = 1, ..., m, \tag{16}$$

$$\hat{q}\,'_j(t) + f_j(t, \ \varepsilon, \ \vec{\hat{q}}\,) = 0, \ \text{for } j = m+1, ..., n \text{ and } t \in \Omega \tag{17}$$

$$\hat{q}_j(0) = v_j(0), \ \text{for } j = m+1, ..., n \tag{18}$$

and

$$\varepsilon\,\tilde{q}_i{}'(t) + f_i(t,\,\varepsilon,\,\vec{\hat{q}} + \vec{\tilde{q}}\,) - f_i(t,\,\varepsilon,\,\vec{\hat{q}}\,) = -\varepsilon\,\hat{q}_i{}'(t),\ \text{for } i = 1, ..., m, \tag{19}$$

$$\tilde{q}_j{}'(t) + f_j(t,\,\varepsilon,\,\vec{\hat{q}} + \vec{\tilde{q}}\,) - f_j(t,\,\varepsilon,\,\vec{\hat{q}}\,) = 0,\ \text{for } j = m+1, ..., n, \tag{20}$$

$$\text{for } \ t \in \Omega \ \text{ with } \ \tilde{q}_l(0) = 0,\ \text{for } l = 1, ..., n. \tag{21}$$

Let $t \in \Omega$. Using (9) and (16), for $i = 1, ..., m$,

$$a_{i1}(t)(\hat{q}_1(t) - r_1(t)) + \cdots + a_{in}(t)(\hat{q}_n(t) - r_n(t)) = 0. \tag{22}$$

Using (10) and (17), for $j = m+1, ...n$,

$$\hat{q}_j{}'(t) - r_j'(t) + a_{j1}(t)(\hat{q}_1(t) - r_1(t)) + \cdots + a_{jn}(t)(\hat{q}_n(t) - r_n(t)) = 0 \tag{23}$$

where $a_{lk}(t) = \dfrac{\partial f_l}{\partial u_k}(t,\,\varepsilon,\,\vec{\chi}_{f_l}(t)),\ \ l, k = 1, ..., n$ are intermediate values. Note that for all $t \in \overline{\Omega}$, $a_{kl}(t) \le 0$ for $k, l = 1, ..., n$, $k \ne l$ and $\min_{t,k} \left(\sum_{l=1}^{n} a_{kl}(t)\right) \ge \alpha > 0$. From (22), for $i = 1, ..., m$,

$$\sum_{l=1}^{n} a_{il}(t)\hat{q}_l(t) = \sum_{l=1}^{n} a_{il}(t) r_l(t) \tag{24}$$

and from (23), for $j = m+1, ...n$,

$$\hat{q}_j{}'(t) + \sum_{l=1}^{n} a_{jl}(t)\hat{q}_l(t) = r_j'(t) + \sum_{l=1}^{n} a_{jl}(t) r_l(t). \tag{25}$$

From (11), (24) and (25), $\|\ \vec{\hat{q}}\ \| \le C$, $\|\ \vec{\hat{q}}\,'\ \| \le C$ and $\|\ \vec{\hat{q}}\,''\ \| \le C$. From (19), (20) and (21), for $i = 1, ..., m,\ j = m+1, ..., n$ and for $l = 1, ..., n$,

$$\varepsilon\,\tilde{q}_i{}'(t) + a_{i1}^*(t)\tilde{q}_1(t) + \cdots + a_{in}^*(t)\tilde{q}_n(t) = -\varepsilon\,\hat{q}_i{}'(t) \tag{26}$$

$$\tilde{q}_j{}'(t) + a_{j1}^*(t)\tilde{q}_1(t) + \cdots + a_{jn}^*(t)\tilde{q}_n(t) = 0 \tag{27}$$

$$\tilde{q}_l(0) = 0 \tag{28}$$

where $a_{kl}^*(t) = \dfrac{\partial f_k}{\partial u_l}(t,\,\varepsilon,\,\vec{\eta}_{f_k}(t)),\ \ k, l = 1, ..., n$, are intermediate values. Note that for all $t \in \overline{\Omega}$,

$$a_{kl}^*(t) \le 0 \text{ for } k, l = 1, ..., n, k \ne l \text{ and } \min_{t,k}\left(\sum_{l=1}^{n} a_{kl}^*(t)\right) \ge \alpha > 0. \tag{29}$$

We now estimate the bounds of $\vec{\tilde{q}}$ and its derivatives. From (26) to (28) and (29), we note that $\vec{\tilde{q}}$ satisfies a problem similar to (15). Hence using the bounds of $\vec{\hat{q}}$ and

Lemma 2, $\| \vec{\tilde{q}} \| \leq C\,\varepsilon$. Using the bounds of $\tilde{q}_k$, for $k = 1, ..., n$ in (26) and (27), $|\tilde{q}_k'(t)| \leq C$ for $k = 1, ..., n$. Differentiating (26) and (27) once and using the bounds of $\tilde{q}_k$ and $\tilde{q}_k'$ for $k = 1, ..., n$, $|\tilde{q}_i''(t)| \leq C\,\varepsilon^{-1}$ for $i = 1, ..., m$ and $|\tilde{q}_j''(t)| \leq C$ for $j = m+1, ..., n$.

From the bounds of $\hat{q}_k$ and $\tilde{q}_k, k = 1, ..., n$, and their derivatives the bounds on $v_k, k = 1, ..., n$, and their derivatives follow. ◻

From (13), for $i = 1, ..., m,\ j = m+1, ..., n$ and for $k = 1, ..., n$,

$$\begin{aligned} \varepsilon\, w_i'(t) + s_{i1}(t)w_1(t) + \cdots + s_{in}(t)w_n(t) &= 0, \\ w_j'(t) + s_{j1}(t)w_1(t) + \cdots + s_{jn}(t)w_n(t) &= 0,\ t \in \Omega, \end{aligned} \tag{30}$$

$$w_k(0) = u_k(0) - v_k(0) \tag{31}$$

where $s_{kl}(t) = \dfrac{\partial f_k}{\partial u_l}(t,\ \varepsilon,\ \vec{\theta}_{f_k}(t)),\ k, l = 1, 2,$ are intermediate values.
The layer function $B(t)$ related with the solution $\vec{u}(t)$ of (8) is defined by

$$B(t) = e^{-\alpha t/\varepsilon},\ t \in \overline{\Omega}.$$

Lemma 4 *Let conditions* (6) *and* (7) *hold. Then for any* $t \in \overline{\Omega}$, *for* $i = 1, ..., m$ *and* $j = m+1, ..., n$,

$$|w_i(t)| \leq C_1 B(t),\quad |w_j(t)| \leq C_1\,\varepsilon(1 - B(t))$$

$$|w_i'(t)| \leq C_1\varepsilon^{-1}B(t) + C_1(1 - B(t)),\ |w_j'(t)| \leq C_1 B(t) + C_1\,\varepsilon(1 - B(t))$$

$$|w_i''(t)| \leq C_1\,\varepsilon^{-2}B(t) + C_1\,\varepsilon^{-1}(1 - B(t))$$

$$|w_j''(t)| \leq C_1\,\varepsilon^{-1}B(t) + C_1(1 - B(t)).$$

Proof Note that for all $t \in \overline{\Omega}$,

$$s_{kl}(t) \leq 0 \text{ for } k, l = 1, ..., n, k \neq l \quad \text{and } \min_{t,k}\left(\sum_{l=1}^{n} s_{kl}(t)\right) \geq \alpha > 0. \tag{32}$$

Define for $i = 1, ..., m$ and for $j = m+1, ..., n$, $\psi_i^{\pm}(t) = C_1 B(t) \pm w_i(t)$ and $\psi_j^{\pm}(t) = C_1\varepsilon(1 - B(t)) \pm w_j(t)$. Then $\psi_i^{\pm}(0) = C_1 \pm w_i(0)$ for $i = 1, ..., m$ and $\psi_j^{\pm}(0) = 0$ for $j = m+1, ..., n$. Hence choosing $C_1 > C + ||\vec{a}||$, $\vec{\psi}^{\pm}(0) \geq \vec{0}$. Now consider,

$$(\vec{L}\vec{\psi}^{\pm})_k(t) = \begin{cases} C_1 B(t)\left(\sum_{l=1}^{n} s_{kl}(t) - \alpha\right), & \text{if } k = 1, ..., m \\ C_1\alpha B(t) + C_1\varepsilon(1 - B(t))\sum_{l=1}^{n} s_{kl}(t), & \text{if } k = m+1, ..., n. \end{cases}$$

Using (32), $\vec{L}\vec{\psi}^{\pm} \geq \vec{0}$ on Ω. Thus using Lemma 1, $\vec{\psi}^{\pm} \geq \vec{0}$ on $\overline{\Omega}$. Hence, for $i = 1, ..., m$ and for $j = m+1, ..., n$,

$$|w_i(t)| \leq C_1 B(t) \text{ and } |w_j(t)| \leq C_1\, \varepsilon(1 - B(t)).$$

Using the bounds on w_k, $k = 1, ..., n$, the bounds on w_k' for $k = 1, ..., n$, follow from (30). Differentiating (30) once and using the bounds of w_k and w_k', $k = 1, .., n$, the bounds on w_k'', $k = 1, .., n$, follow. □

3 Shishkin mesh and discrete problem

On $\overline{\Omega}$, a piecewise uniform Shishkin mesh with N mesh-intervals is constructed as follows. Let $\Omega^N = \{t_j\}_{j=1}^N$ then $\overline{\Omega}^N = \{t_j\}_{j=0}^N$. The interval $\overline{\Omega}$ is subdivided into 2 sub-intervals $[0, \tau]$ and $(\tau, 1]$ such that $\overline{\Omega} = [0, \tau] \cup (\tau, 1]$. The transition parameter τ is defined by

$$\tau = \min\left\{\frac{1}{2}, \frac{\varepsilon}{\alpha} \ln N\right\}.$$

On $[0, \tau]$, a uniform mesh with $\frac{N}{2}$ mesh points is placed and on $(\tau, 1]$ another uniform mesh with $\frac{N}{2}$ mesh points is placed.

The discrete IVP associated with (8) is defined to be

$$\vec{T}^N \vec{U}(t_j) = E\, D^- \vec{U}(t_j) + \vec{f}(t_j, \varepsilon, \vec{U}(t_j)) = \vec{0}, \quad \text{for } t_j \in \Omega^N \tag{33}$$

$$\vec{U}(0) = \vec{u}(0). \tag{34}$$

Here $D^- \Psi(t_j) = \dfrac{\Psi(t_j) - \Psi(t_{j-1})}{h_j}$, $h_j = t_j - t_{j-1}$.

4 Error analysis

Let $\vec{L}^N$ be a linear discrete operator such that

$$\vec{L}^N \vec{\phi}(t_j) = E\, D^- \vec{\phi}(t_j) + P(t_j)\vec{\phi}(t_j), \;\; t_j \in \Omega^N \quad \text{with } \; \vec{\phi}(0) = \vec{\gamma} \tag{35}$$

where $\vec{\gamma}$ is a vector constant and $P(t)$ is as defined in (14).

Lemma 5 *Let $P(t)$ satisfy the conditions in (14). If $\vec{\phi}(0) \geq \vec{0}$ and $\vec{L}^N \vec{\Phi} \geq \vec{0}$ on Ω^N then $\vec{\Phi} \geq \vec{0}$ on $\overline{\Omega}^N$.*

Proof Choose k^*, j^* such that $\Phi_{k^*}(t_{j^*}) = \min_{j,k} \Phi_k(t_j)$. Suppose $\Phi_{k^*}(t_{j^*}) < 0$ then $j^* \neq 0$ and $\Phi_{k^*}(t_{j^*}) - \Phi_{k^*}(t_{j^*-1}) \leq 0$. Let $t_{j^*} \in \Omega^N$. Consider,

$$(\vec{L}^N \vec{\phi})_{k^*}(t_{j^*}) = \begin{cases} \varepsilon\, D^- \Phi_{k^*}(t_{j^*}) + \sum_{l=1}^{n} p_{k^* l}(t_{j^*}) \Phi_l(t_{j^*}), \text{ if } k^* = 1, ..., m \\ D^- \Phi_{k^*}(t_{j^*}) + \sum_{l=1}^{n} p_{k^* l}(t_{j^*}) \Phi_l(t_{j^*}), \\ \qquad\qquad\qquad\qquad \text{if } k^* = m+1, ..., n. \end{cases}$$

Using (14), $(\vec{L}^N \vec{\phi})_{k^*}(t_{j^*}) < 0$, a contradiction. Hence $\Phi_{k^*}(t_{j^*}) \geq 0$ which gives $\vec{\phi} \geq \vec{0}$ on $\overline{\Omega}^N$. □

Lemma 6 *Let $P(t)$ satisfy the conditions in (14). Then for all $t_j \in \overline{\Omega}^N$,*

$$\| \vec{\phi}(t_j) \| \leq \max \left\{ \| \vec{\phi}(0) \|, \frac{1}{\alpha} \| \vec{L}^N \vec{\Phi} \| \right\}.$$

Proof Let $K = \max\{\| \vec{\phi}(0) \|, \frac{1}{\alpha} \| \vec{L}^N \vec{\Phi} \|\}$ and $\vec{\Theta}^{\pm}(t_j) = K\,\vec{e} \pm \vec{\phi}(t_j)$. Then $\vec{\Theta}^{\pm}(0) \geq \vec{0}$ and $\vec{L}^N \vec{\Theta}^{\pm}(t_j) = M P(t_j)\vec{e} \pm \vec{L}^N \vec{\phi}(t_j)$. Using (14), $\vec{L}^N \vec{\Theta}^{\pm} \geq \vec{0}$ on Ω^N. Hence from Lem ma 5, $\vec{\Theta}^{\pm} \geq \vec{0}$ on $\overline{\Omega}^N$, which proves the result. □

Let $\vec{Y}_1$ and $\vec{Y}_2$ be any two mesh functions such that $\vec{Y}_1(0) = \vec{Y}_2(0)$. For $t_j \in \Omega^N$,

$$\begin{aligned} &(\vec{T}^N \vec{Y}_1 - \vec{T}^N \vec{Y}_2)(t_j) \\ &\quad = E\, D^-(\vec{Y}_1 - \vec{Y}_2)(t_j) + \vec{f}(t_j, \varepsilon, \vec{Y}_1(t_j)) - \vec{f}(t_j, \varepsilon, \vec{Y}_2(t_j)) \\ &\quad = E\, D^-(\vec{Y}_1 - \vec{Y}_2)(t_j) + J(\vec{f}, \vec{u}) \\ &\quad = (\vec{T}^N)'(\vec{Y}_1 - \vec{Y}_2)(t_j) \end{aligned} \tag{36}$$

where $J(\vec{f}, \vec{u}) = \left(\frac{\partial f_k}{\partial u_l}(t_j, \varepsilon, \vec{M}(t_j)) \right)_{n \times n}$ is an intermediate value and $(\vec{T}^N)'$ is the Frechet derivative of $\vec{T}^N$. Since $(\vec{T}^N)'$ is linear, it satisfy Lemma 6. Hence, on Ω^N

$$\| \vec{Y}_1 - \vec{Y}_2 \| \leq C\ \| (\vec{T}^N)'(\vec{Y}_1 - \vec{Y}_2) \| = C\ \| \vec{T}^N \vec{Y}_1 - \vec{T}^N \vec{Y}_2 \|. \tag{37}$$

Theorem 1 *Let $\vec{u}$ be the solution of (8) and $\vec{U}$ be the solution of (33)-(34). Then for $t_j \in \overline{\Omega}^N$,*

$$\| \vec{U} - \vec{u} \| \leq\ C\, N^{-1} \ln N. \tag{38}$$

Proof Let $t_j \in \Omega^N$. Since $\vec{U}(0) = \vec{u}(0)$, from (37),

$$\| \vec{U} - \vec{u} \| \leq\ C\ \| \vec{T}^N \vec{U} - \vec{T}^N \vec{u} \|. \tag{39}$$

Using (33),

$$\parallel \vec{T}^N \vec{u}(t_j) \parallel = \parallel (\vec{T}^N \vec{u} - \vec{T}^N \vec{U})(t_j) \parallel .$$

Consider,

$$\begin{aligned} \parallel (\vec{T}^N \vec{u} - \vec{T}^N \vec{U})(t_j) \parallel &= \parallel \vec{T}^N \vec{u}(t_j) \parallel = \parallel (\vec{T}^N \vec{u} - \vec{T} \vec{u})(t_j) \parallel \\ &= E \parallel (D^- - D)\vec{u}(t_j) \parallel \\ &\leq E(\parallel (D^- - D)\vec{v}(t_j) \parallel \\ &\quad + \parallel (D^- - D)\vec{w}(t_j) \parallel) \end{aligned} \tag{40}$$

where $D = \frac{d}{dt}$. From (40),

$$|(\vec{T}^N \vec{u} - \vec{T}^N \vec{U})_i(t_j)| \leq \begin{cases} \varepsilon(|(D^- - D)v_i(t_j)| + |(D^- - D)w_i(t_j)|), \\ \qquad\qquad\qquad \text{for } i = 1, ..., m \\ |(D^- - D)v_i(t_j)| + |(D^- - D)w_i(t_j)|), \\ \qquad\qquad\qquad \text{for } i = m+1, ..., n. \end{cases} \tag{41}$$

Note that for any smooth function ϕ,

$$|(D^- - D)\phi(t_j)| \leq \max_{s \in I_j} |\phi''(s)| \frac{t_j - t_{j-1}}{2} \tag{42}$$

$$|(D^- - D)\phi(t_j)| \leq 2 \max_{s \in I_j} |\phi'(s)| \tag{43}$$

where $I_j = [t_{j-1}, t_j]$. Since $t_j - t_{j-1} \leq 2N^{-1}$ for any choice of τ, using (42), for $i = 1, ..., m$,

$$\varepsilon \, |(D^- - D)v_i(t_j)| \leq C \, \varepsilon \, N^{-1} |v_i''(s)|_{I_j} \tag{44}$$

and for $k = m+1, ..., n$,

$$|(D^- - D)v_k(t_j)| \leq C \, N^{-1} |v_k''(s)|_{I_j}. \tag{45}$$

Hence from Lemma 3, (44) and (45), for $i = 1, ..., m$ and for $k = m+1, ..., n$,

$$\varepsilon |(D^- - D)v_i(t_j)| \leq C \, N^{-1} \text{ and } |(D^- - D)v_k(t_j)| \leq C \, N^{-1}.$$

Now we estimate the error in the component $\vec{w}$. The argument depends on whether $\tau = \frac{1}{2}$ or $\tau = \frac{\varepsilon}{\alpha} \ln N$. If $\tau = \frac{1}{2}$ then $\varepsilon^{-1} \leq C \ln N$. In this case $t_j - t_{j-1} = N^{-1}$. Using (42), for $i = 1, ..., m$,

$$\varepsilon \, |(D^- - D)w_i(t_j)| \leq C \, \varepsilon \, N^{-1} |w_i''(s)|_{I_j} \tag{46}$$

and for $k = m + 1, ..., n$,

$$|(D^- - D)w_k(t_j)| \le C N^{-1} |w_k''(s)|_{I_j}. \tag{47}$$

Hence from Lemma 4, (46) and (47), for $i = 1, ..., m$ and for $k = m + 1, ..., n$,

$$\varepsilon |(D^- - D)w_i(t_j)| \le C N^{-1} \ln N \text{ and } |(D^- - D)w_k(t_j)| \le C N^{-1} \ln N.$$

Now let $\tau = \frac{\varepsilon}{\alpha} \ln N$. In this case we prove the results on the sub-intervals $[0, \tau]$ and $(\tau, 1]$ separately. Consider $t_j \in [0, \tau]$. Then $t_j - t_{j-1} = \frac{2\varepsilon N^{-1} \ln N}{\alpha}$. Hence using (42) and Lemma 4, for $i = 1, ..., m$ and for $k = m + 1, ..., n$,

$$\varepsilon |(D^- - D)w_i(t_j)| \le C N^{-1} \ln N \text{ and } |(D^- - D)w_k(t_j)| \le C N^{-1} \ln N.$$

Now consider $t_j \in (\tau, 1]$. We proceed the proof by dividing the argument into two cases $\varepsilon \ge N^{-1}$ and $\varepsilon \le N^{-1}$. Consider the first case $\varepsilon \ge N^{-1}$. Using (42) and Lemma 4, for $i = 1, ..., m$ and for $k = m + 1, ..., n$,

$$\varepsilon |(D^- - D)w_i(t_j)| \le C N^{-1} \text{ and } |(D^- - D)w_k(t_j)| \le C N^{-1}.$$

Consider the second case $\varepsilon \le N^{-1}$. Using (43), Lemma 4 and the fact that $B(t_{j-1}) = N^{-1}$, for $i = 1, ..., m$ and for $k = m + 1, ..., n$,

$$\varepsilon |(D^- - D)w_i(t_j)| \le C N^{-1} \text{ and } |(D^- - D)w_k(t_j)| \le C N^{-1}.$$

Using all the above estimates in (41), for all $t_j \in \overline{\Omega}^N$,

$$\| (\vec{T}^N \vec{u} - \vec{T}^N \vec{U})(t_j) \| \le C N^{-1} \ln N. \tag{48}$$

Using (48) in (39),

$$\| \vec{U} - \vec{u} \| \le C N^{-1} \ln N.$$

Hence the proof. □

5 The continuation method

The system of nonlinear ordinary differential equations in (8) is modified to an artificial system of nonlinear partial differential equations given by

$$\begin{aligned} &E\, \vec{u}_t(t, x) + \vec{u}_x(t, x) + \vec{f}(t, \varepsilon, \vec{u}(t, x)) = \vec{0}, \ (t, x) \in (0, 1] \times (0, X], \\ &\vec{u}(0, x) = \vec{u}(0), \ 0 < x \le X \text{ and } \vec{u}(t, 0) = \vec{u}_{init}(t), \ 0 < t \le 1. \end{aligned} \tag{49}$$

The discrete problem corresponding to (49) is given by, for $j = 1, \ldots, N$ and $k = 1, \ldots K$,

$$\begin{aligned}
&E\, D_t^- \vec{U}(t_j, x_k) + D_x^- \vec{U}(t_j, x_k) + \vec{f}(t_j,\ \varepsilon,\ \vec{U}(t_j, x_{k-1})) = \vec{0}, \\
&\vec{U}(0, x_k) = \vec{u}(0) \text{ for all } k \text{ and } \vec{U}(t_j, 0) = \vec{u}_{init}(t_j) \text{ for all } t_j \in \overline{\Omega}^N.
\end{aligned} \tag{50}$$

The initial guess $\vec{u}_{init}(t)$ is taken to be $(1-t)\vec{u}(0)$. The choices of $h_x = x_k - x_{k-1}$ and K are determined as follows. Define, for $k = 1, 2, \ldots, K$,

$$\begin{aligned}
e_i(k) &= \max_{1 \le j \le N} \left(\frac{|U_i(t_j, x_k) - U_i(t_j, x_{k-1})|}{h_x} \right) \\
e(k) &= \max_i e_i(k).
\end{aligned} \tag{51}$$

The step size h_x is chosen sufficiently small so that

$$e(k) \le e(k-1) \ \text{ for all } k,\ 1 < k \le K. \tag{52}$$

The number of iterations K is chosen such that

$$e(K) \le tol \tag{53}$$

where tol is a suitably prescribed small tolerance. The following algorithm is used to computed the numerical solution.

Start from x_0 with the initial step size $h_x = 1.0$. If, at some value of k, (52) is not satisfied, then discard the current step and restart from x_{k-1} with half the step size and continue halving the step size until finding a h_x for which (52) is satisfied. If (52) is satisfied at each step h_x, then continue the process until either (53) is satisfied or $K = 100$. If (53) is not satisfied, then it is assumed that the stepping process is stalled due to a large choice of the initial step. In this case, the entire process is repeated again from x_0 with the initial step size $h_x/2$. If (53) is satisfied, the resulting values of $\vec{U}(t_j, x_K)$ are taken as the approximations to the solution of the continuous problem.

6 Numerical illustrations

Three examples are presented in this section. In the first example the nonlinear terms are free from ε whereas in the second and third examples ε also occurs in the nonlinear terms. The continuation method designed in Sect. 5 is used to solve the examples. The tolerance $'tol'$ is taken to be 0.00001. Notations D^N, C_p^N and p^N denote the parameter-uniform maximum pointwise error, parameter-uniform error constant and

Table 1 Values of D^N, p^N and C_p^N for $\alpha = 0.9$.

ε	Number of mesh points N						
	64	128	256	512	1024	2048	4096
2^{-2}	0.0179	0.0096	0.0050	0.0026	0.0013	0.0007	0.0003
2^{-4}	0.0272	0.0195	0.0121	0.0075	0.0046	0.0026	0.0013
2^{-6}	0.0272	0.0195	0.0121	0.0075	0.0043	0.0025	0.0014
2^{-8}	0.0272	0.0195	0.0121	0.0075	0.0043	0.0025	0.0014
2^{-10}	0.0272	0.0195	0.0121	0.0075	0.0043	0.0025	0.0014
2^{-12}	0.0272	0.0195	0.0121	0.0075	0.0043	0.0025	0.0014
2^{-14}	0.0272	0.0195	0.0121	0.0075	0.0043	0.0025	0.0014
D^N	0.0272	0.0195	0.0121	0.0075	0.0046	0.0026	0.0014
p^N	0.4809	0.6832	0.7000	0.6962	0.8518	0.9072	
C_p^N	0.5278	0.4764	0.3738	0.2899	0.2254	0.1574	0.1057

parameter-uniform rate of convergence respectively and given by

$$D^N = \max_{\varepsilon} D_{\varepsilon}^N \quad \text{where } D_{\varepsilon}^N = \| \vec{U}^N - \vec{U}^{2N} \|,$$

$$p^N = \log_2 \frac{D^N}{D^{2N}}, \quad C_p^N = \frac{D^N N^{p^\star}}{1 - 2^{-p^\star}} \quad \text{where } p^\star = \min_N p^N.$$

Example 1 Consider the IVP

$$\begin{aligned} &\varepsilon u_1'(t) + (u_1(t))^3 + 2u_1(t) - e^{-u_1(t)} - 0.1u_2(t) + 1 = 0, \\ &u_2'(t) + (u_2(t))^5 + 3u_2(t) - \cos(u_2(t)) - u_1(t) - 1 = 0, \ t \in (0, 1] \end{aligned}$$

with $u_1(0) = 1, \ u_2(0) = 0$.

For the above IVP, the values of D^N, C_p^N, p^N are presented in Table 1 and a graph of the numerical solution for $\varepsilon = 2^{-2}, 2^{-4}, 2^{-6}$ and $N = 128$ is portrayed in Fig. 1.

Example 2 Consider the IVP,

$$\begin{aligned} &\varepsilon u_1'(t) + (u_1(t))^5 + (2 + \varepsilon)u_1(t) - e^{-u_1(t)} - u_2(t) + \varepsilon \ = 0 \\ &u_2'(t) + (u_2(t))^3 + 3u_2(t) + \sin(u_2(t)) - u_1(t) + 1 - \varepsilon^3 = 0, \ t \in (0, 1] \end{aligned}$$

with $u_1(0) = 0, \ u_2(0) = 1$.

For the above IVP, the values of D^N, C_p^N, p^N are presented in Table 2 and a graph of the numerical solution for $\varepsilon = 2^{-2}, 2^{-4}, 2^{-6}$ and $N = 128$ is portrayed in Fig. 2.

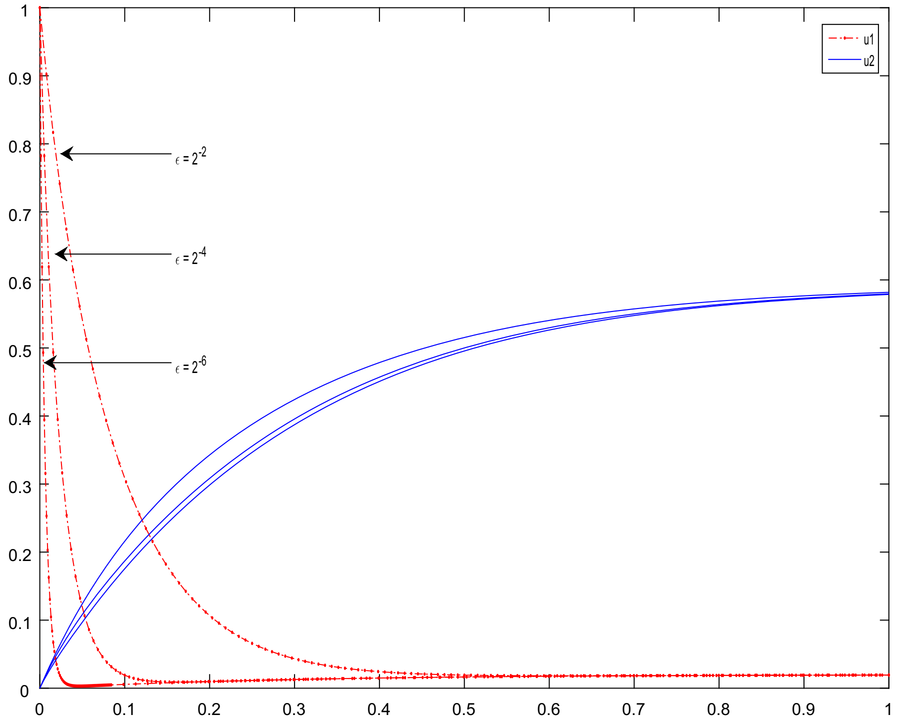

Fig. 1 Numerical solution of the IVP in Example 1

Table 2 Values of D^N, p^N and C_p^N for $\alpha = 0.9$.

ε	Number of mesh points N						
	64	128	256	512	1024	2048	4096
2^{-2}	0.0096	0.0052	0.0027	0.0014	0.0007	0.0003	0.0002
2^{-4}	0.0167	0.0115	0.0070	0.0042	0.0026	0.0014	0.0007
2^{-6}	0.0168	0.0114	0.0070	0.0042	0.0024	0.0014	0.0008
2^{-8}	0.0168	0.0114	0.0070	0.0042	0.0024	0.0014	0.0008
2^{-10}	0.0167	0.0114	0.0070	0.0042	0.0024	0.0014	0.0008
2^{-12}	0.0167	0.0114	0.0070	0.0042	0.0024	0.0014	0.0008
2^{-14}	0.0167	0.0114	0.0070	0.0042	0.0024	0.0014	0.0008
2^{-16}	0.0167	0.0114	0.0070	0.0042	0.0024	0.0014	0.0008
2^{-18}	0.0167	0.0114	0.0070	0.0042	0.0024	0.0014	0.0008
D^N	0.0168	0.0115	0.0070	0.0042	0.0026	0.0014	0.0008
p^N	0.5485	0.7067	0.7431	0.6913	0.8611	0.9186	
C_p^N	0.5202	0.5202	0.4662	0.4074	0.3690	0.2971	0.2299

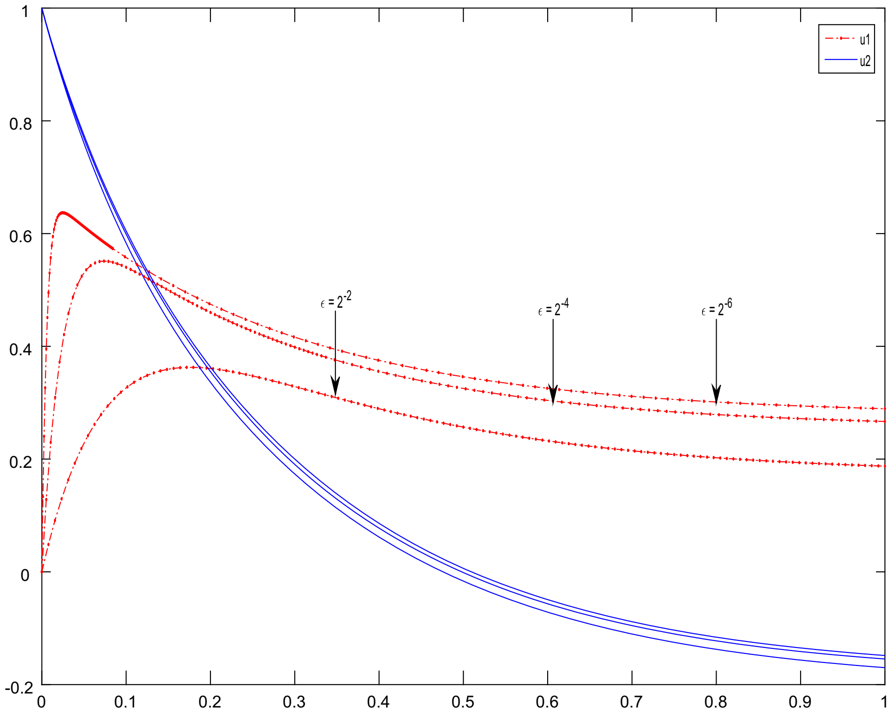

Fig. 2 Numerical solution of the IVP in Example 2

Example 3 Consider the IVP, for $t \in (0, 1]$,

$$
\begin{aligned}
&\varepsilon\, u_1'(t) + (u_1(t))^5 + (4+\varepsilon)u_1(t) - e^{-u_1(t)} - u_2(t) - u_3(t) + \varepsilon &&= 0\\
&\varepsilon\, u_2'(t) + (u_2(t))^3 + 5u_2(t) + \sin(u_2(t)) - u_1(t) - \varepsilon u_4(t) &&= 0\\
&u_3'(t) + (u_3(t))^7 + (5+\varepsilon^2)u_3(t) - u_2(t) - (1+\varepsilon)u_4(t) &&= 0\\
&u_4'(t) + (u_4(t))^5 + 7u_4(t) - u_1(t) - (1+\varepsilon^3)u_2(t) - u_3(t) - \varepsilon^5 &&= 0
\end{aligned}
$$

with $u_1(0) = 0,\ u_2(0) = 1,\ u_3(0) = 1$ and $u_4(0) = 0$.

For the above IVP, the values of D^N, C_p^N, p^N are presented in Table 3 and a graph of the numerical solution for $\varepsilon = 2^{-5}$ and $N = 128$ is portrayed in Fig. 3.

7 Conclusion

From the tables, it is evident that the maximum pointwise error (D^N) decreases when the number of mesh points (N) increases and the maximum pointwise error stabilizes for each N as ε approaches zero. Further, from the tables, we also observe that the proposed method is almost first order parameter-uniform convergent. This is in agreement with Theorem 1.

Table 3 Values of D^N, p^N and C_p^N for $\alpha = 1.9$.

ε	Number of mesh points N				
	64	128	256	512	1024
2^{-2}	0.0297	0.0170	0.0091	0.0047	0.0024
2^{-4}	0.0250	0.0179	0.0110	0.0067	0.0039
2^{-6}	0.0251	0.0179	0.0110	0.0067	0.0039
2^{-8}	0.0251	0.0179	0.0110	0.0067	0.0039
2^{-10}	0.0251	0.0179	0.0110	0.0067	0.0039
2^{-12}	0.0251	0.0179	0.0110	0.0067	0.0039
2^{-14}	0.0251	0.0179	0.0110	0.0067	0.0039
2^{-16}	0.0251	0.0179	0.0110	0.0067	0.0039
D^N	0.0297	0.0179	0.0110	0.0067	0.0039
p^N	0.7026	0.7047	0.7026	0.7973	
C_p^N	1.4318	1.4024	1.4003	1.4003	1.3113

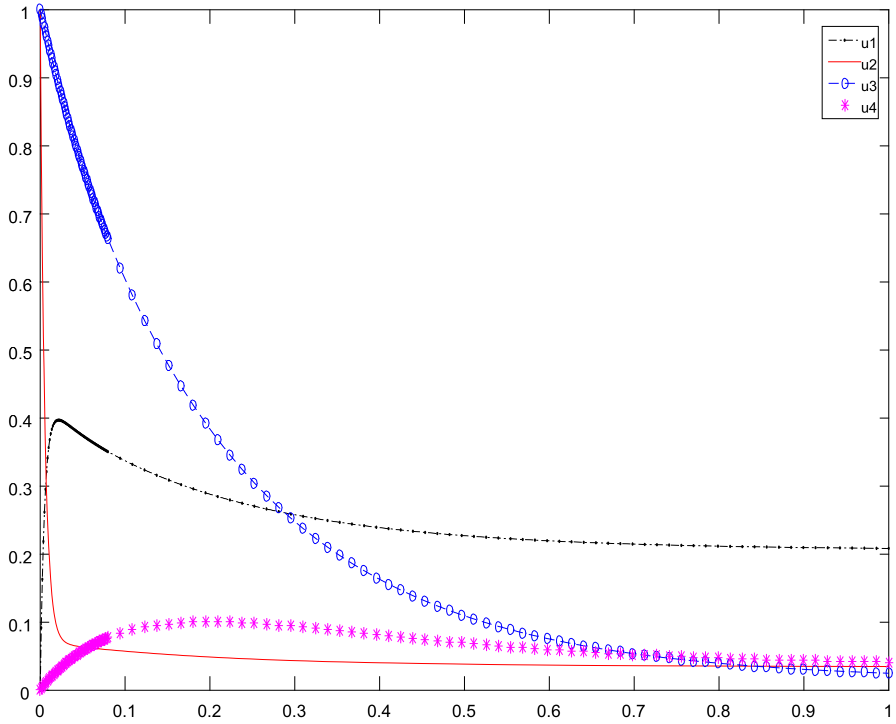

Fig. 3 Numerical solution of the IVP in Example 3

We notice that the solution component u_1 of $\vec{u}$ in Figs. 1 and 2 and the components u_1 and u_2 of $\vec{u}$ in Fig. 3, representing the fast states in the two-time scale system, exhibit initial layer in the neighbourhood of $t = 0$ and varies steadily as t increases whereas the solution component u_2 of $\vec{u}$ in Figs. 1 and 2 and the components u_3 and u_4 of $\vec{u}$ in Fig. 3, representing the slow states in the two-time scale system, exhibit no layer throughout the domain [0, 1] as reported in [14]- [20]. Moreover, we perceive that the component u_1 of $\vec{u}$ in Figs. 1 and 2 and the components u_1 and u_2 of $\vec{u}$ in Fig. 3 changes very rapidly near $t = 0$ as ε approaches zero.

This means that the fast states changes rapidly near $t = 0$ and remains smooth away from $t = 0$. On the other hand, the slow states remains smooth throughout the domain. Thus we witness that by setting the time scale separation of the states $\varepsilon = 0$ in the system, fast states will be removed from the system which is not acceptable.

Note that the time step is largely determined by the fast states due to the very small time constants related with the generators and their controls in the step-by-step simulation of power system dynamics. Hence from the present study we deduce that the negligence of the fast network transients in the two-time scale system will lead to inaccuracies in the system and the degenerate system will be damping which will produce unsatisfactory results.

Acknowledgements The first author wishes to acknowledge the financial assistance extended through Dr. D. S. Kothari Postdoctoral Fellowship by the University Grants Commission, Government of India. And both authors thank the Department of Science and Technology, Government of India, for the computational facilities under DST-PURSE phase II scheme.

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