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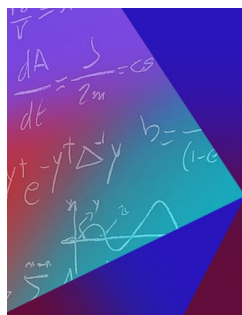
Painlevé analysis, Prolle–Singer approach, symmetries and integrability of damped Hénon–Heiles system

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ABSTRACT

We consider a modified damped version of Hénon–Heiles system and investigate its integrability. By extending the Painlevé analysis of ordinary differential equations we find that the modified Hénon–Heiles system possesses the Painlevé property for three distinct parametric restrictions. For each of the identified cases, we construct two independent integrals of motion using the well known Prelle–Singer method. We then derive a set of nontrivial non-point symmetries for each of the identified integrable cases of the modified Hénon–Heiles system. We infer that the modified Hénon–Heiles system is integrable for three distinct parametric restrictions. Exact solutions are given explicitly for two integrable cases.

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I. INTRODUCTION

The study of nonlinear differential equations has attracted much attention due to various reasons during the past several decades.^{1–3} Nonlinear differential equations are not solvable or integrable in general and they possess various exciting mathematical structures and physical features. Given a nonlinear differential equation, there exists no unique method to determine whether it is integrable or not. There exist several definitions for integrability such as Lax integrability, algebraic integrability, Liouville integrability, etc. in the literature.² Several analytic and numerical methods have been devised by different research groups to deal with nonlinear differential equations in general and integrability in particular.^{1,2,4–6} Among them, the Painlevé analysis proposed by Ablowitz, Ramani, and Segur for ordinary differential equations (ODE) and extended to partial differential equations (PDE) by Weiss, Tabor, and Carnevale played a vital role in identifying the integrable cases of both nonlinear ODEs and PDEs during the past few decades.^{3,7–10} The Painlevé test is based on the realization that the integrability of differential equations is known to be related to the singularity structure of the solution. A nonlinear differential equation is said to possess the Painlevé property if its general solution is single valued about the movable singular point in the case of ODE and movable singularity manifold for PDEs. A nonlinear dynamical system governed either by ODE or PDE passes the Painlevé test then it is expected to be integrable and have a solution that can be described by an appropriate Laurent series expansion locally.^{3,11,12} The effectiveness of the Painlevé analysis has been widely demonstrated by identifying several new integrable systems.^{12–14} Another important problem of nonlinear differential equations is to find its integrals of motion. If an N th order nonlinear ODE admits $N - 1$ functionally independent integrals then it is completely integrable. Also, the existence of integrals of motion enables one to reduce the order of the differential equation. Several analytic methods have been devised to find integrals of motion for nonlinear differential equations.^{2,15–18} Darboux has proposed a systematic method to construct integral of motion for Hamiltonian systems with two degrees of freedom. Chandrasekhar has successfully exploited the idea of Darboux to construct

second and third integrals of motion for Hamiltonian systems with three degrees of freedom.^{19,20} Recent investigations reveal that the direct method proposed by Prelle and Singer provides an efficient tool to find integrals of motion for coupled nonlinear second order ODEs.^{15,21,22} Also, it is known that there exists a deep connection between the existence of nontrivial symmetries and integrals of motion of nonlinear differential equations.^{23,24}

It is known that the celebrated Hénon–Heiles (HH) system described by the Hamiltonian

$$H = \frac{1}{2} \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right) + \frac{1}{2} (Ax^2 + By^2) + \left(\alpha x^2 - \frac{1}{3} \beta y^2 \right) y, \quad (1.1)$$

with equations of motion

$$\frac{d^2x}{dt^2} + Ax + 2\alpha xy = 0, \quad (1.2)$$

$$\frac{d^2y}{dt^2} + By + \alpha x^2 - \beta y^2 = 0, \quad (1.3)$$

where A, B, α and β are constants is completely integrable in the sense of Liouville for the following three distinct cases:

- (i) $A = B, \beta = -\alpha$.
- (ii) A, B arbitrary, $\beta = -6\alpha$.
- (iii) $B = 16A, \beta = -16\alpha$.

If an integrable nonlinear dynamical system undergoes perturbation then the resulting system may not be integrable in general. With this aim, we introduce a damping term in the equations of motion of HH system in the following form:

$$\frac{d^2x}{dt^2} + \phi_1 \frac{dx}{dt} + Ax + 2\alpha xy = 0, \quad (1.4)$$

$$\frac{d^2y}{dt^2} + \phi_2 \frac{dy}{dt} + By + \alpha x^2 - \beta y^2 = 0, \quad (1.5)$$

where ϕ_1 and ϕ_2 are constants in addition to with A, B, α and β . We refer to the above as the modified Hénon–Heiles system equations. We also note here that one can always use the scaling transformation available for the independent variable t to fix the parameter $\alpha = 1$ without loss of generality and also to agree with the original Hénon–Heiles system in the absence of damping. However, we prefer to keep “ α ” as arbitrary here for convenience in order to compare with the various works in the literature on the generalization of Hénon–Heiles system and one can always fix $\alpha = 1$ whenever necessary.

The aim of this article is to identify the parametric restrictions at which (i) the general solution of it is single valued about the movable singular point (ii) admits two functionally independent integrals of motion and (iii) admits more than one set of nontrivial non-point symmetries through the Painlevé analysis, Prelle–Singer method and Lie symmetry analysis, respectively. The plan of the article is as follows. In Sec. II, we extend the Painlevé analysis of ODEs to the modified Hénon–Heiles system governed by the above system [Eqs. (1.4) and (1.5)] of two-coupled second order ODEs and show that its solution is single valued about the movable singular point and admit four arbitrary constants for three distinct parametric restrictions. In Sec. III, we apply the Prelle–Singer method to the system of two coupled second order ODEs associated with the modified Hénon–Heiles system and derive two independent integrals for each of the identified cases ensuring their integrability. In Sec. IV, we derive a set of nontrivial non-point symmetries for each of the identified integrable cases of the modified Hénon–Heiles system. An attempt is made to find exact solution, if it exists, for each one of the three identified cases in Sec. V. A summary of the results of our investigation is given in Sec. VI.

II. PAINLEVÉ ANALYSIS OF THE MODIFIED HÉNON-HEILES SYSTEM

To start with we give a brief summary of the Painlevé analysis for a scalar n th order nonlinear ODE having the form¹²

$$F\left(t, x, \frac{dx}{dt}, \dots, \frac{d^n x}{dt^n}\right) = 0, \quad (2.1)$$

where F is analytic in t and rational in other arguments. In the Painlevé analysis, one looks for solution of Eq. (2.1) expressed as a Laurent series

$$x(t) = (t - t_0)^{q_j} \sum_k a_k (t - t_0)^k, \quad \text{Re}(q_j) < 0, \quad 0 < |t - t_0| < R \quad (2.2)$$

in the neighborhood of an arbitrary singular point t_0 .

The Painlevé analysis for ODEs essentially consists of the following three steps:

- (a) determination of leading-order behaviour of the Laurent series,
- (b) determination of resonances, that is the powers at which arbitrary constants of the solution of (2.2) enter into the Laurent series expansion; and
- (c) verification that a sufficient number of arbitrary constants exist without the introduction of movable critical points. We would like to remind that the Painlevé analysis provides only necessary condition for integrability of the considered nonlinear ODE. The sufficient condition for integrability of it has to be established by other means, for example by constructing its sufficient number of integrals of motion.

The remaining part of the section contains the details of the Painlevé analysis of the modified Hénon–Heiles system governed by

$$\frac{d^2x}{dt^2} + \phi_1 \frac{dx}{dt} + Ax + 2\alpha xy = 0, \quad (2.3)$$

$$\frac{d^2y}{dt^2} + \phi_2 \frac{dy}{dt} + By + \alpha x^2 - \beta y^2 = 0. \quad (2.4)$$

A. Leading-order behaviors

Let us assume that the leading order behavior of $x(t)$ and $y(t)$ be

$$x(t) \approx a_0 \tau^p, \quad y(t) \approx b_0 \tau^q, \quad \tau = (t - t_0) \rightarrow 0 \quad (2.5)$$

in a neighborhood of the movable singularity t_0 , which results to the leading-order equations

$$a_0 p(p-1) \tau^{p-2} + 2\alpha a_0 b_0 \tau^{p+q} = 0, \quad (2.6)$$

$$b_0 q(q-1) \tau^{q-2} + \alpha a_0^2 \tau^{2p} - \beta b_0^2 \tau^{2q} = 0. \quad (2.7)$$

From Eqs. (2.6) and (2.7) we identify the following two distinct sets of possibilities.

Case 1.

$$p = -2, \quad q = -2, \quad a_0 = \pm \frac{3}{\alpha} \sqrt{2 + \frac{\beta}{\alpha}}, \quad b_0 = -\frac{3}{\alpha}. \quad (2.8)$$

Case 2.

$$p = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{48\alpha}{\beta}}, \quad q = -2, \quad a_0 = \text{arbitrary}, \quad b_0 = \frac{6}{\beta}. \quad (2.9)$$

B. Resonance

To identify the resonance values, we substitute

$$x(t) \approx a_0 \tau^p + \omega_1 \tau^{p+r}, \quad y(t) \approx b_0 \tau^q + \omega_2 \tau^{q+r}, \quad \tau = (t - t_0) \rightarrow 0 \quad (2.10)$$

into Eqs. (2.3) and (2.4), retaining only the leading-order terms. This results into the following system of linear algebraic equations,

$$M_2(r) \Omega = 0, \quad \Omega = (\omega_1, \omega_2), \quad (2.11)$$

where $M_2(r)$ is a 2×2 matrix depending on r .

For Case 1, the form of $M_2(r)$ is

$$M_2(r) = \begin{bmatrix} (r-2)(r-3) + 2\alpha b_0 & 2\alpha a_0 \\ 2\alpha a_0 & (r-2)(r-3) - 2\beta b_0 \end{bmatrix}.$$

To have nontrivial solutions for (ω_1, ω_2) , we require $\det(M_2(r)) = 0$,

$$\Rightarrow (r+1)(r-6) \left(r^2 - 5r + 12 + \frac{6\beta}{\alpha} \right) = 0 \quad (2.12)$$

and so

$$r = -1, 6, \frac{5}{2} \pm \frac{1}{2} \left[1 - 24 \left(1 + \frac{\beta}{\alpha} \right) \right]^{\frac{1}{2}}. \quad (2.13)$$

In a similar manner for case 2 we find that the resonances occur at

$$r = -1, 0, (1 - 2p), 6. \quad (2.14)$$

Demanding that the resonance value r be a non-negative integer other than -1 , Eqs. (2.13) and (2.14) allow the following parametric cases and resonance values.

$$\text{Case 1(a)} : p = -2, \quad q = -2; \quad \beta = -\alpha; \quad r = -1, 2, 3, 6. \quad (2.15)$$

$$\text{Case 1(b)} : p = -2, \quad q = -2; \quad \beta = -\frac{4}{3}\alpha; \quad r = -1, 1, 4, 6. \quad (2.16)$$

$$\text{Case 1(c)} : p = -2, \quad q = -2; \quad \beta = -2\alpha; \quad r = -1, 0, 5, 6. \quad (2.17)$$

$$\text{Case 2(a)} : p = -\frac{3}{2}, \quad q = -2; \quad \beta = -\frac{16}{5}\alpha; \quad r = -1, 0, 4, 6. \quad (2.18)$$

$$\text{Case 2(b)} : p = -1, \quad q = -2; \quad \beta = -6\alpha; \quad r = -1, 0, 3, 6. \quad (2.19)$$

$$\text{Case 2(c)} : p = -\frac{1}{2}, \quad q = -2; \quad \beta = -16\alpha; \quad r = -1, 0, 2, 6. \quad (2.20)$$

C. Evaluation of arbitrary constants

Obviously the movable singular point t_0 is arbitrary, associated with resonance value -1 . To check if the appropriate constants are arbitrary corresponding to other resonance values, we introduce the following series expansions,

$$x(t) = \sum_{k=0}^6 a_k \tau^{p+k}, \quad y(t) = \sum_{k=0}^6 b_k \tau^{q+k}, \quad \tau = (t - t_0), \quad (2.21)$$

in all the terms of (2.3) and (2.4). As a result, we obtain a system of 2-coupled recurrence relations. For example, for case 1, the recurrence relations read

$$(k-2)(k-3)a_k + (k-3)\phi_1 a_{k-1} + Aa_{k-2} + 2\alpha \sum_l a_l b_{k-l} = 0, \quad (2.22)$$

$$(k-2)(k-3)b_k + (k-3)\phi_2 b_{k-1} + Bb_{k-2} + \alpha \sum_l a_l a_{k-l} - \beta \sum_l b_l b_{k-l} = 0. \quad (2.23)$$

Solving the above set of recurrence relations successively one can obtain the various a_k and b_k explicitly. For example, for case 1(a), when $k = 0, 1$ respectively we find [Here, $a_{-1} = b_{-1} = a_{-2} = b_{-2} = 0$]

$$a_0 = b_0 = -\frac{3}{\alpha}, \quad a_1 = -\frac{3}{5\alpha}(2\phi_1 - 3\phi_2), \quad b_1 = \frac{3}{5\alpha}(3\phi_1 - 2\phi_2). \quad (2.24)$$

For $k = 2$ we have

$$2\alpha(b_0 a_2 + a_0 b_2) = (\phi_1 - 2\alpha b_1)a_1 - Aa_0, \quad (2.25)$$

$$2\alpha(a_0 a_2 + b_0 b_2) = -\alpha(a_1^2 + b_1^2) + \phi_2 b_1 - Bb_0. \quad (2.26)$$

Solving the above equations we find

$$a_2 = \text{arbitrary}, \quad b_2 = -\frac{25A + 26\phi_1^2 - 63\phi_1\phi_2 + 36\phi_2^2 + 50a_2\alpha}{50\alpha}, \quad (2.27)$$

$$B = A - 6\phi_1\phi_2 + \frac{17}{5}\phi_2^2 + \frac{13}{5}\phi_1^2. \quad (2.28)$$

For $k = 3$ we have

$$2\alpha(b_0 a_3 + a_0 b_3) = -2\alpha(a_1 b_2 + a_2 b_1) - Aa_1, \quad (2.29)$$

$$2\alpha(a_0a_3 + b_0b_3) = -2\alpha(a_1a_2 + b_1b_2) - Bb_1. \quad (2.30)$$

Solving the above equations we find

$$a_3 = -b_3 + \frac{\phi_1^3}{250\alpha}, \quad b_3 = \text{arbitrary}, \quad \phi_2 = \phi_1. \quad (2.31)$$

Proceeding further we find a_4, b_4, a_5 and b_5 explicitly. However the coefficient either a_6 or b_6 is arbitrary only if

$$B = A = \frac{6}{25}\phi_1^2. \quad (2.32)$$

We thus infer that the modified Hénon–Heiles system admits a single valued solution with four arbitrary constants for the parametric restrictions

$$\beta = -\alpha, \quad \phi_2 = \phi_1 = \phi, \quad A = B = \frac{6}{25}\phi^2. \quad (2.33)$$

Proceeding in a similar manner we find that for the following restrictions the modified Hénon–Heiles system admits Laurent series solutions with four arbitrary constants:

$$\text{Case 2(b)} : \quad A = B, \quad \beta = -6\alpha, \quad \phi_1 = \phi_2 = \phi, \quad B = \frac{6}{25}\phi^2, \quad (2.34)$$

$$\text{Case 2(c)} : \quad A = B, \quad \beta = -16\alpha, \quad \phi_1 = \phi_2 = \phi, \quad B = \frac{6}{25}\phi^2, \quad (2.35)$$

We would like to mention that for the remaining cases (1b), (1c) and (2a) there exists a Laurent series solution with only three or two arbitrary constants and consequently special exact solutions may exist for these cases.

D. Remark

We also mention here the fact that on taking the condition $\phi_1 = \phi_2 = \phi$ in (2.3) and (2.4) it leads directly to Painlevé integrability as follows: If one changes the variables x, y to x_1, x_2 such that

$$x = x_1 e^{-\frac{\phi}{2}t}, \quad y = x_2 e^{-\frac{\phi}{2}t}, \quad (2.36)$$

one obtains from (2.3) and (2.4) a system of equations without a friction term where the exponential dependence on time only enters multiplicatively in the nonlinear terms,

$$\frac{d^2x_1}{dt^2} = \left(\frac{\phi^2}{4} - A\right)x_1 - 2\alpha x_1 x_2 e^{-\frac{\phi}{2}t}, \quad \frac{d^2x_2}{dt^2} = \left(\frac{\phi^2}{4} - B\right)x_2 - (\alpha x_1^2 - \beta x_2^2)e^{-\frac{\phi}{2}t}. \quad (2.37)$$

Then, the Painlevé analysis directly applies to (2.37) above to show that all solutions are Laurent series, since a Taylor series expansion of $e^{-\frac{\phi}{2}t}$ would not affect the lower order calculations, where free constants enter, and the Painlevé property is established.

III. THE PRELLE-SINGER METHOD

Given a scalar or coupled second order ODE there exist methods in the literature to find its integrals of motion, if they exist. In this article, we restrict our attention to the Prelle–Singer method. To start with we provide a brief account of the Prelle–Singer method for coupled second order ODEs.^{15,21,22}

A. The Prelle–Singer method for two-coupled second order ODEs

Consider a system of two-coupled second order ODEs having the form

$$\ddot{x} = \frac{P_1(t, x, y, \dot{x}, \dot{y})}{Q_1(t, x, y, \dot{x}, \dot{y})} = \Psi_1(x, y, \dot{x}, \dot{y}), \quad (3.1)$$

$$\ddot{y} = \frac{P_2(t, x, y, \dot{x}, \dot{y})}{Q_2(t, x, y, \dot{x}, \dot{y})} = \Psi_2(x, y, \dot{x}, \dot{y}). \quad (3.2)$$

where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, $\ddot{y} = \frac{d^2y}{dt^2}$, $P_i, Q_i, i = 1, 2$ are polynomials in $(t, x, y, \dot{x}, \dot{y})$ with real or complex coefficients.

Let us assume that the system (3.1) and (3.2) admits a first integral of the form

$$I(t, x, y, \dot{x}, \dot{y}) = C, \quad (3.3)$$

where C is constant and so

$$dI = I_t dt + I_x dx + I_y dy + I_{\dot{x}} d\dot{x} + I_{\dot{y}} d\dot{y} = 0, \quad (3.4)$$

where each subscript denotes partial derivative with respect to that variable. Equations (3.1) and (3.2) can be rewritten as

$$\frac{P_1}{Q_1} dt - d\dot{x} = 0, \quad (3.5)$$

$$\frac{P_2}{Q_2} dt - d\dot{y} = 0. \quad (3.6)$$

Adding a null term $S_1(t, x, y, \dot{x}, \dot{y})\dot{x}dt - S_1(t, x, y, \dot{x}, \dot{y})dx$ and $S_2(t, x, y, \dot{x}, \dot{y})\dot{y}dt - S_2(t, x, y, \dot{x}, \dot{y})dy$ to (3.5) and (3.6), respectively, we obtain the following equations

$$(\Psi_1 dt + S_1 \dot{x})dt - S_1 dx - d\dot{x} = 0, \quad (3.7)$$

$$(\Psi_2 dt + S_2 \dot{y})dt - S_2 dy - d\dot{y} = 0. \quad (3.8)$$

Let $R_1(t, x, y, \dot{x}, \dot{y})$ and $R_2(t, x, y, \dot{x}, \dot{y})$ be the appropriate integrating factors for (3.7) and (3.8), respectively. Hence we have

$$dI = R_1(\Psi_1 + S_1 \dot{x})dt + R_2(\Psi_2 + S_2 \dot{y})dt - R_1 S_1 dx - R_2 S_2 dy - R_1 d\dot{x} - R_2 d\dot{y} = 0. \quad (3.9)$$

On the solutions, we require that (3.4) and (3.9) be proportional. Comparing Eq. (3.4) with (3.9) we have,

$$I_t = R_1(\Psi_1 + S_1 \dot{x}) + R_2(\Psi_2 + S_2 \dot{y}), \quad (3.10)$$

$$I_x = -R_1 S_1, \quad I_y = -R_2 S_2, \quad (3.11)$$

$$I_{\dot{x}} = -R_1, \quad I_{\dot{y}} = -R_2. \quad (3.12)$$

The compatibility conditions,

$$I_{tx} = I_{xt}, \quad I_{ty} = I_{yt}, \quad I_{t\dot{x}} = I_{\dot{x}t}, \quad I_{t\dot{y}} = I_{\dot{y}t}, \quad I_{xy} = I_{yx}, \\ I_{x\dot{x}} = I_{\dot{x}x}, \quad I_{x\dot{y}} = I_{\dot{y}x}, \quad I_{y\dot{x}} = I_{\dot{x}y}, \quad I_{y\dot{y}} = I_{\dot{y}y}, \quad I_{\dot{x}\dot{y}} = I_{\dot{y}\dot{x}},$$

on the Eqs. (3.10)–(3.12) leads to the following conditions,

$$D[S_1] = -\Psi_{1x} - \frac{R_2}{R_1} \Psi_{2x} + \frac{R_2}{R_1} S_1 \Psi_{2\dot{x}} + S_1 \Psi_{1\dot{x}} + S_1^2, \quad (3.13)$$

$$D[S_2] = -\Psi_{2y} - \frac{R_1}{R_2} \Psi_{1y} + \frac{R_1}{R_2} S_2 \Psi_{1\dot{y}} + S_2 \Psi_{2\dot{y}} + S_2^2, \quad (3.14)$$

$$D[R_1] = \frac{\partial R_1}{\partial t} + \dot{x} \frac{\partial R_1}{\partial x} + \dot{y} \frac{\partial R_1}{\partial y} + \ddot{x} \frac{\partial R_1}{\partial \dot{x}} + \ddot{y} \frac{\partial R_1}{\partial \dot{y}} \\ = -(R_1 \Psi_{1\dot{x}} + R_2 \Psi_{2\dot{x}} + R_1 S_1), \quad (3.15)$$

$$D[R_2] = \frac{\partial R_2}{\partial t} + \dot{x} \frac{\partial R_2}{\partial x} + \dot{y} \frac{\partial R_2}{\partial y} + \ddot{x} \frac{\partial R_2}{\partial \dot{x}} + \ddot{y} \frac{\partial R_2}{\partial \dot{y}} \\ = -(R_2 \Psi_{2\dot{y}} + R_1 \Psi_{1\dot{y}} + R_2 S_2), \quad (3.16)$$

$$S_1 R_{1y} = -R_1 S_{1y} + S_2 R_{2x} + R_2 S_{2x}, \quad (3.17)$$

$$R_{1x} = S_1 R_{1\dot{x}} + R_1 S_{1\dot{x}}, \quad R_{1y} = S_2 R_{2\dot{x}} + R_2 S_{2\dot{x}}, \quad (3.18)$$

$$R_{2y} = S_2 R_{2\dot{y}} + R_2 S_{2\dot{y}}, \quad R_{2x} = S_1 R_{1\dot{y}} + R_1 S_{1\dot{y}}, \quad (3.19)$$

$$R_{1\dot{y}} = R_{2\dot{x}}, \quad (3.20)$$

where D is the total differential operator given by

$$D = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \ddot{x} \frac{\partial}{\partial \dot{x}} + \ddot{y} \frac{\partial}{\partial \dot{y}}.$$

Integrating Eqs. (3.10)–(3.12) we obtain the corresponding integral of motion,

$$I = \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 - \int \left[R_2 + \frac{\partial}{\partial \dot{y}} (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4) \right] d\dot{y}, \quad (3.21)$$

where

$$\begin{aligned} \kappa_1 &= \int (R_1(\Psi_1 + S_1 \dot{x}) + R_2(\Psi_2 + S_2 \dot{y})) dt, \quad \kappa_2 = - \int \left(R_1 S_1 + \frac{\partial}{\partial x} (\kappa_1) \right) dx, \\ \kappa_3 &= - \int \left(R_2 S_2 + \frac{\partial}{\partial y} (\kappa_1 + \kappa_2) \right) dy, \quad \kappa_4 = - \int \left(R_1 + \frac{\partial}{\partial \dot{x}} (\kappa_1 + \kappa_2 + \kappa_3) \right) d\dot{x}. \end{aligned}$$

The determining Eqs. (3.13)–(3.20) are highly coupled and cannot be solved directly. Hence we use the following method to solve it.

To start with, we can identify the following two identities using (3.13)–(3.16),

$$D[R_1 S_1] = D(R_1) S_1 + R_1 D(S_1) = -(R_1 \Psi_{1x} + R_2 \Psi_{2x}), \quad (3.22)$$

$$D[R_2 S_2] = D(R_2) S_2 + R_2 D(S_2) = -(R_1 \Psi_{1y} + R_2 \Psi_{2y}). \quad (3.23)$$

To eliminate S_1 and S_2 we again take total derivative of Eqs. (3.15) and (3.16) with respect to “ t ” and then using Eqs. (3.22) and (3.23) we get

$$\begin{aligned} &R_{1tt} + 2\dot{x}R_{1tx} + 2\dot{y}R_{1ty} + 2\Psi_1 R_{1t\dot{x}} + 2\Psi_2 R_{1t\dot{y}} + \dot{x}^2 R_{1xx} + 2\dot{x}\dot{y} R_{1xy} \\ &+ \dot{y}^2 R_{1yy} + \Psi_{1t} R_{1\dot{x}} + \Psi_{2t} R_{1\dot{y}} + \dot{x}\Psi_{1x} R_{1\dot{x}} + \dot{y}\Psi_{1y} R_{1\dot{x}} + 2\dot{x}\Psi_1 R_{1x\dot{x}} \\ &+ 2\dot{y}\Psi_1 R_{1y\dot{x}} + 2\dot{x}\Psi_2 R_{1x\dot{y}} + 2\dot{y}\Psi_2 R_{1y\dot{y}} + \dot{x}\Psi_{2x} R_{1\dot{y}} + \dot{y}\Psi_{2y} R_{1\dot{y}} + \Psi_1 R_{1x} \\ &+ \Psi_2 R_{1y} + \Psi_1 \Psi_{1\dot{x}} R_{1\dot{x}} + \Psi_1^2 R_{1\dot{x}\dot{x}} + \Psi_2 \Psi_{1\dot{y}} R_{1\dot{x}} + \Psi_1 \Psi_{2\dot{x}} R_{1\dot{y}} + \Psi_2 \Psi_{2\dot{y}} R_{1\dot{y}} \\ &+ \Psi_{1\dot{x}} (R_{1t} + \dot{x}R_{1x} + \dot{y}R_{1y} + \Psi_1 R_{1\dot{x}} + \Psi_2 R_{1\dot{y}}) + \Psi_2^2 R_{1\dot{y}\dot{y}} + 2\Psi_1 \Psi_2 R_{1\dot{x}\dot{y}} \\ &+ \Psi_{2\dot{x}} (R_{2t} + \dot{x}R_{2x} + \dot{y}R_{2y} + \Psi_1 R_{2\dot{x}} + \Psi_2 R_{2\dot{y}}) - R_1 \Psi_{1x} - R_2 \Psi_{2x} \\ &+ R_1 (\Psi_{1t\dot{x}} + \dot{x}\Psi_{1x\dot{x}} + \dot{y}\Psi_{1y\dot{x}} + \Psi_1 \Psi_{1\dot{x}\dot{x}} + \Psi_2 \Psi_{1\dot{x}\dot{y}}) \\ &+ R_2 (\Psi_{2t\dot{x}} + \dot{x}\Psi_{2x\dot{x}} + \dot{y}\Psi_{2y\dot{x}} + \Psi_1 \Psi_{2\dot{x}\dot{x}} + \Psi_2 \Psi_{2\dot{x}\dot{y}}) = 0, \end{aligned} \quad (3.24)$$

$$\begin{aligned} &R_{2tt} + 2\dot{x}R_{2tx} + 2\dot{y}R_{2ty} + 2\Psi_1 R_{2t\dot{x}} + 2\Psi_2 R_{2t\dot{y}} + \dot{x}^2 R_{2xx} + 2\dot{x}\dot{y} R_{2xy} + \dot{y}^2 R_{2yy} \\ &+ \Psi_{1t} R_{2\dot{x}} + \Psi_{2t} R_{2\dot{y}} + \dot{x}\Psi_{1x} R_{2\dot{x}} + \dot{y}\Psi_{1y} R_{2\dot{x}} + 2\dot{x}\Psi_1 R_{2x\dot{x}} + 2\dot{y}\Psi_1 R_{2y\dot{x}} \\ &+ 2\dot{x}\Psi_2 R_{2x\dot{y}} + 2\dot{y}\Psi_2 R_{2y\dot{y}} + \dot{x}\Psi_{2x} R_{2\dot{y}} + \dot{y}\Psi_{2y} R_{2\dot{y}} + \Psi_1 R_{2x} + \Psi_2 R_{2y} \\ &+ \Psi_1 \Psi_{1\dot{x}} R_{2\dot{x}} + \Psi_1^2 R_{2\dot{x}\dot{x}} + \Psi_2 \Psi_{1\dot{y}} R_{2\dot{x}} + \Psi_1 \Psi_{2\dot{x}} R_{2\dot{y}} + \Psi_2 \Psi_{2\dot{y}} R_{2\dot{y}} \\ &+ \Psi_{1\dot{y}} (R_{1t} + \dot{x}R_{1x} + \dot{y}R_{1y} + \Psi_1 R_{1\dot{x}} + \Psi_2 R_{1\dot{y}}) + \Psi_2^2 R_{2\dot{y}\dot{y}} \\ &+ 2\Psi_1 \Psi_2 R_{2\dot{x}\dot{y}} + \Psi_{2\dot{y}} (R_{2t} + \dot{x}R_{2x} + \dot{y}R_{2y} + \Psi_1 R_{2\dot{x}} + \Psi_2 R_{2\dot{y}}) \\ &- R_1 \Psi_{1y} - R_2 \Psi_{2y} + R_1 (\Psi_{1t\dot{y}} + \dot{x}\Psi_{1x\dot{y}} + \dot{y}\Psi_{1y\dot{y}} + \Psi_1 \Psi_{1\dot{x}\dot{y}} + \Psi_2 \Psi_{1\dot{y}\dot{y}}) \\ &+ R_2 (\Psi_{2t\dot{y}} + \dot{x}\Psi_{2x\dot{y}} + \dot{y}\Psi_{2y\dot{y}} + \Psi_1 \Psi_{2\dot{x}\dot{y}} + \Psi_2 \Psi_{2\dot{y}\dot{y}}) = 0. \end{aligned} \quad (3.25)$$

Also differentiating Eq. (3.15) with respect to $\dot{x}$ and Eq. (3.16) with respect to $\dot{y}$, we have

$$\begin{aligned} &R_{1t\dot{x}} + \dot{x}R_{1x\dot{x}} + \dot{y}R_{1y\dot{x}} + \Psi_1 R_{1\dot{x}\dot{x}} + \Psi_2 R_{2\dot{x}\dot{x}} + 2R_{1x} + 2\Psi_{2\dot{x}} R_{2\dot{x}} \\ &+ 2\Psi_{1\dot{x}} R_{1\dot{x}} + R_2 \Psi_{2\dot{x}\dot{x}} + R_1 \Psi_{1\dot{x}\dot{x}} = 0, \end{aligned} \quad (3.26)$$

$$\begin{aligned} &R_{2t\dot{y}} + \dot{x}R_{2x\dot{y}} + \dot{y}R_{2y\dot{y}} + \Psi_1 R_{1\dot{y}\dot{y}} + \Psi_2 R_{2\dot{y}\dot{y}} + 2R_{2y} + 2\Psi_{2\dot{y}} R_{2\dot{y}} \\ &+ 2\Psi_{1\dot{y}} R_{1\dot{y}} + R_2 \Psi_{2\dot{y}\dot{y}} + R_1 \Psi_{1\dot{y}\dot{y}} = 0. \end{aligned} \quad (3.27)$$

Differentiating (3.15) with respect to $\dot{y}$ or (3.16) with respect to $\dot{x}$ and using (3.18) and (3.19), we get a single equation

$$\begin{aligned} &R_{1t\dot{y}} + \dot{x}R_{1x\dot{y}} + \dot{y}R_{1y\dot{y}} + \Psi_1 R_{1\dot{x}\dot{y}} + \Psi_2 R_{1\dot{y}\dot{y}} + R_{1y} + R_{2x} + \Psi_{2\dot{x}} R_{2\dot{y}} \\ &+ \Psi_{1\dot{x}} R_{1\dot{y}} + \Psi_{2\dot{y}} R_{2\dot{x}} + \Psi_{1\dot{y}\dot{y}} R_{1\dot{x}} + \Psi_{1\dot{y}} R_{1\dot{x}} + \Psi_{2\dot{y}} R_{1\dot{y}} = 0. \end{aligned} \quad (3.28)$$

Using Eqs. (3.18), (3.19), and (3.26)–(3.28), Eq. (3.24) simplifies as

$$\begin{aligned} R_{1tt} + 2\dot{x}R_{1tx} + 2\dot{y}R_{1ty} + \Psi_1 R_{1t\dot{x}} + \Psi_2 R_{1t\dot{y}} + \dot{x}^2 R_{1xx} + 2\dot{x}\dot{y}R_{1xy} - R_1 \Psi_{1x} \\ - R_2 \Psi_{2x} + \dot{y}^2 R_{1yy} + \Psi_{1t} R_{1\dot{x}} + \Psi_{2t} R_{1\dot{y}} + \dot{x}\Psi_{1x} R_{1\dot{x}} + \dot{y}\Psi_{1y} R_{1\dot{x}} + \dot{x}\Psi_{1x} R_{1\dot{x}} \\ + \dot{y}\Psi_{1y} R_{1\dot{x}} + \dot{x}\Psi_{2x} R_{1\dot{y}} + \dot{y}\Psi_{2y} R_{1\dot{y}} + \dot{x}\Psi_{2x} R_{1\dot{y}} + \dot{y}\Psi_{2y} R_{1\dot{y}} - \Psi_1 R_{1x} \\ - \Psi_2 R_{2x} + \Psi_{1\dot{x}}(R_{1t} + \dot{x}R_{1x} + \dot{y}R_{1y}) + R_1(\Psi_{1t\dot{x}} + \dot{x}\Psi_{1x\dot{x}} + \dot{y}\Psi_{1y\dot{x}}) \\ + \Psi_{2\dot{x}}(R_{2t} + \dot{x}R_{2x} + \dot{y}R_{2y}) + R_2(\Psi_{2t\dot{x}} + \dot{x}\Psi_{2x\dot{x}} + \dot{y}\Psi_{2y\dot{x}}) = 0, \end{aligned} \quad (3.29)$$

Now, using Eqs. (3.18), (3.19), and (3.26)–(3.28), Eq. (3.25) simplifies to the form

$$\begin{aligned} R_{2tt} + 2\dot{x}R_{2tx} + 2\dot{y}R_{2ty} + \Psi_1 R_{2t\dot{x}} + \Psi_2 R_{2t\dot{y}} + \dot{x}^2 R_{2xx} + 2\dot{x}\dot{y}R_{2xy} - R_1 \Psi_{1y} \\ - R_2 \Psi_{2y} + \Psi_{1t} R_{2\dot{x}} + \Psi_{2t} R_{2\dot{y}} + \dot{y}^2 R_{2yy} + \dot{x}\Psi_{1x} R_{2\dot{x}} + \dot{y}\Psi_{1y} R_{2\dot{x}} + \dot{x}\Psi_{1x} R_{2\dot{x}} \\ + \dot{y}\Psi_{1y} R_{2\dot{x}} + \dot{x}\Psi_{2x} R_{2\dot{y}} + \dot{y}\Psi_{2y} R_{2\dot{y}} + \dot{x}\Psi_{2x} R_{2\dot{y}} + \dot{y}\Psi_{2y} R_{2\dot{y}} - \Psi_1 R_{1y} \\ - \Psi_2 R_{2y} + \Psi_{1\dot{y}}(R_{1t} + \dot{x}R_{1x} + \dot{y}R_{1y}) + R_1(\Psi_{1t\dot{y}} + \dot{x}\Psi_{1x\dot{y}} + \dot{y}\Psi_{1y\dot{y}}) \\ + \Psi_{2\dot{y}}(R_{2t} + \dot{x}R_{2x} + \dot{y}R_{2y}) + R_2(\Psi_{2t\dot{y}} + \dot{x}\Psi_{2x\dot{y}} + \dot{y}\Psi_{2y\dot{y}}) = 0. \end{aligned} \quad (3.30)$$

Now solving the Eqs. (3.29) and (3.30) one can find R_1 and R_2 . The null forms S_1 and S_2 can be found by substituting the obtained values of R_1 and R_2 in Eqs. (3.15) and (3.16). Hence integrals of motion for the system (3.1) and (3.2) can be found by using the resultant forms of R_1, R_2, S_1 and S_2 in Eq. (3.21).

B. The Prolle-Singer method for modified Hénon-Heiles system

We rewrite the modified Hénon-Heiles system given by Eqs. (2.3) and (2.4) for $\phi_1 = \phi_2 = \phi$ as

$$\ddot{x} = \Psi_1(x, y, \dot{x}) = -\phi\dot{x} - Ax - 2\alpha xy, \quad (3.31)$$

$$\ddot{y} = \Psi_2(x, y, \dot{y}) = -\phi\dot{y} - By - \alpha x^2 + \beta y^2. \quad (3.32)$$

Let us assume that the above equations admit an integral of motion $I(t, x, y, \dot{x}, \dot{y})$. Following the procedure outlined in the previous sub-section we get

$$\begin{aligned} R_{1tt} + 2\dot{x}R_{1tx} + 2\dot{y}R_{1ty} + \Psi_1 R_{1t\dot{x}} + \Psi_2 R_{1t\dot{y}} + \dot{x}^2 R_{1xx} + 2\dot{x}\dot{y}R_{1xy} - \Psi_2 R_{2x} \\ + (A + 2\alpha y)R_1 + 2\alpha x R_2 + \dot{y}^2 R_{1yy} - \phi\dot{x}R_{1\dot{x}} - 2\alpha x\dot{y}R_{1\dot{x}} + \dot{x}\Psi_{1x} R_{1\dot{x}} \\ + \dot{y}\Psi_{1y} R_{1\dot{x}} + \dot{x}\Psi_{2x} R_{1\dot{y}} + \dot{y}\Psi_{2y} R_{1\dot{y}} - 2\alpha x\dot{x}R_{1\dot{y}} - \Psi_1 R_{1x} \\ + (2\beta y - B)\dot{y}R_{1\dot{y}} - \phi(R_{1t} + \dot{x}R_{1x} + \dot{y}R_{1y}) = 0, \end{aligned} \quad (3.33)$$

$$\begin{aligned} R_{2tt} + 2\dot{x}R_{2tx} + 2\dot{y}R_{2ty} + \Psi_1 R_{2t\dot{x}} + \Psi_2 R_{2t\dot{y}} + \dot{x}^2 R_{2xx} + 2\dot{x}\dot{y}R_{2xy} + 2\alpha x R_1 \\ + (B + 2\beta y)R_2 - (A + 2\alpha y)\dot{x}R_{2\dot{x}} - 2\alpha x\dot{y}R_{2\dot{x}} + \dot{x}\Psi_{1x} R_{2\dot{x}} + \Psi_{1\dot{y}} R_{2\dot{x}} \\ + \dot{y}^2 R_{2yy} + \dot{x}\Psi_{2x} R_{2\dot{y}} + \dot{y}\Psi_{2y} R_{2\dot{y}} - (B - 2\beta y)\dot{y}R_{2\dot{y}} - \Psi_1 R_{1y} \\ - 2\alpha x\dot{x}R_{2\dot{y}} - \Psi_2 R_{2y} - \phi(R_{2t} + \dot{x}R_{2x} + \dot{y}R_{2y}) = 0. \end{aligned} \quad (3.34)$$

A comprehensive examination reveals that obtaining a general solution for the aforementioned equations with respect to R_1 and R_2 is generally unattainable. Nonetheless, a specific solution is sufficient for determining the integrals of motion. Therefore, we assume that

$$\begin{aligned} R_1 &= [(\varepsilon_{11}x + \varepsilon_{12}y + \varepsilon_{13})\dot{x} + (\lambda_{11}x + \lambda_{12}y + \lambda_{13})\dot{y} + (\mu_{11}x + \mu_{12}y + \mu_{13})]f(t), \\ R_2 &= [(\varepsilon_{21}x + \varepsilon_{22}y + \varepsilon_{23})\dot{x} + (\lambda_{21}x + \lambda_{22}y + \lambda_{23})\dot{y} + (\mu_{21}x + \mu_{22}y + \mu_{23})]f(t). \end{aligned}$$

where $\varepsilon_{ij}, \lambda_{ij}, \mu_{ij}, i = 1, 2, j = 1, 2, 3$ are unknown constants and $f(t)$ is an unknown function of t which has to be determined.

Substituting the values of R_1 and R_2 into (3.33) and (3.34), solving the resulting equations we can obtain the following two sets of values for case 1(a).

$$\begin{aligned} (i) \quad &\lambda_{1j} = \varepsilon_{2j} = \lambda_{2k} = \varepsilon_{1k} = \mu_{12} = \mu_{13} = 0, \quad k = 1, 2, \quad j = 1, 2, 3. \\ &\varepsilon_{13} = -1, \quad \lambda_{23} = -1, \quad \mu_{11} = \mu_{21} = -\frac{2}{5}\phi, \quad \mu_{22} = \mu_{23} = 0, \quad f(t) = e^{\frac{6}{5}\phi t}. \\ (ii) \quad &\varepsilon_{1j} = \varepsilon_{2k} = \lambda_{1k} = \lambda_{2j} = \mu_{11} = \mu_{13} = 0, \quad k = 1, 2, \quad j = 1, 2, 3. \\ &\mu_{22} = \mu_{23} = 0, \quad \lambda_{13} = \varepsilon_{23} = -1, \quad \mu_{12} = \mu_{21} = -\frac{2}{5}\phi, \quad f(t) = e^{\frac{6}{5}\phi t}. \end{aligned}$$

Below we give the explicit forms of R_1, R_2, S_1, S_2 and their integrals of motion for case (1a).

Case 1(a). $A = B, \beta = -\alpha, B = \frac{6}{25}\phi^2$.
Here the explicit forms of R_1 and R_2 read

$$\begin{aligned} (i) \quad R_1 &= -\left(\dot{x} + \frac{2}{5}\phi x\right)e^{\frac{6}{5}\phi t}, & R_2 &= -\left(\dot{y} + \frac{2}{5}\phi y\right)e^{\frac{6}{5}\phi t}, \\ (ii) \quad R_1 &= -\left(\dot{y} + \frac{2}{5}\phi y\right)e^{\frac{6}{5}\phi t}, & R_2 &= -\left(\dot{x} + \frac{2}{5}\phi x\right)e^{\frac{6}{5}\phi t}, \end{aligned}$$

Making use of the above R_1 and R_2 in Eqs. (3.15) and (3.16) we obtain the following values of S_1 and S_2 respectively.

$$\begin{aligned} (i) \quad S_1 &= \frac{2}{5}\phi + \frac{2\alpha xy}{\dot{x} + \frac{2}{5}\phi x}, & S_2 &= \frac{2}{5}\phi + \frac{\alpha(x^2 + y^2)}{\dot{y} + \frac{2}{5}\phi y}, \\ (ii) \quad S_1 &= \frac{2}{5}\phi + \frac{\alpha(y^2 + x^2)}{\dot{y} + \frac{2}{5}\phi y}, & S_2 &= \frac{2}{5}\phi + 2\alpha \frac{xy}{\dot{x} + \frac{2}{5}\phi x}, \end{aligned}$$

Thus we obtain the following integrals of motion for (3.31) and (3.32),

$$\begin{aligned} (i) \quad I_1 &= \left(\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{2}{5}\phi(x\dot{x} + y\dot{y}) + \frac{2}{25}\phi^2(x^2 + y^2) + \alpha y\left(x^2 + \frac{1}{3}y^2\right)\right)e^{\frac{6}{5}\phi t}, \\ (ii) \quad I_2 &= \left(\dot{x}\dot{y} + \frac{2}{5}\phi(x\dot{y} + x\dot{x}) + \frac{4}{25}\phi^2xy + \alpha x\left(y^2 + \frac{1}{3}x^2\right)\right)e^{\frac{6}{5}\phi t}. \end{aligned}$$

Proceeding further we obtain nontrivial forms for R_1 and R_2 associated with case 2(b).

The explicit forms of R_1, R_2, S_1, S_2 and the integrals of motion obtained for the parametric restrictions in case 2(b) are given below.

Case 2(b). $A = B, \beta = -6\alpha, B = \frac{6}{25}\phi^2$.

The explicit forms of R_1, R_2, S_1 and S_2 and the related integrals read as

$$\begin{aligned} (i) \quad R_1 &= -\left(\dot{x} + \frac{2}{5}\phi x\right)e^{\frac{6}{5}\phi t}, & R_2 &= -\left(\dot{y} + \frac{2}{5}\phi y\right)e^{\frac{6}{5}\phi t}, \\ S_1 &= \frac{2}{5}\phi + \frac{2\alpha xy}{\dot{x} + \frac{2}{5}\phi x}, & S_2 &= \frac{2}{5}\phi + \frac{\alpha(x^2 + 6y^2)}{\dot{y} + \frac{2}{5}\phi y}, \\ I_1 &= \left(\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{2}{5}\phi(x\dot{x} + y\dot{y}) + \frac{2}{25}\phi^2(x^2 + y^2) + \alpha y(x^2 + 2y^2)\right)e^{\frac{6}{5}\phi t}, \\ (ii) \quad R_1 &= 4\left(2\dot{x}y - x\dot{y} + \frac{2}{5}\phi xy\right)e^{\frac{8}{5}\phi t}, & R_2 &= -4x\left(\dot{x} + \frac{2}{5}\phi x\right)e^{\frac{8}{5}\phi t}, \\ S_1 &= \frac{\dot{x}\dot{y} + \frac{4}{5}\phi x\dot{y} + \frac{4}{25}\phi^2xy + \alpha x^3 + 2\alpha xy^2}{x\dot{y} - 2\dot{x}y - \frac{2}{5}\phi xy}, \\ S_2 &= \frac{-\left(\dot{x}^2 + \frac{2}{5}\phi x\dot{x} - 2\alpha x^2y\right)}{x\dot{y} + \frac{2}{5}\phi x^2}, \\ I_2 &= \left(4\dot{x}(x\dot{y} - y\dot{x}) + \frac{8}{5}\phi x(x\dot{y} - \dot{x}y) + \alpha x^2(x^2 + 4y^2)\right)e^{\frac{8}{5}\phi t}. \end{aligned}$$

It is easy to check that the obtained integrals are independent in both the cases.

In order to construct the integral for case 2(c) we consider R_1 and R_2 as quadratic polynomials in $\dot{x}$ and $\dot{y}$ as

$$R_1 = [(\varepsilon_1x + \varepsilon_2y + \varepsilon_3)\dot{x}^2 + (\varepsilon_4x + \varepsilon_5y + \varepsilon_6)\dot{y}^2 + (\varepsilon_7x + \varepsilon_8y + \varepsilon_9)\dot{x}\dot{y} + \dots]f(t)$$

$$R_2 = [(\theta_1x + \theta_2y + \theta_3)\dot{x}^2 + (\theta_4x + \theta_5y + \theta_6)\dot{y}^2 + (\theta_7x + \theta_8y + \theta_9)\dot{x}\dot{y} + \dots]f(t),$$

where $\varepsilon_i, \theta_i, i = 1, 2, \dots$ are constants and $f(t)$ is an unknown function. A detailed calculation shows that there exists nontrivial solutions for R_1, R_2, S_1 and S_2 in this case also. Their explicit forms for the case 2(c) are given below.

Case 2(c). $A = B$, $\beta = -16\alpha$, $B = \frac{6}{25}\phi^2$.

$$\begin{aligned} (i) \quad R_1 &= -\left(\dot{x} + \frac{2}{5}\phi x\right)e^{\frac{6}{5}\phi t}, \quad R_2 = -\left(\dot{y} + \frac{2}{5}\phi y\right)e^{\frac{6}{5}\phi t}, \\ S_1 &= \frac{2}{5}\phi + \frac{2\alpha xy}{\dot{x} + \frac{2}{5}\phi x}, \quad S_2 = \frac{2}{5}\phi + \frac{\alpha(x^2 + 16y^2)}{\dot{y} + \frac{2}{5}\phi y}, \\ I_1 &= \left(\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{2}{5}\phi(x\dot{x} + y\dot{y}) + \frac{2}{25}\phi^2(x^2 + y^2) + \alpha y\left(x^2 + \frac{16}{3}y^2\right)\right)e^{\frac{6}{5}\phi t}, \\ (ii) \quad R_1 &= -\left(36\dot{x}^3 + \frac{216}{5}\dot{x}^2\phi x + \frac{432}{25}\dot{x}\phi^2x^2 + \frac{288}{125}\phi^3x^3 + 72\alpha x^2y\dot{x} + 24\alpha x^3\phi y - 12\alpha x^3\dot{y}\right)e^{\frac{12}{5}\phi t}, \\ R_2 &= 12\alpha x^3\left(\dot{x} + \frac{2}{5}\phi x\right)e^{\frac{12}{5}\phi t}, \quad S_1 = \frac{U(x, y, \dot{x}, \dot{y})}{V(x, y, \dot{x}, \dot{y})}, \quad S_2 = \frac{3\dot{x}^2 + 2\alpha\phi x\dot{x} - 4\alpha x^2y + \frac{8}{25}\phi^2x^2}{x(\dot{x} + \frac{2}{5}\phi x)}. \end{aligned}$$

where

$$\begin{aligned} U(x, y, \dot{x}, \dot{y}) &= \frac{72}{5}\dot{x}^2\left(\dot{x} + \frac{6}{5}\phi x\right) + 72\alpha xy\dot{x}(\dot{x} + \phi x) - \frac{96}{5}\phi x^3\left(\dot{y} - \frac{6}{125}\phi^3\right) - 12\alpha x^2(\alpha x^3 + 3\dot{x}y) - 48\alpha x^3y\left(\alpha y - \frac{8}{25}\phi^2\right), \\ V(x, y, \dot{x}, \dot{y}) &= 36\dot{x}^2\left(\dot{x} + \frac{6}{5}\phi x\right) + 36\alpha x^2y(2\dot{x} + \dot{y}) + 12\alpha x^3(2\phi y - \dot{y}) + \frac{144}{25}\phi^2x^2\left(3\dot{x} + \frac{2}{5}\phi x\right), \\ I_2 &= \left(9\dot{x}^3\left(\dot{x} + \frac{8}{5}\phi x\right) + \frac{144}{125}\phi^3x^3\left(2\dot{x} + \frac{1}{5}\phi x\right) + 36x^2\dot{x}^2\left(\alpha y + \frac{6}{25}\phi^2\right) \right. \\ &\quad \left. + 12\alpha x^3\dot{x}(2\phi y - \dot{y}) - 2\alpha^2x^4(x^2 + 6y^2) + \frac{24}{5}\phi\alpha x^4\left(\frac{4}{5}\phi y - \dot{y}\right)\right)e^{\frac{12}{5}\phi t}. \end{aligned}$$

One can note that we can use the method suggested in Ref. 25 to remove the time-dependent part of the integrals, which helps to prove the integrability nature of the present cases.

IV. INVARIANCE AND LIE SYMMETRY ANALYSIS OF THE MODIFIED HÉNON-HEILES SYSTEM

The modified Hénon–Heiles system (3.31) and (3.32) is also a Lagrangian system. The Euler–Lagrange equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y}, \quad (4.1)$$

where L is the Lagrangian given by

$$L = \left[\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(Ax^2 + By^2) - \left(\alpha x^2y - \frac{1}{3}\beta y^3\right)\right]e^{\phi t}. \quad (4.2)$$

The integrability properties of the modified Hénon–Heiles system could also be studied through the symmetry properties of it.^{23,24}

Let (4.1) be invariant under the one parameter (ε) Lie group of infinitesimal transformations given by

$$x \rightarrow X = x + \varepsilon \eta_1(t, x, y, \dot{x}, \dot{y}), \quad (4.3)$$

$$y \rightarrow Y = y + \varepsilon \eta_2(t, x, y, \dot{x}, \dot{y}), \quad (4.4)$$

$$t \rightarrow T = t + \varepsilon \xi(t, x, y, \dot{x}, \dot{y}). \quad (4.5)$$

This leads to the following invariance condition to be satisfied

$$\ddot{\eta}_1 - \dot{x}\ddot{\xi} - 2\dot{\xi}\Psi_1 = \mathbf{X}(\Psi_1), \quad \ddot{\eta}_2 - \dot{y}\ddot{\xi} - 2\dot{\xi}\Psi_2 = \mathbf{X}(\Psi_2). \quad (4.6)$$

where Ψ_1 and Ψ_2 are as given in (3.31) and (3.32), respectively, and the infinitesimal operator $\mathbf{X}$ is given by

$$\mathbf{X} = \xi \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y} + (\dot{\eta}_1 - \dot{\xi}x) \frac{\partial}{\partial \dot{x}} + (\dot{\eta}_2 - \dot{\xi}y) \frac{\partial}{\partial \dot{y}}. \quad (4.7)$$

After a detailed calculation, we observe that

$$\xi = 0, \quad \eta_1 = \eta_1(x, y, \dot{x}, \dot{y})e^{\lambda t}, \quad \eta_2 = \eta_2(x, y, \dot{x}, \dot{y})e^{\lambda t}, \quad \lambda = \text{constant},$$

and the nontrivial symmetries occur for the P-cases only. The details of the above investigation will be published elsewhere. Using Noether's theorem, the first integral of motion can be identified for each set of Lie symmetries which takes the form

$$\xi = 0, \quad \eta_1 = \left(\dot{x} + \frac{2}{5}\phi x \right) e^{\frac{\phi}{5}t}, \quad \eta_2 = \left(\dot{y} + \frac{2}{5}\phi y \right) e^{\frac{\phi}{5}t}.$$

Similarly, the second integral of motion can be identified for each set of non-trivial symmetries of the following forms,

Case 1(a): $A = B = \frac{6}{25}\phi^2, \beta = -\alpha$.

$$\xi = 0, \quad \eta_1 = \left(\dot{y} + \frac{2}{5}\phi y \right) e^{\frac{\phi}{5}t}, \quad \eta_2 = \left(\dot{x} + \frac{2}{5}\phi x \right) e^{\frac{\phi}{5}t}$$

Case 2(b): $A = B = \frac{6}{25}\phi^2, \beta = -6\alpha$.

$$\xi = 0, \quad \eta_1 = \left(4x\dot{y} - 8y\dot{x} - \frac{8}{5}\phi xy \right) e^{\frac{3}{5}\phi t}, \quad \eta_2 = \left(4\dot{x}x + \frac{8}{5}\phi x^2 \right) e^{\frac{3}{5}\phi t}$$

Case 2(c): $A = B = \frac{6}{25}\phi^2, \beta = -16\alpha$.

$$\xi = 0, \quad \eta_1 = \left(36\dot{x}^3 + \frac{216}{5}\phi x\dot{x} \left(\dot{x} + \frac{2}{5}\phi x \right) + \frac{288}{125}\phi^3 x^3 + 24\alpha x^2 y (3\dot{x} + \phi x) - 12\alpha x^3 \dot{y} \right) e^{\frac{7}{5}\phi t}, \quad \eta_2 = -12\alpha x^3 \left(\dot{x} + \frac{2}{5}\phi x \right) e^{\frac{7}{5}\phi t}.$$

A. Separability

In this section, we show how the generalized symmetries obtained can be used to find coordinates transformations in which the equations of motion become separable. This is possible when the generalized symmetries are linear in velocities. We say that a differentiable function $U(t, x, y, \dot{x}, \dot{y})$ is an invariant for the transformations (4.3)–(4.5) if $XU = 0$, which is a first order partial differential equation. Then its characteristic equations are

$$\frac{dt}{\xi} = \frac{dx}{\eta_1} = \frac{dy}{\eta_2} = \frac{d\dot{x}}{(\eta_1 - \xi\dot{x})} = \frac{d\dot{y}}{(\eta_2 - \xi\dot{y})}. \quad (4.8)$$

One can also find integrals of motion to the modified Hénon–Heiles system by solving the characteristic Eq. (4.8). Since our aim is to find suitable coordinates transformations leading to separability we consider only the following part of Eq. (4.8). We illustrate this for case (1)

$$\frac{dx}{\left(\dot{y} + \frac{2}{5}\phi y \right) e^{\frac{\phi}{5}t}} = \frac{dy}{\left(\dot{x} + \frac{2}{5}\phi x \right) e^{\frac{\phi}{5}t}}. \quad (4.9)$$

Solving which we get

$$(x^2 - y^2) e^{-\frac{4}{5}\phi t} = \text{constant}, \quad (4.10)$$

leading to the coordinates transformations $u = (x + y) e^{-\frac{2}{5}\phi t}$, $v = (x - y) e^{-\frac{2}{5}\phi t}$.

In the above coordinates the Eqs. (2.3) and (2.4) separates into

$$\frac{d^2 u}{dt^2} + \phi \frac{du}{dt} + Au + \alpha u^2 e^{\frac{2}{5}\phi t} = 0, \quad (4.11)$$

$$\frac{d^2 v}{dt^2} + \phi \frac{dv}{dt} + Av - \alpha v^2 e^{\frac{2}{5}\phi t} = 0. \quad (4.12)$$

Equation (4.11) can be transformed into

$$l^2 \frac{d^2 u}{dl^2} + 2l \frac{du}{dl} + \frac{6}{25}u = -\left(\frac{\alpha}{\phi^2} \right) u^2 l^{\frac{2}{5}}, \quad (4.13)$$

by a substitution $l = e^{\phi t}$. Equation (4.13) can be further transformed into an Emden–Fowler equation of the form

$$\frac{d^2 w}{dz^2} = -\frac{25\alpha}{\phi^2} z^{-3} w^2, \quad (4.14)$$

by the substitution $l = z^5$, $u = z^{-3} w(z)$. The solution of Eq. (4.14) can be written parametrically in terms of the Weierstrass elliptic function.²⁶

V. EXACT SOLUTION OF MODIFIED HÉNON-HEILES SYSTEM

It is well known that finding an exact solution for a given nonlinear PDE or ODE is a difficult task. Also, there exists no unique analytic method to construct an exact solution of nonlinear differential equations, in general. Recent investigations demonstrate that tanh or sech method can provide an effective tool for nonlinear ODEs and PDEs toward constructing specific/particular solutions.²⁷ In this section we extend the tanh or sech method to modified Hénon–Heiles system. With this aim, we look for exact solution of modified Hénon–Heiles system having the form

$$x(t) = A_1 + A_2 \tanh(bt + \delta) + A_3 \tanh^2(bt + \delta),$$

$$y(t) = B_1 + B_2 \tanh(bt + \delta) + B_3 \tanh^2(bt + \delta).$$

where $A_i, B_i, i = 1, 2, 3$ and b are unknown constants to be determined. Substituting $x(t)$ and $y(t)$ in the equations of the modified Hénon–Heiles system we find that each one becomes a fourth degree polynomial in $\tanh(bt + \delta)$. Then equating different powers of $\tanh(bt + \delta)$ in both the equations to zero we obtain a system of algebraic equations in A_i and $B_i, i = 1, 2, 3$. Solving them consistently yields the following exact solutions.

Exact solution for integrable cases:

$$(i) \quad A = B = \frac{6}{25}\phi^2, \quad \beta = -\alpha,$$

$$x(t) = \frac{3\phi^2}{100\alpha} \left[3 + \tanh\left(-\frac{\phi}{10}t + \delta\right) \right] \left[-1 + \tanh\left(-\frac{\phi}{10}t + \delta\right) \right],$$

$$y(t) = -\frac{3\phi^2}{100\alpha} \left[\tanh^2\left(-\frac{\phi}{10}t + \delta\right) + 2 \tanh\left(-\frac{\phi}{10}t + \delta\right) + 5 \right].$$

$$(ii) \quad A = B = \frac{6}{25}\phi^2, \quad \beta = -6\alpha,$$

$$x(t) = \frac{3I\phi^2}{25\alpha} \left[1 + \tanh\left(-\frac{3\phi}{10}t + \delta\right) \right],$$

$$y(t) = \frac{-3\phi^2}{100\alpha} \left[1 + \tanh\left(-\frac{3\phi}{10}t + \delta\right) \right] \left[3 \tanh\left(-\frac{3\phi}{10}t + \delta\right) - 1 \right].$$

$$(iii) \quad A = \frac{21}{100}\phi^2, B = -\frac{24}{100}\phi^2, \quad \beta = -16\alpha, \quad \alpha = -\frac{3}{50}\phi^2,$$

$$x(t) = -\frac{I\sqrt{14}}{2} \left[\tanh\left(-\frac{\phi}{10}t + \delta\right) + 1 \right]^2,$$

$$y(t) = \frac{1}{4} + \tanh\left(-\frac{\phi}{10}t + \delta\right) + \frac{1}{2} \tanh^2\left(-\frac{\phi}{10}t + \delta\right).$$

Exact solution for other cases:

$$(iv) \quad A = B = -\frac{4}{9}\phi^2, \quad \beta = -\frac{4}{3}\alpha,$$

$$x(t) = \frac{\phi^2}{6\sqrt{6}\alpha} \left[\tanh\left(\frac{\phi}{6}t + \delta\right) + 1 \right] \left[\tanh\left(\frac{\phi}{6}t + \delta\right) - 1 \right],$$

$$y(t) = \frac{-\phi^2}{12\alpha} \left[\tanh\left(\frac{\phi}{6}t + \delta\right) + 1 \right] \left[\tanh\left(\frac{\phi}{6}t + \delta\right) - 3 \right].$$

$$(v) \quad A = \frac{9}{100}\phi^2, B = -\frac{6}{25}\phi^2, \quad \beta = -\frac{16}{5}\alpha,$$

$$x(t) = \frac{3I\sqrt{30}\phi^2}{500\alpha} \left[\tanh\left(-\frac{\phi}{10}t + \delta\right) + 1 \right]^2,$$

$$y(t) = -\frac{3\phi^2}{200\alpha} \left[-3 + 4 \tanh\left(-\frac{\phi}{10}t + \delta\right) + 2 \tanh^2\left(-\frac{\phi}{10}t + \delta\right) \right].$$

We see that cases (i) and (ii) above have an exact solution for the integrable cases (1a) and (2b) respectively. Though the integrability of parametric cases (1b) and (2a) is not established, an exact solution for the same is given in (iv) and (v) respectively, which may correspond to the cases admitting lesser number of arbitrary constants in the Laurent series. It appears that case 1(c) may not admit the exact solution expressed in tanh or sech function. One may also be able to find particular forms of elliptic functions, generalizing the above procedure which we hope to explore in future.

VI. SUMMARY AND CONCLUDING REMARKS

In this article, by applying the Painlevé analysis of ordinary differential equations to modified Hénon–Heiles system we report that it admits the Painlevé property for three distinct parametric restrictions. For each of the identified cases, we construct two independent integrals of motion using the well known Prolle–Singer method. Thus we infer that the modified Hénon–Heiles system is integrable for three distinct parametric restrictions. We have shown that there exists an exact solution for two of the isolated three integrable cases of modified Hénon–Heiles system expressed in terms of tanh function. We observe that when

$$\beta = -16\alpha, \quad \phi_2 = \phi_1 = \phi \quad (6.1)$$

leads to the integrable case $2(c)$, where

$$A = B = \frac{6}{25}\phi^2.$$

But we are able to find an exact solution expressed in tanh function for the case in (6.1) when $A = \frac{21}{100}\phi^2$, $B = -\frac{24}{100}\phi^2$, $\alpha = -\frac{3}{50}\phi^2$. Also, we have isolated two more parametric restrictions of modified Hénon–Heiles system admitting Laurent series solution with two arbitrary constants and two of them admit an exact solution expressed in tanh function. From the investigation, we observe that the inclusion of damping term in the equations of motion leads to further parametric restriction in the linear terms of the equations. The connection between the Laurent series solution and the tanh solution of the integrable case needs further investigation.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

C. Uma Maheswari: Methodology (equal); Supervision (equal); Writing – review & editing (equal). **N. Muthuchamy:** Methodology (equal); Writing – original draft (equal). **V. K. Chandrasekar:** Supervision (equal); Writing – review & editing (equal). **R. Sahadevan:** Conceptualization (equal); Methodology (equal). **M. Lakshmanan:** Conceptualization (equal); Methodology (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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