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ORIGINAL RESEARCH

Construction of quantum codes from λ-constacyclic codes over the ring $\frac{\mathbb{F}_p[u,v]}{\langle v^3-v, u^3-u, uv-vu\rangle}$

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Abstract
In this paper we construct quantum codes over $\mathbb{F}_p$ by using cyclic and λ-cyclic codes over the ring $R = \mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p + v\mathbb{F}_p + uv\mathbb{F}_p + u^2v\mathbb{F}_p + v^2\mathbb{F}_p + uv^2\mathbb{F}_p + u^2v^2\mathbb{F}_p$ where $v^3 = v$, $u^3 = u$, $uv = vu$ and p is an odd prime integer. Using the idempotent decomposition method, we have given the parameters of the quantum code. Moreover, the structure of cyclic and λ-constacyclic codes over R is studied. Some quantum codes over $\mathbb{F}_p$ are given.

Keywords Quantum code · Cyclic code · Quasicyclic code · Gray map · Constacyclic code

AMS subject classification 94B15 · 94B05

1 Introduction

Quantum codes play a significant role in quantum computing and quantum communication. A Quantum error-correcting code ($QECC$) is a mapping from k qubits to n

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qubits. Like the bit, the qubit is a unit element in quantum information. In a QECC, k qubits are encoded, and the remaining $n-k$ qbits form a redundancy used to protect information from error [1]. Quantum error-correcting codes are used in quantum information and quantum computing systems to reduce error. Originally quantum codes were studied by Shor [2] and by Steane in [3]. Calderbank et al.[4] constructed a quantum error-correction via codes over the field $GF(4)$. Constructions over finite rings were studied in [5–14] and constructions of new families of nonbinary CSS codes in [15].

Qian et al., presented a new method to obtain self-orthogonal codes over finite field $\mathbb{F}_2$ from cyclic codes over the local ring $\mathbb{F}_2 + u\mathbb{F}_2$ where $u^2 = 0$ in [16]. Qian constructed quantum error-correcting codes of arbitrary lengths from the cyclic codes over the semi-local ring $\mathbb{F}_2 + v\mathbb{F}_2$ where $v^2 = v$ in [5]. Subsequently, that approach was generalized to the ring $\mathbb{F}_p + v\mathbb{F}_p$ with $v^2 = v$ in [17] by Ashraf et al. Quantum codes over $\mathbb{F}_2$ from cyclic codes over the finite ring $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ with $u^2 = u$, $v^2 = v$ and $uv = vu$ in[9] by Dertli et al. Later, in [7] Ashraf et al. constructed quantum codes from cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$ with $u^2 = u, v^2 = v, uv = vu$ and $q = p^m$ where p is an odd prime. Singh et al.[18] constructed binary self-orthogonal codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2$ with $u^3 = u$. In [19], quantum codes from the cyclic codes over $\mathbb{F}_p[u, v, w]/\langle v^2 - 1, u^2 - 1, w^2 - 1, uv - vu, vw - wv, uw - wu\rangle$ was studied by Islam et al., Quantum codes over $\mathbb{F}_p$ from cyclic codes over $\frac{\mathbb{F}_p[u,v]}{< u^2-1, v^3-v, uv-vu >}$ was studied in [20] by Ashraf et al., Gao et al., constructed quantum codes using cyclic codes over the ring $\mathbb{F}_q + v_1\mathbb{F}_q + \cdots + v_r\mathbb{F}_q$ in [21]. Using the algebraic structure of constacyclic and negacyclic codes, a few good quantum codes were constructed in [22–26].

In the present paper, we consider the semi-local ring $\frac{\mathbb{F}_p[u,v]}{\langle v^3-v, u^3-u, uv-vu\rangle}$, of order p^9, for p an odd prime. We study cyclic and constacyclic codes over that ring, and construct QECC over $\mathbb{F}_p$, by using a suitable Gray map. The parameters of the QECC are determined explicitly by decomposing R into local pieces by the idempotent technique.

This paper is organized as follows: Sect. 2 is about preliminaries. In Sect. 3, a new Gray map from $\mathbb{F}_p[u, v]/\langle v^3 - v, u^3 - u, uv - vu\rangle$ to $\mathbb{F}_p^9$ is defined which is bijective and distance preserving; the structure of cyclic codes is determined. We give a construction of quantum codes using cyclic codes in Sect. 4 and present some examples. The structure of λ-constacyclic codes is determined and using $(1 - 2u_k)$-constacyclic codes, we construct a class of quantum codes in Sect. 5 and present some numerical examples.

2 Preliminaries

Let $R = \mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p + v\mathbb{F}_p + uv\mathbb{F}_p + u^2v\mathbb{F}_p + v^2\mathbb{F}_p + uv^2\mathbb{F}_p + u^2v^2\mathbb{F}_p$ where $v^3 = v, u^3 = u, uv = vu$. Then R is a commutative ring with identity of cardinality p^9 and characteristic p, R is not a local ring as the fact that the set of non-unit elements in R is not a subring implies that it is not an ideal.

Consider R^n as R-module. A subset C of R^n is called a *linear code* of length n if it is an R-submodule of R^n. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in C$.

Then the inner product of x and y is defined by $x \cdot y = \sum_{i=1}^{n} x_i y_i$. If the inner product is zero, then x and y are orthogonal. If C is a linear code, then the dual code $C^{\perp} = \{x \mid x \cdot y = 0 \text{ for all } y \in C\}$ of C is also linear. A code C is self-orthogonal if $C \subseteq C^{\perp}$. C is said to be a self-dual code if $C = C^{\perp}$.

If C is invariant under the cyclic shift operator $\delta : R^n \to R^n$ by $\delta(c_1, c_2, \ldots, c_n) = (c_n, c_1, \ldots, c_{n-1})$, then the code C is called a *cyclic code* of length n. A code $C \subseteq \mathbb{F}_p^n$ is cyclic if and only if the corresponding polynomial representation of the code forms an ideal in the ring $\mathbb{F}_p[x]/\langle x^n - 1\rangle$. The generator polynomial of C divides $x^n - 1$. A code C is said to be a *quasi-cyclic of index m* over $\mathbb{F}_p$ if $c = (c^1, c^2, \ldots, c^m) \in C$, where c^i is of length n for every i. $\gamma(c) = (\delta(c^1), \delta(c^2), \ldots, \delta(c^m)) \in C$ where δ represents the cyclic shift of a code.

The Hamming weight of a codeword c is the number of nonzero entries in the codeword and is denoted by $w_H(c)$. The minimum Hamming distance of a code C is $d_H(C) = \min\{d_H(c, c') \mid c, c' \in C \text{ and } c \neq c'\}$ where $d_H(c, c') = w_H(c - c')$.

3 Gray map

Let $R = \mathbb{F}_p[u, v]/\langle v^3 - v, u^3 - u, uv - vu\rangle = \mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p + v\mathbb{F}_p + uv\mathbb{F}_p + u^2v\mathbb{F}_p + v^2\mathbb{F}_p + uv^2\mathbb{F}_p + u^2v^2\mathbb{F}_p$, where $\mathbb{F}_p$ is a field with characteristic p. Then the elements of R are of the form

$$x_1 + uy_1 + u^2z_1 + vx_2 + uvy_2 + vu^2z_2 + v^2x_3 + uv^2y_3 + v^2u^2z_3$$

where $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in \mathbb{F}_p$.

Let $u_1 = 1/4(u^2+u)(v^2+v)$, $u_2 = 1/4(u^2-u)(v^2+v)$, $u_3 = 1/4(u^2+u)(v^2-v)$, $u_4 = 1/4(u^2-u)(v^2-v)$, $u_5 = 1/2(u^2-u)(1-v^2)$, $u_6 = 1/2(u^2+u)(1-v^2)$, $u_7 = 1/2(1-u^2)(v^2-v)$, $u_8 = 1/2(1-u^2)(v^2+v)$, $u_9 = (1-u^2)(1-v^2)$. Then it can be easily seen that

- $u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8 + u_9 = 1$;
- $u_i u_j = 0$ for $i \neq j$ and $u_i^2 = u_i$ for $i = 1, \ldots, 9$;
- $R = u_1R \oplus u_2R \oplus u_3R \oplus u_4R \oplus u_5R \oplus u_6R \oplus u_7R \oplus u_8R \oplus u_9R$

$$\cong u_1\mathbb{F}_p \oplus u_2\mathbb{F}_p \oplus u_3\mathbb{F}_p \oplus u_4\mathbb{F}_p \oplus u_5\mathbb{F}_p \oplus u_6\mathbb{F}_p \oplus u_7\mathbb{F}_p \oplus u_8\mathbb{F}_p \oplus u_9\mathbb{F}_p.$$

The Gray map $\varphi : R \to \mathbb{F}_p^9$ is defined by

$$\varphi\left(\sum_{i=1}^{9} l_i u_i\right) = (l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9)$$

where $l_1, l_2, \ldots, l_9 \in \mathbb{F}_p$. It is easy to see that φ is a $\mathbb{F}_p$-linear map and it can be extended component-wise in the following way. Let $c = (c_1, c_2, \ldots, c_n) \in R^n$ where $c_i = \sum_{j=1}^{9} b_j^i u_j$ then we can define $\varphi : R^n \to \mathbb{F}_p^{9n}$ by

$$\varphi(c_1, c_2, \ldots, c_n) = (b_1^1, b_1^2, \ldots, b_1^n, b_2^1, b_2^2 \ldots, b_2^n, b_3^1, b_3^2 \ldots, b_3^n, b_4^1, b_4^2 \ldots, b_4^n, b_5^1, b_5^2 \cdots, b_5^n, b_6^1, b_6^2, \ldots, b_6^n, b_7^1, b_7^2, \ldots, b_7^n, b_8^1, b_8^2, \ldots, b_8^n, b_9^1, b_9^2 \ldots, b_9^n).$$

The Gray weight of a code is defined as $w_G(\sum_{1=1}^{9} l_i u_i) = w_H(l_1, \ldots, l_9)$. So, the Gray map φ is a weight preserving map from (R^n, Gray weight) to ($\mathbb{F}_p^{9n}$, Hamming weight).

Theorem 3.1 *If C is a self-orthogonal code, then $\varphi(C)$ is also self-orthogonal.*

Proof Let $\varphi(c), \varphi(c') \in \varphi(C)$. Let us take

$$c = (x_1u_1 + y_1u_2 + z_1u_3 + x_2u_4 + y_2u_5 + z_2u_6 + x_3u_7 + y_3u_8 + z_3u_9),$$
$$c' = (x_1'u_1 + y_1'u_2 + z_1'u_3 + x_2'u_4 + y_2'u_5 + z_2'u_6 + x_3'u_7 + y_3'u_8 + z_3'u_9) \in C.$$

Then

$$c.c' = (x_1x_1')u_1 + (y_1y_1')u_2 + (z_1z_1')u_3 + (x_2x_2')u_4 + (y_2y_2')u_5 + (z_2z_2')u_6 + (x_3x_3')u_7 + (y_3y_3')u_8 + (z_3z_3')u_9.$$

Since $c \cdot c' = 0$, $x_1x_1' = y_1y_1' = z_1z_1' = x_2x_2' = y_2y_2' = z_2z_2' = x_3x_3' = y_3y_3' = z_3z_3' = 0$. Hence
$\varphi(c)\varphi(c') = (x_1x_1') + (y_1y_1') + (z_1z_1') + (x_2x_2') + (y_2y_2') + (z_2z_2') + (x_3x_3') + (y_3y_3') + (z_3z_3')$
$= 0$.
Therefore, $\varphi(C)$ is a self-orthogonal code. □

Let $A_i \subseteq R$ for $i = 1, 2$, then

$$A_1 \oplus A_2 = \{a_1 + a_2 \mid a_i \in A_i\} \text{ and } A_1 \otimes A_2 = \{(a_1, a_2) \mid a_i \in A_i\}.$$

Let C be a linear code of length n over R. Define
$C_j = \{m_j \in \mathbb{F}_p^n \mid u_1m_1 + u_2m_2 + u_3m_3 + u_4m_4 + u_5m_5 + u_6m_6 + u_7m_7 + u_8m_8 + u_9m_9,$ for some $m_i \in C, i \neq j\}$, for $j = \{1, 2, \ldots, 9\}$.
Clearly, $C_j's$ are p-ary linear codes of length n. Let C be a linear code of length n over R and $C_i = \{r_i \in R^n \mid$ there exists some $r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_9 \in R^n$ such that $\sum_{i=1}^{9} u_i r_i \in C\}$. Then C_i is a linear code of length n for $1 \leq i \leq 9$. Also, C can be expressed as $C = \oplus_{i=1}^{9} u_i C_i$.

Corollary 3.1 *Let $C = \oplus_{i=1}^{9} u_i C_i$ be a linear code of length n over R. Then the generator matrix for C is $M = \begin{pmatrix} u_1M_1 \\ u_2M_2 \\ \vdots \\ u_9M_9 \end{pmatrix}$, where $M_i's$ $(i = 1, 2, \ldots, 9)$ are generator matrices of the $C_i's$ $(i = 1, 2, \ldots, 9)$, respectively.*

Corollary 3.2 *Let $C = \oplus_{i=1}^{9} u_i C_i$ be a linear code of length n over R. Then $\varphi(C)$ is a $[9n, \sum_{i=1}^{9} k_i, d_H(C)]$, where C_i is a $[n, k_i, d_H(C_i)]$ linear code over $\mathbb{F}_q$ for $1 \leq i \leq 9$ and $d_H(C) = min\{d_H(C_i) \mid i = 1, 2, \ldots, 9\}$.*

Theorem 3.2 *Let $C = \oplus_{i=1}^{9} u_i C_i$ be a linear code of length n over R. Then $C^{\perp} = \oplus_{i=1}^{9} u_i C_i^{\perp}$.*

Proof Let $\overline{C}_i = \{r_i \in \mathbb{F}_q^n \mid$ there exists $r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_9 \in \mathbb{F}_q^n$ such that $\sum_{i=1}^{9} u_i r_i \in C^{\perp}\}$. Then $C^{\perp}$ has the unique expression $C^{\perp} = \oplus_{i=1}^{9} u_i \overline{C}_i$. It is easy to see $\overline{C}_1 \subseteq C_1^{\perp}$. If $z \in C_1^{\perp}$, then $z \cdot x_1 = 0$ for all $x_1 \in C_1$. Let $s = \sum_{i=1}^{9} u_i x_i \in C$. Then $u_1 z s = u_1 x_1 z = 0$, and which implies $u_1 z \in C^{\perp}$. From the construction of $C^{\perp}$, we have $z \in \overline{C}_1$. Therefore, $C_1^{\perp} \subseteq \overline{C}_1$. Hence, $\overline{C}_1 = C_1^{\perp}$. Then by induction we have $C_i^{\perp} = \overline{C}_i$ for $i = 2, \ldots, 9$. Consequently, $C^{\perp} = \oplus_{i=1}^{9} u_i C_i^{\perp}$. □

Theorem 3.3 *A cyclic code C over R satisfies $\gamma(\varphi(c)) = \varphi(\delta(c))$.*

Proof Let C be a cyclic code. Let γ and δ denote the quasi-cyclic shift and cyclic shift operators. Let $c = (c_1, c_2, \ldots, c_n) \in C$, then $\delta(c) = (c_n, c_1, \ldots, c_{n-1})$. Consider $c_i = \sum_{j=1}^{9} b_j^i u_j$ where b_j^i is an element in $\mathbb{F}_p$.

$$\begin{aligned}\varphi(\delta(c)) = {} & (b_1^{n-1}, b_1^1, \ldots, b_1^{n-2}, b_2^{n-1}, \ldots, b_2^{n-2}, b_3^{n-1}, \ldots, b_3^{n-2}, b_4^{n-1}, \ldots, b_4^{n-2}, b_5^{n-1}, \ldots,\\ & b_5^{n-2}, b_6^{n-1}, \ldots, b_6^{n-2}, b_7^{n-1}, \ldots, b_7^{n-2}, b_8^{n-1}, \ldots, b_8^{n-2}, b_9^{n-1}, \ldots, b_9^{n-2})\\ \text{and } \varphi(c) = {} & (b_1^1, b_1^2, \ldots, b_1^{n-1}, b_2^1, \ldots, b_2^{n-1}, b_3^1, \ldots, b_3^{n-1}, b_4^1, \ldots, b_4^{n-1}, b_5^1, \ldots, b_5^{n-1},\\ & b_6^1, \cdots, b_6^{n-1}, b_7^1, \ldots, b_7^{n-1}, b_8^1, \ldots, b_8^{n-1}, b_9^1, \ldots, b_9^{n-1})\end{aligned}$$

This implies

$$\begin{aligned}\gamma(\varphi(c)) = {} & (b_1^{n-1}, b_1^1, \cdots, b_1^{n-2}, b_2^{n-1}, \ldots, b_2^{n-2}, b_3^{n-1}, \ldots, b_3^{n-2}, b_4^{n-1}, \ldots, b_4^{n-2}, b_5^{n-1}, \ldots,\\ & b_5^{n-2}, b_6^{n-1}, \ldots, b_6^{n-2}, b_7^{n-1}, \ldots, b_7^{n-2}, b_8^{n-1}, \ldots, b_8^{n-2}, b_9^{n-1}, \ldots, b_9^{n-2})\end{aligned}$$

Hence $\gamma(\varphi(c)) = \varphi(\delta(c))$. □

Theorem 3.4 *If C is a linear code of length n over R, then $\varphi(C) = C_1 \otimes C_2 \otimes \cdots \otimes C_9$.*

Proof Let $x = (x_1, x_2, \ldots, x_9) \in C_1 \otimes C_2 \otimes \cdots \otimes C_9$ where $x_i = (x_i^1, x_i^2, \ldots, x_i^n) \in C_i$ for $i = 1, 2, \ldots, 9$. Since $C = \oplus_{i=1}^{9} u_i C_i$, is a linear code, there exists an element $a = u_1(x_1^1, x_1^2 \cdots, x_1^n) + u_2(x_2^1, x_2^2 \cdots, x_2^n) + \cdots + u_9(x_9^1, x_9^2 \cdots, x_9^n) \in C$ such that $\varphi(a) = x$. Therefore,
$C_1 \otimes C_2 \otimes \cdots \otimes C_9 \subseteq \varphi(C)$. Since $\varphi(C) \subseteq C_1 \otimes C_2 \otimes \cdots \otimes C_9$, hence we have $\varphi(C) = C_1 \otimes C_2 \otimes \cdots \otimes C_9$. □

4 Construction of quantum codes from cyclic codes

Theorem 4.1 [27]*[CSS Construction] Let C_1 and C_2 be linear codes of parameters $[n, k_1, d_1]$ and $[n, k_2, d_2]$ with $C_2^{\perp} \subseteq C_1$ and let $d = min\{wt(v) \mid v \in$*

$(C_1 \setminus C_2^{\perp}) \bigcup (C_2 \setminus C_1^{\perp})\} \geq min\{d_1, d_2\}$, *then an* $[[n, k_1+k_2-n, min\{d_1, d_2\}]]$ *quantum code exists. If* $C^{\perp} \subseteq C$, *then there exist an* $[[n, 2k_1 - n, d_1]]$ *quantum code.*

Theorem 4.2 *The code* $C = \oplus_{i=1}^{9} u_i C_i$ *over* R *is cyclic code of length* n *if and only if* $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8$ *and* C_9 *are cyclic codes over* $\mathbb{F}_p$ *of length* n.

Proof Assume that C is a cyclic code over R. Let $l^i = (l_1^i, l_2^i, \cdots, l_n^i) \in C_i$ for $1 \leq i \leq 9$ where $l_j^i \in \mathbb{F}_p$. Then $m_i = \sum_{j=1}^{9} u_j l_i^j \in R$ and hence $(m_1, m_2, \ldots, m_n) = \sum_{i=1}^{9} u_i l^i \in C$. Since C is cyclic, $\delta(m_1, m_2, \ldots, m_n) = \sum_{i=1}^{9} u_i \delta(l^i) \in C$. Therefore, $\delta(l^i) \in C_i$ for $1 \leq i \leq 9$ Hence C_i is a cyclic code, for $1 \leq i \leq 9$.

Conversely, we assume that C_i is a cyclic code, for $1 \leq i \leq 9$. Let $(m_1, m_2, \ldots, m_n) \in C$ where $m_i = \sum_{j=1}^{9} u_j l_i^j$ where $l^i = (l_1^i, l_2^i, \ldots, l_n^i) \in C_i$. Since C_i is cyclic, $\delta(l^i) \in C_i$. Therefore, $\sum_{i=1}^{9} u_i \delta(l^i) \in C$. That is, $\delta(m_1, m_2, \ldots, m_n) \in C$. Hence, C is cyclic. □

Corollary 4.1 *Let* $C^{\perp} = \oplus_{i=1}^{9} u_i C_i^{\perp} \subseteq R^n$. *Then* $C^{\perp}$ *is cyclic code if and only if* $C_1^{\perp}, C_2^{\perp}, \ldots, C_9^{\perp}$ *are cyclic codes over* $\mathbb{F}_p$ *of length* n.

The proof of this corollary is similar to that of the above theorem and we omitted.

Theorem 4.3 C *is a linear code which contains its dual code if and only if so do* $C_1, C_2, \ldots, C_9$.

Proof For each i, let

$x_i \in C_i^{\perp}$, then $\sum_{i=1}^{9} x_i u_i \in C^{\perp}$. Since $C^{\perp} \subset C$, $\sum_{i=1}^{9} x_i u_i \in C$ and hence $x_i \in C_i$. Therefore, $C_i^{\perp} \subseteq C_i$ for $1 \leq i \leq 9$.

Conversely, let $x = \sum_{i=1}^{9} x_i u_i \in C^{\perp}$. For $1 \leq i \leq 9$, since $x_i \in C_i^{\perp} \subseteq C_i$, it follows that $x \in C$. □

Corollary 4.2 *Let* $C = \langle u_1 f_1(x), u_2 f_2(x), \ldots, u_9 f_9(x) \rangle$ *be a cyclic code over* R, *then* $C^{\perp} = \langle u_1 h_1^*(x), u_2 h_2^*(x), \ldots, u_9 h_9^*(x) \rangle$ *is a cyclic code over* R *of length* n *where* $h_i^*(x)$ *is the reciprocal polynomial for* $h_i(x) = x^n - 1/g_i(x)$, $h_i^*(x) = x^{deg(h_i(x))} h_i(x^{-1})$ *for* $1 \leq i \leq 9$.

Theorem 4.4 *If* C *is a cyclic code over* R, *then* $C = \langle u_1 f_1(x), u_2 f_2(x), \ldots, u_9 f_9(x) \rangle$ *where* $f_i(x)$ *is the generator polynomial of* C_i *for all* $i = 1, 2, \ldots, 9$.

Proof Since $C = \oplus_i^9 u_i C_i$ and C is cyclic, by Theorem 4.2, C_i is also cyclic for all i and is generated by the polynomial $f_i(x)$ for each i, which implies that,

$$\begin{aligned} C = \; & u_1\langle f_1(x)\rangle + u_2\langle f_2(x)\rangle + u_3\langle f_3(x)\rangle + u_4\langle f_4(x)\rangle + u_5\langle f_5(x)\rangle + u_6\langle f_6(x)\rangle \\ & + u_7\langle f_7(x)\rangle + u_8\langle f_8(x)\rangle + u_9\langle f_9(x)\rangle. \end{aligned}$$

Therefore,

$$C \subseteq \langle u_1 f_1(x), u_2 f_2(x), \ldots, u_9 f_9(x) \rangle.$$

For reverse inclusion, let $g(x) = \sum_{i=1}^{9} u_i g_i(x)$ in $\langle u_1 f_1(x), \ldots, u_9 f_9(x)\rangle$ and $f_i(x)$ divide $g_i(x)$. Then

$$\langle u_1 f_1(x), u_2 f_2(x), \ldots, u_9 f_9(x)\rangle \subseteq C$$

Hence,

$$C = \langle u_1 f_1(x), u_2 f_2(x), \ldots, u_9 f_9(x)\rangle$$

which implies that $\mid C \mid = \prod_{i=1}^{9} \mid C_i \mid = p^{9n-(\sum_{j=1}^{9} k_i)}$, and $\mid C^{\perp} \mid = p^{(\sum_{j=1}^{9} k_i)}$. □

Note that the generator polynomial of the above cyclic code is of the form $f(x) = \sum u_i f_i(x)$. Since $u_i f_i(x) = u_i f(x)$ for all i, $u_i f_i(x) \in \langle f(x)\rangle$ and hence $f_i(x) | x^n - 1$. This implies that there exists $g_i(x) \in \mathbb{F}_p[x]/\langle x^n - 1\rangle$ such that $f_i(x)g_i(x) = x^n - 1$ for all $i = 1, 2, \ldots, 9$. Since $\sum u_i = 1$ and $u_i f_i(x)g_i(x) = u_i(x^n - 1)$, $f_1(x)g_1(x)u_1 + f_2(x)g_2(x)u_2 + \cdots + f_9(x)g_9(x)u_9 = x^n - 1$. Since $u_i u_j = 0$ for $i \neq j$, $(g_1(x)u_1 + g_2(x)u_2 + \cdots + g_9(x)u_9)f(x) = x^n - 1$. Hence $f(x) | x^n - 1$. Thus, we have

Theorem 4.5 *Let $C = \oplus_{i=}^{9} C_i$ be a cyclic code. If $f_i(x)$ is a generator polynomial for C_i, $1 \leq i \leq 9$, then there exists a polynomial $f(x) = \sum_{i=1}^{9} u_i f_i(x)$ generates C and divides $x^n - 1$.*

Corollary 4.3 [4] *A linear cyclic code C contains its dual code if and only if $x^n - 1 \equiv 0 (\mod f(x)f^*(x))$ where $f(x)$ is the generator polynomial of C and $f^*(x)$ is the reciprocal polynomial of $f(x)$.*

Theorem 4.6 *Let $C = \langle u_1 f_1(x), u_2 f_2(x), \ldots, u_9 f_9(x)\rangle$ be a cyclic code of n. Then $C^{\perp} \subseteq C$ if and only if $x^n - 1 \equiv 0 (\mod f_i(x)(f^*)_i(x))$ where $i = 1, 2, \ldots, 9$.*

Proof Let $x^n - 1 \equiv 0 \mod f_i(x)(f^*)_i(x)$. Then Corollary 4.3 implies $C_i^{\perp} \subseteq C_i$, $u_i C_i^{\perp} \subseteq u_i C_i$ and

$$u_1 C_1^{\perp} \oplus u_2 C_2^{\perp} \oplus \cdots \oplus u_9 C_9^{\perp} \subseteq u_1 C_1 \oplus u_2 C_2 \oplus \cdots \oplus u_9 C_9,$$
$$C^{\perp} \subseteq C.$$

Conversely, suppose that $\oplus_{i=1}^{9} C_i^{\perp} \subseteq \oplus_{i=}^{9} C_i$. Then $C_i^{\perp} \subseteq C_i$ for $i = 1, \cdots, 9$. Hence, $x^n - 1 \equiv 0 (\mod f_i(x)(f^*)_i(x))$. □

Corollary 4.4 *Let $\oplus_{i=1}^{9} C_i^{\perp}$ be a cyclic code. If $h_i(x)$ is the generator polynomial for $C_i^{\perp}$, $1 \leq i \leq 9$ then there exists a polynomial $h(x) = \sum_{i=1}^{9} u_i h_i(x)$ generates $C^{\perp}$ and divides $x^n - 1$.*

From Theorems 4.1, 4.2 and 4.3, we have the following result on quantum codes construction directly.

Theorem 4.7 *Let C be a cyclic code over R. If $C_i^{\perp} \subseteq C_i$ then there exists a quantum error code with parameters $[[9n, 2k - 9n, d_G]]$, where d_G denotes the Gray weight of the code C and k is the dimension of $\varphi(C)$.*

Theorem 4.8 [27]*(Quantum Singleton Bound) Let* $C = [[n, k, d]]_q$ *be a quantum error-correction code. Then* $k + 2d \leq n + 2$*, if the equality holds, then it is MDS.*

Example 4.1 Let C be a cyclic code over $R = \frac{\mathbb{F}_5[u,v]}{\langle v^3-v, u^3-u, uv-vu\rangle}$ with length $n = 30$ and $x^{30}-1 = (x+1)^5(x+4)^5(x^2+x+1)^5(x^2+4x+1)^5$. Let $f_1(x) = (x^2+4x+1)$ and $f_i(x) = (x+1)$ for $i = 2, 3, \cdots, 9$, then it is easy to see that $x^{30} - 1 \equiv 0(\mod f_i(x)(f^*)_i(x))$. Then $\varphi(C)$ is a linear code with parameters $[270, 260, 2]$. Hence we obtain a quantum code $[[270, 250, 2]]$.

Example 4.2 Let C be a cyclic code over $R = \frac{\mathbb{F}_5[u,v]}{\langle v^3-v, u^3-u, uv-vu\rangle}$ with length $n = 22$ and $x^{22}-1 = (x+1)(x+4)(x^5+x^4+4x^3+4x^2+3x+1)(x^5+2x^4+4x^3+x^2+x+4)$ $(x^5 + 3x^4 + 4x^3 + 4x^2 + x + 1)(x^5 + 4x^4 + 4x^3 + x^2 + 3x + 4)$.

Let $f_i(x) = (x^5 + x^4 + 4x^3 + 4x^2 + 3x + 1)$ for $i = 1, 2, \cdots, 5$ *and* $f_i(x) = (x^5 + 3x^4 + 4x^3 + 4x^2 + x + 1)$ for $i = 6, 7, 8, 9$. Since $x^{22} - 1 \equiv 0(\mod f_i(x)(f^*)_i(x))$, $\varphi(C)$ is a linear code with parameters $[198, 153, 2]$. We get a quantum code of parameters $[[198, 108, 2]]$.

Example 4.3 Let C be a cyclic code over $R = \frac{\mathbb{F}_{13}[u,v]}{\langle v^3-v, u^3-u, uv-vu\rangle}$ with length $n = 26$ and $x^{26} - 1 = (x + 1)^{13}(x + 12)^{13}$. Let $f_i(x) = (x + 1)$ for $i = 1, 2, \ldots, 9$. Since $x^{26} - 1 \equiv (\mod f_i(x)(f^*)_i(x))$, $\varphi(C)$ is a linear code with parameters $[234, 225, 2]$. Thus, we obtain a quantum code with parameters $[[234, 216, 2]]$.

Example 4.4 Let C be a cyclic code over $R = \frac{\mathbb{F}_5[u,v]}{\langle v^3-v, u^3-u, uv-vu\rangle}$ with length $n = 33$ and
$x^{33}-1 = (x+4)(x^2+x+1)(x^5+2x^4+4x^3+x^2+x+4)(x^5+4x^4+4x^3+x^2+3x+4)$
$(x^{10} + x^9 + 2x^8 + x^7 + 4x^6 + x^5 + 3x^4 + 4x^3 + 3x + 1)$
$(x^{10} + 3x^9 + 4x^7 + 3x^6 + x^5 + 4x^4 + x^3 + 2x^2 + x + 1)$.
Let $f_i(x) = (x^{10}+x^9+2x^8+x^7+4x^6+x^5+3x^4+4x^3+3x+1)$ for $i = 1, 2, \ldots, 5$ and $f_i(x) = (x^{10}+3x^9+4x^7+3x^6+x^5+4x^4+x^3+2x^2+x+1)$ for $i = 6, 7, 8, 9$. Since $x^{33} - 1 \equiv 0(\mod f_i(x)(f^*)_i(x))$, $\varphi(C)$ is a linear code with parameters $[297, 207, 3]$ and hence a Quantum code with parameters $[[297, 117, 3]]$ is obtained.

Example 4.5 Let C be a cyclic code over $R = \frac{\mathbb{F}_7[u,v]}{\langle v^3-v, u^3-u, uv-vu\rangle}$ with length $n = 40$,
$x^{40} - 1 = (x + 1)(x + 6)(x^2 + 1)(x^2 + 3x + 1)(x^2 + 4x + 1)(x^4 + x^3 + x^2 + x + 1)$
$(x^4 + x^3 + 6x^2 + 3x + 1)(x^4 + 3x^3 + 4x^2 + 4x + 1)(x^4 + 3x^3 + 6x^2 + x + 1)$
$(x^4 + 4x^3 + 4x^2 + 3x + 1)(x^4 + 4x^3 + 6x^2 + 6x + 1)(x^4 + 6x^3 + x^2 + 6x + 1)$
$(x^4 + 6x^3 + 6x^2 + 4x + 1)$.
Let $f_i(x) = (x^4 + x^3 + 6x^2 + 3x + 1)$, for $i = 1, \ldots, 5$ and $f_i(x) = (x^4 + 3x^3 + 6x^2 + x + 1)$ for $i = 6, \ldots, 9$. Clearly $x^{40} - 1 \equiv 0(\mod f_i(x)(f^*)_i(x))$ and hence $\varphi(C)$ is a linear code with parameters $[360, 324, 2]$. Thus, we obtain a quantum code with parameters $[[360, 288, 2]]$.

Using Magma Computational Algebra System, we found a few quantum codes over the fields $\mathbb{F}_7, \mathbb{F}_5$ and list them in the following table. Here n is length of a code and $f_i(x)$ is the generating polynomial for C_i where $i = 1, \ldots, 9$, $[n, k, d]$ are parameters for $\varphi(C)$, $[[N, K, D]]_p$ is the quantum code over the field of characteristic p.

n	$f_i(x)$	$[n, k, d]$	$[[N, K, D]]_p$
42	$(x^6 + 4x^5 + x^3 + 2x^2 + x + 3)$	[378, 324, 4]	$[[378, 270, 4]]_7$
24	$(x^2 + x + 2)(x^2 + 2x + 4)$	[216, 180, 3]	$[[216, 144, 3]]_5$
11	$(x^5 + 2x^4 - x^3 + x^2 + x - 1)$	[99, 54, 5]	$[[99, 9, 5]]_5$
31	$(x^3 + 2x + 4)(x^3 + 2x^2 + x + 4)$	[279, 225, 4]	$[[279, 171, 4]]_5$
31	$(4 + x + x^3)$	[279, 252, 3]	$[[279, 225, 3]]_5$
20	$(x + 3)(x + 1)(x + 4)^2$	[180, 144, 3]	$[[180, 108, 3]]_5$
18	$(x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1)$	[162, 108, 3]	$[[162, 54, 3]]_3$
16	$(x^6 + x^5 + x^4 + 2x^3 + 2x + 1)$	[144, 90, 3]	$[[144, 36, 3]]_3$

5 Construction of quantum codes from λ-constacyclic codes

Let $\lambda = (1 - 2u_k)$ or -1 in R, then the λ-constacyclic code C over R^n is defined by whenever $(e_0, e_1, \ldots, e_{n-1}) \in C$ then $(\lambda e_{n-1}, e_0, e_1, \ldots, e_{n-2}) \in C$. We can identify an element in C as a polynomial representation over $\frac{R[x]}{\langle x^n - \lambda \rangle}$ by the following way

$$\theta : C \to \frac{R[x]}{\langle x^n - \lambda \rangle} \text{ by } \theta(e_0, e_1, \ldots, e_{n-1}) = e_0 + e_1 x + \cdots + e_{n-1}x^{n-1}.$$

A subset C of R is said to be a λ-*constacyclic code* if and only if the image of C is an ideal. The generator polynomial of C is a divisor of $x^n - \lambda$. Note that $\lambda^n = 1$ when n is even $\lambda^n = \lambda$ when n is odd. We represent constacyclic shift$(1 - 2u_k)$ by mu, negacyclic shift$(\lambda = -1)$ by η and cyclic $(\lambda = 1)$ shift by δ.

Theorem 5.1 *The linear code $C = \oplus_{i=1}^{9} C_i$ over R is a $(1 - 2u_k)$-constacyclic code of length n if and only if C_k is a negacyclic code and C_i are all cyclic codes over $\mathbb{F}_p$ of length n, where $k \neq i$.*

Proof Assume that C is a $(1-2u_k)$-constacyclic code over R. Let $l^i = (l_0^i, l_2^i, \ldots, l_{n-1}^i)$ $\in C_i$ for $1 \leq i \leq 9$ where $l_j^i \in \mathbb{F}_p$. Then $m_i = \sum_{j=1}^{9} u_j l_i^j \in R$. Since C is $(1 - 2u_k)$-constacyclic,

$$\mu(m_0, m_2, \ldots, m_{n-1}) = ((1 - 2u_k)(u_1 l_{n-1}^1 + u_2 l_{n-1}^2 + \cdots u_9 l_{n-1}^9), \ldots, u_1 l_{n-2}^1 + u_2 l_{n-2}^2 + \cdots u_9 l_{n-2}^9),$$

Since $(u_1 + u_2 + \cdots + u_9)(1 - 2u_k) = -u_k + \sum_{i=1, i\neq k}^{9} u_i$, then $\eta(l^k) \in C_k$, $\delta(l^i) \in C_i \in C_i$, where $k \neq i$.

Hence C_k is a negacyclic code and C_i are cyclic codes, where $k \neq i$.

Conversely, we assume that C_k is a negacyclic code and C_i are cyclic codes, where $k \neq i$. Let $(m_0, m_1, \ldots, m_{n-1}) \in C$ where $m_i = m_i = \sum_{j=1}^{9} u_j l_i^j$ for all i where $l^i = (l_0^i, l_1^i, \ldots, l_{n-1}^i) \in C_i$. Since C_k is negacyclic and C_i is cyclic, $\eta(l^k) \in C_k$ and $\delta(l^i) \in C_i$, we have $u_k \eta(l^k) + \sum_{i=1, i\neq k}^{9} u_i \delta(l^i) = u_k(-l_{n-1}^k, l_0^k, \ldots, l_{n-2}^k) + \sum_{i=1, i\neq k}^{9} u_i(l_{n-1}^i, l_0^i, \ldots, l_{n-2}^i) = (1 - 2u_k)((l_{n-1}^k, l_0^k, \ldots, l_{n-2}^k) + (l_{n-1}^2, l_0^2, \ldots, l_{n-2}^1) + \cdots + (l_{n-1}^9, l_0^9, \ldots, l_{n-2}^9)) = ((1 - 2u_k)(u_1 l_{n-1}^1 + u_2 l_{n-1}^2 + \cdots u_9 l_{n-1}^9), \ldots, u_1 l_{n-2}^1 + u_2 l_{n-2}^2 + \cdots u_9 l_{n-2}^9)$. That is, $(1 - 2u_k)(m_{n-1}, m_0, \ldots, m_{n-2}) \in C$. Hence, C is $(1 - 2u_k)$-constacyclic. □

Theorem 5.2 *C is a λ-constacyclic code if and only if $C^{\perp}$ is a λ-constacyclic code.*

Proof Assume that C is a λ-constacyclic code of length n. Let $x \in C$, $y \in C^{\perp}$. Let $\Lambda(x)_{\lambda}^{i}\lambda = (x_0, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_{n-1})$. Since C is a λ-constacyclic code, $\Lambda(x)_{\lambda}^{n-1} \in C$ we have $\langle \Lambda(x)_{\lambda}^{n-1}, y\rangle = \langle \Lambda(y)_{\lambda}^{n-1}, x\rangle = 0$ which implies $\Lambda(y)_{\lambda}^{n-1} \in C^{\perp}$. Hence $C^{\perp}$ is λ-constacyclic. □

Theorem 5.3 *If C is a $(1-2u_k)$-constacyclic code over R, then $C = \langle u_1 f_1(x), u_2 f_2(x), \ldots, u_9 f_9(x)\rangle$ where $f_i(x)$ is the generator polynomial for C_i for all $i = 1, 2, \ldots, 9$.*

Proof Proof is similar to that of Theorem 4.4. □

Theorem 5.4 *Let $C = \oplus_{i=1}^{9} C_i$ be a $(1-2u_k)$-constacyclic code. If $f_i(x)$ is the generator polynomial for C_i for $1 \le i \le 9$, then there exists a polynomial $f(x) = \sum_{i=1}^{9} u_i f_i(x)$ generates C and divides $x^n - (1-2u_k)$.*

Proof Let $f(x) = \sum_{i=1}^{9} u_i f_i(x)$. Let $k = 1$. Since C is a $(1-2u_1)$-constacyclic code of length n, by above theorem $\langle f_1(x)\rangle = C_1$, $\langle f_i(x)\rangle = C_i$ for $i \in \{2, 3, \ldots, 9\}$, $u_i f(x) \in C$ for all i thus $\langle f(x)\rangle \subseteq C$, hence $C = \langle f(x)\rangle$, $f_1(x)|x^n+1$, $f_i(x)|x^n-1$. This imples that there exist $g_1(x), g_i(x) \in \mathbb{F}_p[x]$ such that $f_1(x)g_1(x) = x^n + 1$ and $f_i(x)g_i(x) = x^n - 1$ for all $i = 2, 3, \ldots, 9$ and hence $u_1 f_1(x)g_1(x) = u_1(x^n + 1)$, $u_i f_i(x)g_i(x) = u_i(x^n - 1)$ for all $i = 2, 3, \ldots, 9$.

Since $\sum u_i = 1$, we have $f_1(x)g_1(x)u_1 + f_2(x)g_2(x)u_2 + \cdots + f_9(x)g_9(x)u_9 = u_1(x^n + 1) + (1 - u_1)(x^n - 1)$

$(g_1(x)u_1 + g_2(x)u_2 + \cdots + g_9(x)u_9)f(x) = x^n - (1-2u_1)$ and hence $f(x)|x^n - (1 - 2u_1)$. It is shown that in a similar way for $k = 2, \ldots, 9$.

Corollary 5.1 *Let $C = \langle u_1 f_1(x), u_2 f_2(x), \ldots, u_9 f_9(x)\rangle$ be a $(1-2u_k)$-constacyclic code over R, then $C^{\perp} = \langle u_1 h_1^*(x), u_2 h_2^*(x), \ldots, u_9 h_9^*(x)\rangle$ be a $(1-2u_k)$-constacyclic code over R of length n where $h_i^*(x)$ is the reciprocal polynomial for $h_i(x) = \frac{x^n-\lambda}{g_i(x)}$ for $1 \le i \le 9$, for some λ depends on k.*

Corollary 5.2 [4] *A linear λ-constacyclic code C contains its dual code if and only if $x^n - \lambda \equiv 0(\mod f(x)f^*(x))$ where $f(x)$ is generator polynomial and $f^*(x)$ is reciprocal of the polynomial $f(x)$.*

Theorem 5.5 *Let $C = \langle u_1 f_1(x), u_2 f_2(x), \ldots, u_9 f_9(x)\rangle$ be a $(1-2u_k)$-constacyclic of length n, and $C^{\perp} \subseteq C$ if and only if $x^n - \lambda \equiv 0(\mod f_i(x)f_i^*(x))$ where $i = 1, 2, \ldots, 9$.*

Proof Proof is similar to Theorem 4.6. □

Theorem 5.6 *Let C be a $(1-2u_k)$-constacyclic code over R. If $C_i^{\perp} \subseteq C_i$ then there exists a quantum error code with parameters $[[9n, 2k - 9n, d_G]]$, where d_G denote the Gray weight of the code C and k is the dimension of $\varphi(C)$.*

Example 5.1 Let C be a $(1-2u_1)$-constacyclic code over $R = \frac{\mathbb{F}_5[u,v]}{\langle v^3-v, u^3-u, uv-vu\rangle}$ with length $n = 21, x^{21}-1 = (x+3)^7(x+5)^7(x+6)^7, x^{21}+1 = (x+1)^7(x+2)^7(x+4)^7$.

Let $f_1(x) = x^4 + 3x^3 + 5x^2 + 5x + 2$ $f_i(x) = x^4 + 4x^3 + 5x^2 + 2x + 2$, for $i = 2, 3, \ldots, 9$. Clearly $x^{21} - 1 \equiv 0(\mod f_1(x)f_1^*(x))$ and $x^{21} + 1 \equiv 0(\mod f_i(x)f_i^*(x))$, for all i then $\varphi(C)$ is a linear code with parameters [189, 153, 3]. Thus, we obtain a quantum code with parameters [[189, 117, 3]]. □

Example 5.2 Let C be a $(1-2u_1)$-constacyclic code over $R = \frac{\mathbb{F}_5[u,v]}{\langle v^3-v, u^3-u, uv-vu\rangle}$, with length $n = 15, x^{15} + 1 = (x+1)^5(x^2+4x+1)^5, x^{15} - 1 = (x-1)^5(x^2+x+1)^5$

Let $f_1(x) = (x^2+4x+1)^2$ $f_i(x) = (x^2+4x+1)^2$, for $i = 2, 3, \ldots, 9$. Clearly, $x^{15} - 1 \equiv 0(\mod f_1(x)f_1^*(x))$, $x^{15} + 1 \equiv 0(\mod f_i(x)f_i^*(x))$, for all i then $\varphi(C)$ is a linear code with parameters [135, 99, 3]. Thus, we obtain a quantum code with parameters [[135, 63, 3]].

Example 5.3 Let C be a $(1-2u_1)$-constacyclic code over $R = \frac{\mathbb{F}_{11}[u,v]}{\langle v^3-v, u^3-u, uv-vu\rangle}$, with length $n = 21$, $x^{21} + 1 = (x+1)(x^2+10x+1)(x^3+4x^2+6x+1)(x^3+6x^2+4x+1)(x^6+5x^5+10x^4+10x^2+7x+1)(x^6+7x^5+10x^4+10x^2+5x+1)$, $x^{21} - 1 = (x+10)(x^2+x+1)(x^3+5x^2+4x+10)(x^3+7x^2+6x+10)(x^6+4x^5+10x^4+10x^2+6x+1)(x^6+6x^5+10x^4+10x^2+4x+1)$.

let $f_1(x) = x^6+5x^5+10x^4+10x^2+7x+1$ $f_i(x) = x^6+6x^5+10x^4+10x^2+4x+1$, for $i = 2, 3, \ldots, 9$. Clearly $x^{21} - 1 \equiv 0(\mod f_1(x)f_1^*(x))$, $x^{21} + 1 \equiv 0(\mod f_i(x)f_i^*(x))$, for all i then $\varphi(C)$ is a linear code with parameters [189, 135, 3]. Thus, we obtain a quantum code with parameters [[189, 81, 3]].

We have obtained some quantum codes by using $(1-2u_1)$-constacyclic codes and list them in the following table where n is the length of the code and $f_i(x)$ represents the generator polynomial for C_i. Polynomial representation in this tabular is given by writing coefficients, for example 2, 0, 2, 3 represents $2x^3 + 2x + 3$.

n	$f_1(x)$	$f_i(x)$	$[n,k,d]$	$[[N,K,D]]_p$
39	1, 11, 3, 11, 1	1, 2, 3, 2, 1	[351, 315, 3]	$[[351, 279, 3]]_{13}$
31	1, 2, 4, 4, 3, 2, 1	1, 3, 3, 1, 4, 3, 1	[279, 225, 4]	$[[279, 171, 4]]_5$
15	1, 3, 3, 3, 1	1, 2, 3, 2, 1	[135, 99, 3]	$[[135, 63, 3]]_5$

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References

1. Calderbank, A.R., Shor, P.W.: Good quantum error-correcting codes exist. Phys. Rev. A **54**, 1098–1105 (1996)
2. Shor, P.W.: Scheme for reducing decoherence in quantum memory. Phys. Rev. A **52**, 2493–2496 (1995)
3. Steane, A.M.: Simple quantum error-correcting codes. Phys. Rev. A **54**, 4741–4751 (1996)

4. Calderbank, A.R., Rains, E.M., Shor, P.M., Sloane, N.J.A.: Quantum error-correction via codes over $GF(4)$. IEEE Trans. Inf. Theory **44**, 1369–1387 (1998)
5. Qian, J.: Quantum codes from cyclic codes over $\mathbb{F}_2 + v\mathbb{F}_2$. J. Inf. Comput. Sci. **10**(6), 1715–1722 (2013)
6. Ashraf, M., Mohammad, G.: Quantum codes from cyclic codes over $\mathbb{F}_3 + v\mathbb{F}_3$. Int. J. Quantum Inf. **12**(6), 1450042 (2014)
7. Ashraf, M., Mohammad, G.: Quantum codes from cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$. Quantum Inf. Process. **15**(10), 4089–4098 (2016)
8. Dertli, A., Cengellenmis, Y.: Quantum codes obtained from cyclic codes over A_3. Acta Univ. Apulensis **51**, 31–39 (2017)
9. Dertli, A., Cengellenmis, Y., Eren, S.: On quantum codes obtained from cyclic codes over A_2. Int. J. Quantum Inf. **13**(3), 1550031 (2015)
10. Gao, J.: Quantum codes from cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q + v^2\mathbb{F}_q + v^3\mathbb{F}_q$. Int. J. Quantum Inf. **13**(8), 1550063 (2015)
11. Yin, X., Ma, W.: Gray map and quantum codes over the ring. In: Int. Joint Conf. of IEEE TrustCom-11 (2011)
12. Feng, K., Ling, S., Xing, C.: Asymptotic bounds on quantum codes from algebraic geometry codes. IEEE Trans. Inf. Theory **52**(3), 986–991 (2006)
13. Kai, X., Zhu, S.: Quaternary construction of quantum codes from cyclic codes over $\mathbb{F}_4 + u\mathbb{F}_4$. Int. J. Quantum Inf. **9**(2), 689–700 (2011)
14. Li, R., Li, X.: Binary construction of quantum codes of minimum distance three and four. IEEE Trans. Inf. Theory **50**(6), 1331–1335 (2004)
15. Gaurdia, G., Palazzo Jr., R.: Constructions of new families of nonbinary CSS codes. Discrete Math **310**(21), 2935–2945 (2010)
16. Qian, J., Ma, W., Gou, W.: Quantum codes from cyclic codes over finite ring. Int. J. Quantum Inf. **7**(6), 1277–1283 (2009)
17. Ashraf, M., Mohammad, G.: Construction of quantum codes from cyclic codes over $\mathbb{F}_p + v\mathbb{F}_p$. Int. J. Inf. Coding Theory **3**(2), 137–144 (2015)
18. Singh, A.K., Pattanayek, S., Kumar, P.: On quantum codes from cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2$. Asian-Eur. J. Math. **11**(1), 1850009 (2018)
19. Islam, H., Prakash, O.: Quantum codes from the cyclic codes over $\mathbb{F}_p[u, v, w]/\langle v^2 - 1, u^2 - 1, w^2 - 1, uv - vu, vw - wv, uw - wu\rangle$. JAMC **60**(1), 625–635 (2019)
20. Ashraf, M., Mohammad, G.: Quantum codes over $\mathbb{F}_p$ from cyclic codes over $\mathbb{F}_p[u, v]/\langle u^2 - 1, v^3 - v, uv - vu\rangle$. Cryptogr. Commun. **11**(2), 325–335 (2019)
21. Gao, Y., Gao, J., Fu, F.-W.: Quantum codes from cyclic codes over the ring $\mathbb{F}_q + v_1\mathbb{F}_q + \cdots + v_r\mathbb{F}_q$. AAECC **30**(2), 161–174 (2018)
22. Gao, J., Wang, Y.: Quantum codes derived from negacyclic codes. Int. J. Theor. Phys. **57**, 682–686 (2018)
23. Gao, J., Wang, Y.: u-Constacyclic codes over $\mathbb{F}_q + u\mathbb{F}_q$ and their applications of constructing new non-binary quantum codes. Quantum Inf. Process. **17**(1), 4 (2018)
24. Li, J., Gao, J., Wang, Y.: Quantum codes from $(1 - -2v)$-constacyclic codes over the ring $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$. Discrete Math. Algorithms Appl. **14**(3), 1850046 (2015)
25. Wang, L., Zhu, S.: New quantum MDS codes derived from constacyclic codes. Quantum Inf. Process. **10**(4), 881–889 (2018)
26. Bag, T., Ashraf, M., Mohammad, G., Upadhyay, A.K.: Quantum codes from $(1 - -2u_1 - 2u_2 - \cdots - 2u_m)$-skew constacyclic codes over the ring $\mathbb{F}_q + u_1\mathbb{F}_q + \cdots + u_{2m}\mathbb{F}_q$. Quantum Inf. Process. **18**(9), 270–285 (2019)
27. Grassl, M., Beth, T.: On optimal quantum codes. Int. J. Quantum Inf. **2**, 55–64 (2004)