

DIOPHANTINE EQUATIONS AND NUMBER PATTERNS



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This is to certify that this thesis entitled “**DIOPHANTINE EQUATIONS AND NUMBER PATTERNS**” in partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** in **MATHEMATICS** is a record of original research work carried out by **Ms. SHARADHA KUMAR** (Reg. No. BDU1910180180 / Mathematics / Full Time / November 2019 / Dated 28.08.2019) during her period of study from 2019 to 2022 under the Full-Time Programme at National College (Autonomous) in Bharathidasan University, Tiruchirappalli, Tamil Nadu, India under my supervision and guidance and the thesis has not been submitted for the award of any other degree / Associationship / Fellowship or similar title of this or any other University.

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Notations and Definitions

NOTATIONS

- Polygonal number of rank n with sides m

$$t_{m,n} = n \left[1 + \frac{(n-1)(m-2)}{2} \right]$$

- Gnomonic number of rank n

$$GNO_n = 2n - 1$$

- Pentagonal pyramidal number of rank n

$$P_n^m = \frac{3n^2 + n^3(m-2) - n(m-5)}{6}$$

- Pronic number of rank n

$$PR_n = n(n+1)$$

- Star number of rank n

$$S_n = 6n^2 - 6n + 1$$

- Triangular number of rank n

$$t_{3,n} = \frac{n(n+1)}{2}$$

- Centered Polygonal number of rank n with sides m

$$Ct_{m,n} = \frac{mn(n+1)}{2} + 1$$

- Pyramidal number of rank n with sides m

$$P_n^m = \frac{1}{6} [n(n+1)] [(m-2)n + (5-m)]$$

- Centered Pyramidal number of rank n with sides m

$$CP_{m,n} = \frac{m(n-1)n(n+1) + 6n}{6}$$

- Stella octangular number of rank n

$$SO_n = n(2n^2 - 1)$$

- Centered hexagonal pyramidal number of rank n

$$CP_{6,n} = n^3$$

- Rombic Dodecagonal number of rank n

$$R_n = 4n^3 - 6n^2 + 4n - 1$$

- Centered triangular pyramidal number of rank n

$$CP_n^3 = \frac{n(n^2 + 1)}{2}$$

- Centered square pyramidal number of rank n

$$CS_n^4 = \frac{n(2n^2 + 1)}{3}$$

- Pentagonal pyramidal number of rank n with size 5

$$P_n^5 = \frac{n^2(n+1)}{2}$$

- Triangular pyramidal number of rank n

$$P_n^3 = \frac{n(n+1)(n+2)}{6}$$

- Centered nonagonal pyramidal number of rank n

$$CP_{9,n} = \frac{n(3n^2 - 1)}{2}$$

- Centered icositetragonal pyramidal number of rank n

$$CP_{24,n} = \frac{24n^3 - 18n}{6}$$

- Centered Hexagonal Number of rank H

$$ct_{6,H} = 3H(H-1)+1$$

- Centered Octagonal Number of rank M

$$ct_{8,M} = 4M(M-1)+1$$

- Centered Decagonal Number of rank M

$$ct_{10,M} = 5M(M-1)+1$$

- Centered Dodecagonal Number of rank D

$$ct_{12,D} = 6D(D-1)+1$$

- Hexagonal Number of rank h

$$t_{6,h} = 2h^2 - h$$

DEFINITIONS

Diophantine m -tuple

A set of m distinct positive integers $\{a_1, a_2, a_3, \dots, a_m\}$ is said to have the property $D(n), n \in \mathbb{Z} - \{0\}$ if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$ and such a set is called a Diophantine m -tuple with property $D(n)$.

Dio m -tuple

A set of m distinct positive integers $\{a_1, a_2, a_3, \dots, a_m\}$ is said to have the property $D(n), n \in \mathbb{Z} - \{0\}$ if $a_i a_j + a_i + a_j + n$ is a perfect square for all $1 \leq i < j \leq m$ and such a set is called a Dio m -tuple with property $D(n)$.

Nasty Number

A positive integer n is a Nasty number if $n = ab = cd$ and $a + b = c - d$ or $a - b = c + d$ where a, b, c, d are non-zero distinct positive integers (Bert Miller 1980).

Bernoulli and Euler Polynomials

The Bernoulli and Euler polynomials can explicitly be defined as,

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k$$

B_k - Bernoulli numbers

and
$$E_n(x) = \sum_{k=1}^n \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}$$

E_k - Euler numbers

Chapter – I

Introduction

CHAPTER – I

INTRODUCTION

State of the art of the research topic

The theory of numbers is one of the attractive and significant branches of pure mathematics concerning mainly to the study of the integers. It has varieties of subject areas and each area has its own history deserving special recognition. One of the oldest and largest subject areas of number theory is diophantine equations playing an important and significant role. A diophantine equation is a polynomial equation in two or more variables for which only the integer solutions are searched. In fact, it is worth mentioning that diophantine problems have fewer equations than unknown variables and involve finding integers that work correctly for all equations. In other words, the study of diophantine equations concerns with the search for non-zero distinct integer solutions to polynomial equations or systems of equations. The formulations of general theories of diophantine equations other than the theory of quadratic forms was an achievement of the twentieth century as the individual equations seem to be a puzzle and have been considered throughout history.

Mathematics is like a banyan tree and the Number Theory is one of its oldest branches as its history spans for atleast 5000 years. Number theory is a vast and fascinating field of Mathematics, sometimes called “Higher Arithmetic”, consisting of the study of the properties of whole numbers. Gauss often known as the “Prince of Mathematics” called Mathematics the “Queen of the Sciences” and considered Number Theory the “Queen of Mathematics”.

Number theory is populated by a variety of exotic flora and fauna that developed from the study of integers. Generally speaking, Number theory is defined

as the study of the property of numbers [4-6, 9, 10, 14-16] where by “Numbers” we mean integers and more specially, positive integers, which are the building blocks of the real number system. It has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition.

Number theory provides a fertile ground for both professionals and amateurs. In addition to known results, number theory abounds with unsolved problems. Although many of its results can be stated in simple and elegant terms, their proofs are sometimes long and complicated. Many unsolved problems that have been daunting mathematicians for centuries provide unlimited opportunities to expand the frontiers of mathematical knowledge.

Number theory has several branches and each branch has its own history [88, 90, 120-122] and they deserve special recognition. The history of number theory can be divided into three parts, progress of number theory before Christian era, its development in the next 1500 years and from sixteenth century to present. The Modern Era in the subject begins with Pierre de Fermat (Generally acknowledged to be the father of modern number theory).

A vast domain of the theory of numbers goes back to Diophantus of Alexandria. The subject of diophantine equations is one of the oldest and largest branches of Number theory. The word Diophantine refers to the Hellenistic mathematician of the 3rd century Diophantus of Alexandria, Egypt who made a study of diophantine equations and introduced symbolism into algebra. The study of diophantine equations is the study of solutions of polynomial equations or systems of equations in integers, rational numbers or sometimes more general number rings. One of the fascinations of the subject is that the problems are usually easy to state and when they can be solved, sometimes involve sophisticated mathematical tools.

Diophantine equations play an important and significant role in Number theory. It is worth mentioning that diophantine problems have fewer equations than unknown variables and involve finding integers that work correctly for all equations. In more technical language, they define an algebraic curve, algebraic surface or more general object and ask about the lattice points on it. Fermat [2] solved new problems, posed many challenges to other mathematicians, invented new methods and in general, was much more advanced than contemporary mathematicians. The formulation of general theories of diophantine equations (further to theory of quadratic forms) was an achievement of the twentieth century [19, 23, 69, 74, 86, 87, 89] as the individual equations seem to be a puzzle and have been considered throughout history.

Diophantine equations are numerous rich because of its variety [18, 25, 28, 32-34, 38-40, 44-46, 48, 49, 54, 55, 60, 61, 76-82, 85, 97]. There is no universal method or algorithm for determining whether an arbitrary diophantine equation has a solution or finding all the solutions, if it exists. Such an algorithm does exist for the solution of first-order diophantine equations. However, the impossibility of obtaining a general solution was proven by Yuri Matiyasevich in 1970. There is a general theory for quadratic diophantine equations in many variables. There are many quadratic diophantine problems which kindled the interest among mathematicians. But there are very few diophantine problems for which the complete solution is known [1, 26, 29, 36, 42, 43].

There are several diophantine equations that have no solutions, trivial solutions, finitely many solutions or infinite number of solutions [30, 37, 47, 52]. For example $3x^2 + xy = 14$ and $x^2 - 2y^4 = -1$ [59, 87] have finite number of solutions and the solutions are $(x, y) = ((1, 1), (2, 1), (-1, -11), (-2, -1), (\mp 7, \pm 19), (\pm 14, \pm 41), (\pm 2, \pm 1) \& (\pm 1, \pm 11))$ and $(x, y) = ((\pm 1, \pm 1), (\pm 239, \pm 39))$ respectively. The binary quadratic equation

$x^2 - y^2 + x + y + xy = 2$ [31] represents a hyperbola and has infinitely many solutions. The pellian equation $x^2 = 19y^2 - 1$ [75] has no solution in integers and Fermat proved that $x^4 \pm y^4 = z^2$ have no non-trivial solutions. The diophantine equations $x^4 \pm 2y^4 = z^2$, $x^4 + y^4 = 7z^2$ [87] have no integer solutions. In particular, $x^5 + Dy^5 = 1$ [87] has only one solution $(-1,1)$ when $D = 2$ and no integer solution when $D = 4, 8, 16$. It is obvious that only a few details are known about the theory of representation of integers by binary cubic forms. Many special cases of the equation $x^2 + c = y^n$ [11-13] where x and y are positive integers and $n \geq 3$ have been considered over the years, but most results for general n are of recent origin.

N. Elkies in 1988 found the example $2682240^4 + 15365639^4 + 1879760^4 = 20615673^4$ and there by disproving Euler's conjecture that the Diophantine equation $w^4 + x^4 + y^4 = z^4$ has only trivial solutions in which two of the variables are zero. Also one may refer [22, 24, 27, 52, 60, 62]. Titu Andreescue and Dorin Andrica proved that the bi-quadratic equation $(x^2 + 1)(y^2 + 1) + 2(x - y)(1 - xy) = 4(1 + xy)$ has only eight solutions $(x, y) = \{(1, 2), (-3, 0), (0, 3), (-2, 1), (-1, 0), (0, -1), (-3, 2), (-2, 3)\}$. The equations $y^2 = 8x^4 + 1$ and $x^2 - 2y^4 = -1$ have a finite number of solutions and the solutions are $(x, y) = \{(0, \pm 1), (\pm 1, \pm 3)\}$ and $(x, y) = \{(\pm 1, \pm 1), (\pm 239, \pm 13)\}$ respectively. The only integral solutions to $y^2 = x^4 + x^3 + x^2 + x + 1$ is $(x, y) = \{(-1, \pm 1), (0, \pm 1), (3, \pm 11)\}$ and $x = y = 1$ is the only solution for the equation $x^2 = 3y^4 - 2$.

The Greek Mathematician Diophantus of Alexandria [3] studied the following problem: Find four (positive rational) numbers such that the product of any two of them increased by one is a perfect square. He obtained the following solution

$\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}$. The first set of four positive integers with the above property was found by Fermat and it was Euler who gave the solution $\{a, b, a+b+2r, 4r(r+a)(r+b)\}$ where $ab+1=r^2$ [8]. In other words, a set of m distinct positive integers $(a_1, a_2, a_3, a_4, a_5, \dots, a_m)$ is called a diophantine- m -tuple with property $D(n)$, if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$ [7, 8, 20, 21, 35, 119]. Non existence of diophantine quadruple with the property $D(4k+2)$ was proved in [8] and yielded the interest of constructing diophantine m -tuple $m=4$ with suitable properties.

In the 17th century, the study of Diophantine equations was taken by Fermat who gave us the famous problem of proving Fermat's last theorem. Fermat also looked at Diophantine problem, but he was more interested in whole number solutions than fractions. Weil's brilliant solution has recently stimulated renewed interest in Number Theory. Mathematicians in India were interested in finding integral solutions of Diophantine equations since the Vedic era.

Number is the essence of mathematical calculations. Numbers have varieties of patterns and varieties of range and richness. Any sequence of numbers represented by a mathematical function may be considered as pattern. In fact, mathematics can be considered as a characterization of patterns. For clear understanding, any regularity that can be illustrated by a scientific theory is a pattern. In other words, a pattern is a group of numbers, shapes or objects that follow a rule. A careful observer of patterns may note that there is a one to one correspondence between the polygonal numbers and the number of sides of the polygon.

One of the fascinating variety of numbers is Ramanujan numbers (or R -number) named after the great Indian Mathematician Srinivasa Ramanujan. The number that

can be expressed as the sum of the squares of two numbers in two different ways are called the Second Order Ramanujan numbers and for simplicity, written as R_2 numbers. In general, if R_n is an R -numbers of order n then its numerical relation can be represented as $R_n = x_1^n + x_2^n = y_1^n + y_2^n$. Obviously, there are many R -numbers in each order, which are special cases of Diophantine problem covering a wide area in Number Theory [69, 85, 86, 88, 91].

Apart from the above patterns we have some more fascinating patterns of numbers namely Armstrong numbers, Harshad numbers, Multiple Harshad numbers, Nivenmorphic numbers, Sphenic numbers, Sphenic Palindrome numbers. For illustrations, one may refer [41, 70-73].

Results from Number Theory have countless applications in mathematics as well as in practical applications including security, memory management, authentication, coding theory, etc. Aspects of elementary number theory pertaining to the golden ratio and golden spiral are shown to be related to and therefore of importance in the simulation of chemical phenomena.

Objectives and scope of the research work

The proposed research work focuses its concentration to search for non-trivial integral solutions of diophantine equations of degree two to five with multivariables. Some procedures have been developed to find an infinite number of non-trivial integral solutions to a few interesting diophantine equations of polynomial types and search for interesting properties among the solutions. A few results about special Diophantine triple, Diophantine Quadruple, Dio-triples and Dio-quadruples and special triples are also studied.

In addition, the objective of this thesis is to find non-zero distinct integral solution to special forms of double equations, triple equations and equality among polygonal numbers.

This Dissertation consists of XII Chapters. Chapter I provides the historical background and necessary literature survey for the variety of problems studied for the corresponding integral solutions and related properties presented in the chapters II to XII.

Chapter-II deals with Quadratic diophantine equations in twelve sections

II.1 to II.12

Section II.1 to II.4

The following binary quadratic equations are analyzed for finding its non-zero distinct integer solutions. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented.

$$\text{II.1[57]} \quad 3x^2 - 8y^2 = 40$$

$$\text{II.2[97]} \quad y^2 = 5\alpha^2(x^2 + 1)$$

$$\text{II.3[98]} \quad 9x^2 - 8y^2 = 49$$

$$\text{II.4 [58]} \quad x^2 = 3(y^2 + y) + 1$$

Section II.5 [99]

The homogeneous cone represented by the ternary quadratic diophantine equation $x^2 + 6xy + 15y^2 = 15z^2$ is studied for finding its non-zero distinct integer solutions. A few interesting properties among the solutions are also exhibited.

Section II.6 [100]

The method of determining different solutions in integers to $x^2 + y^2 = z^2 + 141$ by reducing it to $\beta^2 = D\alpha^2 + 141$ ($D > 0$ and square-free) through employing transformations. A special case has been illustrated along with the corresponding properties. Also, given an integer solution, a process of obtaining sequence of integer solutions based on its given solution is exhibited.

Section II.7 [59]

This section aims at determining non-zero distinct integer solutions satisfying the homogeneous cone represented by the ternary quadratic equation $3x^2 - 8y^2 = 25z^2$. A few interesting relations among the solutions are presented. A general formula for generating sequence of integer solutions to the given cone based on a given solution is illustrated.

Section II.8 [101]

The homogeneous ternary quadratic diophantine equation given by $z^2 = D(x^2 - y^2)$, $D = \text{odd prime}$ is analyzed for its non-zero distinct integer solutions through different methods. Also, formulae for generating sequence of integer solutions based on the given solutions are presented.

Section II.9 [123]

The ternary quadratic equation $x^2 = 25y^2 + 29z^2$ representing a homogeneous cone is analysed for its non-zero distinct integral points. A few interesting properties among the solutions and polygonal numbers are presented.

Section II.10 [126]

The homogeneous ternary quadratic equation given by $z^2 = (2k^2 - 2k + 22)x^2 + y^2$ is analysed for its non-zero distinct integer solutions through different methods. A few interesting properties between the solutions are presented. Also, formulae for generating sequence of integer solutions based on the given solution are presented.

Section II.11 [122]

The homogeneous quadratic diophantine equation with four unknowns given by $2xy + 3z^2 = 8w^2$ is analysed for obtaining its different sets of non-zero distinct integer solutions through employing linear transformations. Also, formulae for generating sequence of integer solutions based on the given solution are presented.

Section II.12 [127]

The homogeneous quadratic diophantine equation with five unknowns given by $4w^2 - x^2 - y^2 + z^2 = 16t^2$ is analyzed for determining its non-zero distinct integer solutions through employing linear transformations.

Chapter III analyses cubic diophantine equations in four sections III.1 to III.4**Section III.1 [60]**

The non-homogeneous cubic equation with three unknowns represented by $3(x^2 + y^2) - 5xy + x + y + 1 = 111z^3$ is analyzed for its patterns of non-zero distinct integer solutions. A few interesting relations among the solutions are presented.

Section III.2 [84]

An attempt is made to solve the cubic equation with four unknowns given by $x^3 + y^3 + 6(x + y)z^2 = 4w^3$ in integers. Some special relations between the solutions are given.

Section III.3 [104]

The homogeneous cubic equation with four unknowns represented by the Diophantine equation $x^3 + y^3 + (x + y)(x - y)^2 = 16zw^2$ is analyzed for its patterns of non-zero distinct integral solutions. Various interesting relations between the solutions and special numbers namely polygonal numbers are exhibited.

Section III.4 [124]

This paper concerns with the problem of obtaining non-zero distinct integer solutions to the non-homogeneous cubic equation with three unknowns given $x^3 + y^3 + x + y = 2z(2z^2 - \alpha^2 + 1)$. A few interesting relations among the solutions are presented. Also, a formula for generating sequence of integer solutions to the considered cubic equation based on its given solution is exhibited.

Chapter-IV focuses on bi-quadratic diophantine equations in four sections**IV.1 to IV.4****Section IV.1 [103]**

We obtain infinitely many non-zero integer solutions to the non-homogeneous ternary bi-quadratic equation $x^2 + 7xy + y^2 = z^4$.

Section IV.2 [105]

This section aims at determining non-zero distinct integer solutions to the algebraic equation of degree four with three unknowns given by $(a+1)(x^2 + y^2) - (2a+1)xy = [p^2 + (4a+3)q^2]z^4, a \geq 0$

Section IV.3 [102]

This section concerns with the problem of determining non-trivial integral solutions of the non-homogeneous bi-quadratic equation with four unknowns given by

$8xy + 5z^2 = 5w^4$. We obtain infinitely many non-zero integer solutions of the equation, by introducing the linear transformations $x = u + v, y = u - v, z = v$.

Section IV.4 [125]

On the Homogeneous Bi-Quadratic Equation with Four Unknowns given by $x^4 + y^4 + (x - y)z^3 = 2(k^2 + 3s^2)^2 w^2$ aims at determining non-zero distinct integer solutions.

Chapter-V [106] searches for the integral solutions to Quintic equation

The quintic non-homogeneous equation with five unknowns represented by the Diophantine equation $3(x + y)(x^3 - y^3) = 7(z^2 - w^2)p^3$ is analyzed for its patterns of non-zero distinct integral solutions.

Chapter VI deals with Double Diophantine Equations in six sections VI.1 to VI.6

Section VI.1 [86]

The system of double equations given by $x - yz = 3w^2, xy = T^3$ is studied for obtaining its non-zero distinct solutions in integers.

Section VI.2 [107]

The pair of equations given by $x + y = z + w, y + z = (x - w)^2$ is studied for obtaining its non-zero distinct solutions in integers.

Section VI.3 [61]

In this section, different methods to obtain non-zero distinct integer solutions to the system of double equations $x + y = z + w, y + z = (x + w)^2$ are illustrated.

Section VI.4 [110]

This section illustrates the method of obtaining non-zero integral solutions to the system of two linear equations to be made squares represented by $an+b=p^2, bn+a=q^2$ for the choices of a and b given by (i) $a = 1, b = 7$ and (ii) $a = 2, b = 7$.

Section VI.5 [87]

Two different methods for obtaining non-zero distinct integer solutions to the pair of equations $x + y = z + w, y + z = (x - w)^3$ are illustrated.

Section VI.6 [109]

The problem of obtaining non-zero distinct integer solutions to the pair of equations $x + y = z + w, y + z = (x + w)^3$ is analysed.

Chapter VII analyses Triple Diophantine Equations in three sections VII.1 to VII.3**Section VII.1 [111]**

An attempt is made to obtain non-zero distinct integer quintuples (x, y, a, b, c) satisfying the system of three equations $x + y = a^2, 2x + y = b^2, x + 2y = a^2 - c^2$. Different sets of integer solutions are presented.

Section VII.2 [112]

Non-zero distinct integer quintuples (x, y, a, b, c) satisfying the system of three equations $x + y = 2a^2, 2x + y = 5a^2 + b^2, x + 2y = c^3$ are determined.

Section VII.3 [17]

Triple equations with five unknowns represented by $x + y = 2a^2, 2x + y = 5a^2 - b^2, x + 2y = 5c^3$ are analyzed for non-zero distinct integral solutions.

Chapter VIII focuses on Diophantine 3-Tuples in four sections VIII.1 to VIII.4

Section VIII.1 [114]

This paper deals with the study of constructing sequences of diophantine triples (a, b, c) such that the product of any two elements of the set added by a polynomial with integer coefficient is a perfect square.

Section VIII.2 [113]

The construction of sequences of diophantine triples (a, b, c) through pronic numbers is studied.

Section VIII.3

This section has three parts VIII.3A [88], VIII.3B [90], VIII.3C [62]

The formulation of sequences of diophantine triples (a, b, c) through Euler polynomials, Bernoulli polynomials & Euler and Bernoulli polynomials is considered in sections VIII.3A, VIII.3B, VIII.3C respectively.

Section VIII.4 [115]

This paper concerns with the formulation of sequences of Diophantine 3-tuples with property $D(k^2 + 10k - 3)$ through matrix method.

**Chapter IX [116] deals with formulation of special Dio 3-Tuples through
polynomials with suitable property**

An attempt has made in constructing sequences of dio 3-tuples (a, b, c) such that the product of any two elements of the set added with the sum or minus the sum of the same elements and increased by a polynomial with integer coefficient is a perfect square.

Chapter X has two sections X.1 and X.2

Section X.1 [117] presents diophantine quadruples (a, b, c, d) generated from two given pronic numbers such that the product of any two members of the set increased by one is a perfect square.

Section X.2 [108] has two subsections X.2A and X.2B

The Sub-section X.2A deals with the study of formulating special Dio-quadruples (a, b, c, d) generated through Euler polynomials such that the product of any two of the set minus their sum and increased by two is a perfect square. Sub-section X.2B concerns with constructing special Dio-quadruples (a, b, c, d) generated through Euler polynomials such that the product of any two of the set minus their sum and increased by five is a perfect square.

Chapter XI deals with special family of 3-tuples in two sections XI.1 to XI.2

Section XI.1 [120] concerns with the study of formulating 3-tuples consisting of polygonal and pyramidal numbers such that, in each three tuple, the sum of any two members is a perfect square.

Section XI.2 [121] deals with the study of formulation of special family of 3-tuples (a, b, c) such that the product of any two elements of the set added with their sum is a perfect square.

Chapter XII focuses on Equality of Polygonal Numbers in two sections XII.1 to XII.2

Section XII.1 [118] illustrates formulas for the ranks of Triangular numbers, Hexagonal numbers, star numbers satisfying the relation $t_{3,N} = t_{6,h} = S_n$.

Section XII.2 [119] exhibits formulas for the ranks of Triangular numbers, Hexagonal numbers, Centered Hexagonal numbers, Centered Octagonal numbers, Centered Decagonal numbers and Centered Dodecagonal numbers satisfying the relations $t_{3,N} = t_{6,h} = ct_{6,H}$, $t_{3,N} = t_{6,h} = ct_{8,M}$, $t_{3,N} = t_{6,h} = ct_{10,M}$, $t_{3,N} = t_{6,h} = ct_{12,D}$.

Chapter –II

Quadratic Diophantine Equations

CHAPTER – II

QUADRATIC DIOPHANTINE EQUATIONS

Chapter-II deals with Quadratic diophantine equations in twelve sections

II.1 to II.12

Section II.1 to II.4

The following binary quadratic equations are analyzed for finding its non-zero distinct integer solutions. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented.

II.1 $3x^2 - 8y^2 = 40$

II.2 $y^2 = 5\alpha^2(x^2 + 1)$

II.3 $9x^2 - 8y^2 = 49$

II.4 $x^2 = 3(y^2 + y) + 1$

Section II.5 to II.10

The method of determining different solutions in integers to ternary quadratic Diophantine equations are discussed in the following sections

II.5 $x^2 + 6xy + 15y^2 = 15z^2$

II.6 $x^2 + y^2 = z^2 + 141$

II.7 $3x^2 - 8y^2 = 25z^2$

II.8 $z^2 = D(x^2 - y^2), D = \text{odd prime}$

II.9 $x^2 = 25y^2 + 29z^2$

II.10 $z^2 = (2k^2 - 2k + 22)x^2 + y^2$

Section II.11

The homogeneous quadratic diophantine equation with four unknowns given by $2xy + 3z^2 = 8w^2$ is analysed for obtaining its different sets of non-zero distinct integer solutions through employing linear transformations. Also, formulae for generating sequence of integer solutions based on the given solution are presented.

Section II.12

The homogeneous quadratic diophantine equation with five unknowns given by $4w^2 - x^2 - y^2 + z^2 = 16t^2$ is analyzed for determining its non-zero distinct integer solutions through employing linear transformations.

II.1. On the Pell-Like Equation $3x^2 - 8y^2 = 40$

The binary quadratic equation representing hyperbola is given by

$$3x^2 - 8y^2 = 40 \quad (2.1)$$

Taking $x = X + 8T$, $y = X + 3T$ (2.2)

in (2.1), it simplifies to the equation

$$X^2 = 24T^2 - 8 \quad (2.3)$$

The smallest positive integer solution (T_0, X_0) of (2.3) is

$$T_0 = 1, X_0 = 4$$

To obtain, the other solutions of (2.3), consider the pellian equation

$$X^2 = 24T^2 + 1 \quad (2.4)$$

whose smallest positive integer solution is

$$\tilde{T}_0 = 1, \tilde{X}_0 = 5$$

The general solution $(\tilde{T}_n, \tilde{X}_n)$ of (2.4) is given by

$$\tilde{X}_n + \sqrt{24}\tilde{T}_n = (5 + \sqrt{24})^{n+1}, n = 0, 1, 2, \dots \quad (2.5)$$

Since, irrational roots occur in pairs, we have

$$\tilde{X}_n - \sqrt{24}\tilde{T}_n = (5 - \sqrt{24})^{n+1}, n = 0, 1, 2, \dots \quad (2.6)$$

From (2.5) and (2.6), solving for $\tilde{X}_n, \tilde{T}_n$, we have

$$\tilde{X}_n = \frac{1}{2} \left[(5 + \sqrt{24})^{n+1} + (5 - \sqrt{24})^{n+1} \right] = \frac{1}{2} f_n$$

$$\tilde{T}_n = \frac{1}{2\sqrt{24}} \left[(5 + \sqrt{24})^{n+1} - (5 - \sqrt{24})^{n+1} \right] = \frac{1}{2\sqrt{24}} g_n$$

Applying Brahmagupta lemma between the solutions (T_0, X_0) and $(\tilde{T}_n, \tilde{X}_n)$, the

general solution (T_{n+1}, X_{n+1}) of (2.3) is found to be

$$T_{n+1} = X_0 \tilde{T}_n + T_0 \tilde{X}_n$$

$$X_{n+1} = X_0 \tilde{X}_n + 24T_0 \tilde{T}_n$$

$$\Rightarrow T_{n+1} = \frac{2}{\sqrt{24}} g_n + \frac{1}{2} f_n \quad (2.7)$$

$$X_{n+1} = 2f_n + \frac{\sqrt{24}}{2} g_n \quad (2.8)$$

Using (2.7) and (2.8) in (2.2) we have

$$x_{n+1} = X_{n+1} + 8T_{n+1} = 6f_n + \frac{28}{\sqrt{24}} g_n \quad (2.9)$$

$$y_{n+1} = X_{n+1} + 3T_{n+1} = \frac{7}{2} f_n + \frac{18}{\sqrt{24}} g_n \quad (2.10)$$

Thus, (2.9) and (2.10) represent the integer solutions of the hyperbola (2.1).

A few numerical examples are given in the following table 2.1

Table 2.1: Examples

n	x_{n+1}	y_{n+1}
-1	12	7
0	116	71
1	1148	703
2	11364	6959

In the above table x -values are even and y -values are odd

Recurrence relations for x and y are:

$$x_{n+3} - 10x_{n+2} + x_{n+1} = 0, n = -1, 0, 1, \dots$$

$$y_{n+3} - 10y_{n+2} + y_{n+1} = 0, n = -1, 0, 1, \dots$$

A few interesting relations among the solutions are given below:

$$\triangleright 8y_{n+1} + 5x_{n+1} - x_{n+2} = 0$$

$$\triangleright 8y_{n+3} + 5x_{n+1} - 49x_{n+2} = 0$$

$$\triangleright 8y_{n+2} + x_{n+1} - 5x_{n+2} = 0$$

$$\triangleright 80y_{n+1} + 49x_{n+1} - x_{n+3} = 0$$

$$\triangleright 16y_{n+2} + x_{n+1} - x_{n+3} = 0$$

$\triangleright$ Each of the following expressions represents a cubical integer:

$$\text{i. } \frac{1}{120} \left[(426x_{3n+3} - 42x_{3n+4}) + 3(426x_{n+1} - 42x_{n+2}) \right]$$

$$\text{ii. } \frac{1}{1200} \left[(4218x_{3n+3} - 42x_{3n+5}) + 3(4218x_{n+1} - 42x_{n+3}) \right]$$

$$\text{iii. } \frac{1}{150} \left[(522x_{3n+3} - 84y_{3n+4}) + 3(522x_{n+1} - 84y_{n+2}) \right]$$

$$\text{iv. } \frac{1}{30} \left[(54x_{3n+3} - 84y_{3n+3}) + 3(54x_{n+1} - 84y_{n+1}) \right]$$

$$\text{v. } \frac{1}{1470} \left[(5166x_{3n+3} - 84y_{3n+5}) + 3(5166x_{n+1} - 84y_{n+3}) \right]$$

$\triangleright$ Each of the following expressions represents bi-quadratic integer:

$$\text{i. } \frac{1}{120^2} \left[(51120x_{4n+4} - 5040x_{4n+5}) + 4(426x_{n+1} - 42x_{n+2})^2 - 28800 \right]$$

$$\text{ii. } \frac{1}{1200^2} \left[(5061600x_{4n+4} - 50400x_{4n+6}) + 4(4218x_{n+1} - 42x_{n+3})^2 - 2880000 \right]$$

$$\text{iii. } \frac{1}{30^2} \left[(1620x_{4n+4} - 2520y_{4n+4}) + 4(54x_{n+1} - 84y_{n+1})^2 - 1800 \right]$$

$$\text{iv. } \frac{1}{150^2} \left[(78300x_{4n+4} - 2600y_{4n+5}) + 4(544x_{n+1} - 84y_{n+2})^2 - 45000 \right]$$

$$\text{v. } \frac{1}{1470^2} \left[(7594020x_{4n+4} - 123480y_{4n+6}) + 4(5166x_{n+1} - 84y_{n+3})^2 - 4321800 \right]$$

➤ Each of the following expressions represents Nasty number:

i. $\frac{1}{20}[240 + 426x_{2n+2} - 42x_{2n+3}]$

ii. $\frac{1}{200}[2400 + 4218x_{2n+2} - 42x_{2n+4}]$

iii. $\frac{1}{5}[60 + 54x_{2n+2} - 84y_{2n+2}]$

iv. $\frac{1}{25}[300 + 522x_{2n+2} - 84y_{2n+3}]$

v. $\frac{1}{245}[2940 + 5166x_{2n+2} - 84y_{2n+4}]$

➤ Each of the following expressions represents Quintic integer

i. $\frac{1}{20}(71x_{5n+5} - 7x_{5n+6}) + 30P_{f_n-1}^3$ where $f_n = \frac{1}{20}(71x_{n+1} - 7x_{n+2})$

ii. $\frac{1}{600}(2109x_{5n+5} - 21x_{5n+7}) + 30P_{f_n-1}^3$ where $f_n = \frac{1}{600}(2109x_{n+1} - 21x_{n+3})$

iii. $\frac{1}{15}(27x_{5n+5} - 42x_{5n+7}) + 30P_{f_n-1}^3$ where $f_n = \frac{1}{15}(27x_{n+1} - 42x_{n+3})$

iv. $\frac{1}{75}(261x_{5n+5} - 42y_{5n+6}) + 30P_{f_n-1}^3$ where $f_n = \frac{1}{75}(261x_{n+1} - 42y_{n+2})$

v. $\frac{1}{735}(2583x_{5n+5} - 42y_{5n+7}) + 30P_{f_n-1}^3$ where $f_n = \frac{1}{735}(2583x_{n+1} - 42y_{n+3})$

REMARKABLE OBSERVATIONS

I. Employing linear combinations among the solutions of (2.1), one may generate integer solutions for other choices of hyperbola which are presented in table 2.2 below:

Table 2.2: Hyperbola

S. No.	Hyperbola	(X_n, Y_n)
1	$24X_n^2 - Y_n^2 = 1382400$	$\left[(426x_{n+1} - 42x_{n+2}), (216x_{n+2} - 2088x_{n+1}) \right]$
2	$24X_n^2 - Y_n^2 = 138240000$	$\left[(4218x_{n+1} - 42x_{n+3}), (216x_{n+3} - 20664x_{n+1}) \right]$
3	$24X_n^2 - Y_n^2 = 86400$	$\left[(54x_{n+1} - 84y_{n+1}), (432y_{n+1} - 252x_{n+1}) \right]$
4	$24X_n^2 - Y_n^2 = 2160000$	$\left[(522x_{n+1} - 84y_{n+2}), (432y_{n+2} - 2556x_{n+1}) \right]$
5	$24X_n^2 - Y_n^2 = 207446400$	$\left[(5166x_{n+1} - 84y_{n+3}), (432y_{n+3} - 25308x_{n+1}) \right]$

II. Employing linear combination among the solutions for other choices of parabola which are presented in table 2.3 below:

Table 2.3: Parabola

S. No.	Parabola	(X_n, Y_n)
1	$2880X_n - Y_n^2 = 1382400$	$\left[(240 + 426x_{2n+2} - 42x_{2n+3}), (216x_{n+2} - 2088x_{n+1}) \right]$
2	$28800X_n - Y_n^2 = 138240000$	$\left[(2400 + 4218x_{2n+2} - 42x_{2n+4}), (216x_{n+3} - 20664x_{n+1}) \right]$
3	$720X_n - Y_n^2 = 86400$	$\left[(60 + 54x_{2n+2} - 84y_{2n+2}), (432y_{n+1} - 252x_{n+1}) \right]$
4	$3600X_n - Y_n^2 = 2160000$	$\left[(300 + 522x_{2n+2} - 84y_{2n+3}), (432y_{n+2} - 2556x_{n+1}) \right]$
5	$35280X_n - Y_n^2 = 207446400$	$\left[(2940 + 5166x_{2n+2} - 84y_{2n+4}), (432y_{n+3} - 25308x_{n+1}) \right]$

PROPERTIES

III. Let $\{m_{s+1}\}$ and $\{n_{s+1}\}$ be sequence of positive integers defined by

$$(i) \quad n_{s+1} = \frac{y_{s+1} - 1}{2}, m_{s+1} = \frac{x_{s+1}}{2}$$

It is seen that

$$128t_{3,s+1} + 96 \text{ is a Nasty number.}$$

(ii) Define $n_{s+1} = \frac{y_{s+1} + 1}{2}, m_{s+1} = \frac{x_{s+1}}{2}$

It is noted that

$$t_{25, m_{s+1}} + 11m_{s+1} - 8(8t_{3, n_{s+1}} + 1) + 64n_{s+1} = 40$$

(iii) Consider $n_{s+1} = \frac{y_{s+1} - 1}{2}, m_{s+1} = \frac{x_{s+1}}{4}$

It is noted that

$$t_{98, m_{s+1}} - 64t_{3, n_{s+1}} + 47m_{s+1} = 48$$

(iv) Assume $n_{s+1} = \frac{y_{s+1} - 3}{2}, m_{s+1} = \frac{x_{s+1}}{2}$

It is observed that

$$t_{25, m_{s+1}} - 64t_{3, n_{s+1}} + 11m_{s+1} - 64n_{s+1} = 112$$

II.2 A Remark on the Positive Pell Equation $y^2 = 5\alpha^2(x^2 + 1)$

The binary quadratic equation to be solved for its non-zero distinct integral solution is

$$y^2 = 5\alpha^2(x^2 + 1) \quad (2.11)$$

whose smallest positive integer solution is

$$x_0 = 2, y_0 = 5\alpha$$

Assuming

$$y = \alpha Y \quad (2.12)$$

in (2.11), we get

$$Y^2 = 5(x^2 + 1) \quad (2.13)$$

whose initial solution is $(x_0, y_0) = (2, 5)$

To obtain the other solutions of (2.13), consider the Pell equation

$$Y^2 = 5x^2 + 1 \quad (2.14)$$

whose smallest positive integer solution is $(\tilde{x}_0, \tilde{Y}_0) = (4, 9)$

The general solution of (2.14) is given by

$$\tilde{Y}_n = \frac{1}{2} f_n, \tilde{x}_n = \frac{1}{2\sqrt{5}} g_n$$

where

$$f_n = (9 + 4\sqrt{5})^{n+1} + (9 - 4\sqrt{5})^{n+1}, g_n = (9 + 4\sqrt{5})^{n+1} - (9 - 4\sqrt{5})^{n+1}, n = -1, 0, 1, \dots$$

In view of (2.12), $\tilde{y}_n = \frac{\alpha}{2} f_n$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (2.11) are given by

$$\left. \begin{aligned} x_{n+1} &= f_n + \frac{\sqrt{5}}{2} g_n \\ y_{n+1} &= \frac{5\alpha}{2} f_n + \sqrt{5} \alpha g_n \end{aligned} \right\}, n = -1, 0, 1, \dots$$

The recurrence relations satisfied by x_{n+1} and y_{n+1} are given by

$$x_{n+3} - 18x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 18y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (2.11) are given in the Table 2.4 below:

Table 2.4: Numerical Examples

n	x_{n+1}	y_{n+1}
-1	2	5α
0	38	85α
1	682	1525α
2	12238	27365α
3	219602	491045α

From the above table, we observe some interesting relations among the solutions which are presented below:

Illustration 1

Consider

$$y_0 = 5\alpha \tag{2.15}$$

$$\alpha = 4 \Rightarrow y_0 = 20$$

$$\Rightarrow 1*20 = 2*10 = 4*5$$

Now,

$$1*20 = 2*10 \Rightarrow (1+20)^2 + (10-2)^2 = (1-20)^2 + (10+2)^2 = 505$$

$$1*20 = 4*5 \Rightarrow (20-1)^2 + (4+5)^2 = (20+1)^2 + (5-4)^2 = 442$$

$$2*10 = 4*5 \Rightarrow (2+10)^2 + (4-5)^2 = (2-10)^2 + (4+5)^2 = 145$$

Thus, 505, 442, 145 represents second order Ramanujam Numbers with base numbers as integers.

Illustration 2

Consider (2.15).

Now, $y_0 = 5*\alpha = 5\alpha*1$, $\alpha > 1$

$$\Rightarrow (5+\alpha)^2 + (5\alpha-1)^2 = (5-\alpha)^2 + (5\alpha+1)^2 = 26(1+\alpha^2)$$

$$\text{Also, } (\alpha+i5)^2 + (5\alpha-i)^2 = (\alpha-i5)^2 + (5\alpha+i)^2 = 26(\alpha^2-1)$$

Thus, $26(1+\alpha^2)$ represents second order Ramanujam Numbers with base numbers as integers whereas, $26(\alpha^2-1)$ represents second order Ramanujam Numbers with base numbers as Gaussian integers.

Property I

Let $\{m_{2s+1}\}$ and $\{n_{2s+1}\}$ be sequences of positive integers defined by

$$\left\{ m_{2s+1} = \frac{x_{2s+1} + 2}{5} \right\}, \left\{ n_{2s+1} = \frac{1}{20\alpha}(y_{2s+1} - 5\alpha) \right\}, s = 0, 1, 2, \dots$$

Observations

- $16t_{3, n_{2s+1}} + 8n_{2s+1}^2 = t_{12, m_{2s+1}}$
- $x_{2s+1}^2 = 5(4n_{2s+1} + 1)^2 - 1$

Property II

Let $\{m_{s+1}\}$ and $\{n_{s+1}\}$ be sequences of positive integers defined by

$$\left\{m_{s+1} = \frac{x_{s+1} + 2}{2}\right\}, \left\{n_{s+1} = \frac{y_{s+1}}{5\alpha}\right\}, s = 0, 1, 2, \dots$$

Observations

- $t_{10, m_{s+1}} = 5(n_{s+1}^2 + m_{s+1} - 1)$
- $x_{s+1}^2 = 5n_{s+1}^2 - 1$
- Each of the following expressions is a nasty number.
 - ❖ $\frac{6}{5\alpha}(19y_{2n+2} - y_{2n+3} + 10\alpha)$
 - ❖ $\frac{6}{\alpha}(2y_{2n+2} - 4\alpha x_{2n+2} + 2\alpha)$
- Each of the following expressions is a cubical integer.
 - ❖ $\frac{1}{5\alpha}(19x_{3n+3} - y_{3n+4}) + \frac{3}{5\alpha}(19y_{n+1} - y_{n+2})$
 - ❖ $\frac{1}{\alpha}(2y_{3n+3} - 4\alpha x_{3n+3}) + \frac{3}{\alpha}(2y_{n+1} - 4\alpha x_{n+1})$
- Each of the following expressions is a bi-quadratic integer.
 - ❖ $\frac{1}{5\alpha}(19x_{4n+4} - y_{4n+5}) + \frac{4}{5\alpha}(19y_{2n+2} - y_{2n+3} + 10\alpha) - 2$
 - ❖ $\frac{1}{\alpha}(2y_{4n+4} - 4\alpha x_{4n+4}) + \frac{4}{\alpha}(2y_{2n+2} - 4\alpha x_{2n+2} + 2\alpha) - 2$
 - ❖ $\frac{1}{5\alpha}(19y_{4n+4} - y_{4n+5}) + \frac{4}{25\alpha^2}(19y_{n+1} - y_{n+2})^2 - 2$
 - ❖ $\frac{1}{\alpha}(2y_{4n+4} - 4\alpha x_{4n+4}) + \frac{4}{\alpha^2}(2y_{n+1} - 4\alpha x_{n+1})^2 - 2$

➤ Relations among the solutions

$$\diamond \frac{4}{\alpha} y_{n+1} = -9x_{n+1} + x_{n+2}$$

$$\diamond \frac{4}{\alpha} y_{n+2} = 9x_{n+2} - x_{n+1}$$

$$\diamond \frac{4}{\alpha} y_{n+3} = 161x_{n+2} - 9x_{n+1}$$

$$\diamond 20\alpha x_{n+1} = y_{n+2} - 9y_{n+1}$$

$$\diamond 20\alpha x_{n+2} = 9y_{n+2} - y_{n+1}$$

$$\diamond 20\alpha x_{n+3} = 161y_{n+2} - 9y_{n+1}$$

$$\diamond \alpha x_{n+2} = 4y_{n+1} + 9\alpha x_{n+1}$$

$$\diamond 2y_{n+2} = 40\alpha x_{n+1} + 18y_{n+1}$$

$$\diamond 2y_{n+3} = 322y_{n+1} + 720\alpha x_{n+1}$$

Remarkable observations

1. Employing linear combinations among the solutions of (2.11), one may generate integer solutions for other choices of hyperbolas. Some examples are presented in the Table 2.5.

Table 2.5: Hyperbolas

S. No.	Hyperbolas	(X_n, Y_n)
1	$4X_n^2 - 5Y_n^2 = 400\alpha^2$	$(19y_{n+1} - y_{n+2}, y_{n+2} - 17y_{n+1})$
2	$5X_n^2 - Y_n^2 = 20\alpha^2$	$(2y_{n+1} - 4\alpha x_{n+1}, 10\alpha x_{n+1} - 4y_{n+1})$

2. Employing linear combinations among the solutions of (2.11), one may generate integer solutions for other choices of parabolas. Some examples are presented in the Table 2.6.

Table 2.6: Parabolas

S. No.	Parabolas	(α_n, Y_n)
1	$Y_n^2 = 4\alpha \alpha_n - 80 \alpha^2$	$(10\alpha + 19y_{2n+2} - y_{2n+3}, y_{n+2} - 17y_{n+1})$
2	$Y_n^2 = 5\alpha \alpha_n - 20 \alpha^2$	$(2\alpha + 2y_{2n+2} - 4\alpha x_{2n+2}, 10\alpha x_{n+1} - 4y_{n+1})$

II.3 On the Binary Quadratic Equation $9x^2 - 8y^2 = 49$

Consider the non homogeneous binary quadratic equation

$$9x^2 - 8y^2 = 49 \quad (2.16)$$

Introducing the linear transformations

$$x = X + 8T, y = X + 9T \quad (2.17)$$

in (2.16), it leads to

$$X^2 = 72T^2 + 49 \quad (2.18)$$

with the least positive integer solutions $X_0 = 11, T_0 = 1$

To obtain the other solutions of equation (2.18), Consider the Pellian equation

$$X^2 = 72T^2 + 1$$

whose general solution, $\tilde{X}_n = \frac{1}{2} f_n, \tilde{T}_n = \frac{1}{2\sqrt{72}} g_n$

in which $f_n = [(17 + 2\sqrt{72})^{n+1} + (17 - 2\sqrt{72})^{n+1}]$

$$g_n = [(17 + 2\sqrt{72})^{n+1} - (17 - 2\sqrt{72})^{n+1}], \text{ where } n = -1, 0, 1, 2, \dots$$

Applying Brahmagupta lemma between the solutions of (X_0, T_0) and $(\tilde{X}_n, \tilde{T}_n)$ the general solutions of equation (2.18) are found to be

$$X_{n+1} = \frac{11}{2} f_n + \frac{\sqrt{72}}{2} g_n$$

$$T_{n+1} = \frac{1}{2} f_n + \frac{11}{2\sqrt{72}} g_n$$

In view of (2.17), the corresponding nonzero distinct integral solutions of (2.16) are

$$x_{n+1} = \frac{19}{2} f_n + \frac{80}{\sqrt{72}} g_n$$

$$y_{n+1} = 10 f_n + \frac{171}{2\sqrt{72}} g_n$$

The recurrence relations satisfied by the values of x and y are respectively

$$x_{n+3} - 34x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 34y_{n+2} + y_{n+1} = 0$$

A few numerical examples are presented in the table 2.7 below:

Table 2.7: Numerical examples

n	x_{n+1}	y_{n+1}
-1	19	20
0	643	682
1	21843	23168
2	742019	787030
3	25206803	26735852
4	856289283	908231938

A few interesting properties are given below:

1. The values of x are odd while the values of y are even.
2. Each of the following is a Nasty number

$$\triangleright \frac{6}{49}[342x_{2n+2} - 320y_{2n+2} + 98]$$

$$\triangleright \frac{6}{833}[11574x_{2n+2} - 320y_{2n+3} + 1666]$$

$$\triangleright \frac{6}{784}[10912x_{2n+2} - 320x_{2n+3} + 1568]$$

$$\triangleright \frac{6}{26656}[370688x_{2n+2} - 320x_{2n+4} + 53312]$$

$$\triangleright \frac{6}{28273}[393174x_{2n+2} - 320y_{2n+4} + 56546]$$

3. Each of the following is a Square number

$$\triangleright \frac{1}{49}[393174x_{2n+4} - 370688y_{2n+4} + 98]$$

$$\triangleright \frac{1}{882}[342y_{2n+3} - 11574y_{2n+2} + 1764]$$

$$\triangleright \frac{1}{29988}[342y_{2n+4} - 393174y_{2n+2} + 59976]$$

$$\triangleright \frac{1}{882}[11574y_{2n+4} - 393174y_{2n+3} + 1764]$$

4. Each of the following is a cubical integer

$$\text{i. } \frac{1}{784}[10912x_{3n+3} - 320x_{3n+4} + 32736x_{n+1} - 960x_{n+2}]$$

$$\text{ii. } \frac{1}{26656}[370688x_{3n+3} - 320x_{3n+5} + 1112064x_{n+1} - 960x_{n+3}]$$

$$\text{iii. } \frac{1}{49}[324x_{3n+3} - 320y_{3n+3} + 34722x_{n+2} - 32736y_{n+2}]$$

$$\text{iv. } \frac{1}{833}[11574x_{3n+3} - 320y_{3n+4} + 1026x_{n+2} - 32736y_{n+1}]$$

$$\text{v. } \frac{1}{28273}[393174x_{3n+3} - 320y_{3n+5} + 1029x_{n+3} - 1112064y_{n+1}]$$

5. Each of the following is a bi-quadratic integer

$$\text{i. } \frac{1}{784}[10912x_{4n+4} - 320x_{4n+5} + 43648x_{2n+2} - 1280x_{2n+3} + 4704]$$

$$\text{ii. } \frac{1}{26656}[370688x_{4n+4} - 320x_{4n+6} + 1482752x_{2n+2} - 1280x_{2n+4} + 159936]$$

$$\text{iii. } \frac{1}{49}[342x_{4n+4} - 320y_{4n+4} + 1368x_{2n+2} - 1280y_{2n+2} + 294]$$

$$\text{iv. } \frac{1}{833}[11574x_{4n+4} - 320y_{4n+5} + 46296x_{2n+2} - 1280y_{2n+3} + 4998]$$

$$v. \quad \frac{1}{28273} [393174x_{4n+4} - 320y_{4n+6} + 1572696x_{2n+2} - 1280y_{2n+4} + 169638]$$

6. Each of the following is a quintic integer

$$\triangleright \quad \frac{1}{784} [10912x_{5n+5} - 320x_{5n+6} + 54560x_{3n+3} - 1600x_{3n+4} - 109120x_{n+1} + 3200x_{n+2}]$$

$$\triangleright \quad \frac{1}{26656} [370688x_{5n+5} - 320x_{5n+7} + 1853440x_{3n+3} - 4800x_{3n+5} - 3706880x_{n+1} + 3200x_{n+3}]$$

$$\triangleright \quad \frac{1}{49} [342x_{5n+5} - 320y_{5n+5} + 1710x_{3n+3} - 1600y_{3n+3} - 3420x_{n+1} + 3200y_{n+1}]$$

$$\triangleright \quad \frac{1}{833} [11574x_{5n+5} - 320y_{5n+7} + 57870x_{3n+3} - 1600y_{3n+5} - 115740x_{n+1} + 3200y_{n+3}]$$

$$\triangleright \quad \frac{1}{784} [370688x_{5n+6} - 10912x_{5n+7} + 1853440x_{3n+4} - 54560x_{3n+5} - 3706880x_{n+2} + 109120x_{n+3}]$$

7. Relations among the solutions are given below:

$$\triangleright \quad 16y_{n+1} + 17x_{n+1} - x_{n+2} = 0$$

$$\triangleright \quad 16y_{n+2} + x_{n+1} - 17x_{n+2} = 0$$

$$\triangleright \quad x_{n+1} + x_{n+3} - 34x_{n+2} = 0$$

$$\triangleright \quad 544y_{n+1} + 577x_{n+1} - x_{n+3} = 0$$

$$\triangleright \quad 32y_{n+2} + x_{n+1} - x_{n+3} = 0$$

3. REMARKABLE OBSERVATIONS

3.1. Employing linear combinations among the solutions of (2.16), one may generate integer solutions for other choices of hyperbolas which are presented in the table 2.8 below.

Table 2.8: Illustrations

S. No.	Hyperbola	(X,Y)
1	$X_n^2 - 72Y_n^2 = 2458624$	$(10912x_{n+1} - 320x_{n+2}, 38x_{n+2} - 1286x_{n+1})$
2	$X_n^2 - 72Y_n^2 = 2842169344$	$(370688x_{n+1} - 320x_{n+3}, 38x_{n+3} - 43686x_{n+1})$
3	$X_n^2 - 72Y_n^2 = 9604$	$(342x_{n+1} - 320y_{n+1}, 38y_{n+1} - 40x_{n+1})$
4	$X_n^2 - 72Y_n^2 = 2775556$	$(11574x_{n+1} - 320y_{n+2}, 38y_{n+2} - 1364x_{n+1})$
5	$X_n^2 - 72Y_n^2 = 3197450116$	$(393174x_{n+1} - 320y_{n+3}, 38y_{n+3} - 46336x_{n+1})$

3.2. Employing linear combinations among the solutions of (2.16), one may generate integer solutions for other choices of parabolas which are presented in the table 2.9 below.

Table 2.9: Illustrations

S. No.	Parabola	(X,Y)
1	$784X_n - 72Y_n^2 = 1229312$	$(370688x_{2n+3} - 10912x_{2n+4}, 1286x_{n+3} - 43686x_{n+2})$
2	$26656X_n - Y_n^2 = 1421084672$	$(370688x_{2n+2} - 320x_{2n+4}, 38x_{n+3} - 43686x_{n+1})$
3	$49X_n - 72Y_n^2 = 4802$	$(342x_{2n+2} - 320y_{2n+2}, 38y_{n+1} - 40x_{n+1})$
4	$833X_n - 72Y_n^2 = 1387778$	$(11574x_{2n+2} - 320y_{2n+3}, 38y_{n+2} - 1364x_{n+1})$
5	$28273X_n - 72Y_n^2 = 1598725058$	$(393174x_{2n+2} - 320y_{2n+4}, 38y_{n+3} - 46336x_{n+1})$

3.3. Employing linear combinations among the solutions of (2.16), one may generate integer solutions for other choices of straight lines which are presented in the table 2.10 below.

Table 2.10: Illustrations

S. No.	Straight line	(X,Y)
1.	$17X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 11584x_{n+1} - 10x_{n+3}$
2.	$X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 342x_{n+1} - 320y_{n+1}$
3.	$17X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 11584x_{n+1} - 320y_{n+2}$
4.	$X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 23268x_{n+2} - 682x_{n+3}$
5.	$17X = Y$	$X = 682x_{n+1} - 20x_{n+2}$ $Y = 342x_{n+2} - 10912y_{n+1}$

3.4. Consider $p = x_{n+1} + y_{n+1}, q = x_{n+1}$ observe that $p > q > 0$. Treat p, q as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$, where $\alpha = 2pq, \beta = p^2 - q^2, \gamma = p^2 + q^2$. Let A, P represent the area and perimeter of $T(\alpha, \beta, \gamma)$. Then the following interesting relations are observed.

- a) $16X - 9Y - 7Z - 98 = 0$.
- b) $\frac{2A}{P} = x_{n+1} y_{n+1}$.
- c) $3(Z - Y)$ is a nasty number.
- d) $3(X - \frac{4A}{P})$ is a nasty number.
- e) $X - \frac{4A}{P} + Y$ is written as the sum of two squares.

3.5. From the values of y_{n+1} , one may generate second order Ramanujan numbers with base numbers as real integers and Gaussian integers.

Illustration

Consider $y_0 = 20 = 20 * 1 = 2 * 10 = 4 * 5$

Now, $20 * 1 = 2 * 10 \Rightarrow (20 + 1)^2 + (10 - 2)^2 = (20 - 1)^2 + (10 + 2)^2$

$$\Rightarrow 21^2 + 8^2 = 19^2 + 12^2 = 505$$

In a similar manner,

$$2 * 10 = 4 * 5 \Rightarrow 12^2 + 1^2 = 8^2 + 9^2 = 145$$

$$20 * 1 = 4 * 5 \Rightarrow 21^2 + 1^2 = 19^2 + 9^2 = 442$$

Thus 505, 145, 442 are second order Ramanujan numbers with base numbers as real integers.

Also,

$$20 * 1 = 2 * 10 \Rightarrow (20 + i)^2 + (10 - 2i)^2 = (20 - i)^2 + (10 + 2i)^2 = 495$$

$$2 * 10 = 4 * 5 \Rightarrow (10 + 2i)^2 + (5 - 4i)^2 = (10 - 2i)^2 + (5 + 4i)^2 = 105$$

$$20 * 1 = 4 * 5 \Rightarrow (20 - i)^2 + (5 + 4i)^2 = (20 + i)^2 + (5 - 4i)^2 = 408$$

Here 495, 105, 408 are second order Ramanujan numbers with base numbers as Gaussian integers.

3.6. Let $\{a_{n+1}\}$ and $\{b_{n+1}\}$ be two sequences of positive integers, where

$$a_{n+1} = \frac{x_{n+1} - 1}{2}, b_{n+1} = \frac{y_{n+1}}{2}$$

It is observed that

$$\text{a) } 6(t_{3, a_{n+1}} - 5) \text{ is a Nasty Number.}$$

$$\text{b) } 9t_{3, a_{n+1}} - t_{10, b_{n+1}} \equiv 2 \pmod{3}$$

$$\text{c) } 9t_{3, a_{n+1}} - 4b_{n+1}^2 = 5 \text{ in which } t_{m, n} \text{ represents a polygonal number of rank } n$$

with side m .

REMARK 1

One may also employ the linear transformations $x = X - 8T$, $y = X - 9T$ to solve (2.16) and obtain a different set of solutions.

REMARK 2

The introduction of the linear transformations $x = 7(X \pm 8T)$, $y = 7(X \pm 9T)$ in (2.16) leads to the pellian equation

$$X^2 = 72T^2 + 1$$

whose solutions are well-known. Applying these values in the above transformations, yet another set of integer solutions to (2.16) is obtained.

In this paper, a study is made for determining many integer solutions to the hyperbola represented by the Pell-Like equation $9x^2 - 8y^2 = 49$. As the quadratic equations are rich in variety, the readers of this paper may attempt to obtain integer solutions to other choices of quadratic equations with two or more unknowns.

II.4. Observations on the Pell Equation $x^2 = 3(y^2 + y) + 1$

The hyperbola represented by the non-homogeneous quadratic equation under consideration is

$$x^2 = 3(y^2 + y) + 1 \quad (2.19)$$

Treating (2.19) as a quadratic in y and solving for y ,

we get

$$y = \frac{-3 \pm \sqrt{12x^2 - 3}}{6} \quad (2.20)$$

Let

$$Y^2 = 12x^2 - 3 \quad (2.21)$$

The smallest positive integer solution to (2.21) is $x_0 = 1$, $Y_0 = 3$

To find the other solutions to (2.19), consider the corresponding pellian equation given by

$$Y^2 = 12x^2 + 1 \quad (2.22)$$

whose the general solution $\tilde{x}_n, \tilde{Y}_n$ is

$$\tilde{Y}_n = \frac{1}{2} f_n$$

$$\tilde{x}_n = \frac{1}{4\sqrt{3}} g_n$$

where

$$f_n = (7 + 4\sqrt{3})^{n+1} + (7 - 4\sqrt{3})^{n+1}$$

$$g_n = (7 + 4\sqrt{3})^{n+1} - (7 - 4\sqrt{3})^{n+1}, \quad n = 0, 1, 2, \dots$$

Employing the lemma of Brahmagupta between the solutions (x_0, Y_0) & $(\tilde{x}_n, \tilde{Y}_n)$, the general solution (x_{n+1}, Y_{n+1}) to (2.21) is given by

$$\begin{aligned} x_{n+1} &= x_0 \tilde{Y}_n + Y_0 \tilde{x}_n \\ &= \frac{1}{2} f_n + \frac{\sqrt{3}}{4} g_n \end{aligned} \quad (2.23)$$

$$\begin{aligned} Y_{n+1} &= Y_0 \tilde{Y}_n + D x_0 \tilde{x}_n \\ &= 3 * \frac{1}{2} f_n + \sqrt{3} * g_n \end{aligned}$$

In view of (2.20) and taking the positive sign before the square-root on the R.H.S. of (2.20),

we have

$$y_{n+1} = \frac{1}{12} (3f_n + 2\sqrt{3}g_n - 6) \quad (2.24)$$

Thus, (2.23) and (2.24) represented the integer solutions to (2.19).

A few numerical solutions to (2.19) are presented in Table below:

Table 2.11: Numerical solutions

N	x_{n+1}	y_{n+1}
-1	1	0
0	13	7
1	181	104
2	2521	1455
3	35113	20272

Observations

➤ The x -values are odd primes whereas y -values are alternatively odd and even.

➤ A few interesting relations among the solutions are given below:

- $x_{n+3} - 14x_{n+2} + x_{n+1} = 0$
- $y_{n+3} - 14y_{n+2} + y_{n+1} = 6$
- $12y_{n+1} + 6 = x_{n+2} - 7x_{n+1}$
- $12y_{n+2} + 6 = -x_{n+1} + 7x_{n+2}$
- $12y_{n+3} + 6 = -7x_{n+1} + 97x_{n+2}$

➤ Expressions representing square integers:

- $[15x_{2n+2} - x_{2n+3} + 2]$
- $\frac{1}{14}[209x_{2n+2} - x_{2n+4} + 28]$
- $[2y_{2n+3} - 26y_{2n+2} - 10]$
- $\frac{1}{7}[y_{2n+4} - 181y_{2n+2} - 76]$

➤ Expressions representing cubical integers:

- $[15x_{3n+3} - x_{3n+4} + 3(15x_{n+1} - x_{n+2})]$
- $\frac{1}{14}[209x_{3n+3} - x_{3n+5} + 3(209x_{n+1} - x_{n+3})]$
- $[2y_{3n+4} - 26y_{3n+3} + 6y_{n+2} - 78y_{n+1} - 48]$
- $\frac{1}{7}[y_{3n+5} - 181y_{3n+3} + 3y_{n+3} - 543y_{n+1} - 360]$

➤ Expressions representing biquadratic integers:

- $(15x_{4n+4} - x_{4n+5}) + 4(15x_{n+1} - x_{n+2})^2 - 2$
- $(15x_{4n+4} - x_{4n+5}) + 4(15x_{2n+2} - x_{2n+3} + 2) - 2$
- $\frac{1}{14}(209x_{4n+4} - x_{4n+6}) + \frac{1}{49}(209x_{n+1} - x_{n+3})^2 - 2$
- $\frac{1}{14}(209x_{4n+4} - x_{4n+6}) + \frac{2}{7}(209x_{2n+2} - x_{2n+4} + 28) - 2$
- $(2y_{4n+5} - 26y_{4n+4} - 14) + 16(y_{n+2} - 13y_{n+1} - 6)^2 - 2$

➤ Employing linear combinations among the solutions, one obtains solutions to other choices of hyperbolas.

Choice 1: Let $Y = x_{n+2} - 13x_{n+1}$, $X = 15x_{n+1} - x_{n+2}$

Note that (X, Y) satisfies the hyperbola

$$3X^2 - 4Y^2 = 12$$

Choice 2: Let $Y = x_{n+3} - 181x_{n+1}$, $X = 209x_{n+1} - x_{n+3}$

Note that (X, Y) satisfies the hyperbola

$$3X^2 - 4Y^2 = 48 * 49$$

Choice 3: Let $Y = 15y_{n+1} - y_{n+2} + 7$, $X = 2y_{n+2} - 26y_{n+1} - 12$

Note that (X, Y) satisfies the hyperbola

$$4X^2 - 3Y^2 = 4$$

Choice 4: Let $Y = 209y_{n+1} - y_{n+3} + 104$, $X = y_{n+3} - 181y_{n+1} - 90$

Note that (X, Y) satisfies the hyperbola

$$4X^2 - 3Y^2 = 49 * 16$$

- Employing linear combinations among the solutions, one obtains solutions to other choices of parabolas.

Choice 1: Let $Y = x_{n+2} - 13x_{n+1}$, $X_1 = 15x_{2n+2} - x_{2n+3} + 2$

Note that (Y, X_1) satisfies the parabola

$$3X_1 - 4Y^2 = 12$$

Choice 2: Let $Y = x_{n+3} - 181x_{n+1}$, $X_1 = 209x_{2n+2} - x_{2n+4} + 28$

Note that (Y, X_1) satisfies the parabola

$$21X_1 - 2Y^2 = 21 \cdot 56$$

Choice 3: Let $Y = 209y_{n+1} - y_{n+3} + 104$, $X_1 = y_{2n+4} - 181y_{2n+2} - 76$

Note that (Y, X_1) satisfies the parabola

$$28X_1 - 3Y^2 = 4 \cdot 196$$

Choice 4: Let $Y = 15y_{n+1} - y_{n+2} + 7$, $X_1 = y_{2n+3} - 13y_{2n+2} - 5$

Note that (Y, X_1) satisfies the parabola

$$2X_1 - 3Y^2 = 4$$

- Considering suitable values of x_{n+1} and y_{n+1} , one generates 2nd order Ramanujan numbers with base integers as real integers.

For illustration, consider

$$y_2 = 104 = 1 \times 104 = 2 \times 52 = 4 \times 26 = 8 \times 13 \quad (2.25)$$

Now, $1 \times 104 = 2 \times 52$

$$\rightarrow (104 + 1)^2 + (52 - 2)^2 = (104 - 1)^2 + (52 + 2)^2$$

$$\rightarrow 105^2 + (50)^2 = (103)^2 + 54^2 = 13525$$

$$1 \times 104 = 4 \times 26$$

$$\rightarrow (104+1)^2 + (26-4)^2 = (104-1)^2 + (26+4)^2 = 11509$$

$$1 \times 104 = 8 \times 13$$

$$\rightarrow (104+1)^2 + (13-8)^2 = (104-1)^2 + (13+8)^2 = 11050$$

$$2 \times 52 = 4 \times 26$$

$$\rightarrow (52+2)^2 + (26-4)^2 = (52-2)^2 + (26+4)^2 = 3400$$

$$2 \times 52 = 8 \times 13$$

$$\rightarrow (52+2)^2 + (13-8)^2 = (52-2)^2 + (13+8)^2 = 2941$$

$$4 \times 26 = 8 \times 13$$

$$\rightarrow (26+4)^2 + (13-8)^2 = (26-4)^2 + (13+8)^2 = 925$$

Also,

$$2 \times 52 = 4 \times 26 \rightarrow 27^2 - 25^2 = 15^2 - 11^2$$

$$\rightarrow 27^2 + 11^2 = 15^2 + 25^2 = 850$$

Thus, 13525, 11509, 11050, 3400, 2941, 925, 850 represent 2nd order Ramanujan numbers with base integers as real integers.

- Considering suitable values of x_{n+1} & y_{n+1} , one generates 2nd order Ramanujan numbers with base integers as Guassian integers.

For illustration, consider again y represented by (2.25)

$$\text{Now, } 1 \times 104 = 2 \times 52 \rightarrow (1+i104)^2 + (2-i52)^2 = (1-i104)^2 + (2+i52)^2 = -13520$$

$$\text{Also, } 1 \times 104 = 2 \times 52 \rightarrow (104+i)^2 + (52-i2)^2 = (104-i)^2 + (52+i2)^2 = 13520$$

Note that -13520 & 13520 represent 2nd order Ramanujan numbers with base integers as Gaussian integers.

In a similar manner, other 2nd order Ramanujan numbers are obtained

Formation of sequence of Diophantine 3-tuples

Consider the solution to (2.19) given by

$$x_1 = 13 = a \text{ (say)}, y_1 = 7 = c_0 \text{ (say)}$$

It is observed that

$$ac_0 + k^2 - 91 = k^2, \text{ a perfect square}$$

The pair (a, c_0) represents diophantine 2-tuple with property $D(k^2 - 91)$.

If c_1 is the 3rd tuple, then it satisfies the system of double equations

$$13c_1 + k^2 - 91 = p^2 \quad (2.26)$$

$$7c_1 + k^2 - 91 = q^2 \quad (2.27)$$

Eliminating c_1 between (2.26) and (2.27), we have

$$6(k^2 - 91) = 13q^2 - 7p^2 \quad (2.28)$$

Taking

$$p = X + 13T, q = X + 7T \quad (2.29)$$

in (2.28) and simplifying, we get

$$X^2 = 91T^2 + k^2 - 91$$

which is satisfied by

$$X = k, T = 1$$

In view of (2.29) and (2.26), it is seen that

$$c_1 = 2k + 20$$

Note that $(13, 7, 2k + 20)$ represents diophantine 3-tuple with property $D(k^2 - 91)$.

The process of obtaining sequences of diophantine 3-tuples with property $D(k^2 - 91)$

is illustrated below:

Let M be a 3×3 square matrix given by

$$M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Now

$$(13, 7, 2k + 20)M = (13, 2k + 20, 4k + 59)$$

Note that

$$13 * (2k + 20) + (k^2 - 91) = \text{perfect square}$$

$$13 * (4k + 59) + (k^2 - 91) = \text{perfect square}$$

$$(2k + 20) * (4k + 59) + (k^2 - 91) = \text{perfect square}$$

Therefore, the triple $(13, 2k + 20, 4k + 59)$ represents diophantine 3-tuple with property $D(k^2 - 91)$. The repetition of the above process leads to sequences of diophantine 3-tuples whose general form (a, c_{s-1}, c_s) is given by

$$(13, 13s^2 + (2k - 26)s - 2k + 20, 13s^2 + 2ks + 7), s = 1, 2, 3, \dots$$

A few numerical illustrations are given in Table below:

Table 2.12: Numerical illustrations

K	(a, c_0, c_1)	(a, c_1, c_2)	(a, c_2, c_3)	$D(k^2 - 91)$
0	(13, 7, 20)	(13, 20, 59)	(13, 59, 124)	D(-91)
1	(13, 7, 22)	(13, 22, 63)	(13, 63, 130)	D(-90)
2	(13, 7, 24)	(13, 24, 67)	(16, 114, 136)	D(-87)

It is noted that the triple $(c_{s-1}, c_s + 13, c_{s+1})$, $s = 1, 2, 3, \dots$

forms an arithmetic progression.

In a similar way, one may generate sequences of diophantine 3-tuples with suitable property through the other solutions to (2.19).

Generation of solutions

Let (x_0, y_0) represents any given solution to (2.19).

Consider the second solution (x_1, y_1) to (2.19) given by

$$x_1 = 2h - x_0, y_1 = y_0 + h \quad (2.30)$$

Substituting (2.30) in (2.19) and simplifying, one obtains

$$h = 4x_0 + 6y_0 + 3$$

In view of (2.30), we have

$$x_1 = 7x_0 + 12y_0 + 6, y_1 = 4x_0 + 7y_0 + 3$$

which is written in the form of matrix as

$$(x_1, y_1, 1)^t = \begin{pmatrix} 7 & 12 & 6 \\ 4 & 7 & 3 \\ 0 & 1 & 1 \end{pmatrix} (x_0, y_0, 1)^t$$

where t is the transpose. The repetition of the above process leads to the general solution to (2.19) as

$$(x_{n+1}, y_{n+1}, 1)^t = \begin{pmatrix} Y_n & 3X_n & \frac{3X_n}{2} \\ X_n & Y_n & \frac{Y_n - 1}{2} \\ 0 & 0 & 1 \end{pmatrix} (x_0, y_0, 1)^t, n = 0, 1, 2, \dots$$

where

$$Y_n = \frac{1}{2} \left((7 + 4\sqrt{3})^{n+1} + (7 - 4\sqrt{3})^{n+1} \right)$$

$$X_n = \frac{1}{2\sqrt{3}} \left((7 + 4\sqrt{3})^{n+1} - (7 - 4\sqrt{3})^{n+1} \right)$$

II.5 A Search on the Homogeneous Cone $x^2 + 6xy + 15y^2 = 15z^2$

The quadratic Diophantine equation with the three unknowns to be solved is given by

$$x^2 + 6xy + 15y^2 = 15z^2 \quad (2.31)$$

Substituting,

$$x + 3y = u \quad (2.32)$$

in (2.31), we get

$$u^2 + 6y^2 = 15z^2 \quad (2.33)$$

Different ways of solving (2.33) for u , y and z are presented below. Knowing the values of u and y , the corresponding values of x are obtained from (2.32).

Way 1

Assume

$$z(a, b) = a^2 + 6b^2 \quad (2.34)$$

Write 15 as

$$15 = (3 + i\sqrt{6})(3 - i\sqrt{6}) \quad (2.35)$$

Using (2.34) and (2.35) in (2.32) and applying the method of factorization define

$$(u + i\sqrt{6}y) = (a + i\sqrt{6}b)^2 (3 + i\sqrt{6})$$

From which, on equating the real and imaginary parts, one obtains

$$u = 3a^2 - 18b^2 - 12ab \quad (2.36)$$

$$y(a, b) = a^2 - 6b^2 + 6ab \quad (2.37)$$

In view of (2.32), we have

$$x(a, b) = -30ab \quad (2.38)$$

Thus, (2.34), (2.37) and (2.38) represent the integer solutions to (2.31).

Properties

1. $x(a,1) + 30Pr_a - 30t_{4,a} = 0$
2. $y(a,1) + 5t_{4,a} - 6Pr_a + 6 = 0$
3. $y(1,b) + t_{14,b} - Pr_b + t_{4,b} - 1 = 0$
4. $x(a,1) + y(a,1) - 25t_{4,a} + 24Pr_a \equiv 0 \pmod{14}$
5. $x(1,b) + y(1,b) + S_b \equiv 0 \pmod{14}$

Note 1

One may also represent 15 as

$$15 = \frac{(9+i7\sqrt{6})(9-i7\sqrt{6})}{5^2}$$

After performing a few calculations as above, the corresponding values of x , y and z are given by

$$x(A, B) = -60A^2 + 360B^2 + 690AB$$

$$y(A, B) = 35A^2 - 210B^2 - 90AB$$

$$z(A, B) = 25A^2 + 150B^2$$

Properties

1. $x(A,1) + t_{122,A} - 631Pr_A + 631t_{4,A} - 360 = 0$
2. $x(1,B) - t_{722,B} - 1049Pr_B + 1049t_{4,B} + 60 = 0$
3. $y(A,1) - t_{72,A} + 56Pr_A - 56t_{4,A} \equiv 0 \pmod{7}$
4. $y(1,B) + t_{422,B} + 299Pr_B - 299t_{4,B} \equiv 0 \pmod{5}$
5. $x(A,1) + y(A,1) + t_{52,A} - 576Pr_A + 576t_{4,A} - 150 = 0$

Way 2

(2.33) is written as

$$u^2 + 6y^2 = 15z^2 * 1 \quad (2.39)$$

Assume

$$1 = \frac{(1+i2\sqrt{6})(1-i2\sqrt{6})}{5^2} \quad (2.40)$$

Substituting (2.34), (2.35) and (2.40) in (2.39) and employing the method of factorization, define

$$(u+i\sqrt{6}y) = (3+i\sqrt{6}) (a+i\sqrt{6}b)^2 * \frac{(1+2i\sqrt{6})}{5}$$

On equating the real and imaginary parts, we get

$$u = \frac{1}{5}(-9a^2 + 54b^2 - 84ab) \quad (2.41)$$

$$y = \frac{1}{5}(7a^2 - 42b^2 - 18ab) \quad (2.42)$$

As our interest is on finding integer solutions, replacing a by 5A and b by 5B in (2.34), (2.41) and (2.42), it is seen that

$$u = -45A^2 + 270B^2 - 420AB$$

$$y(A, B) = 35A^2 - 210B^2 - 90AB \quad (2.43)$$

$$z(A, B) = 25A^2 + 150B^2 \quad (2.44)$$

In view of (2.32), one obtains

$$x(A, B) = -150A^2 + 900B^2 - 150AB \quad (2.45)$$

Thus, (2.43), (2.44) and (2.45) represent the integer solutions to (2.31).

Properties

1. $x(A, 1) + t_{302, A} + 299Pr_A - 299t_{4, A} - 900 = 0$
2. $x(1, B) - t_{1800, B} - 749Pr_B + 749t_{4, B} + 150 = 0$
3. $y(A, 1) - t_{72, A} + 56Pr_A - 56t_{4, A} \equiv 0 \pmod{3}$
4. $y(1, B) + t_{422, B} + 299Pr_B - 299t_{4, B} \equiv 0 \pmod{7}$
5. $x(A, 1) + y(A, 1) + t_{232, A} + 354Pr_A - 354t_{4, A} - 690 = 0$

Note 2

The representations of (2.45) and (2.31) in (2.39) may also be considered as follows:

- (i) $15 = (3 + i\sqrt{6})(3 - i\sqrt{6}), \quad 1 = \frac{(5 + i\sqrt{6})(5 - i\sqrt{6})}{49}$
- (ii) $15 = \frac{(9 + i7\sqrt{6})(9 - i7\sqrt{6})}{25}, \quad 1 = \frac{(1 + i2\sqrt{6})(1 - i2\sqrt{6})}{25}$
- (iii) $15 = \frac{(9 + i7\sqrt{6})(9 - i7\sqrt{6})}{25}, \quad 1 = \frac{(5 + i\sqrt{6})(5 - i\sqrt{6})}{49}$

Employing the procedure as above for each of this representations, the corresponding integer solutions to (2.31) thus obtained are presented below.

Solutions obtained from (i):

$$\begin{aligned} x(A, B) &= -7(30A^2 + 180B^2 + 150AB) \\ y(A, B) &= 7(11A^2 - 66B^2 + 6AB) \\ z(A, B) &= 49(A^2 + 6B^2) \end{aligned}$$

Solutions obtained from (ii):

$$\begin{aligned} x(a, b) &= -6a^2 + 3b^2 + 6ab \\ y(a, b) &= a^2 - 6b^2 - 6ab \\ z(a, b) &= a^2 + 6b^2 \end{aligned}$$

Solutions obtained from (iii):

$$x(A, B) = 35(-198A^2 + 1188B^2 - 402AB)$$

$$y(A, B) = 35(53A^2 - 318B^2 - 78AB)$$

$$z(A, B) = 35^2(A^2 + 6B^2)$$

Way 3

(2.33) can be written in the form of ratio as

$$\frac{u + 3z}{6(z - y)} = \frac{z + y}{u - 3z} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (2.46)$$

which is equivalent to the system of double equations

$$\left. \begin{aligned} \beta u + 6\alpha y + (-6\alpha + 3\beta) z &= 0 \\ -\alpha u + \beta y + (3\alpha + \beta) z &= 0 \end{aligned} \right\} \quad (2.47)$$

Solving (2.47) by method of cross multiplication we get,

$$\left. \begin{aligned} u &= 18\alpha^2 - 3\beta^2 + 12\alpha\beta \\ y(\alpha, \beta) &= 6\alpha^2 - \beta^2 - 6\alpha\beta \\ z(\alpha, \beta) &= 6\alpha^2 + \beta^2 \end{aligned} \right\} \quad (2.48)$$

Using (2.32), we have

$$x(\alpha, \beta) = 30\alpha\beta \quad (2.49)$$

Thus, (2.48) and (2.49) represent the integer solutions to (2.31).

Properties

1. $x(\alpha, 1) - 30Pr_\alpha + 30t_{4,\alpha} = 0$
2. $y(\alpha, 1) - t_{14,\alpha} + Pr_\alpha - t_{4,\alpha} + 1 = 0$
3. $y(1, \beta) + t_{4,\beta} + 6Pr_\beta - 6t_{4,\beta} \equiv 0 \pmod{2}$
4. $x(\alpha, 1) + y(\alpha, 1) - t_{14,\alpha} - 29Pr_\alpha + 29t_{4,\alpha} + 1 = 0$
5. $x(1, \beta) + y(1, \beta) - 24Pr_\beta + 25t_{4,\beta} \equiv 0 \pmod{6}$

Case 2

(2.33) is written in the form of ratio as

$$\frac{u+3z}{2(z-y)} = \frac{3(z+y)}{u-3z} = \frac{\alpha}{\beta}, \beta \neq 0$$

which is equivalent to the system of double equations

$$\left. \begin{aligned} \beta u + 2\alpha y + (3\beta - 2\alpha) &= 0 \\ -\alpha u + 3\beta y + (3\beta + 3\alpha)z &= 0 \end{aligned} \right\} \quad (2.50)$$

Solving (2.50) by method of cross multiplication, we get

$$\left. \begin{aligned} u &= 6\alpha^2 - 9\beta^2 + 12\alpha\beta \\ y(\alpha, \beta) &= 2\alpha^2 - 3\beta^2 - 6\alpha\beta \\ z(\alpha, \beta) &= 2\alpha^2 + 3\beta^2 \end{aligned} \right\} \quad (2.51)$$

Using (2.32),

$$x(\alpha, \beta) = 30\alpha\beta \quad (2.52)$$

Thus, (2.51) and (2.52) represent the integer solutions to (2.31).

Properties

1. $y(\alpha, 1) - t_{6,\alpha} + 5Pr_\alpha - 5t_{4,\alpha} \equiv 0 \pmod{3}$
2. $y(1, \beta) + t_{8,\beta} + 8Pr_\beta - 8t_{4,\beta} \equiv 0 \pmod{2}$
3. $x(\alpha, 1) + y(\alpha, 1) - t_{6,\alpha} - 25Pr_\alpha + 25t_{4,\alpha} + 3 = 0$
4. $x(1, \beta) + y(1, \beta) + 27t_{4,\beta} - 24Pr_\beta - 2 = 0$
5. $x(\alpha, 1) - y(\alpha, 1) + t_{6,\alpha} - 35Pr_\alpha + 35t_{4,\alpha} - 3 = 0$

Case 3

(2.33) is written in the form of ratio as

$$\frac{u-3z}{3(z-y)} = \frac{2(z+y)}{u+3z} = \frac{\alpha}{\beta}, \beta \neq 0$$

which is equivalent to the system of double equations

$$\left. \begin{aligned} \beta u + 3\alpha y + (-3\alpha - 3\beta)z &= 0 \\ -\alpha u + 2\beta y + (-3\alpha + 2\beta)z &= 0 \end{aligned} \right\} \quad (2.53)$$

Solving (2.53) by method of cross multiplication, we get

$$\left. \begin{aligned} u &= -9\alpha^2 + 6\beta^2 + 12\alpha\beta \\ y(\alpha, \beta) &= 3\alpha^2 - 2\beta^2 + 6\alpha\beta \\ z(\alpha, \beta) &= 3\alpha^2 + 2\beta^2 \end{aligned} \right\} \quad (2.54)$$

Using (2.32),

$$x = -18\alpha^2 + 12\beta^2 - 6\alpha\beta \quad (2.55)$$

Thus, (2.54) and (2.55) represent the integer solutions to (2.31).

Properties

1. $x(\alpha, 1) + t_{38, \alpha} + 23Pr_{\alpha} - 23t_{4, \alpha} \equiv 0 \pmod{12}$
2. $x(1, \beta) - t_{26, \beta} - 5Pr_{\beta} + 5t_{4, \beta} \equiv 0 \pmod{9}$
3. $y(\alpha, 1) - t_{8, \alpha} - 8Pr_{\alpha} + 8t_{4, \alpha} + 2 = 0$
4. $y(1, \beta) + t_{6, \beta} - 5Pr_{\beta} + 5t_{4, \beta} - 3 = 0$
5. $x(\alpha, 1) + y(\alpha, 1) + 15t_{4, \alpha} - 10 = 0$

Case 4

(2.33) is written in the form of ratio as

$$\frac{u + 3z}{z - y} = \frac{6(z + y)}{u - 3z} = \frac{\alpha}{\beta}, \beta \neq 0$$

which is equivalent to the system of double equations

$$\left. \begin{aligned} \beta u + \alpha y + (-\alpha + 3\beta)z &= 0 \\ -\alpha u + 6\beta y + (3\alpha + 6\beta)z &= 0 \end{aligned} \right\} \quad (2.56)$$

Solving (2.56) by method of cross multiplication, we get

$$\begin{aligned} u &= 3\alpha^2 - 18\beta^2 + 12\alpha\beta \\ \left. \begin{aligned} y(\alpha, \beta) &= \alpha^2 - 6\beta^2 - 6\alpha\beta \\ z(\alpha, \beta) &= \alpha^2 + 6\beta^2 \end{aligned} \right\} \end{aligned} \quad (2.57)$$

Using (2.32),

$$x(\alpha, \beta) = 30\alpha\beta \quad (2.58)$$

Thus, (2.57) and (2.58) represent the integer solutions to (2.31).

Properties

1. $z(\alpha, 1) - t_{4,\alpha} - 6 = 0$
2. $y(\alpha, 1) - 7t_{4,\alpha} + 6\text{Pr}_\alpha \equiv 0 \pmod{3}$
3. $z(1, \beta) - 6t_{4,\alpha} - 1 = 0$
4. $y(1, \beta) + t_{14,\beta} + 11\text{Pr}_\beta - 11t_{4,\beta} - 1 = 0$
5. $y(\alpha, 1) + z(\alpha, 1) + 6\text{Pr}_\alpha - 8t_{4,\alpha} = 0$

Case 5

(2.33) is written in the form of ratio as

$$\frac{u - 3z}{z - y} = \frac{6(z + y)}{u + 3z} = \frac{\alpha}{\beta}, \beta \neq 0$$

which is equivalent to the system of double equations

$$\left. \begin{aligned} \beta u + \alpha y + (-\alpha - 3\beta)z &= 0 \\ -\alpha u + 6\beta y + (-3\alpha + 6\beta)z &= 0 \end{aligned} \right\} \quad (2.59)$$

Solving (2.59) by method of cross multiplication, we get

$$u = -3\alpha^2 + 18\beta^2 + 12\alpha\beta$$

$$\left. \begin{aligned} y(\alpha, \beta) &= \alpha^2 - 6\beta^2 + 6\alpha\beta \\ z(\alpha, \beta) &= \alpha^2 + 6\beta^2 \end{aligned} \right\} \quad (2.60)$$

Using (2.32),

$$x(\alpha, \beta) = -6\alpha^2 + 36\beta^2 - 6\alpha\beta \quad (2.61)$$

Thus, (2.60) and (2.61) represent the integer solutions to (2.31).

Properties

1. $x(\alpha, 1) + t_{14, \alpha} + 11Pr_{\alpha} - 11t_{4, \alpha} \equiv 0 \pmod{4}$
2. $x(1, \beta) - t_{74, \beta} - 29Pr_{\beta} + 29t_{4, \beta} \equiv 0 \pmod{3}$
3. $y(\alpha, 1) - t_{4, \alpha} - 6Pr_{\alpha} + 6t_{4, \alpha} + 6 = 0$
4. $y(1, \beta) + t_{14, \beta} - Pr_{\beta} + t_{4, \beta} - 1 = 0$
5. $x(\alpha, 1) + y(\alpha, 1) + 5t_{4, \alpha} - 30 = 0$

Way 4

Introducing the linear transformations

$$z = X + 6T, \quad y = X + 15T, \quad u = 3U \quad (2.62)$$

in (2.33), it is written as

$$X^2 = 90T^2 + U^2 \quad (2.63)$$

which is satisfied by

$$T = 2rs, \quad U = 90r^2 - s^2, \quad X = 90r^2 + s^2$$

In view of (2.62) and (2.32), the corresponding integer solutions to (2.31) are given by

$$\begin{aligned} x &= -6s^2 - 90rs \\ y &= 90r^2 + s^2 + 30rs \\ z &= 90r^2 + s^2 + 12rs \end{aligned}$$

Also, (2.63) can be expressed as the system of double equations as presented below in Table 2.13:

Table 2.13: System of Double Equations

System	1	2	3	4	5	6	7	8	9
$X + U$	T^2	$3T^2$	$5T^2$	$9T^2$	$15T^2$	$45T^2$	$90T$	$45T$	$30T$
$X - U$	90	30	18	10	6	2	T	$2T$	$3T$

System	10	11	12	13	14	15	16	17	18
$X + U$	$18T$	$15T$	$10T$	$9T$	$6T$	$5T$	$3T$	$2T$	T
$X - U$	$5T$	$6T$	$9T$	$10T$	$15T$	$18T$	$30T$	$45T$	$90T$

Solving each of the above system of equations, the values of X, U and T are obtained.

In view of (2.62) and (2.32), the corresponding integer solutions to (2.31) are obtained. For simplicity, we present below the corresponding solutions in Table 2.14:

Table 2.14: Solutions

System	x	y	z
1	$-270 - 90k$	$2k^2 + 30k + 45$	$2k^2 + 12k + 45$
2	$-90 - 90k$	$6k^2 + 30k + 15$	$6k^2 + 12k + 15$
3	$-54 - 90k$	$10k^2 + 30k + 9$	$10k^2 + 12k + 9$
4	$-30 - 90k$	$18k^2 + 30k + 5$	$18k^2 + 12k + 5$
5	$-18 - 90k$	$30k^2 + 30k + 3$	$30k^2 + 12k + 3$
6	$-6 - 90k$	$90k^2 + 30k + 1$	$90k^2 + 12k + 1$
7	$-96k$	$121k$	$103k$
8	$-102k$	$77k$	$59k$
9	$-108k$	$63k$	$45k$
10	$-120k$	$53k$	$35k$

System	x	y	z
11	$-126k$	$51k$	$33k$
12	$-144k$	$49k$	$31k$
13	$-150k$	$49k$	$31k$
14	$-180k$	$51k$	$33k$
15	$-198k$	$53k$	$35k$
16	$-270k$	$63k$	$45k$
17	$-360k$	$77k$	$59k$
18	$-630k$	$121k$	$103k$

II.6 On the Ternary Quadratic Equation $x^2 + y^2 = z^2 + 141$

Consider the second degree equation with three variables

$$x^2 + y^2 = z^2 + 141 \quad (2.64)$$

The introduction of the transformations

$$\left. \begin{aligned} x &= (2k^2 + 20k - 21)\alpha, \\ z &= (2k^2 + 20k - 20)\alpha, k > 0, \alpha \neq 0 \end{aligned} \right\} \quad (2.65)$$

in (2.64) gives

$$y^2 = (4k^2 + 40k - 41)\alpha^2 + 141 \quad (2.66)$$

which represents the positive pell equation. The initial positive integer solution to (2.66) is $\alpha_0 = 1, y_0 = 2k + 10$

To obtain the other integer solutions to (2.66), consider the corresponding pell equation

$$y^2 = (4k^2 + 40k - 41)\alpha^2 + 1 \quad (2.67)$$

whose least positive integer solution is $(\tilde{\alpha}_0, \tilde{y}_0)$.

The general solution $(\tilde{\alpha}_n, \tilde{y}_n)$ of (2.67) is given by

$$\tilde{y}_n = \frac{1}{2} f_n \quad (2.68)$$

$$\tilde{\alpha}_n = \frac{1}{2\sqrt{4k^2 + 40k - 41}} g_n \quad (2.69)$$

where

$$f_n = \left(\tilde{y}_0 + \sqrt{4k^2 + 40k - 41} \tilde{\alpha}_0 \right)^{n+1} + \left(\tilde{y}_0 - \sqrt{4k^2 + 40k - 41} \tilde{\alpha}_0 \right)^{n+1}, n = -1, 0, 1, 2, \dots$$

$$g_n = \left(\tilde{y}_0 + \sqrt{4k^2 + 40k - 41} \tilde{\alpha}_0 \right)^{n+1} - \left(\tilde{y}_0 - \sqrt{4k^2 + 40k - 41} \tilde{\alpha}_0 \right)^{n+1}, n = -1, 0, 1, 2, \dots$$

Employing the lemma of Brahmagupta between the solutions (α_0, y_0) and $(\tilde{\alpha}_n, \tilde{y}_n)$, the other solutions to (2.66) are represented by

$$\alpha_{n+1} = \alpha_0 \tilde{y}_n + y_0 \tilde{\alpha}_n, n = -1, 0, 1, 2, \dots \quad (2.70)$$

$$y_{n+1} = y_0 \tilde{y}_n + (4k^2 + 40k - 41) \alpha_0 \tilde{\alpha}_n, n = -1, 0, 1, 2, \dots \quad (2.71)$$

To study the properties among the solutions, one has to go for particular values of k .

For simplicity and brevity the choice $k = 1$ in (2.66), (2.67), (2.68) and (2.69) correspondingly leads to

$$y^2 = 3\alpha^2 + 141, \alpha_0 = 1, y_0 = 12$$

$$y^2 = 3\alpha^2 + 1, \tilde{\alpha}_0 = 1, \tilde{y}_0 = 2$$

$$\begin{aligned} \tilde{y}_n &= \frac{1}{2} f_n, f_n = \left[(2 + \sqrt{3})^{n+1} + (2 - \sqrt{3})^{n+1} \right] \\ \tilde{\alpha}_n &= \frac{1}{2\sqrt{3}} g_n, g_n = \left[(2 + \sqrt{3})^{n+1} - (2 - \sqrt{3})^{n+1} \right], n = -1, 0, 1, \dots \end{aligned}$$

$$\alpha_{n+1} = \frac{1}{2} f_n + 2\sqrt{3} g_n \quad (2.72)$$

$$y_{n+1} = 6 f_n + \frac{1}{2} \sqrt{3} g_n \quad (2.73)$$

Substituting $k = 1$ in (2.65) and using (2.72), we get

$$x_{n+1} = \frac{1}{2} f_n + 2\sqrt{3} g_n \quad (2.74)$$

$$z_{n+1} = f_n + 4\sqrt{3} g_n \quad (2.75)$$

Thus, (2.73), (2.74) and (2.75) represent different positive solution in integers to (2.64).

A few numerical examples are given in the following table 2.15 below:

Table 2.15: Numerical Examples

n	x_{n+1}	y_{n+1}	z_{n+1}
-1	1	12	2
0	14	27	28
1	55	96	110
2	206	357	412
3	769	1332	1538
4	2870	4971	5740
5	10711	18552	21422
6	39974	69237	79948

From the above table, one may generate Ramanujan numbers of second order from suitable values of x, y and z .

Illustration

$$\begin{aligned}
 y_2 &= 96 = 2 * 48 = 4 * 24 = 6 * 16 = 8 * 12 \\
 &= 25^2 - 23^2 = 14^2 - 10^2 = 11^2 - 5^2 = 10^2 - 2^2 \\
 25^2 - 23^2 &= 14^2 - 10^2 \Rightarrow 25^2 + 10^2 = 23^2 + 14^2 = 725 \\
 25^2 - 23^2 &= 11^2 - 5^2 \Rightarrow 25^2 + 5^2 = 23^2 + 11^2 = 650 \\
 25^2 - 23^2 &= 10^2 - 2^2 \Rightarrow 25^2 + 2^2 = 23^2 + 10^2 = 629 \\
 14^2 - 10^2 &= 11^2 - 5^2 \Rightarrow 14^2 + 5^2 = 10^2 + 11^2 = 221 \\
 11^2 - 5^2 &= 10^2 - 2^2 \Rightarrow 11^2 + 2^2 = 5^2 + 10^2 = 125
 \end{aligned}$$

Thus, 725, 650, 629, 221, 125 are Ramanujan numbers of second order.

Recurrence relations for x, y and z are:

$$x_{n+3} - 4x_{n+2} + x_{n+1} = 0, n = -1, 0, 1, \dots$$

$$y_{n+3} - 4y_{n+2} + y_{n+1} = 0, n = -1, 0, 1, \dots$$

$$z_{n+3} - 4z_{n+2} + z_{n+1} = 0, n = -1, 0, 1, \dots$$

❖ Some combinations between the solutions are given below:

$$\text{➤ } y_{n+1} - x_{n+2} + 2x_{n+1} = 0$$

$$\text{➤ } y_{n+3} - 7x_{n+2} + 2x_{n+1} = 0$$

$$\text{➤ } 4y_{n+1} - x_{n+3} + 7x_{n+1} = 0$$

$$\text{➤ } 2y_{n+2} - x_{n+3} + x_{n+1} = 0$$

❖ Cubical integer:

$$\text{i. } \frac{1}{47} \left[(8x_{3n+4} - 18x_{3n+3}) + 3(8x_{n+2} - 18x_{n+1}) \right]$$

$$\text{ii. } \frac{1}{188} \left[(8x_{3n+5} - 64x_{3n+3}) + 3(8x_{n+3} - 64x_{n+1}) \right]$$

$$\text{iii. } \frac{1}{47} \left[(8y_{3n+3} - 2x_{3n+3}) + 3(8y_{n+1} - 2x_{n+1}) \right]$$

$$\text{iv. } \frac{1}{94} \left[(8y_{3n+4} - 28y_{3n+3}) + 3(8y_{n+2} - 28x_{n+1}) \right]$$

❖ Bi-quadratic integer:

$$\text{i. } \frac{1}{47^2} \left[(376x_{4n+5} - 846x_{4n+4}) + 4(8x_{n+2} - 18x_{n+1})^2 - 4418 \right]$$

$$\text{ii. } \frac{1}{188^2} \left[(1504x_{4n+6} - 12032x_{4n+4}) + 4(8x_{n+3} - 64x_{n+1})^2 - 70688 \right]$$

$$\text{iii. } \frac{1}{47^2} \left[(376y_{4n+4} - 94x_{4n+4}) + 4(8y_{n+1} - 2x_{n+1})^2 - 4418 \right]$$

$$\text{iv. } \frac{1}{94^2} \left[(752y_{4n+5} - 2632x_{4n+4}) + 4(8y_{n+2} - 28x_{n+1})^2 - 17672 \right]$$

❖ Nasty number:

$$\text{i. } \frac{1}{47} [564 + 48x_{2n+3} - 108x_{2n+2}]$$

$$\text{ii. } \frac{1}{188} [2256 + 48x_{2n+4} - 384x_{2n+2}]$$

$$\text{iii. } \frac{1}{47} [564 + 48y_{2n+2} - 12x_{2n+2}]$$

$$\text{iv. } \frac{1}{94} [1128 + 48y_{2n+3} - 168x_{2n+2}]$$

Remarkable Observations

I. Choices of hyperbola with their solutions generated through the known solutions are in Table 2.16 below:

Table 2.16: Hyperbola

S. No.	Hyperbola	(X_n, Y_n)
1	$3X_n^2 - Y_n^2 = 26508$	$[(8x_{n+2} - 18x_{n+1}), (28x_{n+1} - 2x_{n+2})]$
2	$3X_n^2 - Y_n^2 = 424128$	$[(8x_{n+3} - 64x_{n+1}), (110x_{n+1} - 2x_{n+3})]$
3	$3X_n^2 - Y_n^2 = 26508$	$[(8y_{n+1} - 2x_{n+1}), (24x_{n+1} - 2y_{n+1})]$
4	$3X_n^2 - Y_n^2 = 106032$	$[(8y_{n+2} - 28x_{n+1}), (54x_{n+1} - 2y_{n+2})]$

II. Employing linear combination among the solutions other choices of parabola are presented in Table 2.17 below:

Table 2.17: Parabola

S. No.	Parabola	(X_n, Y_n)
1	$141X_n - Y_n^2 = 26508$	$[(94 + 8x_{2n+3} - 18x_{2n+2}), (28x_{n+1} - 2x_{n+2})]$
2	$564X_n - Y_n^2 = 424128$	$[(376 + 8x_{2n+4} - 64x_{2n+2}), (110x_{n+1} - 2x_{n+3})]$
3	$141X_n - Y_n^2 = 26508$	$[(94 + 8y_{2n+2} - 2x_{2n+2}), (24x_{n+1} - 2y_{n+1})]$
4	$282X_n - Y_n^2 = 106032$	$[(188 + 8y_{2n+3} - 28x_{2n+2}), (54x_{n+1} - 2y_{n+2})]$

Generation of solutions

Let (x_0, y_0, z_0) be a known solution of (2.64).

Consider the second solution (x_1, y_1, z_1) of (2.64) to be

$$x_1 = h - x_0, y_1 = h - y_0, z_1 = h + z_0 \quad (2.76)$$

where h is a non-zero integer to be determined.

Substituting (2.76) in (2.64) and simplifying, we get

$$h = 2(x_0 + y_0 + z_0) \quad (2.77)$$

Using (2.77) in (2.76), the second solution of (2.64) is represented in the matrix form as

$$(x_1, y_1, z_1)^t = M(x_0, y_0, z_0)^t$$

$$\text{where } M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \text{ and } t \text{ is the transpose}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (2.64) in the matrix form as

$$(x_{n+1}, y_{n+1}, z_{n+1})^t = \tilde{M}(x_0, y_0, z_0)^t, \quad (2.78)$$

$$\text{where } \tilde{M} = \begin{pmatrix} \frac{Y_n - (-1)^n}{2} & \frac{Y_n + (-1)^n}{2} & X_n \\ \frac{Y_n + (-1)^n}{2} & \frac{Y_n - (-1)^n}{2} & X_n \\ X_n & X_n & Y_n \end{pmatrix}, n = 0, 1, 2, \dots$$

in which (x_n, y_n) represents the general solution of the pell equation $Y^2 = 2X^2 + 1$.

Thus, given an integer solution (x_0, y_0, z_0) , one may generate sequence of integer solutions to the given equation based on the known solution through employing (2.78).

Remark

In addition to (2.65), one may introduce the transformations

$$x = \frac{1}{2}(k^2 + 11k - 6)\alpha, \quad z = \frac{1}{2}(k^2 + 11k - 4)\alpha$$

in (2.64) leading to

$$y^2 = (k^2 + 11k - 5)\alpha^2 + 141, \quad \alpha_0 = 2, \quad y_0 = 2k + 11$$

Following the procedure presented above, another set of integer solutions to (2.64) are obtained.

II.7 On the Homogeneous Cone $3x^2 - 8y^2 = 25z^2$

Consider the homogeneous cone represented by the ternary quadratic equation

$$3x^2 - 8y^2 = 25z^2 \quad (2.79)$$

we present below different methods of solving (2.79) and thus, obtain different sets of integer solutions to (2.79)

Method 1

Introducing the linear transformations

$$x = X + 8T, \quad y = X + 3T \quad (2.80)$$

$$X^2 + 5z^2 = 24T^2 \quad (2.81)$$

Assume

$$T = T(a, b) = a^2 + 5b^2 \quad (2.82)$$

write 24 as

$$24 = (2 + i2\sqrt{5})(2 - i2\sqrt{5}) \quad (2.83)$$

Using (2.82), (2.83) in (2.81) and employing the method of factorization, define

$$X + i\sqrt{5}z = (2 + i2\sqrt{5})(a + i\sqrt{5}b)^2$$

Equating the real and imaginary parts in the above equation, one obtains

$$X = X(a, b) = 2a^2 - 10b^2 - 20ab \quad (2.84)$$

$$z = z(a, b) = 2a^2 - 10b^2 + 4ab \quad (2.85)$$

Substituting (2.82) and (2.84) in (2.80), we have

$$\left. \begin{aligned} x &= x(a, b) = 10a^2 + 30b^2 - 20ab \\ y &= y(a, b) = 5a^2 + 5b^2 - 20ab \end{aligned} \right\} \quad (2.86)$$

Note that (2.85) and (2.86) represent the non-zero distinct integer solutions to (2.79).

Properties

- $x(1, b) + 2y(1, b) = 40t_{3,b}$
- $y(a, 1) - 10t_{3,n} + 25a \equiv 0 \pmod{5}$
- $z(a, a+1) + 4PR_a + 6GNo_a + 16 = 0$
- $z(1, b) - 5z(1, b) = 40t_{6,b}$
- $x(a, 1) - 2y(a, 1) - 10GNo_a \equiv 0 \pmod{3}$

Method 2

(2.81) is written as

$$X^2 + 5z^2 = 24T^2 * 1 \quad (2.87)$$

Assume

$$1 = \frac{(2+i\sqrt{5})(2-i\sqrt{5})}{9} \quad (2.88)$$

Substituting (2.82), (2.83) and (2.88) in (2.87) and employing the method of factorization, define

$$(X + i\sqrt{5}z) = \frac{(2+i\sqrt{5})}{3} (2+i2\sqrt{5})(a+i\sqrt{5}b)^2 \quad (2.89)$$

Equating the real and imaginary parts in (2.89), we have

$$X = X(a, b) = -2a^2 + 10b^2 - 20ab \quad (2.90)$$

$$z = z(a, b) = 2a^2 - 10b^2 - 4ab \quad (2.91)$$

Substituting (2.90) and (2.82) in (2.80),

$$\left. \begin{aligned} x &= x(a, b) = 6a^2 + 50b^2 - 20ab \\ y &= y(a, b) = a^2 + 25b^2 - 20ab \end{aligned} \right\} \quad (2.92)$$

Thus, (2.91) and (2.92) represents the integer solutions to (2.79).

Properties

- $x(a, a) - t_{a,14} - 5a = 0$
- $y(a, a) - 5z(a, a) - 11S_a - 33GNo_a \equiv 0 \pmod{11}$
- $x(a, a) - y(a, a) - 30PR_a + 15GNo_a + 15 = 0$
- $y(a, a)$ is a nasty number
- $z(b, b+1) + 24t_{3,n} + 6GNo_b + 16 = 0$

Note: It is to be noted that, in addition to (2.88) I may also be represented as shown below:

Way 1:
$$1 = \frac{(1+i4\sqrt{5})(1-i4\sqrt{5})}{81} \quad (2.93)$$

Way 2:
$$1 = \frac{(2+i3\sqrt{5})(2-i3\sqrt{5})}{49} \quad (2.94)$$

Following the procedure as above, the corresponding integer solutions to (2.79) for (2.93) and (2.94) are presented below:

Solutions for (2.93):

$$x = x(A, B) = 306A^2 + 4950B^2 - 900AB$$

$$y = y(A, B) = -99A^2 + 2925B^2 - 900AB$$

$$z = z(A, B) = 90A^2 - 450B^2 - 684AB$$

Solutions for (2.94):

$$x = x(A, B) = 210A^2 + 2870B^2 - 700AB$$

$$y = y(A, B) = -35A^2 + 1645B^2 - 700AB$$

$$z = z(A, B) = 70A^2 - 350B^2 - 364AB$$

Generation of solutions

Here we obtain formula for generating sequence of integer solutions to (2.79) based on its initial solution.

Let (x_0, y_0, z_0) be the initial solution of (2.79).

Formula 1

Let (x_1, y_1, z_1) be the second solution of (2.79),

$$\text{where } x_1 = 33x_0, y_1 = h - 33y_0, z_1 = h - 33z_0 \quad (2.95)$$

Substituting (2.95) in (2.79) and simplifying, we get

$$h = 16y_0 + 50z_0$$

Thus, the second solution (x_1, y_1, z_1) to (2.79) is given by

$$x_1 = 33x_0, y_1 = -17y_0 + 50z_0, z_1 = 16y_0 + 17z_0$$

Express y_1 and z_1 in the form of 2×2 matrix as follows:

$$\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = M \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \quad \text{where } M = \begin{pmatrix} -17 & 50 \\ 16 & 17 \end{pmatrix}$$

Repeating the above process, the general values of y and z are given by

$$\begin{pmatrix} y_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$$

If α, β are the eigen values of M , then

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I)$$

$$M^n = \begin{pmatrix} \frac{1}{33}(8\alpha^n + 25\beta^n) & \frac{25}{33}(\alpha^n - \beta^n) \\ \frac{8}{33}(\alpha^n - \beta^n) & \frac{25}{33}(\alpha^n + 8\beta^n) \end{pmatrix}$$

Hence, the general values of x, y, z satisfying (2.79) are given by

$$\begin{aligned}x_n &= 33^n x_0 \\y_n &= \frac{1}{33}(8\alpha^n + 25\beta^n)y_0 + \frac{25}{33}(\alpha^n - \beta^n)z_0 \\z_n &= \frac{8}{33}(\alpha^n - \beta^n)y_0 + \frac{25}{33}(\alpha^n + 8\beta^n)z_0\end{aligned}$$

Formula 2

Let (x_1, y_1, z_1) be the second solution of (2.79),

$$\text{where } x_1 = 3h - x_0, y_1 = y_0, z_1 = h + z_0 \quad (2.96)$$

Substituting (2.96) in (2.79) and simplifying, we get

$$h = 9x_0 + 25z_0$$

Thus, the second solution (x_1, y_1, z_1) to (2.79) is given by

$$x_1 = 26x_0 + 75z_0, y_1 = y_0, z_1 = 9x_0 + 26z_0$$

Express y_1 and z_1 in the form of 2×2 matrix as follows:

$$\begin{pmatrix} x_1 \\ z_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ z_0 \end{pmatrix} \quad \text{where } M = \begin{pmatrix} 26 & 75 \\ 9 & 26 \end{pmatrix}$$

Repeating the above process, the general values of x and z are given by

$$\begin{pmatrix} x_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}$$

If α, β are the eigen values of M , then

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I)$$

$$M^n = \begin{pmatrix} \frac{1}{2}(\alpha^n + \beta^n) & \frac{5\sqrt{3}}{6}(\alpha^n - \beta^n) \\ \frac{\sqrt{3}}{16}(\alpha^n - \beta^n) & \frac{1}{2}(\alpha^n + \beta^n) \end{pmatrix}$$

Hence, the general values of x, y, z satisfying (2.79) are given by

$$\begin{aligned}x_n &= \frac{1}{2}(\alpha^n + \beta^n)x_0 + \frac{5\sqrt{3}}{6}(\alpha^n - \beta^n)z_0 \\y_n &= y_0 \\z_n &= \frac{\sqrt{3}}{16}(\alpha^n - \beta^n)x_0 + \frac{1}{2}(\alpha^n + \beta^n)z_0\end{aligned}$$

Formula 3

Let (x_1, y_1, z_1) be the second solution of (2.79),

$$\text{where } x_1 = 5x_0 + h, \quad y_1 = h - 5y_0, \quad z_1 = 5z_0 \quad (2.97)$$

Substituting (2.97) in (2.79) and simplifying, we get

$$h = 6x_0 + 16y_0$$

Thus, the second solution (x_1, y_1, z_1) to (2.79) is given by

$$x_1 = 11x_0 + 16y_0, \quad y_1 = 6x_0 + 11y_0, \quad z_1 = 5z_0$$

Express x_1 and y_1 in the form of 2×2 matrix as follows:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad \text{where } M = \begin{pmatrix} 11 & 16 \\ 6 & 11 \end{pmatrix}$$

Repeating the above process, the general values of x and z are given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

If α, β are the eigen values of M , then

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I)$$

$$M^n = \begin{pmatrix} \frac{\sqrt{6}(\alpha^n + \beta^n)}{2\sqrt{5}} & \frac{\sqrt{6}(\alpha^n + \beta^n)}{2\sqrt{5}} \\ \frac{3(\alpha^n - \beta^n)}{4\sqrt{5}} & \frac{\sqrt{6}(\alpha^n + \beta^n)}{2\sqrt{5}} \end{pmatrix}$$

Hence, the general values of x, y, z satisfying (2.79) are given by

$$\begin{aligned}x_n &= \frac{\sqrt{6}(\alpha^n + \beta^n)}{2\sqrt{5}}x_0 + \frac{\sqrt{6}(\alpha^n + \beta^n)}{2\sqrt{5}}y_0 \\y_n &= \frac{3(\alpha^n - \beta^n)}{4\sqrt{5}}x_0 + \frac{\sqrt{6}(\alpha^n + \beta^n)}{2\sqrt{5}}y_0 \\z_n &= 5^n z_0\end{aligned}$$

II.8 On the Integer Solutions to Ternary Quadratic Diophantine Equation

$$z^2 = D(x^2 - y^2), D = \text{odd prime}$$

The ternary quadratic diophantine equation to be solved for its integer solutions is

$$z^2 = D(x^2 - y^2), D = \text{odd prime} \quad (2.98)$$

Since D is an odd prime, each of the expressions $\frac{D+1}{2}, \frac{D-1}{2}$ is an integer.

In view of the identity

$$(a+b)^2 - (a-b)^2 = 4ab$$

it is observed that (1) is satisfied by

$$x = \left(\frac{D+1}{2}\right)K, y = \left(\frac{D-1}{2}\right)K, z = KD$$

The other sets of solutions to (2.98) are illustrated below:

Set: 1

(2.98) is written in the form of ratio as

$$\frac{z}{x+y} = \frac{D(x-y)}{z} = \frac{\alpha}{\beta}, \beta \neq 0$$

which is equivalent to the system of double equations

$$\begin{aligned} \alpha x + \alpha y - \beta z &= 0 \\ -Dx\beta + \beta Dy + \alpha z &= 0 \end{aligned}$$

Applying the method of cross-multiplication to the above system of equations, one obtains

$$\begin{aligned} z &= 2\alpha\beta D \\ y &= -\alpha^2 + D\beta^2 \\ x &= \alpha^2 + D\beta^2 \end{aligned}$$

which satisfy (2.98).

Note: 1

It is observed that (2.98) may also be represented as below:

$$\frac{z}{D(x+y)} = \frac{x-y}{z} = \frac{\alpha}{\beta}, \beta \neq 0$$

Employing the procedure as above, the corresponding solutions to (2.98) are given by:

$$z = 2\alpha\beta D, y = -D\alpha^2 + \beta^2, x = D\alpha^2 + \beta^2$$

Set: 2

(2.98) is written as

$$z^2 + Dy^2 = Dx^2 = Dx^2 * 1 \quad (2.99)$$

Assume x as

$$x = a^2 + Db^2 \quad (2.100)$$

Write 1 as

$$1 = \frac{\left[D - k^2 + i 2k\sqrt{D} \right] \left[D - k^2 - i 2k\sqrt{D} \right]}{(D + k^2)^2} \quad (2.101)$$

Using (2.100) & (2.101) in (2.99) and employing the method of factorization, consider

$$\left(z + i\sqrt{D} y \right) = i\sqrt{D} \left(a + i\sqrt{D} b \right)^2 \cdot \frac{\left[D - k^2 + i 2k\sqrt{D} \right]}{(D + k^2)}.$$

Equating the real & imaginary parts, it is seen that

$$\left. \begin{aligned} z &= \frac{1}{(D + k^2)} \left[-2D (D - k^2) ab - 2kD [a^2 - Db^2] \right] \\ y &= \frac{1}{(D + k^2)} \left[(D - k^2) [a^2 - Db^2] - 4kDab \right] \end{aligned} \right\} \quad (2.102)$$

Since our interest is to find the integer solutions, replacing a by $(D + k^2)A$ & b by $(D + k^2)B$ in (2.100) & (2.102), the corresponding integer solutions to (2.98) are given by

$$\begin{aligned}
z &= (D+k^2) \left[-2D(D-k^2)AB - 2kD[A^2 - DB^2] \right], \\
y &= (D+k^2) \left[(D-k^2)[A^2 - DB^2] - 4kDAB \right], \\
x &= (D+k^2)^2 [A^2 + DB^2]
\end{aligned}$$

Set: 3

Taking

$$z = DW$$

in (2.98), it is written as

$$x^2 - Dw^2 = y^2 = y^2 * 1 \quad (2.103)$$

Assume y as

$$y = a^2 - Db^2$$

Note that 1 may be represented as follows:

$$1 = \frac{(D+k^2+2k\sqrt{D})(D+k^2-2k\sqrt{D})}{(D-k^2)^2}$$

Following the procedure as in Set 2, the corresponding integer solutions to (2.98) are

given by

$$\begin{aligned}
z &= (D-k^2) \left[2D(D+k^2)AB + 2kD[A^2 + DB^2] \right], \\
x &= (D-k^2) \left[(D+k^2)[A^2 + DB^2] + 4kDAB \right], \\
y &= (D-k^2)^2 [A^2 - DB^2]
\end{aligned}$$

Set: 4

(2.103) is written as the system of double equations as below:

$$x + y = w^2, x - y = D$$

Solving the above system, the values of x, y and z satisfying (2.98) are found to be

$$x = 2k^2 + 2k + \frac{D+1}{2}, y = 2k^2 + 2k + \frac{1-D}{2}, z = (2k+1)D$$

GENERATION OF SOLUTIONS

Different formulas for generating sequence of integer solutions based on the given solution are presented below:

Let (x_0, y_0, z_0) be any given solution to (2.98).

Formula: 1

Let (x_1, y_1, z_1) given by

$$x_1 = -(D-1)x_0 + h, \quad y_1 = (D-1)y_0, \quad z_1 = (D-1)z_0 + h, \quad (2.104)$$

be the 2nd solution to (2.98). Using (2.104) in (2.98) and simplifying, one obtains

$$h = 2Dx_0 + 2z_0$$

In view of (2.104), the values of x_1 and z_1 are written in the matrix form as

$$(x_1, z_1)^t = M (x_0, z_0)^t$$

where

$$M = \begin{pmatrix} D+1 & 2 \\ 2D & D+1 \end{pmatrix} \text{ and } t \text{ is the transpose}$$

The repetition of the above process leads to the n^{th} solutions x_n, z_n given by

$$(x_n, z_n)^t = M^n (x_0, z_0)^t$$

If α, β are the distinct eigen values of M , then

$$\alpha = D+1+2\sqrt{D}, \quad \beta = D+1-2\sqrt{D}$$

We know that

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I), \quad I = 2 \times 2 \text{ identity matrix}$$

Thus, the general formulas for integer solutions to (2.98) are given by

$$\begin{aligned}x_n &= \left(\frac{\alpha^n + \beta^n}{2} \right) x_0 + \left(\frac{\alpha^n - \beta^n}{2\sqrt{D}} \right) z_0, \\y_n &= (D-1)^n y_0, \\z_n &= \frac{\sqrt{D}}{2} (\alpha^n - \beta^n) x_0 + \left(\frac{\alpha^n + \beta^n}{2} \right) z_0\end{aligned}$$

Formula: 2

Let (x_1, y_1, z_1) given by

$$x_1 = (D+1)x_0, \quad y_1 = -(D+1)y_0 + h, \quad z_1 = h - (D+1)z_0, \quad (2.105)$$

be the 2nd solution to (2.98). Using (2.105) in (2.98) and simplifying, one obtains

$$h = 2Dy_0 + 2z_0$$

In view of (2.105), the values of y_1 and z_1 are written in the matrix form as

$$(y_1, z_1)^t = M^n (y_0, z_0)^t$$

where

$$M = \begin{pmatrix} D-1 & 2 \\ 2D & -(D-1) \end{pmatrix} \text{ and } t \text{ is the transpose}$$

The repetition of the above process leads to the nth solutions y_n, z_n given by

$$(y_n, z_n)^t = M^n (y_0, z_0)^t$$

If α, β are the distinct eigen values of M , then

$$\alpha = D+1, \quad \beta = -(D+1)$$

Thus, the general formulas for integer solutions to (2.98) are given by

$$\begin{aligned}x_n &= (D+1)^n x_0, \\y_n &= \left(\frac{D\alpha^n + \beta^n}{D+1} \right) y_0 + \left(\frac{\alpha^n - \beta^n}{D+1} \right) z_0, \\z_n &= D \frac{(\alpha^n - \beta^n)}{D+1} y_0 + \left(\frac{\alpha^n + D\beta^n}{D+1} \right) z_0\end{aligned}$$

Formula: 3

Let (x_1, y_1, z_1) given by

$$x_1 = -3Dx_0 + 2h, y_1 = 3Dy_0 + h, z_1 = 3Dz_0, \quad (2.106)$$

be the 2nd solution to (2.98). Using (2.106) in (2.98) and simplifying, one obtains

$$h = 4Dx_0 + 2Dy_0$$

In view of (2.106), the values of x_1 and y_1 is written in the matrix form as

$$(x_1, y_1)^t = M^n (x_0, y_0)^t$$

where

$$M = \begin{pmatrix} 5D & 4D \\ 4D & 5D \end{pmatrix} \text{ and } t \text{ is the transpose}$$

The repetition of the above process leads to the nth solutions x_n, y_n given by

$$(x_n, y_n)^t = M^n (x_0, y_0)^t$$

If α, β are the distinct eigen values of M , then

$$\alpha = D, \beta = 9D$$

Thus, the general formulas for integer solutions to (2.98) are given by

$$x_n = \frac{\alpha^n + \beta^n}{2} x_0 + \frac{\beta^n - \alpha^n}{2} y_0, \\ y_n = \frac{(\beta^n - \alpha^n)}{2} x_0 + \frac{\alpha^n + \beta^n}{2} y_0, \quad z_n = (3D)^n z_0$$

II.9 On Finding Integer Solutions to the Homogeneous Cone $x^2 = 25y^2 + 29z^2$

The quadratic Diophantine equation with three unknowns studied for its non-zero distinct integer solutions is given by

$$x^2 = 25y^2 + 29z^2 \quad (2.107)$$

We illustrate below different sets of integral solutions of (2.107).

Set I

It is observed that (2.107) is of the form

$$x^2 = y^2 + Dz^2 \quad (2.108)$$

where $D = 29$. Employing the most cited solutions of (2.108), one may obtain

$$x = 29m^2 + n^2$$

$$y = \frac{1}{5}(29m^2 - n^2)$$

$$z = 2mn, m, n \in N.$$

Since our interest centers on finding integral solutions, it is possible to choose m, n such that x, y and z are integers. For the sake of clear understanding, the values of m, n with the corresponding solutions are presented in Table 2.18 below:

Table 2.18: Values of m, n with solutions

Choices	m	N	x, y, z
1	$5M$	$5N$	$725M^2 + 25N^2, 145M^2 - 5N^2, 50MN$
2	$5k - 4$	$5k - 3$	$750k^2 - 1190k + 473, 140k^2 - 226k + 91, 50k^2 - 70k + 24$
3	$5k - 4$	$5k - 2$	$750k^2 - 1180k + 468, 140k^2 - 228k + 92, 50k^2 - 60k + 16$
4	$5k - 3$	$5k - 4$	$750k^2 - 910k + 277, 140k^2 - 166k + 49, 50k^2 - 70k + 24$
5	$5k - 3$	$5k - 1$	$750k^2 - 880k + 262, 140k^2 - 172k + 52, 50k^2 - 40k + 6$
6	$5k - 2$	$5k - 4$	$750k^2 - 620k + 132, 140k^2 - 108k + 20, 50k^2 - 60k + 16$
7	$5k - 2$	$5k - 1$	$750k^2 - 590k + 117, 140k^2 - 114k + 23, 50k^2 - 30k + 4$
8	$5k - 1$	$5k - 3$	$750k^2 - 320k + 38, 140k^2 - 52k + 4, 50k^2 - 40k + 6$
9	$5k - 1$	$5k - 2$	$750k^2 - 310k + 33, 140k^2 - 54k + 5, 50k^2 - 30k + 4$

A few interesting properties among the solutions for each of the above choices in Table 2.18 are presented below:

Properties

Choice 1

1. $y - t_{292,M} + t_{12,N} \equiv 0 \pmod{4}$
2. $x - t_{1452,M} - t_{52,N} \equiv 0 \pmod{4}$
3. $x - y - t_{1162,M} - t_{62,N} = 579M + 29N$

Choice 2

1. $x - y - t_{1222,k} \equiv 27 \pmod{355}$
2. $z - t_{102,k} \equiv 3 \pmod{21}$
3. $y - t_{282,k} \equiv 4 \pmod{87}$

Choice 3

1. $x - t_{802,k} - t_{702,k} \equiv 36 \pmod{432}$
2. $y - t_{202,k} - t_{82,k} \equiv 2 \pmod{90}$
3. $z - 2t_{52,k} \equiv 4 \pmod{12}$

Choice 4

1. $x - z - t_{1402,k} \equiv 112 \pmod{141}$
2. $y - 2t_{142,k} \equiv 21 \pmod{28}$
3. $y + z - t_{382,k} \equiv 26 \pmod{47}$

Choice 5

$$1. \quad x - y - 2t_{612,k} \equiv 10 \pmod{100}$$

$$2. \quad z - 5t_{22,k} \equiv 1 \pmod{5}$$

$$3. \quad y - 7t_{42,k} \equiv 13 \pmod{39}$$

Choice 6

$$1. \quad x - 50t_{32,k} \equiv 52 \pmod{80}$$

$$2. \quad y - 10t_{30,k} \equiv 20 \pmod{22}$$

$$3. \quad z - 2t_{52,k} \equiv 4 \pmod{12}$$

Choice 7

$$1. \quad x - 3t_{502,k} \equiv 117 \pmod{157}$$

$$2. \quad y - 20t_{16,k} \equiv 5 \pmod{6}$$

$$3. \quad z - 25t_{6,k} \equiv 4 \pmod{5}$$

Choice 8

$$1. \quad x - 250t_{8,k} \equiv 38 \pmod{180}$$

$$2. \quad y - 20t_{16,k} \equiv 4 \pmod{68}$$

$$3. \quad z - 10t_{12,k} = 6$$

Choice 9

$$1. \quad x - 10t_{152,k} \equiv 33 \pmod{430}$$

$$2. \quad y - 7t_{42,k} \equiv 5 \pmod{79}$$

$$3. \quad z - t_{42,k} - t_{62,k} \equiv 4 \pmod{18}$$

Set II

Express (2.107) as the system of double equations as presented in Table 2.19 below:

Table 2.19: System of double equations

System	I	II	III
$x + 5y$	z^2	$29z^2$	$29z$
$x - 5y$	29	1	Z

Solving each of the above system of double equations, one obtains the corresponding integer solutions to (2.107) as exhibited below:

Solutions to System I

$$x = 50k^2 + 30k + 19$$

$$y = 10k^2 + 6k - 2$$

$$z = 10k + 3$$

Properties

1. $x - t_{42,k} - t_{62,k} \equiv 19 \pmod{78}$
2. $y + z - 5t_{22,k} \equiv 1 \pmod{25}$
3. $x + y - t_{122,k} \equiv 17 \pmod{95}$

Solutions to System II

$$x = 1450k^2 + 870k + 131$$

$$y = 290k^2 + 174k + 26$$

$$z = 10k + 3$$

Properties

1. $x - 25t_{118,k} \equiv 131 \pmod{2295}$
2. $y - 58t_{12,k} \equiv 26 \pmod{406}$
3. $x + z - 29t_{102,k} \equiv 134 \pmod{2301}$

Solutions to System III

$$x = 75\alpha, y = 14\alpha, z = 5\alpha$$

Properties

$$1. \quad x^2 - 75t_{152,\alpha} \equiv 0 \pmod{5550}$$

$$2. \quad y^2 - 14t_{30,\alpha} \equiv 0 \pmod{182}$$

$$3. \quad z^2 - t_{52,\alpha} \equiv 0 \pmod{24}$$

Set III

Write (2.107) as

$$25y^2 + 29z^2 = x^2 * 1 \quad (2.109)$$

$$\text{Let } x = 25a^2 + 29b^2 \quad (2.110)$$

Write 1 on the right hand side of (2.109) as

$$1 = \frac{(14 + i\sqrt{29})(14 - i\sqrt{29})}{15^2} \quad (2.111)$$

Substituting (2.110) and (2.111) in (2.109) and employing the factorization method, define

$$5y + i\sqrt{29}z = \frac{1}{15}(5a + i\sqrt{29}b)^2(14 + i\sqrt{29})$$

Equating real and imaginary parts, we've

$$\left. \begin{aligned} 5y &= \frac{1}{15}[350a^2 - 406b^2 - 290ab] \\ z &= \frac{1}{5}[25a^2 - 29b^2 + 140ab] \end{aligned} \right\} \quad (2.112)$$

As our interest is finding integer solutions, we choose a and b suitably so that x, y, z are integers,

Replacing a by $15a$ and b by $15b$ in (2.110) and (2.112), the corresponding integer solutions to (2.107) are given by

$$\begin{aligned}x &= x(a, b) = 5625a^2 + 6525b^2 \\y &= y(a, b) = 1050a^2 - 1218b^2 - 870ab \\z &= z(a, b) = 375a^2 - 435b^2 + 2100ab\end{aligned}$$

Properties

1. $z(a, 1) - t_{402, a} - t_{352, a} \equiv 2038 \pmod{2473}$
2. $x(a, 1) - 625t_{20, a} \equiv 1525 \pmod{5000}$
3. $y(a, 1) - 50t_{44, a} \equiv 82 \pmod{130}$

II.10 On the Homogeneous Cone $z^2 = (2k^2 - 2k + 22)x^2 + y^2$

The ternary quadratic equation to be solved for its integer solutions is

$$z^2 = (2k^2 - 2k + 22)x^2 + y^2 \quad (2.113)$$

We present below different methods of solving (2.113):

Method: 1

(2.113) is written in the form of ratio as

$$\frac{z+y}{(2k^2 - 2k + 22)x} = \frac{x}{z-y} = \frac{r}{s}, s \neq 0 \quad (2.114)$$

which is equivalent to the system of double equations

$$\begin{aligned} (2k^2 - 2k + 22)rx - sy - sz &= 0 \\ sx + ry - rz &= 0 \end{aligned}$$

Applying the method of cross-multiplication to the above system of equations,

$$\begin{aligned} x &= x(r, s) = 2rs \\ y &= y(r, s) = (2k^2 - 2k + 22)r^2 - s^2 \\ z &= z(r, s) = (2k^2 - 2k + 22)r^2 + s^2 \end{aligned}$$

which satisfy (2.113)

Note: 1

It is observed that (2.113) may also be represented in the form of ratio as below:

$$(i) \frac{z+y}{2x} = \frac{(k^2 - k + 11)x}{z-y} = \frac{r}{s}, s \neq 0$$

The corresponding solutions to (2.113) are given as:

$$x = 2rs, y = 2r^2 - (k^2 - k + 11)s^2, z = 2r^2 + (k^2 - k + 11)s^2$$

$$(ii) \frac{z+y}{(k^2 - k + 11)x} = \frac{2x}{z-y} = \frac{r}{s}, s \neq 0$$

The corresponding solutions to (2.113) are given as:

$$x = 2rs, y = (k^2 - k + 11)r^2 - 2s^2, z = (k^2 - k + 11)r^2 + 2s^2$$

Method: 2

(2.113) is written as the system of double equation in Table 2.20 as follows:

Table 2.20: System of Double Equations

System	1	2	3	4
$z + y$	$2x$	$(k^2 - k + 11)x^2$	$(2k^2 - 2k + 22)x$	$(k^2 - k + 11)x$
$z - y$	$(k^2 - k + 11)x$	2	x	$2x$

Solving each of the above system of double equations, the value of x, y & z satisfying (2.113) are obtained. For simplicity and brevity, in what follows, the integer solutions thus obtained are exhibited.

Solutions for system: I

$$x = 2s, y = -(k^2 - k + 9)s, z = (k^2 - k + 13)s$$

Solutions for system: II

$$x = 2s, y = 2s^2(k^2 - k + 11) - 1, z = 2s^2(k^2 - k + 11) + 1$$

Solution for system: III

$$x = 2s, y = (2k^2 - 2k + 21)s, z = (2k^2 - 2k + 23)s$$

Solution for system: IV

$$x = 2s, y = s(k^2 - k + 11) - 2s, z = s(k^2 - k + 11) + 2s$$

Method: 3

(2.113) is written as

$$y^2 + (2k^2 - 2k + 22)x^2 = z^2 = z^2 * 1 \quad (2.115)$$

Assume z as

$$z = a^2 + (2k^2 - 2k + 22)b^2 \quad (2.116)$$

Write 1 as

$$1 = \frac{\left[(2k^2 - 2k + 22)r^2 - s^2 + i2rs\sqrt{2k^2 - 2k + 22} \right] * \left[(2k^2 - 2k + 22)r^2 - s^2 - i2rs\sqrt{2k^2 - 2k + 22} \right]}{\left((2k^2 - 2k + 22)r^2 + s^2 \right)^2} \quad (2.117)$$

Using (2.116) & (2.117) in (2.115) and employing the method of factorization, consider

$$y + i\sqrt{2k^2 - 2k + 22}x = \frac{\left(a + ib\sqrt{2k^2 - 2k + 22} \right)^2 \left[(2k^2 - 2k + 22)r^2 - s^2 + i\sqrt{2k^2 - 2k + 22} 2rs \right]}{(2k^2 - 2k + 22)r^2 + s^2}$$

Equating real & imaginary parts, it is seen that

$$\left. \begin{aligned} y &= \frac{1}{(2k^2 - 2k + 22)r^2 + s^2} \left[\left\{ (2k^2 - 2k + 22)r^2 - s^2 \right\} \left\{ a^2 - (2k^2 - 2k + 22)b^2 \right\} - 4abrs \left\{ 2k^2 - 2k + 22 \right\} \right] \\ x &= \frac{1}{(2k^2 - 2k + 22)r^2 + s^2} \left[2ab \left\{ (2k^2 - 2k + 22)r^2 - s^2 \right\} + 2rs \left\{ a^2 - (2k^2 - 2k + 22)b^2 \right\} \right] \end{aligned} \right\} \quad (2.118)$$

Since our interest is to find the integer solutions, replacing a by

$$\left[(2k^2 - 2k + 22)r^2 + s^2 \right] A \text{ \& } b \text{ by } \left[(2k^2 - 2k + 22)r^2 + s^2 \right] B \text{ in (2.118) \& (2.116),}$$

the corresponding integer solutions to (2.113) are given by

$$\begin{aligned} x &= x(A, B) = \left((2k^2 - 2k + 22)r^2 + s^2 \right) \left[\left(A^2 - (2k^2 - 2k + 22)B^2 \right) 2rs + 2AB \left((2k^2 - 2k + 22)r^2 - s^2 \right) \right] \\ y &= y(A, B) = \left((2k^2 - 2k + 22)r^2 + s^2 \right) \left[\left(A^2 - (2k^2 - 2k + 22)B^2 \right) \left[(2k^2 - 2k + 22)r^2 - s^2 \right] \right. \\ &\quad \left. - 4ABrs(2k^2 - 2k + 22) \right] \\ z &= z(A, B) = \left((2k^2 - 2k + 22)r^2 + s^2 \right)^2 \left(A^2 + (2k^2 - 2k + 22)B^2 \right) \end{aligned}$$

Following the above procedure, one may obtain difference sets of integer solutions to (2.113).

Method: 4

(2.113) is written as

$$z^2 - (2k^2 - 2k + 22)x^2 = y^2 = y^2 * 1 \quad (2.119)$$

Assume y as

$$y = a^2 - (2k^2 - 2k + 22)b^2 \quad (2.120)$$

Write 1 as

$$1 = \frac{\left((2k^2 - 2k + 22)r^2 + s^2 + \sqrt{2k^2 - 2k + 22} 2rs \right) \left((2k^2 - 2k + 22)r^2 + s^2 - \sqrt{2k^2 - 2k + 22} 2rs \right)}{\left((2k^2 - 2k + 22)r^2 - s^2 \right)^2} \quad (2.121)$$

Using (2.120) & (2.121) in (2.119) and employing the method of factorization, consider

$$z + \sqrt{2k^2 - 2k + 22} x = \frac{\left[(2k^2 - 2k + 22)r^2 + s^2 + 2rs\sqrt{2k^2 - 2k + 22} \right] * \left[\begin{matrix} a^2 + (2k^2 - 2k + 22)b^2 \\ + 2ab\sqrt{2k^2 - 2k + 22} \end{matrix} \right]}{(2k^2 - 2k + 22)r^2 - s^2}$$

Equating rational and irrational parts, it is seen that,

$$\left. \begin{aligned} x &= \frac{(a^2 + (2k^2 - 2k + 22)b^2) 2rs + 2ab((2k^2 - 2k + 22)r^2 + s^2)}{(2k^2 - 2k + 22)r^2 - s^2} \\ z &= \frac{(a^2 + (2k^2 - 2k + 22)b^2)((2k^2 - 2k + 22)r^2 + s^2) + 4abrs(2k^2 - 2k + 22)}{(2k^2 - 2k + 22)r^2 - s^2} \end{aligned} \right\} \quad (2.122)$$

Since our interest to find the integer solution, replacing a by

$((2k^2 - 2k + 22)r^2 - s^2)A$ & b by $((2k^2 - 2k + 22)r^2 - s^2)B$ in (2.122) & (2.120),

the corresponding integer solutions to (2.113) are given by

$$\begin{aligned}
x &= x(A, B) = \left((2k^2 - 2k + 22)r^2 - s^2 \right) \left[\left(A^2 + (2k^2 - 2k + 22)B^2 \right) 2rs + 2AB \left((2k^2 - 2k + 22)r^2 + s^2 \right) \right] \\
y &= y(A, B) = \left((2k^2 - 2k + 22)r^2 - s^2 \right)^2 \left[A^2 - (2k^2 - 2k + 22)B^2 \right] \\
z &= z(A, B) = \left((2k^2 - 2k + 22)r^2 - s^2 \right) \left[\left(A^2 + (2k^2 - 2k + 22)B^2 \right) \left((2k^2 - 2k + 22)r^2 + s^2 \right) \right. \\
&\quad \left. + 4ABrs (2k^2 - 2k + 22) \right]
\end{aligned}$$

Following the above procedure, one may obtain difference sets of integer solutions to (2.113).

GENERATION OF SOLUTIONS

Different formulas for generating sequence of integer solutions based on the given solution are presented below:

Let (x_0, y_0, z_0) be any given solution to (2.113).

Formula: 1

Let (x_1, y_1, z_1) given by

$$x_1 = 3x_0, y_1 = 3y_0 + h, z_1 = 3z_0 + 2h \quad (2.123)$$

be the 2nd solution to (2.113). Using (2.123) in (2.113) and simplifying, one obtains

$$h = 2y_0 - 4z_0$$

In view of (2.123), the values of y_1 and z_1 are written in the matrix form as

$$(y_1, z_1)^t = M (y_0, z_0)^t$$

$$\text{where } M = \begin{bmatrix} 5 & -4 \\ 4 & -5 \end{bmatrix}$$

and t is the transpose

The repetition of the above proses leads to the n^{th} solutions y_n, z_n given by

$$(y_n, z_n)^t = M^n (y_0, z_0)^t$$

If α, β are the distinct eigen values of M , then

$$\alpha = 3, \beta = -3$$

We know that

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I), \quad I = 2 \times 2 \text{ Identity matrix}$$

Thus, the general formulas for integer solutions to (2.113) are given by

$$\begin{aligned} x_n &= 3^n x_0 \\ \begin{pmatrix} y_n \\ z_n \end{pmatrix} &= \frac{1}{3} \begin{bmatrix} 4\alpha^n - \beta^n & -2\alpha^n + 2\beta^n \\ 2\alpha^n - 2\beta^n & -\alpha^n + 4\beta^n \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \end{aligned}$$

Formula: 2

Let (x_1, y_1, z_1) given by

$$x_1 = h - (2k^2 - 2k + 23)x_0, \quad y_1 = h - (2k^2 - 2k + 23)y_0, \quad z_1 = (2k^2 - 2k + 23)z_0 \quad (2.124)$$

be the 2nd solution to (2.113). Using (2.124) in (2.113) and simplifying, one obtains

$$h = (4k^2 - 4k + 44)x_0 + 2y_0$$

In view of (2.124), the values of x_1 and y_1 are written in the matrix form as

$$(x_1, y_1)^t = M (x_0, y_0)^t$$

$$\text{where } M = \begin{bmatrix} 2k^2 - 2k + 21 & 2 \\ 4k^2 - 4k + 44 & -(2k^2 - 2k + 21) \end{bmatrix}$$

And t is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, y_n given by

$$(x_n, y_n)^t = M^n (x_0, y_0)^t$$

If α, β are the distinct eigen values of M , then

$$\alpha = 2k^2 - 2k + 23, \beta = -(2k^2 - 2k + 23)$$

Thus, the general formulas for integer solutions to (2.113) are given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \frac{1}{(2k^2 - 2k + 23)} \begin{bmatrix} (2k^2 - 2k + 22)\alpha^n + \beta^n & \alpha^n - \beta^n \\ (2k^2 - 2k + 22)(\alpha^n - \beta^n) & \alpha^n + (2k^2 - 2k + 22)\beta^n \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$z_n = (2k^2 - 2k + 23)^n z_0$$

Formula: 3

Let (x_1, y_1, z_1) given by

$$x_1 = h - (2k^2 - 2k + 21)x_0, \quad y_1 = (2k^2 - 2k + 21)y_0, \quad z_1 = (2k^2 - 2k + 21)z_0 + h \quad (2.125)$$

be the 2nd solution to (2.113). Using (2.125) in (2.113) and simplifying, one obtains

$$h = 2z_0 + (4k^2 - 4k + 44)x_0$$

In view of (2.125), the values of x_1 and z_1 are written in the matrix form as

$$(x_1, z_1)^t = M (x_0, z_0)^t$$

$$\text{where } M = \begin{bmatrix} 2k^2 - 2k + 23 & 2 \\ 4k^2 - 4k + 44 & 2k^2 - 2k + 23 \end{bmatrix}$$

and t is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, z_n given by

$$(x_n, z_n)^t = M^n (x_0, z_0)^t$$

If α, β are the distinct eigen values of M , then

$$\begin{aligned} \alpha &= 2k^2 - 2k + 23 + 2\sqrt{2k^2 - 2k + 22}, \\ \beta &= 2k^2 - 2k + 23 - 2\sqrt{2k^2 - 2k + 22} \end{aligned}$$

Thus, the general formulas for integer solutions to (2.113) are given by

$$y_n = (2k^2 - 2k + 21)^n y_0$$

$$\begin{pmatrix} x_n \\ z_n \end{pmatrix} = \frac{1}{2} \begin{bmatrix} \alpha^n + \beta^n & \frac{\alpha^n - \beta^n}{2\sqrt{2k^2 - 2k + 22}} \\ \sqrt{2k^2 - 2k + 22}(\alpha^n - \beta^n) & \alpha^n + \beta^n \end{bmatrix} \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}$$

II.11 On the Homogeneous Quadratic Diophantine

Equation with Four Unknowns $2xy + 3z^2 = 8w^2$

The homogeneous quadratic equation with four unknowns to be solved for its integer solutions is

$$2xy + 3z^2 = 8w^2 \quad (2.126)$$

We present below different sets of distinct integer solutions to (2.126) through employing linear transformations.

Introduction of the linear transformations

$$x = u + v, y = u - v, z = v, (u \neq v \neq 0) \quad (2.127)$$

in (2.126) leads to

$$v^2 + 2u^2 = 8w^2 \quad (2.128)$$

Assume

$$w = a^2 + 2b^2 \quad (2.129)$$

Set I

Write 8 as

$$8 = (i2\sqrt{2})(-i2\sqrt{2}) \quad (2.130)$$

Using (2.129) and (2.130) in (2.128) and employing the method of factorization, define

$$v + i\sqrt{2}u = (i2\sqrt{2})(a + i\sqrt{2}b)^2$$

On equating the real and imaginary parts, one obtains

$$v = 8ab, u = 2a^2 - 4b^2$$

In view of (2.127), note that

$$\left. \begin{aligned} x &= 2a^2 - 4b^2 + 8ab \\ y &= 2a^2 - 4b^2 - 8ab \\ z &= 8ab \end{aligned} \right\} \quad (2.131)$$

Thus, (2.131) and (2.129) represent the distinct integer solutions to (2.126).

Set II

Note that 8 may be expressed as the product of complex conjugates as below:

$$8 = \frac{(8 + i2\sqrt{2})(8 - i2\sqrt{2})}{9} \quad (2.132)$$

Following the procedure as in Set I, the corresponding integer solutions to (2.126) are given below:

$$\begin{aligned} x &= 3(10a^2 - 20b^2 + 8ab) \\ y &= 3(-6a^2 + 12b^2 + 24ab) \\ z &= 3(8a^2 - 16b^2 - 8ab) \\ w &= 9(a^2 + 2b^2) \end{aligned}$$

Set III

(2.128) is written as

$$v^2 + 2u^2 = 8w^2 = 8w^2 * 1 \quad (2.133)$$

Consider 1 as

$$1 = \frac{(1 + i2\sqrt{2})(1 - i2\sqrt{2})}{9} \quad (2.134)$$

Using (2.134), (2.130) and (2.129) in (2.133) and employing the method of factorization, define

$$v + i\sqrt{2}u = (i2\sqrt{2})(a + i\sqrt{2}b)^2 \frac{(1 + i2\sqrt{2})}{3}$$

In this case, the corresponding integer solutions to (2.126) are found to be

$$\begin{aligned}x &= 3(-6a^2 + 12b^2 - 24ab) \\y &= 3(10a^2 - 20b^2 - 8ab) \\z &= 3(-8a^2 + 16b^2 - 8ab) \\w &= 9(a^2 + 2b^2)\end{aligned}$$

It is worth to note that, by substituting (2.134), (2.132) and (2.129) in (2.133) and performing the analysis as above, one obtains a different set of integer solutions to (2.126).

Remark

It is worth mentioning here that, in (2.134), 1 may be represented as the product of complex conjugates, in general, as exhibited below:

$$1 = \frac{(2r^2 - s^2 + i2\sqrt{2}rs)(2r^2 - s^2 - i2\sqrt{2}rs)}{(2r^2 + s^2)^2}$$

Set IV

Introduction of the linear transformations

$$x = X + 8T + 6V, y = X + 8T - 6V, z = 6V, w = X + 2T \quad (2.135)$$

in (2.126) leads to

$$X^2 = 16T^2 + 6V^2 \quad (2.136)$$

After performing a few calculations, the above equation is satisfied by the following three choices of solutions:

- i. $X = 20k, T = k, V = 8k$
- ii. $X = 28k, T = 5k, V = 8k$
- iii. $X = 24R^2 + 4S^2, T = 6R^2 - S^2, V = 8RS$

In view of (2.135), the corresponding integer solutions to (2.126) are represented as follows:

Solutions for (i):

$$x = 76k, y = -20k, z = 48k, w = 22k$$

Solutions for (ii):

$$x = 116k, y = 20k, z = 48k, w = 38k$$

Solutions for (iii):

$$x = 72R^2 - 4S^2 + 48RS, y = 72R^2 - 4S^2, V - 48RS, z = 48RS, w = 36R^2 + 2S^2$$

Note: Suppose, instead of (2.135), the linear transformations are taken as

$$x = X - 8T + 6V, y = X - 8T - 6V, z = 6V, w = X - 2T$$

then, the corresponding three choices of solutions to (2.126) are as follows:

Solutions for (i):

$$x = 60k, y = -36k, z = 48k, w = 18k$$

Solutions for (ii):

$$x = 36k, y = -60k, z = 48k, w = 18k$$

Solutions for (iii):

$$x = -24R^2 + 12S^2 + 48RS, y = -24R^2 + 12S^2 - 48RS, z = 48RS, w = 12R^2 + 6S^2$$

GENERATION OF SOLUTIONS

Three different formulas for generating sequence of integer solutions based on the given solution are presented below:

Let (x_0, y_0, z_0, w_0) be any given solution to (2.126)

Formula: 1

Let (x_1, y_1, z_1, w_1) given by

$$x_1 = x_0, y_1 = y_0, z_1 = 2h - z_0, w_1 = h + w_0 \quad (2.137)$$

be the 2nd solution to (2.126). Using (2.137) in (2.126) and simplifying, one obtains

$$h = 3z_0 + 4w_0$$

In view of (2.126), the values of z_1 and w_1 are written in the matrix form as

$$(z_1, w_1)^t = M (z_0, w_0)^t \quad (2.138)$$

where

$$M = \begin{pmatrix} 5 & 8 \\ 3 & 5 \end{pmatrix} \text{ and } t \text{ is the transpose}$$

The repetition of the above process leads to the n^{th} solutions z_n, w_n given by

$$(z_n, w_n)^t = M^n (z_0, w_0)^t$$

We know that

$$M^n = \frac{\alpha^n}{(\alpha - \beta)} (M - \beta I) + \frac{\beta^n}{(\beta - \alpha)} (M - \alpha I),$$

$I = 2 \times 2$ Identity matrix and α, β are the distinct eigen values of M .

For M given above in (2.138), it is seen that

$$\alpha = 5 + 2\sqrt{6}, \beta = 5 - 2\sqrt{6}$$

Thus, the generation formula to obtain sequence of integer solutions to (2.126) is given by

$$\begin{aligned} x_n &= x_0, y_n = y_0 \\ z_n &= \left(\frac{\alpha^n + \beta^n}{2} \right) z_0 + \frac{2}{\sqrt{6}} (\alpha^n - \beta^n) w_0 \\ w_n &= \frac{3}{4\sqrt{6}} (\alpha^n - \beta^n) z_0 + \left(\frac{\alpha^n + \beta^n}{2} \right) w_0 \end{aligned}$$

Formula: 2

Let (x_1, y_1, z_1, w_1) given by

$$x_1 = 5x_0, y_1 = 5y_0, z_1 = 5z_0 + h, w_1 = h - 5w_0 \quad (2.139)$$

be the 2nd solution to (2.126). For this choice, the generation formula for getting sequence of integer solutions to (2.126) is obtained as below:

$$\begin{aligned} x_n &= 5^n x_0, y_n = 5^n y_0 \\ \text{where } z_n &= \left(\frac{\alpha^n + \beta^n}{2} \right) z_0 + \frac{2}{\sqrt{6}} (\alpha^n - \beta^n) w_0 \\ w_n &= \frac{\sqrt{6}}{8} (\alpha^n - \beta^n) z_0 + \left(\frac{\alpha^n + \beta^n}{2} \right) w_0 \end{aligned}$$

$$\alpha = 11 + 4\sqrt{6}, \beta = 11 - 4\sqrt{6}$$

Formula: 3

Let (x_1, y_1, z_1, w_1) given by

$$x_1 = 3h - x_0, y_1 = h - y_0, z_1 = -z_0 + h, w_1 = h + w_0 \quad (2.140)$$

be the 2nd solution to (2.126). Using (2.140) in (2.126) and simplifying, one obtains

$$h = 2x_0 + 6y_0 + 6z_0 + 16w_0$$

In view of (2.140), we have

$$\begin{aligned} x_1 &= 5x_0 + 18y_0 + 18z_0 + 48w_0 \\ y_1 &= 2x_0 + 5y_0 + 6z_0 + 16w_0 \\ z_1 &= 2x_0 + 6y_0 + 5z_0 + 16w_0 \\ w_1 &= 2x_0 + 6y_0 + 6z_0 + 17w_0 \end{aligned}$$

which is written in the form of matrix as

$$(x_1, y_1, z_1, w_1)^t = \begin{pmatrix} 5 & 18 & 18 & 48 \\ 2 & 5 & 6 & 16 \\ 2 & 6 & 5 & 16 \\ 2 & 6 & 6 & 17 \end{pmatrix} (x_0, y_0, z_0, w_0)^t$$

where t is the transpose. The repetition of the above process leads to the general solution to (2.126) as

$$x_{n+1} = \frac{Y_n - (-1)^n 2}{3} x_0 + (Y_n + (-1)^n) y_0 + (Y_n + (-1)^n) z_0 + 8 X_n w_0$$

$$y_{n+1} = \frac{Y_n + (-1)^n}{9} x_0 + \frac{Y_n - (-1)^n 2}{3} y_0 + \frac{Y_n + (-1)^n}{3} z_0 + \frac{8 X_n}{3} w_0$$

$$z_{n+1} = \frac{Y_n + (-1)^n}{9} x_0 + \frac{Y_n + (-1)^n}{3} y_0 + \frac{Y_n - (-1)^n 2}{3} z_0 + \frac{8 X_n}{3} w_0$$

$$w_{n+1} = \frac{X_n}{3} x_0 + X_n y_0 + X_n z_0 + Y_n w_0$$

where

$$Y_n = \frac{1}{2} \left((17 + 6\sqrt{8})^{n+1} + (17 - 6\sqrt{8})^{n+1} \right),$$

$$X_n = \frac{1}{2\sqrt{8}} \left((17 + 6\sqrt{8})^{n+1} - (17 - 6\sqrt{8})^{n+1} \right), n = 0, 1, 2, \dots$$

II.12 On Homogeneous Quadratic with Five Unknowns $4w^2 - x^2 - y^2 + z^2 = 16t^2$

The second degree diophantine equation with five unknowns to be solved is

$$4w^2 - x^2 - y^2 + z^2 = 16t^2 \quad (2.141)$$

The process of obtaining different sets of non-zero distinct integer solutions to (2.141) is exhibited below:

Set 1

The substitution of the linear transformations

$$x = 4P + 12Q, y = 8Y, z = 4(P - Q), w = 4(P + Q), t = 2T \quad (2.142)$$

in (2.141) leads to the space pythagorean equation

$$P^2 = Q^2 + Y^2 + T^2 \quad (2.143)$$

which is satisfied by

$$P = a^2 + b^2 + c^2, T = a^2 - b^2 - c^2, Q = 2ab, Y = 2ac \quad (2.144)$$

In view of (2.142), one has the integer solutions to (2.141) given by

$$\begin{aligned} x &= 4(a^2 + b^2 + c^2 + 6ab), y = 16ac, z = 4(a^2 + b^2 + c^2 - 2ab), \\ w &= 4(a^2 + b^2 + c^2 + 2ab), t = 2(a^2 - b^2 - c^2) \end{aligned}$$

Set 2

Introducing the linear transformations

$$x = (8a^2 - 1)s, y = 4aY, z = s, w = 4a^2s, t = aT \quad (2.145)$$

in (2.141), it simplifies to the Pythagorean equation

$$s^2 = Y^2 + T^2 \quad (2.146)$$

whose solutions may be taken as

$$s = p^2 + q^2, T = p^2 - q^2, Y = 2pq \quad (2.147)$$

In view of (2.145), the integer solutions to (2.141) are given by

$$x = (8a^2 - 1)(p^2 + q^2), y = 8apq, z = (p^2 + q^2), w = 4a^2(p^2 + q^2), t = a(p^2 - q^2)$$

Note 1

The solutions to (2.146) is also taken as

$$s = p^2 + q^2, Y = p^2 - q^2, T = 2pq$$

In this case, the integer solutions to (2.141) are given by

$$x = (8a^2 - 1)(p^2 + q^2), y = 4a(p^2 - q^2), z = (p^2 + q^2), w = 4a^2(p^2 + q^2), t = 2apq$$

Set 3

Taking

$$x = 4(P + Q), y = 4(P - Q), w = 4P, z = 4Q \quad (2.148)$$

in (2.141), it reduces to

$$Q^2 + t^2 = 2P^2 \quad (2.149)$$

After some algebra, it is seen that (2.149) is satisfied by

$$\begin{aligned} t &= a^2 - b^2 + 2ab, \\ Q &= a^2 - b^2 - 2ab, \\ P &= a^2 + b^2 \end{aligned}$$

In view of (2.148), it is seen that

$$\begin{aligned} x &= 8a(a - b), \\ y &= 8b(a + b), \\ z &= 4(a^2 - b^2 - 2ab), \\ w &= 4(a^2 + b^2) \end{aligned}$$

Thus, the above values of x, y, z, w, t satisfies (2.141).

Note 2

After performing a few calculations, (2.149) is also satisfied by

$$\begin{aligned}t &= 2a^2 - b^2, \\Q &= 2a^2 + b^2 + 4ab, \\P &= 2a^2 + b^2 + 2ab\end{aligned}$$

In view of (2.148), the corresponding values of x, y, z, w are found to be

$$\begin{aligned}x &= 4(4a^2 + 2b^2 + 6ab), \\y &= -8ab, \\z &= 4(2a^2 + b^2 + 4ab), \\w &= 4(2a^2 + b^2 + 2ab)\end{aligned}$$

Set 4

The choice

$$z = x + 4t \tag{2.150}$$

in (2.141) leads to

$$y^2 - 4w^2 = 8xt \tag{2.151}$$

which is expressed as the system of double equations as shown in Table 2.21 below:

Table 2.21: System of double equations

System	1	2	3	4
$y + 2w$	$8x$	$4x$	$8t$	$2x$
$y - 2w$	t	$2t$	x	$4t$

Solving each of the above systems, one obtains the values of x, y, w, t . In view of (2.150), the corresponding value of z is obtained. For simplicity, the integer solutions to the corresponding system of equations are exhibited below:

Solutions to system 1

$$x = s, y = 4s + 2k, z = s + 16k, w = 2s - k, t = 4k$$

Solutions to system 2

$$x = s, y = 2s + 2k, z = s + 8k, w = s - k, t = 2k$$

Solutions to system 3

$$x = 4s, y = 2s + 4k, z = 4s + 4k, w = 2k - s, t = k$$

Solutions to system 4

$$x = 2s, y = 2s + 2k, z = 2s + 4k, w = s - k, t = k$$

Chapter – III

Cubic Diophantine Equations

CHAPTER – III

CUBIC DIOPHANTINE EQUATIONS

Chapter III analyses cubic diophantine equations in four sections III.1 to III.4

Section III.1

The non-homogeneous cubic equation with three unknowns represented by $3(x^2 + y^2) - 5xy + x + y + 1 = 111z^3$ is analyzed for its patterns of non-zero distinct integer solutions. A few interesting relations among the solutions are presented.

Section III.2

An attempt is made to solve the cubic equation with four unknowns given by $x^3 + y^3 + 6(x + y)z^2 = 4w^3$ in integers. Some special relations between the solutions are given.

Section III.3

The homogeneous cubic equation with four unknowns represented by the Diophantine equation $x^3 + y^3 + (x + y)(x - y)^2 = 16zw^2$ is analyzed for its patterns of non-zero distinct integral solutions. Various interesting relations between the solutions and special numbers namely polygonal numbers are exhibited.

Section III.4

This paper concerns with the problem of obtaining non-zero distinct integer solutions to the non-homogeneous cubic equation with three unknowns given $x^3 + y^3 + x + y = 2z(2z^2 - \alpha^2 + 1)$. A few interesting relations among the solutions are presented. Also, a formula for generating sequence of integer solutions to the considered cubic equation based on its given solution is exhibited.

III.1 On the non-homogeneous ternary cubic equation

$$3(x^2 + y^2) - 5xy + x + y + 1 = 111z^3$$

The ternary non-homogeneous cubic equation to be solved is

$$3(x^2 + y^2) - 5xy + x + y + 1 = 111z^3 \quad (3.1)$$

Introducing the linear transformations

$$x = u + v, \quad y = u - v \quad (3.2)$$

in (3.1), it gives

$$U^2 + 11v^2 = 111z^3 \quad (3.3)$$

$$\text{where } U = u + 1 \quad (3.4)$$

Assume

$$z = z(a, b) = a^2 + 11b^2 \quad (3.5)$$

Solving (3.3) through various methods and using (3.2), different sets of integer solutions to (3.1) are obtained.

A. Method 1

Consider

$$111 = (10 + i\sqrt{11})(10 - i\sqrt{11}) \quad (3.6)$$

Substituting (3.5), (3.6) in (3.3) and applying the method of factorization,

$$(U + i\sqrt{11}v)(U - i\sqrt{11}v) = (10 + i\sqrt{11})(10 - i\sqrt{11}) * (a + i\sqrt{11}b)^3 (a - i\sqrt{11}b)^3$$

Equating the positive and negative terms in the above equation, we have

$$(U + i\sqrt{11}v) = (10 + i\sqrt{11})(a + i\sqrt{11}b)^3 \quad (3.7)$$

$$(U - i\sqrt{11}v) = (10 - i\sqrt{11})(a - i\sqrt{11}b)^3 \quad (3.8)$$

Equating the real and imaginary parts in either (3.7) or (3.8), we have

$$U = 10a^3 - 330ab^2 - 33a^2b + 121b^3 \quad (3.9)$$

$$v = a^3 - 33ab^2 + 30a^2b - 110b^3 \quad (3.10)$$

Substitution of (3.9) in (3.4) gives

$$u = 10a^3 - 330ab^2 - 33a^2b + 121b^3 - 1 \quad (3.11)$$

Substituting the above values of u and v in (3.2), we get

$$\left. \begin{aligned} x &= x(a, b) = 11a^3 - 363ab^2 - 3a^2b + 11b^3 - 1 \\ y &= y(a, b) = 9a^3 - 297ab^2 - 63a^2b + 231b^3 - 1 \end{aligned} \right\} \quad (3.12)$$

Thus, (3.5) and (3.12) represents the integer solutions of (3.1).

Properties

- ❖ $x(1, b) - 11CP_{b,6} + t_{728,b} + 365b - 10 = 0$
- ❖ $y(a, a) + 120CP_{a,6} + 1 = 0$
- ❖ $x(a, a+1) + R_a + 340CP_{a,6} + 702PR_a - 188GNO_a - 197 = 0$
- ❖ $x(a, 1) - y(a, 1) - SO_a - 60PR_a + 125a + 220 = 0$
- ❖ $y(a, -a) + 456CP_{a,6} + 1 = 0$

Note 1

Apart from (3.6), 111 is also expressed as

$$111 = \frac{(13 + i5\sqrt{11})(13 - i5\sqrt{11})}{4} \quad (3.13)$$

In this case, the corresponding solutions to (3.1) are given by

$$\begin{aligned} x &= x(A, B) = 72A^3 - 504A^2B - 2376AB^2 + 1848B^3 - 1 \\ y &= y(A, B) = 32A^3 - 816A^2B - 1056AB^2 + 2992B^3 - 1 \\ z &= z(A, B) = 4A^2 + 44B^2 \end{aligned}$$

B. Method 2

(3.3) is written as

$$U^2 + 11v^2 = 111z^3 * 1 \quad (3.14)$$

Assume

$$1 = \frac{(5 + i\sqrt{11})(5 - i\sqrt{11})}{36} \quad (3.15)$$

Substituting (3.5), (3.6) and (3.15) in (3.14) and employing the method of factorization, define

$$(U + i\sqrt{11}v) = \frac{1}{6}(10 + i\sqrt{11})(5 + i\sqrt{11})(a + i\sqrt{11}b) \quad (3.16)$$

Equating the real and imaginary parts in (3.16), we have

$$\left. \begin{aligned} U &= \frac{1}{6}(39a^3 - 1287ab^2 - 495a^2b + 1815b^3) \\ v &= \frac{1}{6}(15a^3 - 495ab^2 + 117a^2b - 429b^3) \end{aligned} \right\} \quad (3.17)$$

Replacing a by $6A$ and b by $6B$ in (3.17) and (3.5) we get

$$U = 1404A^3 - 46332AB^2 - 17820A^2B + 65340B^3 \quad (3.18)$$

$$v = 540A^3 - 17820AB^2 + 4212A^2B - 15444B^3 \quad (3.19)$$

$$z = 36A^2 + 396B^2 \quad (3.20)$$

Substitution of (3.18) in (3.4) gives

$$u = 1404A^3 - 46332AB^2 - 17820A^2B + 65340B^3 - 1 \quad (3.21)$$

Substituting the above values of u and v in (3.2), it is seen that

$$\left. \begin{aligned} x &= x(A, B) = 1944A^3 - 64152AB^2 - 13608A^2B + 49896B^3 - 1 \\ y &= y(A, B) = 864A^3 - 28512AB^2 - 22032A^2B + 80784B^3 - 1 \end{aligned} \right\} \quad (3.22)$$

Thus, (3.20) and (3.22) represents the integer solutions of (3.1).

Properties

$$\diamond x(A, A) + 25920CP_{A,6} + 1 = 0$$

$$\diamond y(B+1, B) - 106272CP_{B,3} + 47952PR_B + 23328GNO_B + 44497 = 0$$

$$\diamond x(1, B) - 24948SO_B + t_{128306,B} + 52811B - 1943 = 0$$

$$\diamond x(1, B) - y(1, B) + 30888CP_{B,6} + 35640PR_B - 22032(GNO_B) - 23112 = 0$$

$$\diamond y(A, 1) - 864CP_{A,6} - S_A + 22038PR_A + 3234GO_A - 77548 = 0$$

Note 2

It is to be noted that, in addition to (3.15), 1 may also be represented as

$$1 = \frac{(1 + i3\sqrt{11})(1 - i3\sqrt{11})}{100} \quad (3.23)$$

For this choice, the corresponding integer solutions to (3.1) are given by

$$x = x(A, B) = 800A^3 - 26400AB^2 - 109200A^2B + 400400B^3 - 1$$

$$y = y(A, B) = -5400A^3 + 178200AB^2 - 95400A^2 + 349800B^3 - 1$$

$$z = z(A, B) = 100A^2 + 1100B^2$$

Note 3

In (3.14), employing (3.13) along with (3.15) and (3.23) in turn, one obtains two more sets of integer solutions to (3.1) which are exhibited below:

Set 1

$$x = x(A, B) = 6912A^3 - 228096AB^2 - 176256A^2 + 646272B^3 - 1$$

$$y = y(A, B) = -4032A^3 + 133056AB^2 - 184896A^2B + 677952B^3 - 1$$

$$z = z(A, B) = 144A^2 + 1584B^2$$

Set 2

$$x = x(A, B) = -43200A^3 + 1425600AB^2 - 763200A^2B + 2798400B^3 - 1$$

$$y = y(A, B) = -78400A^3 + 2587200AB^2 - 398400A^2B + 1460800B^3 - 1$$

$$z = z(A, B) = 400A^2 + 4400B^2$$

In this paper, a search is performed to obtain different sets of integer solutions to the ternary cubic equation given by (3.1). To conclude, one may search for other choices of integer solutions to (3.1).

III.2 On the Cubic Equation $x^3 + y^3 + 6(x + y)z^2 = 4w^3$

The cubic equation with four unknowns to be solved is

$$x^3 + y^3 + 6(x + y)z^2 = 4w^3 \quad (3.24)$$

Introducing the linear transformations

$$x = u + v, \quad y = u - v, \quad w = u \quad (3.25)$$

in (3.24), it is written as

$$u^2 = 3v^2 + 6z^2 \quad (3.26)$$

Again, the substitution of

$$u = 3U, \quad v = X + 6T, \quad z = X - 3T \quad (3.27)$$

in (3.26), leads to

$$U^2 = 18T^2 + X^2 \quad (3.28)$$

whose solutions are

$$T = 2rs, \quad X = 18r^2 - s^2, \quad U = 18r^2 + s^2 \quad (3.29)$$

From (3.29), (3.27), (3.25), the solutions of (3.24) are given below:

$$x = x(r, s) = 72r^2 + 2s^2 + 12rs$$

$$y = y(r, s) = 36r^2 + 4s^2 - 12rs$$

$$z = z(r, s) = 18r^2 - s^2 - 6rs$$

$$w = w(r, s) = 54r^2 + 3s^2$$

Properties

- $x(1, s) - 2y(1, s) + 6PR_s - 21GNO_s - 21 = 0$
- $y(r, s) - 2z(r, s)$ is a Nasty number
- $x(r, 1) - 4z(r, 1) - 18GNO_r \equiv 0 \pmod{2}$
- $w(r, 1) - 9S_r - 27GNO_r = 21$
- $z(r, 1) - 11r + 1 = t_{38, r}$

Remark: One may also consider the transformation (3.27) as

$$u = 3U, v = X - 6T, z = X + 3T \quad (3.30)$$

In this case, the corresponding values of x, y, z, w satisfying (3.24) are represented by

$$x = x(r, s) = 72r^2 + 2s^2 - 12rs$$

$$y = y(r, s) = 36r^2 + 4s^2 + 12rs$$

$$z = z(r, s) = 18r^2 - s^2 + 6rs$$

$$w = w(r, s) = 54r^2 + 3s^2$$

In addition to the above solutions, other sets of solutions to (3.24) may be obtained as illustrated below:

Note that (3.28) is represented as the pair of equations as in Table 3.1:

Table 3.1: Pair of equations

S. No.	1	2	3	4	5	6
$U + X$	T^2	$9T^2$	$3T^2$	$9T$	$6T$	$18T$
$U - X$	18	2	6	$2T$	$3T$	T

Substituting the corresponding values of U, X and T from the above Table1 and in (3.27) and (3.25) the different sets of integer solutions to (3.24) thus obtained are shown below in Table 3.2:

Table 3.2: Solutions

S. No.	x	y	z	w
1	$8k^2 + 12k + 18$	$4k^2 - 12k + 36$	$2k^2 - 6k - 9$	$6k^2 + 27$
2	$72k^2 + 12k + 2$	$36k^2 - 12k + 4$	$18k^2 - 6k - 1$	$54k^2 + 3$
3	$24k^2 + 12k + 6$	$12k^2 - 12k + 12$	$6k^2 - 6k - 3$	$18k^2 + 9$
4	$54k$	$14k$	k	$33k$
5	$42k$	$12k$	$-3k$	$27k$
6	$86k$	$28k$	$11k$	$57k$

Note: Substituting the values of U, X, T obtained from Table 1 in (3.30) and (3.25), one obtains some more choices of solutions to (3.24).

Also, (3.28) is taken as

$$X^2 + 18T^2 = U^2 * 1 \quad (3.31)$$

Consider

$$U = a^2 + 18b^2 \quad (3.32)$$

and

$$1 = \frac{(3 + i\sqrt{18})(3 - i\sqrt{18})}{81} \quad (3.33)$$

Applying (3.32), (3.33) in (3.31) and factorizing, take

$$X + i\sqrt{18}T = \frac{(a + i\sqrt{18}b)^2 (3 + i\sqrt{18})}{9}$$

from which note that

$$\begin{aligned} X &= \frac{1}{9}(3a^2 - 54b^2 - 72ab) \\ T &= \frac{1}{9}(2a^2 - 36b^2 + 6ab) \end{aligned}$$

Replacing a by $3A$ in the above equations, we have

$$\left. \begin{aligned} X &= 3A^2 - 6b^2 - 24Ab \\ T &= 2A^2 - 4b^2 + 2Ab \end{aligned} \right\} \quad (3.34)$$

and from (3.32),

$$U = 9A^2 + 18b^2 \quad (3.35)$$

Substituting (3.34) and (3.35) in (3.27) and (3.25), the corresponding values of x, y, z, w satisfying (3.24) are given by

$$x = 42A^2 + 24b^2 - 12Ab$$

$$y = 12A^2 + 84b^2 + 12Ab$$

$$z = -3A^2 + 6b^2 - 30Ab$$

$$w = 27A^2 + 54b^2$$

Note: In addition to (3.33), one may also write 1 as

$$1 = \frac{(7 + i 2\sqrt{18})(7 - i 2\sqrt{18})}{121}$$

The repetition of the above process leads to a different set of solutions to (3.24).

In this paper, an attempt has been made to obtain different sets of integer solutions to $x^3 + y^3 + 6(x + y)z^2 = 4w^3$. In conclusion, a search for determining integer solutions to the considered cubic equation with four unknowns may be performed.

III.3 On the Equation of Degree Three with Four Unknowns

$$x^3 + y^3 + (x + y)(x - y)^2 = 16zw^2$$

The homogeneous cubic equation with four unknowns to be solved for its distinct non-zero integral solution is

$$x^3 + y^3 + (x + y)(x - y)^2 = 16zw^2 \quad (3.36)$$

Introduction of the linear transformations

$$x = u + v, y = u - v, z = u \quad (3.37)$$

in (3.36) leads to

$$u^2 + 7v^2 = 8w^2 \quad (3.38)$$

Different methods of obtaining the patterns of integer solutions to (3.36) are illustrated below:

PATTERN: 1

Let

$$w = a^2 + 7b^2 \quad (3.39)$$

where a and b are non-zero integers.

Write 8 as

$$8 = (1 + i\sqrt{7})(1 - i\sqrt{7}) \quad (3.40)$$

Using (3.39), (3.40) in (3.38) and applying the method of factorization, define

$$(u + i\sqrt{7}v) = (1 + i\sqrt{7})(a + i\sqrt{7}b)^2 \quad (3.41)$$

from which we have

$$\left. \begin{aligned} u &= a^2 - 14ab - 7b^2 \\ v &= a^2 + 2ab - 7b^2 \end{aligned} \right\} \quad (3.42)$$

Using (3.42) in (3.37), the values of x, y and z are given by

$$\left. \begin{aligned} x &= x(a, b) = 2a^2 - 12ab - 14b^2 \\ y &= y(a, b) = -16ab \\ z &= z(a, b) = a^2 - 14ab - 7b^2 \end{aligned} \right\} \quad (3.43)$$

Thus (3.39) and (3.43) represent the non-zero integer solutions to (3.36).

OBSERVATIONS

1. $x(a, a) + y(a, a) + t_{34,a} + t_{26,a} + t_{18,a} + t_{10,a} \equiv 0 \pmod{3}$
2. $z(b, b) - y(b, b) + t_{4,2b} = 0$
3. $x(a, a) - y(a, a) + z(a, a) + t_{58,a} = -27a$
4. $x(a, a+1) - y(a, a+1) + z(a, a+1) + 79t_{3,a} = 0$
5. $x(b, b) - y(b, b) - w(b, b) + t_{4,4b} = 0$

PATTERN: 2

Write 8 as

$$8 = \frac{(5 + i\sqrt{7})(5 - i\sqrt{7})}{4} \quad (3.44)$$

Using (3.39), (3.44) in (3.38) and applying the method of factorization, define

$$(u + i\sqrt{7}v) = \left(\frac{5 + i\sqrt{7}}{2} \right) (a + i\sqrt{7}b)^2 \quad (3.45)$$

from which we have

$$\left. \begin{aligned} u &= \frac{1}{2}(5a^2 - 14ab - 35b^2) \\ v &= \frac{1}{2}(a^2 + 10ab - 7b^2) \end{aligned} \right\} \quad (3.46)$$

Since our interest is on finding integer solutions, replacing a by $2A$, b by $2B$ in (3.39) and by using (3.46) in (3.37), the corresponding integer solutions to (3.36) are given by

$$\left. \begin{aligned} x &= x(A, B) = 12A^2 - 8AB - 84B^2 \\ y &= y(A, B) = 8A^2 - 48AB - 56B^2 \\ z &= z(A, B) = 10A^2 - 28AB - 70B^2 \\ w &= w(A, B) = 4A^2 + 28B^2 \end{aligned} \right\} \quad (3.47)$$

Thus (3.47) represent the non-zero integer solutions to (3.36).

OBSERVATIONS

1. $z(B, B) - x(B, B) + w(B, B) + t_{114, B} + t_{82, B} + t_{74, B} + t_{26, B} = -124B$
2. $y(A, A) - x(A, A) + t_{4, 4A} = 0$
3. $x(A, A) - y(A, A) - z(A, A) - t_{74, A} - t_{106, A} - t_{34, A} = 101A$

PATTERN: 3

Write 8 as

$$8 = \frac{(11 + i\sqrt{7})(11 - i\sqrt{7})}{16} \quad (3.48)$$

Using (3.39), (3.48) in (3.38) and applying the method of factorization, define

$$(u + i\sqrt{7}v) = \left(\frac{11 + i\sqrt{7}}{4} \right) (a + i\sqrt{7}b)^2 \quad (3.49)$$

from which we have

$$\left. \begin{aligned} u &= \frac{1}{4}(11a^2 - 14ab - 77b^2) \\ v &= \frac{1}{4}(a^2 + 22ab - 7b^2) \end{aligned} \right\} \quad (3.50)$$

Since our interest is on finding integer solutions, replacing a by $2A$, b by $2B$ in (3.39) and by using (3.50) in (3.37), the corresponding integer solutions to (3.36) are given by

$$\left. \begin{aligned} x &= x(A, B) = 12A^2 + 8AB - 84B^2 \\ y &= y(A, B) = 10A^2 - 36AB - 70B^2 \\ z &= z(A, B) = 11A^2 - 14AB - 77B^2 \\ w &= w(A, B) = 4A^2 + 28B^2 \end{aligned} \right\} \quad (3.51)$$

Thus (3.51) represent the non-zero integer solutions to (3.36).

OBSERVATIONS

1. $x(A, A) + w(A, A) + t_{18,A} + t_{22,A} + t_{30,A} \equiv 0 \pmod{29}$
2. $x(B, B) - z(B, B) - t_{4,4B} = 0$
3. $y(A, A) - z(A, A) + t_{14,A} + t_{22,A} \equiv 0 \pmod{7}$

PATTERN: 4

Write (3.38) as

$$u^2 + 7v^2 = 8w^2 * 1 \quad (3.52)$$

Write 1 as

$$1 = \left(\frac{(3+i\sqrt{7})(3-i\sqrt{7})}{16} \right) \quad (3.53)$$

Using (3.39), (3.40), (3.53) in (3.52) and applying the method of factorization, define

$$(u + i\sqrt{7}v) = (1 + i\sqrt{7}) (a + i\sqrt{7}b)^2 \left(\frac{3 + i\sqrt{7}}{4} \right) \quad (3.54)$$

from which we have

$$\left. \begin{aligned} u &= (-a^2 - 14ab + 7b^2) \\ v &= (a^2 - 2ab - 7b^2) \end{aligned} \right\} \quad (3.55)$$

Using (3.55) in (3.37), the values of x, y and z are given by

$$\left. \begin{aligned} x &= x(a, b) = -16ab \\ y &= y(a, b) = -2a^2 - 12ab + 14b^2 \\ z &= z(a, b) = -a^2 - 14ab + 7b^2 \end{aligned} \right\} \quad (3.56)$$

Thus (3.39) and (3.56) represent the non-zero integer solutions to (3.36).

OBSERVATIONS

1. $y(a, a) - w(a, a) + t_{18,a} \equiv 0 \pmod{7}$
2. $y(a, a) + z(a, a) + t_{12,a} + t_{8,a} = -6a$
3. $x(b, b) - 2w(b, b) + t_{66,b} \equiv 0 \pmod{31}$

PATTERN: 5

Assume 1 as

$$1 = \left(\frac{(1 + i3\sqrt{7})(1 - i3\sqrt{7})}{64} \right) \quad (3.57)$$

Using (3.39), (3.40), (3.57) in (3.52) and applying the method of factorization, define

$$(u + i\sqrt{7}v) = (1 + i\sqrt{7})(a + i\sqrt{7}b)^2 \left(\frac{1 + i3\sqrt{7}}{8} \right) \quad (3.58)$$

from which we have

$$\left. \begin{aligned} u &= \frac{1}{2}(-5a^2 - 14ab + 35b^2) \\ v &= \frac{1}{2}(a^2 - 10ab - 7b^2) \end{aligned} \right\} \quad (3.59)$$

Since our interest is on finding integer solutions, replacing a by $2A$, b by $2B$ in (3.39)

and by using (3.59) in (3.37), the corresponding integer solutions to (3.36) are given by

$$\left. \begin{aligned} x &= x(A, B) = (-8A^2 - 48AB + 56B^2) \\ y &= y(A, B) = (-12A^2 - 8AB + 84B^2) \\ z &= z(A, B) = (-10A^2 - 28AB + 70B^2) \\ w &= w(A, B) = (4A^2 + 28B^2) \end{aligned} \right\} \quad (3.60)$$

Thus (3.60) represent the non-zero integer solutions to (3.36).

OBSERVATIONS

1. $x(A, A) + y(A, A) - t_{4,8A} = 0$
2. $y(B, B+1) - w(B, B+1) - 64t_{3,B} \equiv 0 \pmod{10}$
3. $z(B, B) + w(B, B) - t_{4,4B} - t_{22,B} - t_{38,B} - t_{42,B} \equiv 0 \pmod{5}$

PATTERN: 6

Assume 1 as

$$1 = \left(\frac{(3 + i4\sqrt{7})(3 - i4\sqrt{7})}{121} \right) \quad (3.61)$$

Using (3.39), (3.40), (3.61) in (3.52) and applying the method of factorization, define

$$(u + i\sqrt{7}v) = (1 + i\sqrt{7})(a + i\sqrt{7}b)^2 \left(\frac{3 + i4\sqrt{7}}{11} \right) \quad (3.62)$$

from which we have

$$\left. \begin{aligned} u &= \frac{1}{11}(-25a^2 - 98ab + 175b^2) \\ v &= \frac{1}{11}(7a^2 - 50ab - 49b^2) \end{aligned} \right\} \quad (3.63)$$

Since our interest is on finding integer solutions, replacing a by $11A$, b by $11B$ in (3.39) and by using (3.63) in (3.37), the corresponding integer solutions to (3.36) are given by

$$\left. \begin{aligned} x &= x(A, B) = (-198A^2 - 1628AB + 1386B^2) \\ y &= y(A, B) = (-352A^2 - 528AB + 2464B^2) \\ z &= z(A, B) = (-275A^2 - 1078AB + 1925B^2) \\ w &= w(A, B) = (121A^2 + 847B^2) \end{aligned} \right\} \quad (3.64)$$

Thus (3.64) represent the non-zero integer solutions to (3.36).

PATTERN: 7

Assume 1 as

$$1 = \left(\frac{(1 + i48\sqrt{7})(1 - i48\sqrt{7})}{127^2} \right) \quad (3.65)$$

Using (3.39), (3.40), (3.65) in (3.52) and applying the method of factorization, define

$$(u + i\sqrt{7}v) = (1 + i\sqrt{7})(a + i\sqrt{7}b)^2 \left(\frac{1 + i48\sqrt{7}}{127} \right) \quad (3.66)$$

from which we have

$$\left. \begin{aligned} u &= \frac{1}{127}(-335a^2 - 686ab + 2345b^2) \\ v &= \frac{1}{127}(49a^2 - 670ab - 343b^2) \end{aligned} \right\} \quad (3.67)$$

Since our interest is on finding integer solutions, replacing a by $127A$, b by $127B$ in (3.39)

and by using (3.67) in (3.37), the corresponding integer solutions to (3.36) are given by

$$\left. \begin{aligned} x &= x(A, B) = (-36322A^2 - 172212AB + 254254B^2) \\ y &= y(A, B) = (-48768A^2 - 2032AB + 341376B^2) \\ z &= z(A, B) = (-42545A^2 - 87122AB + 297815B^2) \\ w &= w(A, B) = (16129A^2 + 112903B^2) \end{aligned} \right\} \quad (3.68)$$

Thus (3.68) represent the non-zero integer solutions to (3.36).

PATTERN: 8

Assume 1 as

$$1 = \left(\frac{(3 + i4\sqrt{7})(3 - i4\sqrt{7})}{121} \right) \quad (3.69)$$

Using (3.39), (3.44), (3.69) in (3.52) and applying the method of factorization, define

$$(u + i\sqrt{7}v) = \left(\frac{5 + i\sqrt{7}}{2} \right) (a + i\sqrt{7}b)^2 \left(\frac{3 + i4\sqrt{7}}{11} \right) \quad (3.70)$$

from which we have

$$\left. \begin{aligned} u &= \frac{1}{22}(-13a^2 - 322ab + 91b^2) \\ v &= \frac{1}{22}(23a^2 - 26ab - 161b^2) \end{aligned} \right\} \quad (3.71)$$

Since our interest is on finding integer solutions, replacing a by $22A$, b by $22B$ in (3.39)

and by using (3.71) in (3.37), the corresponding integer solutions to (3.36) are given by

$$\left. \begin{aligned} x &= x(A, B) = (220A^2 - 7656AB - 1540B^2) \\ y &= y(A, B) = (-792A^2 - 6512AB + 5544B^2) \\ z &= z(A, B) = (-286A^2 - 7084AB + 2002B^2) \\ w &= w(A, B) = (484A^2 + 3388B^2) \end{aligned} \right\} \quad (3.72)$$

Thus (3.72) represent the non-zero integer solutions to (3.36).

PATTERN: 9

Assume 1 as

$$1 = \left(\frac{(1 + i48\sqrt{7})(1 - i48\sqrt{7})}{127^2} \right) \quad (3.73)$$

Using (3.39), (3.44), (3.73) in (3.52) and applying the method of factorization, define

$$(u + i\sqrt{7}v) = \left(\frac{5 + i\sqrt{7}}{2} \right) (a + i\sqrt{7}b)^2 \left(\frac{1 + i48\sqrt{7}}{127} \right) \quad (3.74)$$

from which we have

$$\left. \begin{aligned} u &= \frac{1}{254} (-331a^2 - 3374ab + 2317b^2) \\ v &= \frac{1}{254} (241a^2 - 662ab - 1687b^2) \end{aligned} \right\} \quad (3.75)$$

Since our interest is on finding integer solutions, replacing a by $254A$, b by $254B$ in (3.39)

and by using (3.75) in (3.37), the corresponding integer solutions to (3.36) are given by

$$\left. \begin{aligned} x &= x(A, B) = (-22860A^2 - 1025144AB + 160020B^2) \\ y &= y(A, B) = (-145288A^2 - 688848AB + 1017016B^2) \\ z &= z(A, B) = (-84074A^2 - 856996AB + 588518B^2) \\ w &= w(A, B) = (64516A^2 + 451612B^2) \end{aligned} \right\} \quad (3.76)$$

Thus (3.76) represent the non-zero integer solutions to (3.36).

PATTERN: 10

Assume 1 as

$$1 = \left(\frac{(3 + i\sqrt{7})(3 - i\sqrt{7})}{16} \right) \quad (3.77)$$

Using (3.39), (3.48), (3.77) in (3.52) and applying the method of factorization, define

$$(u + i\sqrt{7}v) = \left(\frac{11 + i\sqrt{7}}{4} \right) (a + i\sqrt{7}b)^2 \left(\frac{3 + i\sqrt{7}}{4} \right) \quad (3.78)$$

from which we have

$$\left. \begin{aligned} u &= \frac{1}{8} (13a^2 - 98ab - 91b^2) \\ v &= \frac{1}{8} (7a^2 + 26ab - 49b^2) \end{aligned} \right\} \quad (3.79)$$

Since our interest is on finding integer solutions, replacing a by $8A$, b by $8B$ in (3.39) and by using (3.79) in (3.37), the corresponding integer solutions to (3.36) are given by

$$\left. \begin{aligned} x &= x(A, B) = (160A^2 - 576AB - 1120B^2) \\ y &= y(A, B) = (48A^2 - 992AB - 336B^2) \\ z &= z(A, B) = (104A^2 - 784AB - 728B^2) \\ w &= w(A, B) = (64A^2 + 448B^2) \end{aligned} \right\} \quad (3.80)$$

Thus (3.80) represent the non-zero integer solutions to (3.36).

PATTERN: 11

Assume 1 as

$$1 = \left(\frac{(1+i3\sqrt{7})(1-i3\sqrt{7})}{64} \right) \quad (3.81)$$

Using (3.39), (3.48), (3.81) in (3.52) and applying the method of factorization, define

$$(u+i\sqrt{7}v) = \left(\frac{11+i\sqrt{7}}{4} \right) (a+i\sqrt{7}b)^2 \left(\frac{1+i3\sqrt{7}}{8} \right) \quad (3.82)$$

from which we have

$$\left. \begin{aligned} u &= \frac{1}{16}(-5a^2 - 238ab + 35b^2) \\ v &= \frac{1}{16}(17a^2 - 10ab - 119b^2) \end{aligned} \right\} \quad (3.83)$$

Since our interest is on finding integer solutions, replacing a by $16A$, b by $16B$ in (3.39) and by using (3.83) in (3.37), the corresponding integer solutions to (3.36) are given by

$$\left. \begin{aligned} x &= x(A, B) = (192A^2 - 3968AB - 1344B^2) \\ y &= y(A, B) = (-352A^2 - 3648AB + 2464B^2) \\ z &= z(A, B) = (-80A^2 - 3808AB + 560B^2) \\ w &= w(A, B) = (256A^2 + 1792B^2) \end{aligned} \right\} \quad (3.84)$$

Thus (3.84) represent the non-zero integer solutions to (3.36).

PATTERN: 12

Assume 1 as

$$1 = \left(\frac{(3 + i4\sqrt{7})(3 - i4\sqrt{7})}{121} \right) \quad (3.85)$$

Using (3.39), (3.48), (3.85) in (3.52) and applying the method of factorization, define

$$(u + i\sqrt{7}v) = \left(\frac{11 + i\sqrt{7}}{4} \right) (a + i\sqrt{7}b)^2 \left(\frac{3 + i4\sqrt{7}}{11} \right) \quad (3.86)$$

from which we have

$$\left. \begin{aligned} u &= \frac{1}{44} (5a^2 - 658ab - 35b^2) \\ v &= \frac{1}{44} (47a^2 + 10ab - 329b^2) \end{aligned} \right\} \quad (3.87)$$

Since our interest is on finding integer solutions, replacing a by $44A$, b by $44B$ in (3.39) and by using (3.87) in (3.37), the corresponding integer solutions to (3.36) are given by

$$\left. \begin{aligned} x &= x(A, B) = (2288A^2 - 28512AB - 16016B^2) \\ y &= y(A, B) = (-1848A^2 - 29392AB + 12936B^2) \\ z &= z(A, B) = (220A^2 - 28952AB - 1540B^2) \\ w &= w(A, B) = (1936A^2 + 13552B^2) \end{aligned} \right\} \quad (3.88)$$

Thus (3.88) represent the non-zero integer solutions to (3.36).

III.4 On Non-homogeneous Ternary Cubic Equation

$$x^3 + y^3 + x + y = 2z(2z^2 - \alpha^2 + 1)$$

The non-homogeneous ternary cubic equation to be solved is

$$x^3 + y^3 + x + y = 2z(2z^2 - \alpha^2 + 1) \quad (3.89)$$

Introduction of the linear transformations

$$x = u + v, y = u - v, z = u, u \neq v \neq 0 \quad (3.90)$$

in (3.89) leads to

$$u^2 = 3v^2 + \alpha^2 \quad (3.91)$$

which is the well-known positive Pell equation. The general solution (v_{n+1}, u_{n+1}) to

(3.91) is given by

$$\begin{aligned} v_{n+1} &= \frac{\alpha}{\sqrt{3}} g_n + \frac{\alpha}{2} f_n, \\ u_{n+1} &= \alpha f_n + \frac{\sqrt{3}}{2} \alpha g_n, n = -1, 0, 1, \dots \end{aligned}$$

where

$$f_n = (2 + \sqrt{3})^{n+1} + (2 - \sqrt{3})^{n+1}, g_n = (2 + \sqrt{3})^{n+1} - (2 - \sqrt{3})^{n+1},$$

In view of (3.90), the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ to (3.89) is given by

$$\left. \begin{aligned} x_{n+1} &= \frac{3}{2} \alpha f_n + \frac{5\sqrt{3}}{6} \alpha g_n, \\ y_{n+1} &= \frac{1}{2} \alpha f_n + \frac{\sqrt{3}}{6} \alpha g_n, \\ z_{n+1} &= \alpha f_n + \frac{\sqrt{3}}{2} \alpha g_n, \end{aligned} \right\} n = -1, 0, 1, \dots \quad (3.92)$$

A few numerical examples are presented in Table 3.3 below:

Table 3.3: Numerical examples

n	x_{n+1}	y_{n+1}	z_{n+1}
-1	3α	α	2α
0	11α	3α	7α
1	41α	11α	26α
2	153α	41α	97α
3	571α	153α	362α

From the above Table 3.3, the following results are observed:

- (i) The values of x and y are both even or odd according as α is even or odd.
- (ii) The values of z are even when α is even and alternatively even & odd when α is odd.
- (iii) $x_{n+1} = y_{n+2}$
- (iv) $x_{n+1} + x_{n+3} = 4y_{n+3}$
- (v) $z_{n+1} + z_{n+2} = 3x_{n+1}$
- (vi) $z_{n+3} + z_{n+2} = 3y_{n+3}$
- (vii) $y_{n+1} + x_{n+2} = 4y_{n+2}$
- (viii) $y_{n+3} + y_{n+2} = 2z_{n+2}$
- (ix) $z_{n+3} + 5z_{n+1} = 3(y_{n+3} + y_{n+1})$
- (x) $x_{n+2} + x_{n+1} = y_{n+3} + y_{n+2}$

Each of the following expressions is a perfect square:

- $\alpha(8z_{2n+2} - 2z_{2n+3} + 2\alpha)$
- $\alpha(10z_{2n+2} - 6x_{2n+2} + 2\alpha)$

- $\alpha(18x_{2n+2} - 2z_{2n+4} + 2\alpha)$
- $\alpha(10z_{2n+2} - 6y_{2n+3} + 2\alpha)$
- $\alpha(5y_{2n+2} - x_{2n+2} + 2\alpha)$

Each of the following expressions is a cubical integer:

- $\alpha^2 [5y_{3n+3} - x_{3n+3} + 3(5y_{n+1} - x_{n+1})]$
- $\alpha^2 [10z_{3n+3} - 6y_{3n+4} + 3(10z_{n+1} - 6y_{n+2})]$
- $\alpha^2 [18x_{3n+3} - 2z_{3n+5} + 3(18x_{n+1} - 2z_{n+3})]$
- $\alpha^2 [10z_{3n+3} - 6x_{3n+3} + 3(10z_{n+1} - 6x_{n+1})]$
- $\alpha^2 [8z_{3n+3} - 2z_{3n+4} + 3(8z_{n+1} - 2z_{n+2})]$

Employing the linear combinations between the solutions of (3.89), one obtains integer solutions to special hyperbolas and parabolas :

Illustration 1

The pairs of integers

$$(X, Y) = (4z_{n+2} - 14z_{n+1}, 8z_{n+1} - 2z_{n+2}), (12x_{n+1} - 18z_{n+1}, 10z_{n+1} - 6x_{n+1}), \\ (12y_{n+2} - 18z_{n+1}, 10z_{n+1} - 6y_{n+2}), (3x_{n+1} - 9y_{n+1}, 5y_{n+1} - x_{n+1})$$

satisfy the hyperbola $3Y^2 - X^2 = 12\alpha^2$ correspondingly.

Illustration 2

The pairs of integers

$$(X, Y) = (4z_{n+2} - 14z_{n+1}, 8z_{2n+2} - 2z_{2n+3} + 2\alpha), (12x_{n+1} - 18z_{n+1}, 10z_{2n+2} - 6x_{2n+2} + 2\alpha), \\ (12y_{n+2} - 18z_{n+1}, 10z_{2n+2} - 6y_{2n+3} + 2\alpha), (3x_{n+1} - 9y_{n+1}, 5y_{2n+2} - x_{2n+2} + 2\alpha)$$

satisfy the hyperbola $3\alpha Y - X^2 = 12\alpha^2$ correspondingly.

Generation of Solutions

The process of obtaining a formula for generating sequence of integer solutions based on the given solution is presented below:

Let (u_0, v_0) be any given solution to (3.91).

Let (u_1, v_1) given by

$$u_1 = 2h - u_0, \quad v_1 = h + v_0 \quad (3.93)$$

be the 2nd solution to (3.91). Using (3.93) in (3.91) and simplifying, one obtains

$$h = 4u_0 + 6v_0$$

In view of (3.93), the values of u_1 and v_1 are written in the matrix form as

$$(u_1, v_1)^t = M (u_0, v_0)^t$$

where

$$M = \begin{pmatrix} 7 & 12 \\ 4 & 7 \end{pmatrix} \text{ and } t \text{ is the transpose.}$$

The repetition of the above process leads to the n^{th} solutions u_n, v_n given by

$$(u_n, v_n)^t = M^n (u_0, v_0)^t \quad (3.94)$$

Now, if p, q are the distinct eigen values of M , then

$$p = 7 + 4\sqrt{3}, \quad q = 7 - 4\sqrt{3}$$

We know that

$$M^n = \frac{p^n}{(p-q)}(M - qI) + \frac{q^n}{(q-p)}(M - pI), \quad I = 2 \times 2 \text{ Identity matrix}$$

and in view of (3.94), one obtains the values of u_n, v_n . Employing (3.90), the values of x_n, y_n, z_n satisfying (3.89) are given by

$$\left. \begin{aligned} x_n &= \frac{1}{4\sqrt{3}} \left[(2\sqrt{3}(\alpha^n + \beta^n) + 4(\alpha^n - \beta^n)) x_0 - 2(\alpha^n - \beta^n) y_0 \right], \\ y_n &= \frac{1}{4\sqrt{3}} \left[(2\sqrt{3}(\alpha^n + \beta^n) - 4(\alpha^n - \beta^n)) y_0 + 2(\alpha^n - \beta^n) x_0 \right], \\ z_n &= \frac{1}{4} \left[(\alpha^n + \beta^n + \sqrt{3}(\alpha^n - \beta^n)) x_0 + (\alpha^n + \beta^n - \sqrt{3}(\alpha^n - \beta^n)) y_0 \right] \end{aligned} \right\} \quad (3.95)$$

In the above system (3.95),

$$x_0 = u_0 + v_0, y_0 = u_0 - v_0$$

Chapter – IV

Bi-Quadratic Diophantine Equations

CHAPTER – IV

BI-QUADRATIC DIOPHANTINE EQUATIONS

Chapter-IV focuses on bi-quadratic diophantine equations in four sections

IV.1 to IV.4

Section IV.1

We obtain infinitely many non-zero integer solutions to the non-homogeneous ternary bi-quadratic equation $x^2 + 7xy + y^2 = z^4$.

Section IV.2

This section aims at determining non-zero distinct integer solutions to the algebraic equation of degree four with three unknowns given by

$$(a+1)(x^2 + y^2) - (2a+1)xy = [p^2 + (4a+3)q^2] z^4, a \geq 0$$

Section IV.3

This section concerns with the problem of determining non-trivial integral solutions of the non-homogeneous bi-quadratic equation with four unknowns given by $8xy + 5z^2 = 5w^4$. We obtain infinitely many non-zero integer solutions of the equation, by introducing the linear transformations $x = u + v, y = u - v, z = v$.

Section IV.4

On the Homogeneous Bi-Quadratic Equation with Four Unknowns given by $x^4 + y^4 + (x - y)z^3 = 2(k^2 + 3s^2)^2 w^2$ aims at determining non-zero distinct integer solutions.

IV.1 On Non-Homogeneous Ternary Bi-Quadratic Equation

$$x^2 + 7xy + y^2 = z^4$$

The ternary bi-quadratic diophantine equation to be solved for its non-zero distinct integral solutions is given by

$$x^2 + 7xy + y^2 = z^4 \quad (4.1)$$

Introducing the linear transformations

$$x = u + v, y = u - v, u \neq v \neq 0 \quad (4.2)$$

in (4.2), it leads to

$$9u^2 = 5v^2 + z^4 \quad (4.3)$$

We present below different methods of solving (4.3) and thus obtain different patterns of integral solutions to (4.1).

Pattern 1

It is noted that (4.3) is satisfied by

$$v = 2rs, z^2 = 5r^2 - s^2, 3u = 5r^2 + s^2 \quad (4.4)$$

$$\text{Taking } r = 3R, s = 3S \quad (4.5)$$

in (4.5), we have

$$\left. \begin{aligned} u &= 15R^2 + 3S^2 \\ v &= 18RS \end{aligned} \right\} \quad (4.6)$$

and

$$z^2 = 45R^2 - 9S^2 \quad (4.7)$$

Now, we have to find R, S and z satisfying (4.7).

For this, assume

$$R = \alpha^2 + 9\beta^2 \quad (4.8)$$

Also, 45 can be written as

$$45 = (6 + 3i)(6 - 3i) \quad (4.9)$$

Substituting (4.8) and (4.9) in (4.7) and applying the method of factorization, define

$$(z + 3iS) = (6 + 3i)(\alpha + i3\beta)^2$$

Equating the real and imaginary parts, we get

$$z = 6\alpha^2 - 54\beta^2 - 18\alpha\beta \quad (4.10)$$

$$S = \alpha^2 - 9\beta^2 + 12\alpha\beta \quad (4.11)$$

Substituting (4.8) and (4.11) in (4.6), we get

$$\left. \begin{aligned} u &= 18\alpha^4 + 1458\beta^4 + 648\alpha^2\beta^2 + 72\alpha^3\beta - 648\alpha\beta^3 \\ v &= 18\alpha^4 - 1458\beta^4 + 216\alpha^3\beta + 1944\alpha\beta^3 \end{aligned} \right\} \quad (4.12)$$

Employing (4.12) in (4.2), we have

$$\left. \begin{aligned} x &= 36\alpha^4 + 648\alpha^2\beta^2 + 288\alpha^3\beta + 1296\alpha\beta^3 \\ y &= 2916\beta^4 + 648\alpha^2\beta^2 - 144\alpha^3\beta - 2592\alpha\beta^3 \end{aligned} \right\} \quad (4.13)$$

Thus, (4.10) and (4.13) represent non-zero distinct integer solutions to (4.1).

Note

It is worth to note that, in addition to (4.9), one may write 45 as

$$45 = (3 + 6i)(3 - 6i)$$

Following the procedure as presented above, the corresponding non-zero distinct integer solutions to (4.1) are given by

$$\begin{aligned} x &= 63\alpha^4 - 729\beta^4 + 162\alpha^2\beta^2 + 180\alpha^3\beta + 324\alpha\beta^3 \\ y &= -94\alpha^4 + 5103\beta^4 + 162\alpha^2\beta^2 - 36\alpha^3\beta - 1620\alpha\beta^3 \\ z &= 3\alpha^2 - 27\beta^2 - 36\alpha\beta \end{aligned}$$

Pattern 2

Introduction of the linear transformations

$$u = X + 5T, v = X + 9T, z = 2w \quad (4.14)$$

in (4.3) leads to

$$X^2 - 4w^4 = 45T^2 \quad (4.15)$$

which may be expressed as the system of double equations as presented in Table 4.1 below:

Table 4.1: System of double equations

System	1	2	3	4	5
$X + 2w^2$	$15T$	$9T$	$45T$	$5T^2$	T^2
$X - 2w^2$	$3T$	$5T$	T	9	45

Solving each of the above systems, the values of X, T and w are obtained. In view of (4.2) and (4.14), the corresponding values of x, y and z for each of the systems in Table 4.1 are found. Note that the values of x, y and z thus obtained satisfy (4.1). For the sake of simplicity and brevity, the values of x, y and z satisfying (4.1), that are obtained through the system of equations in Table 4.1, are exhibited in Table 4.2 as follows:

Table 4.2: Solutions

System	Solutions
1	$x = 96k^2, y = -12k^2, z = 6k$
2	$x = 28k^2, y = -4k^2, z = 2k$
3	$x = 660k^2, y = -44k^2, z = 22k$
4	$x_{n+1} = 20k_{n+1}^2 + 48k_{n+1} + 28$ $y_{n+1} = -(8k_{n+1} + 4)$ $z_{n+1} = 3f_n + \frac{15}{\sqrt{20}}g_n$

System	Solutions
	where, $k_{n+1} = \frac{1}{10} \left[-5 \pm \left\{ \frac{15}{2} f_n + \frac{30}{\sqrt{20}} g_n \right\} \right], n = -1, 0, 1, \dots$ $f_n = (9 + 2\sqrt{20})^{n+1} + (9 - 2\sqrt{20})^{n+1}$ $g_n = (9 + 2\sqrt{20})^{n+1} - (9 - 2\sqrt{20})^{n+1}$
5	Set 1: $x = 896, y = -92, z = 22$ Set 2: $x = 252, y = -36, z = 6$ Set 3: $x = 192, y = -28, z = 2$

IV.2 A Search for Integer Solutions to Ternary Bi-Quadratic Equation

$$(a+1)(x^2+y^2)-(2a+1)xy = [p^2+(4a+3)q^2]z^4$$

The fourth degree equation with three unknowns to be solved is

$$(a+1)(x^2+y^2)-(2a+1)xy = [p^2+(4a+3)q^2]z^4 \quad (4.16)$$

Different sets of integer solutions to (4.16) are illustrated below:

Set 1:

The choice

$$x=u+v, y=u-v, u \neq v \neq 0 \quad (4.17)$$

in (4.16) leads to

$$u^2+(4a+3)v^2 = [p^2+(4a+3)q^2]z^4 \quad (4.18)$$

Take

$$z = \alpha^2 + (4a+3)\beta^2 \quad (4.19)$$

Substituting (4.19) in (4.18) and factorizing, the resulting equation is written as the system of double equations

$$(u+i\sqrt{4a+3}v) = (p+i\sqrt{4a+3}q)(\alpha+i\sqrt{4a+3}\beta)^4 \quad (4.20)$$

$$(u-i\sqrt{4a+3}v) = (p-i\sqrt{4a+3}q)(\alpha-i\sqrt{4a+3}\beta)^4 \quad (4.21)$$

On equating the rational and irrational parts either in (4.20) or (4.21), we have

$$\left. \begin{aligned} u &= p\alpha^4 - 6p(4a+3)\alpha^2\beta^2 + p(4a+3)^2\beta^4 - 4(4a+3)\alpha^3\beta q \\ &\quad + 4(4a+3)^2\alpha\beta^3q \\ v &= 4p\alpha^3\beta - 4p(4a+3)\alpha\beta^3 + \alpha^4q - 6(4a+3)\alpha^2\beta^2q \\ &\quad + (4a+3)^2q\beta^4 \end{aligned} \right\} \quad (4.22)$$

From (4.22) and (4.17), we get

$$\left. \begin{aligned} x &= \left\{ \begin{aligned} &\alpha^4 (p+q) - 6(4a+3)(p+q)\alpha^2\beta^2 + (4a+3)^2 (p+q)\beta^4 \\ &+ 4[p - (4a+3)q]\alpha^3\beta + 4[(4a+3)^2 q - (4a+3)p]\alpha\beta^3 \end{aligned} \right\} \\ y &= \left\{ \begin{aligned} &(p-q)\alpha^4 + 6(4a+3)(q-p)\alpha^2\beta^2 + (4a+3)^2 (p-q)\beta^4 \\ &- 4[p + (4a+3)q]\alpha^3\beta + 4[(4a+3)^2 q + (4a+3)p]\alpha\beta^3 \end{aligned} \right\} \end{aligned} \right\} \quad (4.23)$$

Thus, (4.19) and (4.23) represents the integer solutions to (4.16).

Set 2

Observe that (4.18) is written in the form of ratio as

$$\frac{u + pz^2}{qz^2 + v} = \frac{(4a+3)(qz^2 - v)}{u - pz^2} = \frac{\alpha}{\beta}, \beta \neq 0$$

which is equivalent to the system of double equations

$$\begin{aligned} \beta u - \alpha v + (p\beta - \alpha q)z^2 &= 0 \\ u\alpha + (4a+3)\beta v - [p\alpha + (4a+3)q\beta]z^2 &= 0 \end{aligned}$$

Applying the method of cross multiplication we have

$$\left. \begin{aligned} u &= p\alpha^2 - (4a+3)p\beta^2 + 2(4a+3)\alpha\beta q \\ v &= -\alpha^2 q + (4a+3)q\beta^2 + 2\alpha\beta p \end{aligned} \right\} \quad (4.24)$$

$$z^2 = \alpha^2 + (4a+3)\beta^2 \quad (4.25)$$

Note that (4.25) is satisfied by

$$\beta = 2mn, \alpha = (4a+3)m^2 - n^2 \quad (4.26)$$

$$z = (4a+3)m^2 + n^2 \quad (4.27)$$

Substituting the values of α and β from (4.26) in (4.24) we get,

$$\left. \begin{aligned} u &= p \left[(4a+3)^2 m^4 + n^4 - 2(4a+3)m^2 n^2 \right] - 4(4a+3)pm^2 n^2 \\ &\quad + 4(4a+3)[(4a+3)m^3 n - mn^3]q \\ v &= -q \left[(4a+3)^2 m^4 + n^4 - 2(4a+3)m^2 n^2 \right] + 4(4a+3)qm^2 n^2 \\ &\quad + 4[(4a+3)m^3 n - mn^3]p \end{aligned} \right\} \quad (4.28)$$

Substitution of (4.28) in (4.17) gives

$$\left. \begin{aligned} x &= (p-q) \left[(4a+3)^2 m^4 + n^4 - 2(4a+3)m^2 n^2 \right] + 4(4a+3)(q-p)m^2 n^2 \\ &\quad + 4 \left[(4a+3)m^3 n - mn^3 \right] \left[P + (4a+3q) \right] \\ y &= (p+q) \left[(4a+3)^2 m^4 + n^4 - 2(4a+3)m^2 n^2 \right] - 4(4a+3)(p+q)m^2 n^2 \\ &\quad + 4 \left[(4a+3)m^3 n - mn^3 \right] \left[(4a+3)q - p \right] \end{aligned} \right\} \quad (4.29)$$

Thus, (4.27) and (4.29) represent the integer solutions to (4.16).

Note 1

Also, (4.18) is written in the form of ratio as

$$\frac{u + pz^2}{(4a+3)(qz^2 + v)} = \frac{qz^2 - v}{u - pz^2} = \frac{\alpha}{\beta}, \beta \neq 0$$

In this case, the corresponding integer solutions to (4.18) are given by

$$\begin{aligned} x &= 6(4a+3)(p-q)m^2 n^2 + (4a+3)^2 (q-p)m^4 + (q-p)n^4 + 4(4a+3) \left[p + (4a+3)q \right] m^3 n \\ &\quad - 4 \left[p + (4a+3)q \right] mn^3 \\ y &= 6(4a+3)(p+q)m^2 n^2 - (4a+3)^2 (p+q)m^4 - (p+q)n^4 + 4(4a+3) \left[(4a+3)q - p \right] m^3 n \\ &\quad + 4 \left[p - (4a+3)q \right] mn^3 \\ z &= \beta^2 + (4a+3)\alpha^2 \end{aligned}$$

Set 3

Write (4.18) as

$$u^2 + (4a+3)v^2 = \left[p^2 + (4a+3)q^2 \right] z^4 \quad (4.30)$$

Assume

$$1 = \frac{\left[(2a+1) + i\sqrt{4a+3} \right] \left[(2a+1) - i\sqrt{4a+3} \right]}{(2a+2)^2} \quad (4.31)$$

Substituting (4.19) and (4.31) in (4.30) and employing the method of factorization,

define

$$(u+i\sqrt{4a+3}v)(u-i\sqrt{4a+3}v) = (p+i\sqrt{4a+3}q)(p-i\sqrt{4a+3}q)(\alpha+i\sqrt{4a+3}\beta)^4$$

$$(\alpha+i\sqrt{4a+3}\beta)^4 * \frac{[(2a+1)+i\sqrt{4a+3}][(2a+1)-i\sqrt{4a+3}]}{(2a+2)^2}$$

Equating the positive and negative terms in the above equation, we get

$$(u+i\sqrt{4a+3}v) = \frac{1}{(2a+2)} \left\{ \frac{(p+i\sqrt{4a+3}q)(\alpha+i\sqrt{4a+3}\beta)^4}{[(2a+1)+i\sqrt{4a+3}]} \right\} \quad (4.32)$$

$$(u-i\sqrt{4a+3}v) = \frac{1}{(2a+2)} \left\{ \frac{(p-i\sqrt{4a+3}q)(\alpha-i\sqrt{4a+3}\beta)^4}{[(2a+1)-i\sqrt{4a+3}]} \right\} \quad (4.33)$$

Equating the real and imaginary parts in (4.32) or (4.33), we have

$$u = \frac{1}{(2a+2)} \left\{ \begin{aligned} &[(2a+1)p - (4a+3)q]\alpha^4 - 6(4a+3)[(2a+1)p - (4a+3)q]\alpha^2\beta^2 \\ &+ (4a+3)^2[(2a+1)p - (4a+3)q]\beta^4 - 4(4a+3)[(2a+1)q + p]\alpha^3\beta \\ &+ 4(4a+3)^2[(2a+1)q + p]\alpha\beta^3 \end{aligned} \right\} \quad (4.34)$$

$$v = \frac{1}{(2a+2)} \left\{ \begin{aligned} &[(2a+1)q + p]\alpha^4 - 6(4a+3)[(2a+1)q + p]\alpha^2\beta^2 \\ &+ (4a+3)^2[(2a+1)q + p]\beta^4 + 4[(2a+1)p - (4a+3)q]\alpha^3\beta \\ &- 4(4a+3)[(2a+1)p - (4a+3)q]\alpha\beta^3 \end{aligned} \right\}$$

Substituting (4.34) in (4.17) we get,

$$x = \left\{ \begin{aligned} &(p-q)\alpha^4 + (4a+3)^2(p-q)\beta^4 - 6(4a+3)(p-q)\alpha^2\beta^2 \\ &+ 4[-p - (4a+3)q]\alpha^3\beta + 4(4a+3)[p + (4a+3)q]\alpha\beta^3 \end{aligned} \right\}$$

$$y = \frac{1}{(2a+2)} \left\{ \begin{aligned} &[2ap - (6a+4)q]\alpha^4 + 6(4a+3)[(6a+4)q - 2ap]\alpha^2\beta^2 \\ &+ (4a+3)^2[2ap - (6a+4)q]\beta^4 - 4[(6a+4)p + 2a(4a+3)q]\alpha^3\beta \\ &+ 4(4a+3)[(4a+3)2aq + (6a+4)p]\alpha\beta^3 \end{aligned} \right\} \quad (4.35)$$

As our interest is in finding integer solutions, it is seen that replacing α by $(2a+2)M$ and β by $(2a+2)N$ in (4.35) and (4.19), the corresponding integer solutions to (4.16) are obtained and they are given below:

$$\begin{aligned}
x &= (2a+2)^4 \left\{ \begin{aligned} &(p-q)M^4 + (4a+3)^2(p-q)N^4 - 6(4a+3)(p-q)M^2N^2 \\ &+ 4[-p-(4a+3)q]M^3N + 4(4a+3)[p+(4a+3)q]MN^3 \end{aligned} \right\} \\
y &= (2a+2)^3 \left\{ \begin{aligned} &[2ap-(6a+4)q]M^4 + 6(4a+3)[(6a+4)q-2ap]M^2N^2 \\ &+ (4a+3)^2[2ap-(6a+4)q]N^4 - 4[(6a+4)p+2a(4a+3)q]M^3N \\ &+ 4(4a+3)[(4a+3)2aq+(6a+4)p]MN^3 \end{aligned} \right\} \\
z &= (2a+2)^2 [M^2 + (4a+3)N^2]
\end{aligned} \quad (4.36)$$

Note 2

It is to be noted that, in addition to (4.31), 1 may also be represented as below:

$$\begin{aligned}
\text{(i)} \quad 1 &= \frac{[(3-2a)+i3\sqrt{4a+3}][(3-2a)-i3\sqrt{4a+3}]}{(2a+6)^2} \\
\text{(ii)} \quad 1 &= \frac{[(4a-1)+i4\sqrt{4a+3}][(4a-1)-i4\sqrt{4a+3}]}{(4a+7)^2} \\
\text{(iii)} \quad 1 &= \frac{[(9-6a)+i9\sqrt{4a+3}][(9-6a)-i9\sqrt{4a+3}]}{(6a+18)^2} \\
\text{(iv)} \quad 1 &= \frac{[(16a-4)+i16\sqrt{4a+3}][(16a-4)-i16\sqrt{4a+3}]}{(16a+28)^2}
\end{aligned}$$

It is worth mentioning here that, by giving various integer values to a , p and q , one may obtain integer solutions to the corresponding biquadratic equation.

For illustration, the choices

$$a = 1, p = 8, q = 1 \quad (4.37)$$

in (4.16) give

$$2(x^2 + y^2) - 3xy = 71z^4 \quad (4.38)$$

Substituting (4.37) in (4.19), (4.23); (4.27), (4.29) and (4.36) the corresponding three sets of integer solutions to (4.38) are as follows:

Set 1

$$\begin{aligned}x &= 9\alpha^4 - 378\alpha^2\beta^2 + 441\beta^4 + 4\alpha^3\beta - 28\alpha\beta^3 \\y &= 7\alpha^4 - 294\alpha^2\beta^2 + 343\beta^4 - 60\alpha^3\beta + 420\alpha\beta^3 \\z &= \alpha^2 + 7\beta^2\end{aligned}$$

Set 2

$$\begin{aligned}x &= 343r^4 - 294r^2s^2 + 7s^4 + 420r^3s - 60rs^3 \\y &= 441r^4 - 378r^2s^2 + 9s^4 - 28r^3s + 4rs^3 \\z &= 7r^2 + s^2\end{aligned}$$

Set 3

$$\begin{aligned}x &= 1792M^4 - 75264M^2N^2 + 87808N^4 - 15360M^3N + 107520MN^3 \\y &= 384M^4 - 16128M^2N^2 + 18816N^4 - 24064M^3N + 168448MN^3 \\z &= 16M^2 + 112N^2\end{aligned}$$

Further, it is observed that, by choosing suitably the values of a , p and q in (4.16), the solutions presented in [4.17, 4.21-4.24] are correspondingly obtained.

IV.3 On the Non-Homogeneous Bi-Quadratic Equation with Four Unknowns

$$8xy + 5z^2 = 5w^4$$

Consider the bi-quadratic equations with four unknowns

$$8xy + 5z^2 = 5w^4 \quad (4.39)$$

The process of obtaining different choices of non-zero distinct integer solutions to (4.39) is illustrated below:

Choice 1

Introducing the linear transformations

$$x = u + v, y = u - v, z = v \quad (4.40)$$

in (4.39), it is written as

$$8u^2 - 3v^2 = 5w^4 \quad (4.41)$$

Again, employing the linear transformations

$$u = X + 3T, v = X + 8T, \quad (4.42)$$

(4.41) is written as

$$X^2 - 24T^2 = w^4 \quad (4.43)$$

Express (4.43) as the system of double equations as shown in Table 4.3 below:

Table 4.3: System of Double Equations

System	I	II	III	IV
$X + w^2$	$6T^2$	$8T$	$6T$	$12T$
$X - w^2$	4	$3T$	$4T$	$2T$

Consider system I

Solving for X and w , we get

$$X = 3T^2 + 2 \quad (4.44)$$

$$w^2 = 3T^2 - 2 \quad (4.45)$$

Now, observe that (4.45) is a negative Pellian equation whose solutions are obtained as illustrated below:

The smallest positive integer solution to (4.45) is

$$T_0 = 1, w_0 = 1 \quad (4.46)$$

To obtain the other solutions to (4.45), consider the corresponding Pellian equation

$$w^2 = 3T^2 + 1 \quad (4.47)$$

The smallest positive integer solution to (4.47) is

$$\tilde{T}_0 = 1, \tilde{w}_0 = 2$$

whose, general solution is given by

$$\tilde{w}_n = \frac{1}{2} f_n, \quad \tilde{T}_n = \frac{1}{2\sqrt{3}} g_n$$

where, $f_n = (2 + \sqrt{3})^{n+1} + (2 - \sqrt{3})^{n+1}$

$$g_n = (2 + \sqrt{3})^{n+1} - (2 - \sqrt{3})^{n+1}, \quad n = 0, 1, 2, \dots$$

Applying Brahmagupta lemma between the solutions (T_0, w_0) and $(\tilde{T}_n, \tilde{w}_n)$, the other

integer solutions of (4.45) are given by

$$T_{n+1} = \frac{1}{2} f_n + \frac{1}{2\sqrt{3}} g_n \quad (4.48)$$

$$w_{n+1} = \frac{1}{2} f_n + \frac{\sqrt{3}}{2} g_n \quad (4.49)$$

Using (4.48) in (4.44), we have

$$X_{n+1} = 3 \left(\frac{1}{2} f_n + \frac{1}{2\sqrt{3}} g_n \right)^2 + 2$$

In view of (4.42) and (4.40), the values of x, y and z are represented by

$$\left. \begin{aligned} x_{n+1} &= 6 \left(\frac{1}{2} f_n + \frac{1}{2\sqrt{3}} g_n \right)^2 + 11 \left(\frac{1}{2} f_n + \frac{1}{2\sqrt{3}} g_n \right) + 4 \\ y_{n+1} &= -5 \left(\frac{1}{2} f_n + \frac{1}{2\sqrt{3}} g_n \right) \\ z_{n+1} &= 3 \left(\frac{1}{2} f_n + \frac{1}{2\sqrt{3}} g_n \right)^2 + 8 \left(\frac{1}{2} f_n + \frac{1}{2\sqrt{3}} g_n \right) + 2 \end{aligned} \right\} \quad (4.50)$$

Thus, (4.50) and (4.49) represent the integer solutions to (4.39).

Consider system II

Solving for X and w , we get

$$X = \frac{11T}{2}, \quad w^2 = \frac{5T}{2}$$

Taking

$$T = 10k^2$$

we have

$$X = 55k^2$$

$$w = 5k \quad (4.51)$$

In view of (4.42) and (4.40), the values of x, y and z are represented by

$$x = 220k^2, \quad y = -50k^2, \quad z = 135k^2 \quad (4.52)$$

Thus, (4.52) and (4.51) represent the integer solutions to (4.39).

Consider system III

Solving for X and w , we get

$$X = 5T, \quad w^2 = T$$

Taking

$$T = k^2$$

we have

$$\begin{aligned} X &= 5k^2 \\ w &= k \end{aligned} \tag{4.53}$$

In view of (4.42) and (4.40), the values of x , y and z are represented by

$$x = 21k^2, y = -5k^2, z = 13k^2 \tag{4.54}$$

Thus, (4.54) and (4.53) represent the integer solutions to (4.39).

Consider system IV

Solving for X and w , we get

$$X = 7T, w^2 = 5T$$

Taking

$$T = 5k^2$$

we have

$$\begin{aligned} X &= 35k^2 \\ w &= 5k \end{aligned} \tag{4.55}$$

In view of (4.42) and (4.40), the values of x , y and z are represented by

$$x = 125k^2, y = -25k^2, z = 75k^2 \tag{4.56}$$

Thus, (4.55) and (4.56) represent the integer solutions to (4.39).

Choice 2

Introducing the linear transformation

$$z = x - y \tag{4.57}$$

in (4.39), it becomes

$$5x^2 - 2xy + (5y^2 - 5w^4) = 0 \tag{4.58}$$

Considering (4.58) as a quadratic expression in x and solving for x , one obtains

$$x = \frac{1}{5}(y \pm \beta) \quad (4.59)$$

where

$$\beta^2 = 25w^4 - 24y^2 \quad (4.60)$$

Now, (4.60) is written in the form of ratio as

$$\frac{\beta + w^2}{w^2 + y} = \frac{24(w^2 - y)}{\beta - w^2} = \frac{M}{N}, \quad N > 0$$

which is equivalent to the system of double equations

$$N\beta + (N - M)w^2 - My = 0 \quad (4.61)$$

$$-M\beta + (24N + M)w^2 - 24Ny = 0 \quad (4.62)$$

Applying the method of cross- multiplication between (4.61) and (4.62) we get

$$\beta = M^2 - 24N^2 + 48MN \quad (4.63)$$

$$w^2 = M^2 + 24N^2 \quad (4.64)$$

$$y = -M^2 + 24N^2 + 2MN \quad (4.65)$$

Substituting (4.63), (4.65) in (4.59) and taking the positive sign, we get

$$x = 10MN \quad (4.66)$$

and from (4.57),

$$z = M^2 - 24N^2 + 8MN \quad (4.67)$$

Now, observe that (4.64) is satisfied by

$$M = 24s^2 - t^2, \quad N = 2st \quad (4.68)$$

$$w = 24s^2 + t^2 \text{ where } s, t \text{ are integers.} \quad (4.69)$$

Substituting (4.68) in (4.65), (4.66) & (4.67), we get

$$\left. \begin{aligned} x &= 20st(24s^2 - t^2) \\ y &= -(24s^2 - t^2)^2 + 96s^2t^2 + 4st(24s^2 - t^2) \\ z &= (24s^2 - t^2)^2 - 96s^2t^2 + 16st(24s^2 - t^2) \end{aligned} \right\} \quad (4.70)$$

Thus, (4.69) and (4.70) give integer solutions of (4.39).

Note

It is to be noted that (4.60) is also expressed in the form of ratio's as follows:

- i. $\frac{\beta + w^2}{24(w^2 + y)} = \frac{w^2 - y}{\beta - w^2} = \frac{M}{N}, N > 0$
- ii. $\frac{\beta + w^2}{4(w^2 + y)} = \frac{6(w^2 - y)}{\beta - w^2} = \frac{M}{N}, N > 0$
- iii. $\frac{\beta + w^2}{6(w^2 + y)} = \frac{4(w^2 - y)}{\beta - w^2} = \frac{M}{N}, N > 0$
- iv. $\frac{\beta + w^2}{3(w^2 + y)} = \frac{8(w^2 - y)}{\beta - w^2} = \frac{M}{N}, N > 0$
- v. $\frac{\beta + w^2}{8(w^2 + y)} = \frac{3(w^2 - y)}{\beta - w^2} = \frac{M}{N}, N > 0$
- vi. $\frac{\beta + w^2}{2(w^2 + y)} = \frac{12(w^2 - y)}{\beta - w^2} = \frac{M}{N}, N > 0$
- vii. $\frac{\beta + w^2}{12(w^2 + y)} = \frac{2(w^2 - y)}{\beta - w^2} = \frac{M}{N}, N > 0$

Solving each of the above ratio's as presented above, one obtains different sets of integer solutions to (4.39).

However, there are other sets of solutions to (4.64) leading to other choices of solutions to (4.39) that are illustrated below:

We can write (4.64) as the system of double equations as shown in following Table 4.4:

Table 4.4: System of double equations

System	I	II	III	IV	V	VI
$w + M$	N^2	$12N^2$	$6N^2$	$8N$	$6N$	$12N$
$w - M$	24	2	4	$3N$	$4N$	$2N$

For simplicity and brevity, we present below the integer solutions to (4.39) obtained on solving each of the above system of equations.

System I

$$\begin{aligned}
 x &= 40p(p^2 - 6) \\
 y &= -4(p^2 - 6)^2 + 96p^2 + 8p(p^2 - 6) \\
 z &= 4(p^2 - 6)^2 - 96p^2 + 32p(p^2 - 6) \\
 w &= 2p^2 + 12
 \end{aligned}$$

System II

$$\begin{aligned}
 x &= 60N^3 - 10N \\
 y &= -(6N^2 - 1)^2 + 24N^2 + 2N(6N^2 - 1) \\
 z &= (6N^2 - 1)^2 - 24N^2 + 8N(6N^2 - 1) \\
 w &= 6N^2 + 1
 \end{aligned}$$

System III

$$\begin{aligned}
 x &= 10N(3N^2 - 2) \\
 y &= -(3N^2 - 2)^2 + 24N^2 + 2N(3N^2 - 2) \\
 z &= (3N^2 - 2)^2 - 24N^2 + 8N(3N^2 - 2) \\
 w &= 3N^2 + 2
 \end{aligned}$$

System IV

$$x = 100k^2, \quad y = 91k^2, \quad z = 9k^2, \quad w = 11k$$

System V

$$x = 10N^2, \quad y = 25N^2, \quad z = 15N^2, \quad w = 5N$$

System VI

$$x = 50N^2, \quad y = 9N^2, \quad z = 41N^2, \quad w = 7N$$

Choice 3

Substituting

$$x = 5 * 2^{2\alpha-1} y, \quad (\alpha > 1) \tag{4.71}$$

in (4.39),

It is written as

$$\left(2^{2\alpha-1} y\right)^2 + z^2 = w^4 \tag{4.72}$$

Note that, the above equation is similar to the well-known pythagorean equation.

Employing the most cited solutions of the Pythagorean equation, one obtains the following two sets of solution to (4.39):

Set 1

$$\begin{aligned} x &= 5 * 2^{3\alpha} uv (u^2 - v^2) \\ y &= 2uv * 2^\alpha (u^2 - v^2) \\ z &= 2^{2\alpha} \left[4u^2 v^2 - (u^2 - v^2)^2 \right] \\ w &= 2^\alpha (u^2 + v^2) \end{aligned}$$

Set 2

$$x = 5 * 2^{3\alpha} (6p^2q^2 - p^4 - q^4)$$

$$y = 2^{\alpha+1} (6p^2q^2 - p^4 - q^4)$$

$$z = 16 * 2^{2\alpha} * pq(p^2 - q^2)$$

$$w = 2^{\alpha+1} (p^2 + q^2)$$

When $\alpha = 1$ in (4.71), the corresponding solutions to (4.39) are obtained as below:

Set 3

$$x = 10uv(u^2 - v^2)$$

$$y = uv(u^2 - v^2)$$

$$z = 4u^2v^2 - (u^2 - v^2)^2$$

$$w = u^2 + v^2$$

Set 4

$$x = 10(6p^2q^2 - p^4 - q^4)$$

$$y = (6p^2q^2 - p^4 - q^4)$$

$$z = 16pq(p^2 - q^2)$$

$$w = 2(p^2 - q^2)$$

IV.4 On Non-homogeneous Bi-quadratic Equation with Four Unknowns

$$x^4 + y^4 + (x - y)z^3 = 2(k^2 + 3s^2)^2 w^2$$

The non-homogeneous bi-quadratic equation with four unknowns to be solved is

$$x^4 + y^4 + (x - y)z^3 = 2(k^2 + 3s^2)^2 w^2 \quad (4.73)$$

The substitution of the linear transformations

$$x = u + v, \quad y = u - v, \quad z = 2v, \quad u \neq v \neq 0 \quad (4.74)$$

in (4.73) leads to

$$u^2 + 3v^2 = (k^2 + 3s^2) w \quad (4.75)$$

Solving (4.75) in different ways for u, v, w and using (4.74), one obtains different sets of solutions to (4.73).

Way 1

Consider

$$w = w(p, q) = p^2 + 3q^2 \quad (4.76)$$

Using (4.76) in (4.75) and employing the method of factorization, define

$$u + i\sqrt{3}v = (k + i\sqrt{3}s)(p + i\sqrt{3}q)$$

from which, on equating the real and imaginary parts, one has

$$u = kp - 3sq, \quad v = kq + ps \quad (4.77)$$

In view of (4.74), note that

$$\left. \begin{aligned} x &= x(p, q, k, s) = k(p + q) + s(p - 3q) \\ y &= y(p, q, k, s) = k(p - q) - s(p + 3q) \\ z &= z(p, q, k, s) = 2(kq + ps) \end{aligned} \right\} \quad (4.78)$$

Thus, (4.76) and (4.78) represent the integer solutions to (4.73).

Properties

- $x(s+1, q, 3s, s) = 8t_{3,s}$
- $x(s(s+1), q, 3s, s) = 8P_s^5$
- $x(s^2-1, q, 3s, s) = 24P_{s-1}^3$
- $kz(p, 3c^2, 3a^2-b^2, 2ab) - s(x(p, 3c^2, 3a^2-b^2, 2ab) + y(p, 3c^2, 3a^2-b^2, 2ab))$ is a nasty number.
- $p(kz(p, q, k, s) - s(x(p, q, k, s) + y(p, q, k, s))) = q((k+3s)x(p, q, k, s) + (k-3s)y(p, q, k, s))$

Way: 2

Write (4.75) as

$$u^2 + 3v^2 = (k^2 + 3s^2) w^* 1 \quad (4.79)$$

Consider 1 on the R.H.S. of (4.79) as

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \quad (4.80)$$

Using (4.76) & (4.80) in (4.79) and employing the method of factorization, define

$$u + i\sqrt{3}v = \frac{1}{2}(1+i\sqrt{3})(k+i\sqrt{3}s)(p+i\sqrt{3}q) \quad (4.81)$$

On equating the real and imaginary parts, one has

$$\begin{aligned} u &= \frac{1}{2}[kp - 3sq - 3(kq + sp)] \\ v &= \frac{1}{2}[kp - 3sq + kq + sp] \end{aligned}$$

In view of (4.74), one has

$$\left. \begin{aligned} x &= kp - 3sq - kq - sp \\ y &= -2(kq + sp) \\ z &= kp - 3sq + kq + sp \end{aligned} \right\} \quad (4.82)$$

Thus, (4.76) & (4.82) represent the integer solutions to (4.73).

Properties

- ❖ $z(p, q, 3q, p) - x(p, q, 3q, p) - y(p, q, 3q, p) = 4w(p, q)$
- ❖ $z(p, q, p, -q) + x(p, q, p, -q) = 2w(p, q)$
- ❖ $3(2x(3a^2 + b^2, 2ab, 3a^2 + b^2, 2ab) - y(3a^2 + b^2, 2ab, 3a^2 + b^2, 2ab))$ is a nasty number
- ❖ $z(p, -q, 3q + 3, q + 1) + 2y(p, -q, 3q + 3, q + 1) = 24t_{3,q} = ct_{24,q} - 1 = 2(ct_{12,q} - 1)$
- ❖ $z(p, -q, (3q + 3)(2q + 1), (q + 1)(2q + 1))$
 $+ 2y(p, -q, (3q + 3)(2q + 1), (q + 1)(2q + 1)) = 72 P_q^4$
- ❖ $z(p, -q, 3(12q - 11), 12q - 11) + 2y(p, -q, 3(12q - 11), 12q - 11) = 12t_{26,q}$
- ❖ $2(z(p, -q, 3q, q) + 2y(p, -q, 3q, q))$ is a nasty number.

Note: 1

In addition to (4.80), represent 1 on the R.H.S. of (4.79) as below:

$$1 = \frac{(3r^2 - s^2 + i\sqrt{3}2rs)(3r^2 - s^2 - i\sqrt{3}2rs)}{(3r^2 + s^2)^2}$$

The repetition of the above process leads to a different set of solutions to (4.73).

Way: 3

Let

$$w = (p^2 + 3q^2)^2 \quad (4.83)$$

Using (4.83) in (4.75) and employing the method of factorization, define

$$u + i\sqrt{3}v = (k + i\sqrt{3}s)(p + i\sqrt{3}q)^2$$

On equating the real and imaginary parts, one has

$$u = k(p^2 - 3q^2) - 6spq, \quad v = s(p^2 - 3q^2) + 2kpq \quad (4.84)$$

In view of (4.74), note that

$$\left. \begin{aligned} x &= (k+s)(p^2-3q^2) + pq(2k-6s), \\ y &= (k-s)(p^2-3q^2) - pq(2k+6s), \\ z &= 2s(p^2-3q^2) + 4kpq \end{aligned} \right\} \quad (4.85)$$

Thus, (4.83) and (4.85) represent the integer solutions to (4.73).

Note: 2

Using (4.83) & (4.80) in (4.79) and following the analysis given in Way: 2, a different set of integer solutions to (4.73) are obtained.

Chapter – V

Quintic Diophantine Equation

CHAPTER – V

QUINTIC DIOPHANTINE EQUATION

The quintic non-homogeneous equation with five unknowns represented by the Diophantine equation $3(x+y)(x^3-y^3)=7(z^2-w^2)p^3$ is analyzed for its patterns of non-zero distinct integral solutions.

Integral Solutions of Non-Homogeneous Quintic Equation With Five Unknowns

$$3(x+y)(x^3-y^3)=7(z^2-w^2)p^3$$

The non-homogeneous quintic equation with five unknowns to be solved for its distinct non-zero integral solutions is

$$3(x+y)(x^3-y^3)=7(z^2-w^2)p^3 \quad (5.1)$$

METHOD 1

Introduction of the linear transformations

$$x = u + v, y = u - v, z = 3u + v, w = 3u - v \quad (5.2)$$

in (5.1) leads to

$$v^2 + 3u^2 = 7P^3 \quad (5.3)$$

Different methods of obtaining the patterns of integer solutions to (5.1) are illustrated below:

PATTERN: 1

Let

$$P = a^2 + 3b^2 \quad (5.4)$$

where a and b are non-zero integers.

Write 7 as

$$7 = (2 + i\sqrt{3})(2 - i\sqrt{3}) \quad (5.5)$$

Using (5.4), (5.5) in (5.3) and applying the method of factorization, define

$$(v + i\sqrt{3}u) = (2 + i\sqrt{3})(a + i\sqrt{3}b)^3 \quad (5.6)$$

from which we have

$$\left. \begin{aligned} v &= 2a^3 - 18ab^2 - 9a^2b + 9b^3 \\ u &= a^3 - 9ab^2 + 6a^2b - 6b^3 \end{aligned} \right\} \quad (5.7)$$

Using (5.7) and (5.2), the values of x, y, z and w are given by

$$\left. \begin{aligned} x(a, b) &= 3a^3 + 3b^3 - 27ab^2 - 3a^2b \\ y(a, b) &= -a^3 - 15b^3 + 9ab^2 + 15a^2b \\ z(a, b) &= 5a^3 - 9b^3 - 45ab^2 + 9a^2b \\ w(a, b) &= a^3 - 27b^3 - 9ab^2 + 27a^2b \end{aligned} \right\} \quad (5.8)$$

Thus (5.4) and (5.8) represent the non-zero integer solutions to (5.1).

PATTERN: 2

Write 7 as

$$7 = \frac{(5 + i\sqrt{3})(5 - i\sqrt{3})}{4} \quad (5.9)$$

Using (5.4), (5.9) in (5.3) and applying the method of factorization, define

$$(v + i\sqrt{3}u) = \frac{(5 + i\sqrt{3})}{2} (a + i\sqrt{3}b)^3 \quad (5.10)$$

from which we have

$$\left. \begin{aligned} v &= \frac{1}{2} [5a^3 + 9b^3 - 45ab^2 - 9a^2b] \\ u &= \frac{1}{2} [a^3 - 15b^3 - 9ab^2 + 15a^2b] \end{aligned} \right\} \quad (5.11)$$

Using (5.11) and (5.2), the values of x, y, z and w are given by

$$\left. \begin{aligned} x(a, b) &= 3a^3 - 3b^3 - 27ab^2 + 3a^2b \\ y(a, b) &= -2a^3 - 12b^3 + 18ab^2 + 12a^2b \\ z(a, b) &= 4a^3 - 18b^3 - 36ab^2 + 18a^2b \\ w(a, b) &= -a^3 - 27b^3 + 9ab^2 + 27a^2b \end{aligned} \right\} \quad (5.12)$$

Thus (5.4) and (5.12) represent the non-zero integer solutions to (5.1).

PATTERN: 3

Write (5.3) as

$$v^2 + 3u^2 = 7p^3 * 1 \quad (5.13)$$

Write 1 as

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \quad (5.14)$$

Using (5.4), (5.5), (5.14) in (5.13) and applying the method of factorization, define

$$(v + i\sqrt{3}u) = \frac{1}{2}(1+i\sqrt{3})(2+i\sqrt{3})(a+i\sqrt{3}b)^3 \quad (5.15)$$

from which we have

$$\left. \begin{aligned} v &= \frac{1}{2}[-a^3 + 27b^3 + 9ab^2 - 27a^2b] \\ u &= \frac{1}{2}[3a^3 + 3b^3 - 27ab^2 - 3a^2b] \end{aligned} \right\} \quad (5.16)$$

Using (5.16) in (5.2), the values of x, y, z and w are given by

$$\left. \begin{aligned} x(a, b) &= a^3 + 15b^3 - 9ab^2 - 15a^2b \\ y(a, b) &= 2a^3 - 12b^3 - 18ab^2 + 12a^2b \\ z(a, b) &= 4a^3 + 18b^3 - 36ab^2 - 18a^2b \\ w(a, b) &= 5a^3 - 9b^3 - 45ab^2 + 9a^2b \end{aligned} \right\} \quad (5.17)$$

Thus (5.4) and (5.17) represents the non-zero integer solutions to (5.1).

PATTERN: 4

Write 1 as

$$1 = \frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49} \quad (5.18)$$

Using (5.4), (5.5) and (5.18) in (5.13) and applying the method of factorization, define

$$(v+i\sqrt{3}u) = \frac{1}{7}(2+i\sqrt{3})(1+i4\sqrt{3})(a+i\sqrt{3}b)^3 \quad (5.19)$$

from which we have

$$\left. \begin{aligned} v &= \frac{1}{7}[-10a^3 + 81b^3 + 90ab^2 - 81a^2b] \\ u &= \frac{1}{7}[9a^3 + 30b^3 - 81ab^2 - 30a^2b] \end{aligned} \right\} \quad (5.20)$$

Since our interest is on finding integer solutions, replacing a by $7A$, b by $7B$ in (5.4)

and (5.20) & using (5.2), the corresponding integer solutions to (5.1) are given by

$$\left. \begin{aligned} x(A, B) &= 7^2(-A^3 + 111B^3 + 9AB^2 - 111A^2B) \\ y(A, B) &= 7^2(19A^3 - 51B^3 - 171AB^2 + 51A^2B) \\ z(A, B) &= 7^2(17A^3 + 171B^3 - 153AB^2 - 171A^2B) \\ w(A, B) &= 7^2(37A^3 + 9B^3 - 333AB^2 - 9A^2B) \\ p(A, B) &= 7^2(A^2 + 3B^2) \end{aligned} \right\} \quad (5.21)$$

Thus (5.21) represents the non-zero integer solutions to (5.1).

PATTERN: 5

Write 1 as

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \quad (5.22)$$

Using (5.4), (5.9) and (5.22) in (5.13) and applying the method of factorization, define

$$(v+i\sqrt{3}u) = \frac{1}{4}(5+i\sqrt{3})(1+i\sqrt{3})(a+i\sqrt{3}b)^3 \quad (5.23)$$

from which we have

$$\left. \begin{aligned} v &= \frac{1}{2} [a^3 + 27b^3 - 9ab^2 - 27a^2b] \\ u &= \frac{1}{2} [3a^3 - 3b^3 - 27ab^2 + 3a^2b] \end{aligned} \right\} \quad (5.24)$$

Using (5.24) in (5.2), the values of x, y, z and w are given by

$$\left. \begin{aligned} x(a, b) &= 2a^3 + 12b^3 - 18ab^2 - 12a^2b \\ y(a, b) &= a^3 - 15b^3 - 9ab^2 + 15a^2b \\ z(a, b) &= 5a^3 + 9b^3 - 45ab^2 - 9a^2b \\ w(a, b) &= 4a^3 - 18b^3 - 36ab^2 + 18a^2b \end{aligned} \right\} \quad (5.25)$$

Thus (5.4) and (5.25) represents the non-zero integer solutions to (5.1).

METHOD 2

Introduction of the linear transformations

$$x = u + v, y = u - v, z = u + 3v, w = u - 3v \quad (5.26)$$

in (5.1) leads to (5.3).

Following the same process from Pattern 1 to Pattern 5 and using the transformation

(5.26), the sets of solutions to (5.1) are given below in Table 5.1:

Table 5.1: Solutions

Patterns	Solutions
1	$\begin{aligned} x(a, b) &= 3a^3 + 3b^3 - 27ab^2 - 3a^2b \\ y(a, b) &= -a^3 - 15b^3 + 9ab^2 + 15a^2b \\ z(a, b) &= 7a^3 + 21b^3 - 63ab^2 - 21a^2b \\ w(a, b) &= -5a^3 - 33b^3 + 45ab^2 + 33a^2b \\ P(a, b) &= a^2 + 3b^2 \end{aligned}$

Patterns	Solutions
2	$x(a, b) = 3a^3 - 3b^3 - 27ab^2 + 3a^2b$ $y(a, b) = -2a^3 - 12b^3 + 18ab^2 + 12a^2b$ $z(a, b) = 8a^3 + 6b^3 - 72ab^2 - 6a^2b$ $w(a, b) = -7a^3 - 21b^3 + 63ab^2 + 21a^2b$ $P(a, b) = a^2 + 3b^2$
3	$x(a, b) = a^3 + 15b^3 - 9ab^2 - 15a^2b$ $y(a, b) = 2a^3 - 12b^3 - 18ab^2 + 12a^2b$ $z(a, b) = 42b^3 - 42a^2b$ $w(a, b) = 3a^3 - 39b^3 - 27ab^2 + 39a^2b$ $P(a, b) = a^2 + 3b^2$
4	$x(A, B) = 7^2(-A^3 + 111B^3 + 9AB^2 - 111A^2B)$ $y(A, B) = 7^2(19A^3 - 51B^3 - 171AB^2 + 51A^2B)$ $z(A, B) = 7^2(-21A^3 + 273B^3 + 189AB^2 - 273A^2B)$ $w(A, B) = 7^2(39A^3 - 213B^3 - 351AB^2 + 213A^2B)$ $p(A, B) = 7^2(A^2 + 3B^2)$
5	$x(a, b) = 2a^3 + 12b^3 - 18ab^2 - 12a^2b$ $y(a, b) = a^3 - 15b^3 - 9ab^2 + 15a^2b$ $z(a, b) = 3a^3 + 39b^3 - 27ab^2 - 39a^2b$ $w(a, b) = 42a^2b - 42b^3$ $P(a, b) = a^2 + 3b^2$

METHOD 3

Introduction of the linear transformations

$$x = u + v, y = u - v, z = 3uv + 1, w = 3uv - 1 \quad (5.27)$$

in (5.1) leads to (5.3).

Following the same process from Pattern 1 to Pattern 5 and using the transformation (5.27), the sets of solutions to (5.1) are given below in Table 5.2:

Table 5.2: Solutions

Patterns	Solutions
1	$x(a, b) = 3a^3 + 3b^3 - 27ab^2 - 3a^2b$ $y(a, b) = -a^3 - 15b^3 + 9ab^2 + 15a^2b$ $z(a, b) = 6f^2(a, b) + 9f(a, b)g(a, b) - 162g^2(a, b) + 1$ $w(a, b) = 6f^2(a, b) + 9f(a, b)g(a, b) - 162g^2(a, b) - 1$ where $f(a, b) = a^3 - 9ab^2$ and $g(a, b) = a^2b - b^3$ $P(a, b) = a^2 + 3b^2$
2	$x(a, b) = 3a^3 - 3b^3 - 27ab^2 + 3a^2b$ $y(a, b) = -2a^3 - 12b^3 + 18ab^2 + 12a^2b$ $z(a, b) = 48[5f^2(a, b) + 66f(a, b)g(a, b) - 135g^2(a, b)] + 1$ $w(a, b) = 48[5f^2(a, b) + 66f(a, b)g(a, b) - 135g^2(a, b)] - 1$ where $f(a, b) = a^3 - 9ab^2$ and $g(a, b) = a^2b - b^3$ $P(a, b) = a^2 + 3b^2$
3	$x(a, b) = a^3 + 15b^3 - 9ab^2 - 15a^2b$ $y(a, b) = 2a^3 - 12b^3 - 18ab^2 + 12a^2b$ $z(a, b) = 48[-3f^2(a, b) - 78f(a, b)g(a, b) + 81g^2(a, b)] + 1$ $w(a, b) = 48[-3f^2(a, b) - 78f(a, b)g(a, b) + 81g^2(a, b)] - 1$ where $f(a, b) = a^3 - 9ab^2$ and $g(a, b) = a^2b - b^3$ $P(a, b) = a^2 + 3b^2$
4	$x(A, B) = 7^2(-A^3 + 111B^3 + 9AB^2 - 111A^2B)$ $y(A, B) = 7^2(19A^3 - 51B^3 - 171AB^2 + 51A^2B)$ $z(A, B) = 147[-90f^2(A, B) - 429f(A, B)g(A, B) + 2430g^2(A, B)] + 1$ $w(A, B) = 147[-90f^2(A, B) - 429f(A, B)g(A, B) + 2430g^2(A, B)] - 1$ where $f(A, B) = A^3 - 9AB^2$ and $g(A, B) = A^2B - B^3$ $p(A, B) = 7^2(A^2 + 3B^2)$

Patterns	Solutions
5	$x(a,b) = 2a^3 + 12b^3 - 18ab^2 - 12a^2b$ $y(a,b) = a^3 - 15b^3 - 9ab^2 + 15a^2b$ $z(a,b) = 48[3f^2(a,b) - 78f(a,b)g(a,b) - 81g^2(a,b)] + 1$ $w(a,b) = 48[3f^2(a,b) - 78f(a,b)g(a,b) - 81g^2(a,b)] - 1$ where $f(a,b) = a^3 - 9ab^2$ and $g(a,b) = a^2b - b^3$ $P(a,b) = a^2 + 3b^2$

Chapter – VI

Double Diophantine Equations

CHAPTER – VI

DOUBLE DIOPHANTINE EQUATIONS

Chapter VI deals with Double Diophantine Equations in six sections VI.1 to VI.6

Section VI.1

The system of double equations given by $x - yz = 3w^2, xy = T^3$ is studied for obtaining its non-zero distinct solutions in integers.

Section VI.2

The pair of equations given by $x + y = z + w, y + z = (x - w)^2$ is studied for obtaining its non-zero distinct solutions in integers.

Section VI.3

In this section, different methods to obtain non-zero distinct integer solutions to the system of double equations $x + y = z + w, y + z = (x + w)^2$ are illustrated.

Section VI.4

This section illustrates the method of obtaining non-zero integral solutions to the system of two linear equations to be made squares represented by $an + b = p^2, bn + a = q^2$ for the choices of a and b given by (i) $a = 1, b = 7$ and (ii) $a = 2, b = 7$.

Section VI.5

Two different methods for obtaining non-zero distinct integer solutions to the pair of equations $x + y = z + w, y + z = (x - w)^3$ are illustrated.

Section VI.6

The problem of obtaining non-zero distinct integer solutions to the pair of equations $x + y = z + w, y + z = (x + w)^3$ is analysed.

VI.1 On the Simultaneous Equations $x - yz = 3w^2$, $xy = T^3$

Consider the pair of equations

$$x - yz = 3w^2 \quad (6.1)$$

$$xy = T^3 \quad (6.2)$$

The elimination of y between (6.1) and (6.2) gives

$$x^2 - 3w^2x - zT^3 = 0 \quad (6.3)$$

Treating (6.3) as a quadratic in x and solving for x , we have

$$x = \frac{1}{2} \left[3w^2 \pm \sqrt{9w^4 + 4zT^3} \right] \quad (6.4)$$

The square root on the R.H.S of (6.4) is eliminated when

$$(i) \quad w = T = (k+3)\alpha, \quad z = k(k+3)^2\alpha \quad (6.5)$$

$$(ii) \quad w = T = k\alpha, \quad z = k^2(k+3)\alpha \quad (6.6)$$

Now, taking (i), the corresponding values of x and y are given by

$$x = (k+3)^3\alpha^2, \quad y = \alpha \quad (6.7)$$

and for (ii)

$$x = -k^3\alpha^2, \quad y = -\alpha \quad (6.8)$$

Note that, the pairs (6.5), (6.7) and (6.6), (6.8) satisfy (6.1) and (6.2) respectively.

However, there are other choices of integer solutions to (6.1) and (6.2) and they are illustrated as below:

Consider the transformations

$$x = y^2, \quad T = y \quad (6.9)$$

Note that (6.2) is satisfied automatically, Substituting (6.9) in (6.1), we have

$$y^2 - yz = 3w^2 \quad (6.10)$$

which is a quadratic in y and solving for y , we have

$$y = \frac{1}{2} \left(z \pm \sqrt{z^2 + 12w^2} \right) \quad (6.11)$$

which is satisfied by

$$w = 2rs, z = 12r^2 - s^2 \text{ and } y = 12r^2, -s^2 \quad (6.12)$$

In view of (6.9), one obtains

$$x = 144r^4, s^4 \text{ and } T = 12r^2, -s^2 \quad (6.13)$$

Note that, (6.12) and (6.13) exhibits two sets of integer solutions to (6.1) and (6.2).

Also, to eliminate the square root on the R.H.S of (6.11), assume

$$\alpha^2 = z^2 + 12w^2 \quad (6.14)$$

which is represented as the system of double equations as shown below in Table 6.1.

Table 6.1: System of double equations

System	1	2	3	4	5
$\alpha + z$	w^2	$6w$	$12w$	$4w$	$6w^2$
$\alpha - z$	12	$2w$	w	$3w$	2

Solving each of the above systems in Table 1 and performing some algebra, the values of

x, y, z, w and T satisfying (6.1) and (6.2) are presented below in Table 6.2.

Table 6.2: Solutions

System	x	y	z	w	T
1	$4k^4, 36$	$2k^2, -6$	$2k^2 - 6$	$2k$	$2k^2, -6$
2	$9w^2, w^2$	$3w, -w$	$2w$	w	$3w, -w$
3	$144k^2, k^2$	$12k, -k$	$11k$	$2k$	$12k, -k$
4	$16k^2, 9k^2$	$4k, -3k$	k	$2k$	$4k, -3k$
5	$9w^4, 1$	$3w^2, -1$	$3w^2 - 1$	w	$3w^2, -1$

It is to be noted that, one may also write (6.10) as the system of double equations as in Table 6.3 below:

Table 6.3: System of double equations

System	1	2	3	4	5	6
y	1	3	w	w^2	$3w$	$3w^2$
$y - z$	$3w^2$	w^2	$3w$	3	w	1

In this case, the corresponding values of x, y, z, w and T are given by the quintuples

$$(x, y, z, w, T) = (1, 1, 1 - 3w^2, w, 1), (9, 3, 3 - w^2, w, 3), (w^2, w, -2w, w, w), (w^4, w^2, w^2 - 3, w, w^2) \\ (9w^2, 3w, 2w, w, 3w), (9w^4, 3w^2, 3w^2 - 1, w, 3w^2)$$

Further, write (6.14) as

$$z^2 + 12w^2 = \alpha^2 = \alpha^2 * 1 \quad (6.15)$$

Assume

$$\alpha = 4a^2 + 12b^2 \quad (6.16)$$

Write 1 as

$$1 = \frac{(2 + i\sqrt{12})(2 - i\sqrt{12})}{16} \quad (6.17)$$

Substituting (6.16) and (6.17) in (6.15) and employing the method of factorization, define

$$z + i\sqrt{12}w = (2a + i\sqrt{12}b)^2 \frac{(2 + i\sqrt{12})}{4}$$

On equating the real and imaginary parts, we have

$$z = 2a^2 - 6b^2 - 12ab, w = a^2 - 3b^2 + 2ab \quad (6.18)$$

In view of (6.11) and (6.9) we have

$$\left. \begin{aligned} T &= y = 3a^2 + 3b^2 - 6ab, -a^2 - 9b^2 - 6ab \\ x &= 9(a - b)^4, (a + 3b)^4 \end{aligned} \right\} \quad (6.19)$$

Thus (6.18) and (6.19) represent the solutions to (6.1) and (6.2).

It is worth mentioning that, in addition to (6.16), (6.17), α and 1 may also be written as

$$\alpha = 49(a^2 + 12b^2), 1 = \frac{(1 + i2\sqrt{12})(1 - i2\sqrt{12})}{49}$$

For this choice, the solutions of (6.1) and (6.2) are given by

$$x = 784(a - 3b)^4, 441(a + 4b)^4$$

$$T = y = 28(a - 3b)^2, -21(a + 4b)^2$$

$$z = 7(a^2 - 12b^2 - 48ab)$$

$$w = 14(a^2 - 12b^2 + ab)$$

VI.2 On the Pair of Equations $x + y = z + w, y + z = (x - w)^2$

Consider the system of double equations

$$x + y = z + w \quad (6.20)$$

$$y + z = (x - w)^2 \quad (6.21)$$

Four different methods of solving (6.20) and (6.21) are illustrated below:

Method: 1

The introduction of the transformations

$$x = u + v, w = u - v \quad (6.22)$$

in (6.20) and (6.21) leads to

$$z - y = 2v, z + y = 4v^2$$

from which, on solving, we get

$$z = 2v^2 + v, y = 2v^2 - v \quad (6.23)$$

Note that (6.22) and (6.23) satisfy (6.20) and (6.21).

Properties

- Each of the following expressions represents a perfect square
 - ❖ $4xw - (z + y)$
 - ❖ $4xw + (z - y)^2$
- $z^2 - y^2$ is a Cubical integer
- Each of the following expressions represents a Bi-quadratic integer
 - ❖ $4zy + z + y$
 - ❖ $4(zy + z + y) - 3(x - w)^2$

Method: 2

Assume

$$z = u + v, y = u - v, w = s \quad (6.24)$$

Substituting (6.24) in (6.20) and (6.21) and simplifying, note that

$$u = 2v^2 \quad (6.25)$$

The substitution of (6.25) in (6.24) leads to

$$z = 2v^2 + v, y = 2v^2 - v, w = s \quad (6.26)$$

Also, from (6.20),

$$x = 2v + s \quad (6.27)$$

Observe that (6.26) and (6.27) satisfy (6.20) and (6.21)

Properties

- $6((x-w)^4 - 4zy)$ is a Nasty number.
- $2(z^2 + y^2) - (x-w)^2$ is a Bi-quadratic integer.
- $\frac{2(z^4 - y^4)}{(x-w)^2 + 1}$ is a Quintic integer.
- $(2v^2 + v, 2v^2 - v, 8v^2)$ is the diophantine triple with the property $D(v^2)$ as the product of any two members of the set added with v^2 is a perfect square.
- $(2v^2 + v, 2v^2 - v, 8v^2 + 3)$ is the special dio-triple with the property $D(v^2 + 1)$ as the product of any two members of the set added with the same members and increased by $v^2 + 1$ is a perfect square.

Method: 3

Assume w is chosen arbitrarily and take

$$w = s \ (\neq 0) \quad (6.28)$$

Eliminating x between (6.20) and (6.21), we have

$$z^2 - (2y+1)z + y^2 - y = 0$$

Treating the above equation as a quadratic in z and solving for z , one gets

$$z = \frac{1}{2}(2y+1 \pm \sqrt{8y+1}) \quad (6.29)$$

The square-root on the R.H.S of (6.29) is eliminated when

$$y = \frac{n(n+1)}{2} \quad (6.30)$$

$$\text{and } z = \frac{1}{2}(n+1)(n+2), \frac{1}{2}n(n-1) \quad (6.31)$$

Substituting (6.30), (6.31) and (6.28) in (6.20), we have

$$x = s + z - y = \begin{cases} s + n + 1 \\ s - n \end{cases}$$

Thus, there are two sets of solutions to (6.20) and (6.21) represented as below:

Set: 1

$$x = s + n + 1, y = t_{3,n}, z = t_{3,n+1}, w = s$$

Set: 2

$$x = s - n, y = t_{3,n}, z = t_{3,n-1}, w = s$$

where $t_{3,\alpha}$ is the triangular number of rank α .

Method: 4

Consider the transformations

$$x = p + q, y = p - q, z = p + r, w = p - r \quad (6.32)$$

where p, q, r are non-zero distinct integers.

Note that equation (6.20) is automatically satisfied. The substitution of (6.32) in (6.21) leads to

$$q^2 + (2r+1)q + r^2 - r - 2p = 0$$

The above equation is quadratic in q and solving for q , we have

$$q = \frac{1}{2}(-2r-1 \pm \sqrt{8r+1+8p}) \quad (6.33)$$

The square root on the R.H.S of (6.33) is removed by choosing suitably the values of r and p and the corresponding values of q are obtained from (6.33). Substituting these values of p, q, r in (6.32), the values of x, y, z, w satisfying (6.20) and (6.21) are obtained. A few examples are given below:

Example: 1

Take $r = \frac{s(s+1)}{2}, p = \frac{k^2 + (2s+1)k}{2}$

$$\therefore q = \frac{1}{2}(-s^2 + s + 2k), \frac{1}{2}(-s^2 - 3s - 2 - 2k)$$

In view of (6.32), the corresponding 2 sets of solutions to (6.20) and (6.21) are as follows:

Set: 1

$$x = \frac{1}{2}(k^2 + (2s+1)k - s^2 + s + 2k)$$

$$y = \frac{1}{2}(k^2 + (2s+1)k + s^2 - s - 2k)$$

$$z = \frac{1}{2}(k^2 + (2s+1)k + s(s+1))$$

$$w = \frac{1}{2}(k^2 + (2s+1)k - s(s+1))$$

Set: 2

$$x = \frac{1}{2}(k^2 + (2s+1)k - s^2 - 3s - 2 - 2k)$$

$$y = \frac{1}{2}(k^2 + (2s+1)k + s^2 + 3s + 2 + 2k)$$

$$z = \frac{1}{2}(k^2 + (2s+1)k + s(s+1))$$

$$w = \frac{1}{2}(k^2 + (2s+1)k - s(s+1))$$

Example: 2

Consider

$$r = \frac{1}{2}(s^2 + s + 2), \quad p = \frac{1}{2}(k^2 + (2s+1)k - 2)$$

$$\therefore \quad q = \frac{1}{2}(-s^2 + s + 2k - 2), \quad \frac{1}{2}(-s^2 - 3s - 2k - 4)$$

Employing (6.32), the corresponding 2 sets of solutions to (6.20) and (6.21) are as follows:

Set: 3

$$x = \frac{1}{2}(k^2 + (2s+1)k - 2 - s^2 + s + 2k - 2)$$

$$y = \frac{1}{2}(k^2 + (2s+1)k - 2 + s^2 - s - 2k + 2)$$

$$z = \frac{1}{2}(k^2 + (2s+1)k - 2 + s^2 + s + 2)$$

$$w = \frac{1}{2}(k^2 + (2s+1)k - 2 - s^2 - s - 2)$$

Set: 4

$$x = \frac{1}{2} \left(k^2 + (2s+1)k - 2 - s^2 - 3s - 2k - 4 \right)$$

$$y = \frac{1}{2} \left(k^2 + (2s+1)k - 2 + s^2 + 3s + 2k + 4 \right)$$

$$z = \frac{1}{2} \left(k^2 + (2s+1)k - 2 + s^2 + s + 2 \right)$$

$$w = \frac{1}{2} \left(k^2 + (2s+1)k - 2 - s^2 - s - 2 \right)$$

VI.3 On the system of double equations $x + y = z + w$, $y + z = (x + w)^2$

Let x, y, z and w be four non-zero distinct integers such that the equations

$$x + y = z + w \quad (6.34)$$

$$y + z = (x + w)^2 \quad (6.35)$$

are satisfied. Different methods to obtain non-zero distinct integer values to x, y, z and w satisfying (6.34) and (6.35) are exhibited below:

Method 1

Eliminating y between (6.34) and (6.35), the resulting equation is

$$x^2 + (2w + 1)x + (w^2 - 2z - w) = 0 \quad (6.36)$$

Treating (6.36) as a quadratic in x and solving for x , one obtains

$$x = \frac{1}{2} \left[(-2w - 1) \pm \sqrt{8z + 8w + 1} \right] \quad (6.37)$$

The square-root on the R.H.S of (6.37) is eliminated when

$$z = m, w = \frac{1}{2}(n^2 + 3n - 2m + 2) \quad (6.38)$$

From (6.37) and (6.38), we get

$$x = \frac{1}{2}(-n^2 - n + 2m), -\frac{1}{2}(n^2 + 5n - 2m + 6) \quad (6.39)$$

In view of (6.34), note that

$$y = n^2 + 2n - m + 1, n^2 + 4n - m + 4 \quad (6.40)$$

Thus, (6.38), (6.39) and (6.40) give two sets of non-zero distinct integer solutions to the system of equations (6.34) and (6.35).

Method 2

The introduction of the transformations

$$x = u + v, w = u - v, z = 4k, y = 4l, (u \neq v \neq 0), (k \neq l \neq 0) \quad (6.41)$$

in (6.34) and (6.35) leads respectively to the equations

$$v = 2(k - l) \quad (6.42)$$

and

$$u^2 = k + l \quad (6.43)$$

Observe that (6.43) is satisfied when

$$l = m, k = (n+1)^2 - m, u = (n+1) \quad (6.44)$$

and from (6.42), we have

$$v = 2 \left[(n+1)^2 - 2m \right] \quad (6.45)$$

Using (6.44) and (6.45) in (6.41), we get

$$x = 2n^2 + 5n - 4m + 3$$

$$y = 4m$$

$$z = 4n^2 + 8n - 4m + 4$$

$$w = -2n^2 - 3n + 4m - 1$$

which satisfy (6.34) and (6.35).

Method 3

Consider the transformations

$$x = p + q, y = p - q, z = p + s, w = p - s, (p \neq q \neq s \neq 0) \quad (6.46)$$

it is seen that (6.34) is automatically satisfied.

The substitution of (6.46) in (6.35) leads to

$$4p^2 + p[4(q-s)-2] + (q-s)^2 + (q-s) = 0 \quad (6.47)$$

which is a quadratic in p and solving for p , we get,

$$p = \frac{1}{4} \left\{ [2(s-q)+1] \pm \sqrt{1-8q+8s} \right\} \quad (6.48)$$

The square-root on the R.H.S of (6.48) is eliminated when

$$q = m, s = \frac{1}{2}(n^2 - n + 2m) \quad (6.49)$$

From (6.48) and (6.49) we have,

$$p = \frac{1}{4}(n^2 + n), \frac{1}{4}(n^2 - 3n + 2) \quad (6.50)$$

Substituting (6.49) and (6.50) in (6.46), there are two sets of solutions to (6.34) and (6.35) and they are represented as below:

Set 1

$$\begin{aligned} x &= \frac{1}{4}(n^2 + n) + m \\ y &= \frac{1}{4}(n^2 + n) - m \\ z &= \frac{1}{4}(3n^2 - n) + m \\ w &= \frac{1}{4}(-n^2 + 3n) - m \end{aligned}$$

where $n, m \neq 0$

Note that, for the values of x, y, z and w to be in integers, choose n such that

$$n \equiv 0, -1 \pmod{4} \text{ and } m \in \mathbb{Z} - \{0\}$$

Set 2

$$\begin{aligned} x &= \frac{1}{4}[(n-1)(n-2)] + m \\ y &= \frac{1}{4}[(n-1)(n-2)] - m \\ z &= \frac{1}{4}(3n^2 - 5n + 2) + m \\ w &= \frac{1}{4}(-n^2 - n + 2) - m \end{aligned}$$

where $n, m \neq 0$

In this case for integer solutions n should be such that

$$n \equiv 1, 2 \pmod{4} \text{ and } m \in \mathbb{Z} - \{0\}.$$

However, by treating (6.47) as a quadratic in q, s in turn and following the above procedure different sets of values of x, y, z and w satisfying (6.34) and (6.35) are exhibited below in Table 6.4:

Table 6.4: Solutions

Set	x	y	z	w
3	$-4n^2 + 5n + s - 1$	$12n^2 - 7n - s + 1$	$4n^2 - n + s$	$4n^2 - n - s$
4	$-4n^2 - 3n + s$	$12n^2 + n - s$	$4n^2 - n + s$	$4n^2 - n - s$
5	$-4n^2 + 3n + s$	$12n^2 - n - s$	$4n^2 + n + s$	$4n^2 + n - s$
6	$-4n^2 - 5n + s - 1$	$12n^2 + 7n - s + 1$	$4n^2 + n + s$	$4n^2 + n - s$
7	$4n^2 - n + q$	$4n^2 - n - q$	$12n^2 + n + q$	$-4n^2 - 3n - q$
8	$4n^2 - n + q$	$4n^2 - n - q$	$12n^2 - 7n + q + 1$	$-4n^2 + 5n - q - 1$
9	$4n^2 + n + q$	$4n^2 + n - q$	$12n^2 + 7n + q + 1$	$-4n^2 - 5n - q - 1$
10	$4n^2 + n + q$	$4n^2 + n - q$	$12n^2 - n + q$	$-4n^2 + 3n - q$

VI.4 On the System of Two Linear Equations to be Made Squares

$$an + b = p^2, \quad bn + a = q^2$$

Let a, b be two given non-zero distinct positive integers.

The problem under consideration is to find non-zero positive integers n such that

$$an + b = p^2 \tag{6.51}$$

$$bn + a = q^2 \tag{6.52}$$

Eliminating n between (6.51) and (6.52), one obtains

$$bp^2 - aq^2 = b^2 - a^2 \tag{6.53}$$

Introduction of the linear transformations

$$p = X + aT \tag{6.54}$$

$$q = X + bT \tag{6.55}$$

in (6.53) leads to

$$X^2 = abT^2 + a + b \tag{6.56}$$

Knowing the values of a and b , one obtains the corresponding values for X and T .

In view of (6.54) and (6.51), the required values of n are obtained. The above process is illustrated through the following examples:

Examples 1

Let $a = 1, b = 7$

The system of double equations to be solved is

$$n + 7 = p^2 \tag{6.57}$$

$$7n + 1 = q^2 \tag{6.58}$$

$$(3) \Rightarrow q^2 = 7p^2 - 48 \tag{6.59}$$

whose least positive integer solution is

$$p_0 = 4, q_0 = 8$$

To obtain the other values of p, q satisfying (6.59), consider the pell equation

$$q^2 = 7p^2 + 1 \quad (6.60)$$

whose general solution $(\tilde{p}_s, \tilde{q}_s)$ is given by

$$\tilde{q}_s = \frac{1}{2} f_s, \tilde{p}_s = \frac{1}{2\sqrt{7}} g_s$$

where

$$\left. \begin{aligned} f_s &= (8+3\sqrt{7})^{s+1} + (8-3\sqrt{7})^{s+1} \\ g_s &= (8+3\sqrt{7})^{s+1} - (8-3\sqrt{7})^{s+1} \end{aligned} \right\}, s = 0, 1, 2, \dots$$

Applying Brahmagupta lemma between (p_0, q_0) and $(\tilde{p}_s, \tilde{q}_s)$, the other values of p

and q satisfying (6.59) are given by

$$\left. \begin{aligned} p_{s+1} &= 2f_s + \frac{4}{\sqrt{7}} g_s \\ q_{s+1} &= 4f_s + \frac{14}{\sqrt{7}} g_s \end{aligned} \right\}, s = -1, 0, 1, \dots$$

From (6.51), it is seen that

$$n_{s+1} = p_{s+1}^2 - 7, s = -1, 0, 1, \dots$$

Note that the above values of n also satisfy (6.52).

A few numerical examples are given in Table 6.5 below:

Table 6.5: Numerical Examples

s	n_{s+1}
-1	9
0	3129
1	795657

Observe that $n_{s+1} \equiv 0 \pmod{3}$.

However, we have other values of n satisfying (6.51) and (6.52) that are obtained as illustrated below:

In view of (6.56), we have

$$X^2 = 7T^2 + 8 \quad (6.61)$$

whose least positive integer solution is

$$T_0 = 2, X_0 = 6$$

To obtain the other values of X, T satisfying (6.61), consider the pell equation

$$X^2 = 7T^2 + 1$$

Following the procedure as above the values of n are given by

$$n_{s+1} = \left(4f_s + \frac{10}{\sqrt{7}}g_s \right)^2 - 7, \quad s = -1, 0, 1, \dots$$

A few numerical examples are given in Table 6.6 below:

Table 6.6: Numerical Examples

s	n_{s+1}
-1	57
0	15369
1	3904569

Observe that $n_{s+1} \equiv 0 \pmod{3}$

Example 2

Let $a = 2, b = 7$

In view of (6.56), we have

$$X^2 = 14T^2 + 9 \quad (6.62)$$

whose least positive integer solution is

$$T_0 = 12, X_0 = 45$$

To obtain the other values of X, T satisfying (6.62), consider the pell equation

$$X^2 = 14T^2 + 1$$

Following the procedure as in example 1 the values of n satisfying (6.51) and (6.52)

are given by

$$n_{s+1} = \frac{1}{2} \left[\left(\frac{69}{2} f_s + \frac{129}{\sqrt{14}} g_s \right)^2 - 7 \right], \quad s = -1, 0, 1, \dots$$

A few numerical examples are given in Table 6.7 below:

Table 6.7: Numerical Examples

s	n_{s+1}
-1	2377
0	2136241
1	1918343737

Example 3:

Let $a = 1, b = s^2 + 2s$

In view of (6.56), we have

$$X^2 = (s^2 + 2s) T^2 + (s+1)^2 \quad (6.63)$$

whose least positive integer solution is

$$T_0 = s+1, X_0 = (s+1)^2$$

To obtain the other values of X, T satisfying (6.63), consider the pell equation

$$X^2 = (s^2 + 2s) T^2 + 1$$

whose initial solution is given by

$$\tilde{T}_0 = 1, \tilde{X}_0 = s + 1$$

whose general solution $(\tilde{X}_t, \tilde{T}_t)$ is given by

$$\tilde{X}_t = \frac{1}{2} f_t, \quad \tilde{T}_t = \frac{1}{2\sqrt{s^2 + s}} g_t$$

where

$$\left. \begin{aligned} f_t &= \left(s + 1 + \sqrt{s^2 + 2s}\right)^{t+1} + \left(s + 1 - \sqrt{s^2 + 2s}\right)^{t+1} \\ g_t &= \left(s + 1 + \sqrt{s^2 + 2s}\right)^{t+1} - \left(s + 1 - \sqrt{s^2 + 2s}\right)^{t+1} \end{aligned} \right\}, t = 0, 1, 2, \dots$$

Applying Brahmagupta lemma between (X_0, T_0) and $(\tilde{X}_t, \tilde{T}_t)$, the other values of X and T satisfying (6.63) are given by

$$\left. \begin{aligned} T_{t+1} &= \frac{1}{2}(s+1) f_t + \frac{(s+1)^2}{2\sqrt{s^2 + 2s}} g_t \\ X_{t+1} &= \frac{1}{2}(s+1)^2 f_t + \frac{(s^2 + 2s)(s+1)}{2\sqrt{s^2 + 2s}} g_t \end{aligned} \right\}, t = -1, 0, 1, \dots$$

From (6.51), it is seen that

$$n_{t+1} = p_{t+1}^2 - (s^2 + 2s), t = -1, 0, 1, \dots$$

Note that the above values of n also satisfy (6.52).

A few numerical examples are given in Table 6.8 below:

Table 6.8: Numerical Examples

s	t	n_{t+1}
1	-1	33
2	0	4753
3	1	1507969

In conclusion, one may search for the values of n satisfying (6.51) and (6.52) for other choices of a and b .

VI.5 On the Double Equations $x + y = z + w$, $y + z = (x - w)^3$

This paper illustrates two different methods for obtaining non-zero distinct integer solutions to the pair of equations

$$x + y = z + w \quad (6.64)$$

$$y + z = (x - w)^3 \quad (6.65)$$

Method 1

Consider the linear transformations

$$x = u + v, w = u - v, u \neq v \neq 0 \quad (6.66)$$

Substituting (6.66) in (6.64) and (6.65) and simplifying, we have,

$$z = 4v^3 + v, y = 4v^3 - v \quad (6.67)$$

Note that (6.66) and (6.67) satisfy (6.64) and (6.65).

A few interesting relations observed among the solutions are as follows:

I. Each of the following expressions represents a cubical integer.

$$(i) \quad 2z + w - x$$

$$(ii) \quad 2y + x - w$$

$$II. \quad x + z \equiv (y + w) \pmod{4}$$

III. Each of the following triples represents Pythagorean triples

$$(i) \quad (y, (z - y)^2, z)$$

$$(ii) \quad (y, (x - y)^2, z)$$

$$(iii) \quad (y + 2(z - y)^2 + 2z, 2y + (z - y)^2 + 2z, 2y + 2(z - y)^2 + 3z)$$

$$(iv) \quad (y + 2(x - w)^2 + 2z, 2y + (x - w)^2 + 2z, 2y + 2(x - w)^2 + 3z)$$

IV. Each of the following triples represents diophantine 3-tuples with property $D(v^2)$:

$$\text{Triple 1: } (z, c_n, c_{n+1}), c_n = (2n+2)^2 v^3 + (n^2 - 1) v, n = 1, 2, 3, \dots$$

$$\text{Triple 2: } (y, c_n, c_{n+1}), c_n = (2n+2)^2 v^3 - (n^2 - 1) v, n = 1, 2, 3, \dots$$

$$\text{Triple 3: } (x, c_n, c_{n+1}), c_n = (n+1)^2 u + (n^2 - 1) v, n = 1, 2, 3, \dots$$

$$\text{Triple 4: } (w, c_n, c_{n+1}), c_n = (n+1)^2 u - (n^2 - 1) v, n = 1, 2, 3, \dots$$

V. It is worth to note that the value of y represents 3 times centered octagonal pyramidal number where as the value of z represents centered triangular pyramidal number.

Method 2

Introduction of the transformations

$$w = s, \quad x = (k+1)s, \quad k > 0 \quad (6.68)$$

in (6.64) and (6.65) lead to

$$(z - y) = ks, \quad z + y = k^3 s^3$$

from which we have

$$z = \frac{1}{2}(k^3 s^3 + ks), \quad y = \frac{1}{2}(k^3 s^3 - ks) \quad (6.69)$$

Note that (6.68) and (6.69) satisfy (6.64) and (6.65).

Observations

1. It is worth mentioning that the value of z represents centered triangular pyramidal number where as the value of y represents 3 times triangular pyramidal number.
2. The triple $(k^3 w^3 + k, c_s, c_{s+1})$ is a Diophantine 3-tuple with property $D(k^2 w^2)$ where $c_s = (s+1)^2 k^3 w^3 + (s^2 - 1)kw, s = 1, 2, 3, \dots$

VI.6 On the System of Equations $x + y = z + w$, $y + z = (x + w)^3$

This paper illustrates two different methods for obtaining non-zero distinct integer solutions to the pair of equations

$$x + y = z + w \quad (6.70)$$

$$y + z = (x + w)^3 \quad (6.71)$$

Method 1

Consider the linear transformations

$$x = u + v, w = u - v, u \neq v \neq 0 \quad (6.72)$$

Substituting (6.72) in (6.70) and (6.71) and simplifying, we have,

$$z = 4u^3 + v, y = 4u^3 - v \quad (6.73)$$

Note that (6.72) and (6.73) satisfy (6.70) and (6.71).

A few interesting relations observed among the solutions are as follows:

I. Each of the following expressions represents a cubical integer.

$$(i) \quad 2z + w - x$$

$$(ii) \quad 2y + x - w$$

$$II. \quad x + z \equiv (y + w) \pmod{4}$$

III. Each of the following triples represents Pythagorean triples

$$(i) \quad (4\alpha^6 v^3 - v, 4\alpha^3 v^2, 4\alpha^6 v^3 + v)$$

$$(ii) \quad (12\alpha^6 v^3 + 8\alpha^3 v^2 + v, 16\alpha^6 v^3 + 4\alpha^3 v^2, 20\alpha^6 v^3 + 8\alpha^3 v^2 + v)$$

IV. A. Each of the following triples represents diophantine 3-tuples with property

$$D(v^2 + 2nu + n^2):$$

$$\text{Triple 1: } (x, c_s, c_{s+1}), c_s = (s+1)^2 u + (s^2 - 1)v + 2sn, s = 1, 2, 3, \dots$$

$$\text{Triple 2: } (w, c_s, c_{s+1}), c_s = (s+1)^2 u - (s^2 - 1)v + 2sn, s = 1, 2, 3, \dots$$

B. Each of the following triples represents diophantine 3-tuples with property

$$D(v^2 + 8nu^3 + n^2):$$

$$\text{Triple 3: } (y, c_s, c_{s+1}), c_s = 4(s+1)^2 u^3 - (s^2 - 1)v + 2sn, s = 1, 2, 3, \dots$$

$$\text{Triple 4: } (z, c_s, c_{s+1}), c_s = 4(s+1)^2 u^3 + (s^2 - 1)v + 2sn, s = 1, 2, 3, \dots$$

C. The triple $(u+v, u-v, 4u+2s+1)$ represents a dio 3-tuple with property

$$D(v^2 + (2s-2)u + s^2).$$

D. The triple $(4u^3 + v, 4u^3 - v, 16u^3 + 2s + 1)$ represents a dio 3-tuple with property

$$D(v^2 + s^2 + (8s-8)u^3).$$

V. It is worth to note that, each of the values of $y+x, z+w$ represents centered triangular pyramidal number.

VI. The values of $(y+z) - (x+w)$ represents six times centered octagonal pyramidal number.

VII. The values of $x+y+z+w$ represents two times centered triangular pyramidal number.

VIII. $2(z^2 + y^2) - (x-w)^2$ is a sextic integer.

IX. $2(xy + zw) + (z-y)^2$ is a bi-quadratic integer.

X. $2(xy - zw) = (x^2 - w^2)((x+w)^2 - 1)$

XI. When $u = 3r^2 - s^2$, $v = 2rs$, $r \neq s \neq 0$

$$x^2 - xw + w^2 \text{ is a perfect square.}$$

XII. Consider, u, v to represent the sum of the legs and difference between the legs of the Pythagorean triangle then, it is observed that $x^2 + w^2$ is a perfect square.

Chapter – VII

Simultaneous Equations

CHAPTER – VII

SIMULTANEOUS EQUATIONS

Chapter VII analyses Triple Diophantine Equations in three sections VII.1 to VII.3

Section VII.1

An attempt is made to obtain non-zero distinct integer quintuples (x, y, a, b, c) satisfying the system of three equations $x + y = a^2, 2x + y = b^2, x + 2y = a^2 - c^2$. Different sets of integer solutions are presented.

Section VII.2

Non-zero distinct integer quintuples (x, y, a, b, c) satisfying the system of three equations $x + y = 2a^2, 2x + y = 5a^2 + b^2, x + 2y = c^3$ are determined.

Section VII.3

Triple equations with five unknowns represented by $x + y = 2a^2, 2x + y = 5a^2 - b^2, x + 2y = 5c^3$ are analyzed for non-zero distinct integral solutions.

VII.1 On the System of Triple Equations with Five Variables

$$x + y = a^2, \quad 2x + y = b^2, \quad x + 2y = a^2 - c^2$$

Consider the system of equations

$$x + y = a^2 \tag{7.1}$$

$$2x + y = b^2 \tag{7.2}$$

$$x + 2y = a^2 - c^2 \tag{7.3}$$

Eliminating x and y between (7.1) to (7.3), the resulting equation is

$$b^2 = c^2 + 2a^2 \tag{7.4}$$

which is satisfied by

$$a = 2k, c = k, b = 3k \tag{7.5}$$

$$\text{Now, (7.2)-(7.1)} \Rightarrow x = b^2 - a^2 \tag{7.6}$$

$$\text{and (7.1)} \Rightarrow y = a^2 - x \tag{7.7}$$

Using (7.5) in (7.6) and (7.7), the values of x and y satisfying the system of equations (7.1) to (7.3) are given by

$$x = 5k^2, \quad y = -k^2 \tag{7.8}$$

Also, note that (7.4) is satisfied by

$$a = 2rs, \quad c = 2r^2 - s^2, \quad b = 2r^2 + s^2$$

For this choice, the corresponding solutions to (7.1) to (7.3) are represented by

$$x = 4r^4 + s^4, \quad y = 4r^2s^2 - 4r^4 - s^4 \tag{7.9}$$

In addition to the above two solutions (7.8) and (7.9) of (7.1) to (7.3), there are other choices of solutions to the system of equations under consideration that are illustrated below:

Choice 1

Write (7.4) as

$$c^2 + 2a^2 = b^2 * 1 \quad (7.10)$$

$$\text{Assume } b = 9(p^2 + 2q^2) \quad (7.11)$$

$$\text{and } 1 = \frac{(1+i2\sqrt{2})(1-i2\sqrt{2})}{9} \quad (7.12)$$

Substituting (7.11) and (7.12) in (7.10) and applying the method of factorization, define

$$c + i\sqrt{2}a = \frac{(1+i2\sqrt{2})}{3} * 9(p + i\sqrt{2}q)^2$$

Equating the rational and irrational parts in the above equation, one obtains

$$c = 3(p^2 - 2q^2 - 8pq) \quad (7.13)$$

$$a = 3(2p^2 - 4q^2 + 2pq) \quad (7.14)$$

Using the values of a, b, c given by (7.14), (7.11) and (7.13) in (7.6) and (7.7), the required values of x and y satisfying (7.1) to (7.3) are given by

$$x = 81(p^2 + 2q^2)^2 - 36(p^2 - 2q^2 + pq)^2$$

$$y = 72(p^2 - 2q^2 + pq)^2 - 81(p^2 + 2q^2)^2$$

Choice 2

1 can also be written as

$$1 = \frac{(7+i4\sqrt{2})(7-i4\sqrt{2})}{81} \quad (7.15)$$

Substituting (7.11) and (7.15) in (7.10) and applying the method of factorization, define

$$c + i\sqrt{2}a = \frac{(7+i4\sqrt{2})}{9} * 9(p + i\sqrt{2}q)^2$$

Equating the rational and irrational parts in the above equation, one obtains

$$c = (7p^2 - 14q^2 - 16pq) \quad (7.16)$$

$$a = 2(2p^2 - 4q^2 + 7pq) \quad (7.17)$$

Using the values of a, b, c given by (7.17), (7.11) and (7.16) in (7.6) and (7.7), the required values of x and y satisfying (7.1) to (7.3) are obtained as

$$\begin{aligned} x &= 81(p^2 + 2q^2)^2 - 4(2p^2 - 4q^2 + 7pq)^2 \\ y &= 8(2p^2 - 4q^2 + 7pq)^2 - 81(p^2 + 2q^2)^2 \end{aligned}$$

Choice 3

Write (7.4) as

$$1 * c^2 = c^2 = b^2 - 2a^2 \quad (7.18)$$

Assume

$$c = p^2 - 2q^2 \quad (7.19)$$

Write 1 as

$$1 = Y_n^2 - 2X_n^2 \quad (7.20)$$

where

$$\begin{aligned} Y_n &= \frac{1}{2} \left[(3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1} \right] \\ X_n &= \frac{1}{2\sqrt{2}} \left[(3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1} \right], n = 0, 1, 2, \dots \end{aligned}$$

Substituting (7.19) and (7.20) in (7.18) and applying the method of factorization, define

$$b + \sqrt{2}a = (p + \sqrt{2}q)^2 (Y_n + \sqrt{2}X_n)$$

Equating the rational and irrational parts, we have

$$b = Y_n(p^2 + 2q^2) + 4pqX_n \quad (7.21)$$

$$a = 2pqY_n + (p^2 + 2q^2)X_n \quad (7.22)$$

Using the values of a, b, c given by (7.22), (7.21) and (7.19) in (7.6) and (7.7), the required values of x, y satisfying the system (7.1) to (7.3) are given by

$$x = (p^2 + 2q^2)^2 (Y_n^2 - X_n^2) + 4p^2q^2 (4X_n^2 - Y_n^2) + 4pq(p^2 + 2q^2)X_nY_n$$

$$y = 4p^2q^2 (2Y_n^2 - 4X_n^2) + (p^2 + 2q^2)^2 (2X_n^2 - Y_n^2), n = 0, 1, 2, \dots$$

VII.2 On the Simultaneous Equation $x + y = 2a^2$, $2x + y = 5a^2 + b^2$, $x + 2y = c^3$

Let x, y, a, b and c be five non-zero distinct integers such that

$$x + y = 2a^2 \quad (7.23)$$

$$2x + y = 5a^2 + b^2 \quad (7.24)$$

$$x + 2y = c^3 \quad (7.25)$$

Eliminating x and y between (7.23) to (7.25), the resulting equation is

$$a^2 - b^2 = c^3 \quad (7.26)$$

Solving (7.26) through different methods, one obtains different sets of solutions to the system (7.23) to (7.25).

Method 1

It is observed that (7.26) is satisfied by

$$a = m(m^2 - n^2), b = n(m^2 - n^2), c = (m^2 - n^2) \quad (7.27)$$

where $m \neq \pm n$ and $n \neq 1$. Eliminating y between (7.23) and (7.24), the values of x is given by

$$x = 3a^2 + b^2 = (m^2 - n^2)^2 (3m^2 - n^2) \quad (7.28)$$

From (7.23),

$$y = 2a^2 - x = -(m^2 - n^2)^2 (m^2 + n^2) \quad (7.29)$$

Note that, (7.27) to (7.29) satisfy (7.23) to (7.25). A few numerical examples are given in Table 7.1 below:

Table 7.1: Numerical Examples

m	n	a	b	c	x	y
2	3	-10	-15	-5	525	-325
5	-7	-120	168	-24	71424	-42624
11	9	440	360	40	710400	-323200
9	2	693	154	77	1464463	-503965

Method 2

After performing numerical calculations, it is seen that (7.26) is satisfied by

$$a = t_{3,k+1}, b = t_{3,k}, c = (k+1) \quad (7.30)$$

where $t_{3,k}$ the triangular number of rank k .

The corresponding values of x and y satisfying (7.23) to (7.25) are represented by

$$x = 3(t_{3,k+1})^2 + (t_{3,k})^2 = (k+1)^4 + (k+1)^3 + (k+1)^2$$

$$y = -(t_{3,k+1})^2 - (t_{3,k})^2 = -\frac{1}{2}[(k+1)^4 + (k+1)^2]$$

A few numerical examples are given in Table 7.2 below:

Table 7.2: Numerical Examples

k	a	b	c	x	y
2	6	3	3	117	-45
3	10	6	4	336	-136
4	15	10	5	775	-325
5	21	15	6	1548	-666

Method 3

Observe that (7.26) is satisfied by

$$a = \frac{c^3 + 1}{2}, b = \frac{c^3 - 1}{2}$$

Since our interest is on finding integer solutions,

take

$$c = 2k + 1$$

and we have

$$a = 4k^3 + 6k^2 + 3k + 1$$

$$b = 4k^3 + 6k^2 + 3k$$

For this choice, the values of x and y satisfying (7.23) to (7.25) are given by

$$x = 4(4k^3 + 6k^2 + 3k)^2 + 6(4k^3 + 6k^2 + 3k) + 3$$

$$y = -2(4k^3 + 6k^2 + 3k)^2 - 2(4k^3 + 6k^2 + 3k) - 1$$

A few numerical examples are presented in Table 7.3 below:

Table 7.3: Numerical Examples

k	a	b	c	x	y
2	63	62	5	15751	-7813
3	172	171	7	117993	-58825
4	365	364	9	532171	-265721
5	666	665	11	1772893	-885781

Method 4

Introducing the transformations

$$a = u + v, b = u - v, c = 2\alpha \quad (7.31)$$

in (7.26), we have

$$uv = 2\alpha^3 \quad (7.32)$$

One may choose u and v suitably in (7.32) and using (7.31) the corresponding values of x and y satisfying the system of equations (7.23) to (7.25) are obtained.

Choice 1

$$u = \alpha^3, v = 2$$

$$\therefore a = \alpha^3 + 2, b = \alpha^3 - 2$$

$$\text{Thus, } x = 3(\alpha^3 + 2)^2 + (\alpha^3 - 2)^2 = 4\alpha^6 + 8\alpha^3 + 16$$

$$y = -(\alpha^3 + 2)^2 - (\alpha^3 - 2)^2 = -(2\alpha^6 + 8)$$

Choice 2

$$u = 2\alpha^3, v = 1$$

$$\therefore a = 2\alpha^3 + 1, b = 2\alpha^3 - 1$$

$$\text{Thus, } x = 3(2\alpha^3 + 1)^2 + (2\alpha^3 - 1)^2 = 16\alpha^6 + 8\alpha^3 + 4$$

$$y = -(2\alpha^3 + 1)^2 - (2\alpha^3 - 1)^2 = -(8\alpha^6 + 2)$$

Choice 3

$$u = 2\alpha^3, v = \alpha$$

$$\therefore a = 2\alpha^2 + \alpha, b = 2\alpha^2 - \alpha$$

$$\text{Thus, } x = 3(2\alpha^2 + \alpha)^2 + (2\alpha^2 - \alpha)^2 = 16\alpha^4 + 8\alpha^3 + 4\alpha^2$$

$$y = -(2\alpha^2 + \alpha)^2 - (2\alpha^2 - \alpha)^2 = -(8\alpha^4 + 2\alpha^2)$$

Choice 4

$$u = \alpha^2, v = 2\alpha$$

$$\therefore a = \alpha^2 + 2\alpha, b = \alpha^2 - 2\alpha$$

$$\text{Thus, } x = 3(\alpha^2 + 2\alpha)^2 + (\alpha^2 - 2\alpha)^2 = 4\alpha^4 + 8\alpha^3 + 16\alpha^2$$

$$y = -(\alpha^2 + 2\alpha)^2 - (\alpha^2 - 2\alpha)^2 = -(2\alpha^4 + 8\alpha^2)$$

Method 5

The introductions of the transformations

$$a = u + 2k^3v, b = u - 2k^3v, c = 2k\alpha \quad (7.33)$$

in (7.26), leads to

$$uv = \alpha^3 \quad (7.34)$$

One may choose u and v suitably in (7.33) and using (7.34) the corresponding values of x and y satisfying the system of equations (7.23) to (7.25) are obtained.

Choice 5

$$u = \alpha^3, v = 1$$

$$\therefore a = \alpha^3 + 2k^3, b = \alpha^3 - 2k^3$$

$$\text{Thus, } x = 3(\alpha^3 + 2k^3)^2 + (\alpha^3 - 2k^3)^2 = 4\alpha^6 + 8k^3\alpha^3 + 16k^6$$

$$y = 2(\alpha^3 + 2k^3)^2 - (4\alpha^6 + 8k^3\alpha^3 + 16k^6) = -(2\alpha^6 + 8k^6)$$

Choice 6

$$u = 1, v = \alpha^3$$

$$\therefore a = 1 + 2k^3\alpha^3, b = 1 - 2k^3\alpha^3$$

$$\text{Thus, } x = 3(1 + 2k^3\alpha^3)^2 + (1 - 2k^3\alpha^3)^2 = 16k^6\alpha^6 + 8k^3\alpha^3 + 4$$

$$y = 2(1 + 2k^3\alpha^3)^2 - (16k^6\alpha^6 + 8k^3\alpha^3 + 4) = -(8k^6\alpha^6 + 2)$$

Choice 7

$$u = \alpha^2, v = \alpha$$

$$\therefore a = \alpha^2 + 2k^3\alpha, b = \alpha^2 - 2k^3\alpha$$

$$\text{Thus, } x = 3(\alpha^2 + 2k^3\alpha)^2 + (\alpha^2 - 2k^3\alpha)^2 = 4\alpha^4 + 16k^6\alpha^2 + 8k^3\alpha^3$$

$$y = 2(\alpha^2 + 2k^3\alpha)^2 - (4\alpha^4 + 16k^6\alpha^2 + 8k^3\alpha^3) = -(8k^6\alpha^2 + 2\alpha^4)$$

Choice 8

$$u = \alpha, v = \alpha^2$$

$$\therefore a = \alpha + 2k^3\alpha^2, b = \alpha - 2k^3\alpha^2$$

$$\text{Thus, } x = 3(\alpha + 2k^3\alpha^2)^2 + (\alpha - 2k^3\alpha^2)^2 = 4\alpha^2 + 8k^3\alpha^3 + 16k^6\alpha^4$$

$$y = 2(\alpha + 2k^3\alpha^2)^2 - (4\alpha^2 + 8k^3\alpha^3 + 16k^6\alpha^4) = -(2\alpha^2 + 8k^6\alpha^4)$$

VII.3 On a Set of Three Diophantine Equations

$$x + y = 2a^2, \quad 2x + y = 5a^2 - b^2, \quad x + 2y = 5c^3$$

The system of triple equations with five unknowns to be solved for its distinct non-zero integral solutions are

$$x + y = 2a^2 \tag{7.35}$$

$$2x + y = 5a^2 - b^2 \tag{7.36}$$

$$x + 2y = 5c^3 \tag{7.37}$$

Eliminating x and y between (7.35) to (7.37), we get

$$a^2 + b^2 = 5c^3 \tag{7.38}$$

Also, solving (7.35) and (7.36) for x and y , one obtains

$$x = 3a^2 - b^2 \tag{7.39}$$

$$y = b^2 - a^2 \tag{7.40}$$

Now, solving (7.38), the values of a, b, c are obtained. In view of (7.39) and (7.40), the values of x and y satisfy (7.35) to (7.37) are obtained. Thus, the above values of x, y, a, b and c represent the solutions to the system of equations (7.35) to (7.37). The above process is illustrated as follows:

Method 1

Let

$$c = \alpha^2 + \beta^2 \tag{7.41}$$

where α and β are non-zero integers.

Write 5 as

$$5 = (2 + i)(2 - i) \tag{7.42}$$

Using (7.41), (7.42) in (7.38) and applying the method of factorization, define

$$(a + ib) = (2 + i)[(\alpha + i\beta)]^3 \quad (7.43)$$

from which we have

$$\left. \begin{aligned} a &= a(\alpha, \beta) = 2\alpha^3 - 6\alpha\beta^2 - 3\alpha^2\beta + \beta^3 \\ b &= b(\alpha, \beta) = \alpha^3 - 3\alpha\beta^2 + 6\alpha^2\beta - 2\beta^3 \end{aligned} \right\} \quad (7.44)$$

Substituting (7.44) in (7.39) and (7.40), the corresponding values of x and y are obtained.

A few numerical examples are given in the Table 7.4 below:

Table 7.4: Numerical values

α	β	a	b	c	x	y
1	2	-20	-15	5	975	-175
2	3	-101	-28	13	29819	-9417
1	3	-34	-62	10	-376	2688
2	4	-160	-120	20	62400	-11200

Method 2

Write 5 as

$$5 = (1 + 2i)(1 - 2i) \quad (7.45)$$

Using (7.45), (7.41) in (7.38) and applying the method of factorization, define

$$(a + ib) = (1 + 2i)[(\alpha + i\beta)]^3 \quad (7.46)$$

Equating real and imaginary parts, we have

$$\left. \begin{aligned} a &= a(\alpha, \beta) = \alpha^3 - 3\alpha\beta^2 - 6\alpha^2\beta + 2\beta^3 \\ b &= b(\alpha, \beta) = 2\alpha^3 - 6\alpha\beta^2 + 3\alpha^2\beta - \beta^3 \end{aligned} \right\} \quad (7.47)$$

Substituting (7.47) in (7.39) and (7.40), the corresponding values of x and y are obtained.

A few numerical examples are given in the Table 7.5 below:

Table 7.5: Numerical values

α	β	a	b	c	x	y
1	2	-7	-24	5	-429	527
2	3	-64	-83	13	5399	2793
1	3	10	-70	10	-4600	4800
2	4	-56	-192	20	-27456	33728

Method 3

Substituting

$$a = (5k - 3)b \quad (7.48)$$

in (7.38), it gives

$$(5k^2 - 6k + 2)b^2 = c^3 \quad (7.49)$$

Choose b and c such that

$$b = (5k^2 - 6k + 2)u^3 \quad (7.50)$$

$$c = (5k^2 - 6k + 2)u^2 \quad (7.51)$$

Note that (7.49) is satisfied.

Using (7.50) in (7.48), we have

$$a = (5k - 3)(5k^2 - 6k + 2)u^3$$

It is observed that the values of a , b and c represented by

$$a = (5k - 3)(5k^2 - 6k + 2)u^3$$

$$b = (5k^2 - 6k + 2)u^3$$

$$c = (5k^2 - 6k + 2)u^2$$

satisfy (7.38). In view of (7.39) and (7.40) the values of x and y satisfying (7.35) to (7.37) are obtained.

A few numerical examples are given in the Table 7.6 below:

Table 7.6: Numerical values

u	k	a	b	c	x	y
2	1	2	8	4	11	-3
2	2	70	80	40	14600	-4800
2	3	348	232	116	362471	-120263
2	4	986	464	232	2913224	-968832

Method 4

Write (7.38) as

$$a^2 + b^2 = 5c^3 * 1 \quad (7.52)$$

Write 1 as

$$1 = \frac{(3+4i)(3-4i)}{25} \quad (7.53)$$

Using (7.41), (7.42), (7.53) in (7.52) and applying the method of factorization, define

$$(a + ib) = (2 + i) \left[(\alpha + i\beta) \right]^3 \frac{(3 + 4i)}{5} \quad (7.54)$$

from which we have

$$\left. \begin{aligned} a &= a(\alpha, \beta) = \frac{1}{5} \left[2\alpha^3 - 6\alpha\beta^2 - 33\alpha^2\beta + 11\beta^3 \right] \\ b &= b(\alpha, \beta) = \frac{1}{5} \left[11\alpha^3 - 33\alpha\beta^2 + 6\alpha^2\beta - 2\beta^3 \right] \end{aligned} \right\} \quad (7.55)$$

Since our interest is on finding integer solutions, replacing, α by $5A$ and β by $5B$

in (7.41) and (7.55), the corresponding integer solutions to a , b and c are given by

$$\begin{aligned} a(A, B) &= 50A^3 - 150AB^2 - 825A^2B + 275B^3 \\ b(A, B) &= 275A^3 - 825AB^2 + 150A^2B - 50B^3 \\ c(A, B) &= 25(A^2 + B^2) \end{aligned}$$

In view of (7.39) and (7.40), the values of x and y are obtained.

A few numerical examples are given in the Table 7.7 below:

Table 7.7: Numerical values

α	β	A	B	a	b	c	x	y
5	15	1	3	3650	-8050	250	-24835000	51480000
10	15	2	3	-4775	-12200	325	-80438125	126039375
15	10	3	2	-13100	-175	325	514799375	-171579375
10	30	2	6	29200	-64400	1000	-1589440000	3294720000

Method 5

Using (7.41), (7.44), (7.53) in (7.52) and applying the method of factorization, define

$$(a + ib) = (1 + 2i) [(\alpha + i\beta)]^3 \frac{(3 + 4i)}{5} \quad (7.56)$$

Equating real and imaginary parts, we have

$$\left. \begin{aligned} a &= a(\alpha, \beta) = -\alpha^3 + 3\alpha\beta^2 - 6\alpha^2\beta + 2\beta^3 \\ b &= b(\alpha, \beta) = 2\alpha^3 - 6\alpha\beta^2 - 3\alpha^2\beta + \beta^3 \end{aligned} \right\} \quad (7.57)$$

Substituting (7.57) in (7.39) and (7.40), the corresponding values of x and y are obtained.

A few numerical examples are given in the Table 7.8 below:

Table 7.8: Numerical values

α	β	a	b	c	x	y
1	2	15	-20	5	275	175
2	3	28	-101	13	-7849	9417
1	3	62	-34	10	10376	-2688
2	4	120	-160	20	17600	11200

Chapter – VIII

Diophantine 3-Tuples

CHAPTER - VIII

DIOPHANTINE 3-TUPLES

Chapter VIII focuses on Diophantine 3-Tuples in four sections VIII.1 to VIII.4

Section VIII.1

This paper deals with the study of constructing sequences of diophantine triples (a, b, c) such that the product of any two elements of the set added by 11 is a perfect square.

Section VIII.2

The construction of sequences of diophantine triples (a, b, c) through pronic numbers is studied.

Section VIII.3

This section has three parts VIII.3A, VIII.3B, VIII.3C

The formulation of sequences of diophantine triples (a, b, c) through Euler polynomials, Bernoulli polynomials & Euler and Bernoulli polynomials is considered in sections VIII.3A, VIII.3B, VIII.3C respectively.

Section VIII.4

This paper concerns with the formulation of sequences of Diophantine 3-tuples with property $D(k^2 + 10k - 3)$ through matrix method.

VIII.1 Formulation of Sequences of Diophantine 3-Tuples with Property $D(11)$

Sequence 1

Let $a = 7$, $b = 7n^2 + 4n - 1$

It is observed that

$$ab + 11 = (7n + 2)^2$$

Therefore, the pair (a, b) represents Diophantine 2-tuples with property $D(11)$.

Let c_1 be any non-zero polynomial such that

$$ac_1 + 11 = p^2 \tag{8.1}$$

$$bc_1 + 11 = q^2 \tag{8.2}$$

Eliminating c_1 between (8.1) and (8.2), we have

$$bp^2 - aq^2 = (b - a)(11) \tag{8.3}$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + bT \tag{8.4}$$

in (8.3) and simplifying we get

$$X^2 = abT^2 + 11$$

which is satisfied by $T = 1$, $X = 7n + 2$

In view of (8.4) and (8.1), it is seen that

$$c_1 = 7n^2 + 18n + 10$$

Note that (a, b, c_1) represents Diophantine 3-tuples with property $D(11)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 7n^2 + 32n + 35$$

exhibits Diophantine 3 –tuples with property $D(11)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3) where

$$c_3 = 7n^2 + 46n + 74$$

exhibits Diophantine 3 –tuples with property $D(11)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4) where

$$c_4 = 7n^2 + 60n + 127$$

exhibits Diophantine 3-tuples with property $D(11)$.

The repetition of the above process leads to the generation of sequence of Diophantine 3-tuples whose general form is given by (a, c_s, c_{s+1}) where

$$c_s = 7n^2 + (14s + 4)n + (7s^2 + 4s - 1), \quad s = 1, 2, 3, \dots$$

Sequence 2

Let $a = 7, b = 7n^2 - 4n - 1$

It is observed that

$$ab + 11 = (7n - 2)^2$$

Therefore, the pair (a, b) represents Diophantine 2-tuples with property $D(11)$.

Let c_1 be any non-zero polynomial such that

$$ac_1 + 11 = p^2 \tag{8.5}$$

$$bc_1 + 11 = q^2 \tag{8.6}$$

Eliminating c_1 between (8.5) and (8.6), we have

$$bp^2 - aq^2 = (b - a)(11) \tag{8.7}$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + bT \quad (8.8)$$

in (8.7) and simplifying we get

$$X^2 = abT^2 + 11$$

which is satisfied by $T = 1, X = 7n - 2$

In view of (8.8) and (8.5), it is seen that

$$c_1 = 7n^2 + 10n + 2$$

Note that (a, b, c_1) represents Diophantine 3-tuples with property $D(11)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 7n^2 + 24n + 19$$

exhibits Diophantine 3-tuples with property $D(11)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 7n^2 + 38n + 50$$

exhibits Diophantine 3-tuples with property $D(11)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4)

where

$$c_4 = 7n^2 + 52n + 95$$

exhibits Diophantine 3-tuples with property $D(11)$.

The repetition of the above process leads to the generation of sequence of Diophantine

3-tuples whose general form is given by (a, c_s, c_{s+1}) where

$$c_s = 7n^2 + (14s - 4)n + (7s^2 - 4s - 1), \quad s = 1, 2, 3, \dots$$

Sequence 3

Let $a = 2$, $b = 2^{2n-1} + 2^n - 5$

It is observed that

$$ab + 11 = (2^n + 1)^2$$

Therefore, the pair (a, b) represents Diophantine 2-tuples with property $D(11)$.

Let c_1 be any non-zero polynomial such that

$$ac_1 + 11 = p^2 \quad (8.9)$$

$$bc_1 + 11 = q^2 \quad (8.10)$$

Eliminating c_1 between (8.9) and (8.10), we have

$$bp^2 - aq^2 = (b - a)(11) \quad (8.11)$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + bT \quad (8.12)$$

in (8.11) and simplifying we get

$$X^2 = abT^2 + 11$$

which is satisfied by $T = 1$, $X = 2^n + 1$

In view of (8.12) and (8.9), it is seen that

$$c_1 = 2^{2n-1} + 3 \cdot 2^n - 1$$

Note that (a, b, c_1) represents Diophantine 3-tuples with property $D(11)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 2^{2n-1} + 5 \cdot 2^n + 7$$

exhibits Diophantine 3-tuples with property $D(11)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3) where

$$c_3 = 2^{2n-1} + 7 * 2^n + 19$$

exhibits Diophantine 3-tuples with property $D(11)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4) where

$$c_4 = 2^{2n-1} + 9 * 2^n + 35$$

exhibits Diophantine 3-tuples with property $D(11)$.

The repetition of the above process leads to the generation of sequence of Diophantine 3-tuples whose general form is given by (a, c_s, c_{s+1}) where

$$c_s = 2^{2n-1} + (2s+1)2^n + (2s^2 + 2s - 5), s = 1, 2, 3, \dots$$

Now, consider (b, c_1) and employing the above procedure, it is seen that the triple (b, c_1, c_2) where

$$c_2 = 4 * 2^{2n-1} + 8 * 2^n - 14$$

exhibits Diophantine 3-tuples with property $D(11)$.

Taking (b, c_2) and employing the above procedure, it is seen that the triple (b, c_2, c_3) where

$$c_3 = 9 * 2^{2n-1} + 15 * 2^n - 37$$

exhibits Diophantine 3-tuples with property $D(11)$.

Taking (b, c_3) and employing the above procedure, it is seen that the triple (b, c_3, c_4) where

$$c_4 = 16 * 2^{2n-1} + 24 * 2^n - 70$$

exhibits Diophantine 3-tuples with property $D(11)$.

The repetition of the above process leads to the generation of sequence of Diophantine 3-tuples whose general form is given by (b, c_s, c_{s+1}) where

$$c_s = s^2 2^{2n-1} + (s^2 + 2s) 2^n - (5s^2 - 2s - 2), \quad s = 1, 2, 3, \dots$$

VIII.2 On Sequences of Diophantine 3-Tuples Generated through Pronic Numbers

Sequence: 1

Consider the Pronic numbers Pr_n and Pr_{2n} given by

$$Pr_n = n(n+1), \quad Pr_{2n} = 2n(2n+1)$$

Let $a = 4Pr_n$, $b = Pr_{2n}$

It is observed that

$$ab + n^2 = (4n^2 + 3n)^2$$

Therefore, the pair (a, b) represents diophantine 2-tuple with the property $D(n^2)$.

Let c_1 be any non-zero polynomial in x such that

$$ac_1 + n^2 = p^2 \tag{8.13}$$

$$bc_1 + n^2 = q^2 \tag{8.14}$$

Eliminating c_1 between (8.13) and (8.14), we have

$$bp^2 - aq^2 = (b-a)n^2 \tag{8.15}$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + bT \tag{8.16}$$

in (8.15) and simplifying we get

$$X^2 = abT^2 + n^2$$

which is satisfied by $T=1$, $X = 4n^2 + 3n$

In view of (8.16) and (8.13), it is seen that

$$c_1 = 16n^2 + 12n$$

Note that (a, b, c_1) represents diophantine 3-tuple with property $D(n^2)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 36n^2 + 30n$$

exhibits diophantine 3-tuple with property $D(n^2)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 64n^2 + 56n$$

exhibits diophantine 3-tuple with property $D(n^2)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4)

where

$$c_4 = 100n^2 + 90n$$

exhibits diophantine 3-tuple with property $D(n^2)$.

The repetition of the above process leads to the generation of sequence of diophantine 3-tuples whose general form is given by (a, c_s, c_{s+1}) where

$$c_s = (4s^2 + 8s + 4)n^2 + (4s^2 + 6s + 2), \quad s = 1, 2, 3, \dots$$

Now, consider (b, c_1) and employing the above procedure, it is seen that the triple

(b, c_1, c_2) where

$$c_2 = 36n^2 + 24n$$

exhibits diophantine 3-tuple with property $D(n^2)$.

Taking (b, c_2) and employing the above procedure, it is seen that the triple (b, c_2, c_3)

where

$$c_3 = 64n^2 + 40n$$

exhibits diophantine 3-tuple with property $D(n^2)$.

Taking (b, c_3) and employing the above procedure, it is seen that the triple (b, c_3, c_4)

where

$$c_4 = 100n^2 + 60n$$

exhibits diophantine 3-tuple with property $D(n^2)$.

The repetition of the above process leads to the generation of sequence of diophantine 3-tuples whose general form is given by (b, c_s, c_{s+1}) where

$$c_n = (2s+2)^2 n^2 + (2s^2 + 6s + 4)n, \quad s = 1, 2, 3, \dots$$

Sequence: 2

Consider the Pronic numbers Pr_n and Pr_{2n} given by

$$Pr_n = n(n+1), \quad Pr_{2n} = 2n(2n+1)$$

Let $a = Pr_n$, $b = Pr_{2n}$

It is observed that

$$ab + 2n^2 + 2n^3 = (2n^2 + 2n)^2$$

Therefore, the pair (a, b) represents diophantine 2-tuple with the property $D(2n^2 + 2n^3)$.

Let c_1 be any non-zero polynomial in x such that

$$ac_1 + 2n^2 + 2n^3 = p^2 \tag{8.17}$$

$$bc_1 + 2n^2 + 2n^3 = q^2 \tag{8.18}$$

Eliminating c_1 between (8.17) and (8.18), we have

$$bp^2 - aq^2 = (b-a)(2n^2 + 2n^3) \tag{8.19}$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + bT \quad (8.20)$$

in (8.19) and simplifying we get

$$X^2 = abT^2 + 2n^2 + 2n^3$$

which is satisfied by $T = 1, X = 2n^2 + 2n$

In view of (8.20) and (8.17), it is seen that

$$c_1 = 9n^2 + 7n$$

Note that (a, b, c_1) represents diophantine 3-tuple with property $D(2n^2 + 2n^3)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 16n^2 + 14n$$

exhibits diophantine 3-tuple with property $D(2n^2 + 2n^3)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 25n^2 + 23n$$

exhibits diophantine 3-tuple with property $D(2n^2 + 2n^3)$.

The repetition of the above process leads to the generation of sequence of diophantine

3-tuples whose general form is given by (a, c_s, c_{s+1}) where

$$c_s = (s+2)^2 n^2 + (s^2 + 4s + 2)n, \quad s = 1, 2, 3, \dots$$

Now, consider (b, c_1) and employing the above procedure, it is seen that the triple

(b, c_1, c_2) where

$$c_2 = 25n^2 + 17n$$

exhibits diophantine 3-tuple with property $D(2n^2 + 2n^3)$.

Taking (b, c_2) and employing the above procedure, it is seen that the triple (b, c_2, c_3)

where

$$c_3 = 49n^2 + 31n$$

exhibits diophantine 3-tuple with property $D(2n^2 + 2n^3)$.

The repetition of the above process leads to the generation of sequence of diophantine 3-tuples whose general form is given by (b, c_s, c_{s+1}) where

$$c_s = (2s+1)^2 n^2 + (2s^2 + 4s + 1)n, \quad s = 1, 2, 3, \dots$$

Sequence: 3

Consider the Pronic numbers Pr_n and Pr_{4n} given by

$$Pr_n = n(n+1), \quad Pr_{4n} = 4n(4n+1)$$

Let $a = 4Pr_n$, $b = Pr_{4n}$

It is observed that

$$ab + 9n^2 = (8n^2 + 5n)^2$$

Therefore, the pair (a, b) represents diophantine 2-tuple with the property $D(9n^2)$.

Let c_1 be any non-zero polynomial in x such that

$$ac_1 + 9n^2 = p^2 \tag{8.21}$$

$$bc_1 + 9n^2 = q^2 \tag{8.22}$$

Eliminating c_1 between (8.21) and (8.22), we have

$$bp^2 - aq^2 = (b-a)(9n^2) \tag{8.23}$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + bT \quad (8.24)$$

in (8.23) and simplifying we get

$$X^2 = abT^2 + 9n^2$$

which is satisfied by $T = 1, X = 8n^2 + 5n$

In view of (8.24) and (8.21), it is seen that

$$c_1 = 36n^2 + 18n$$

Note that (a, b, c_1) represents diophantine 3-tuple with property $D(9n^2)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 64n^2 + 40n$$

exhibits diophantine 3-tuple with property $D(9n^2)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 100n^2 + 70n$$

exhibits diophantine 3-tuple with property $D(9n^2)$.

The repetition of the above process leads to the generation of sequence of diophantine

3-tuples whose general form is given by (a, c_s, c_{s+1}) where

$$c_s = (2s + 4)^2 n^2 + (4s^2 + 10s + 4)n, \quad s = 1, 2, 3, \dots$$

Now, consider (b, c_1) and employing the above procedure, it is seen that the triple

(b, c_1, c_2) where

$$c_2 = 100n^2 + 40n$$

exhibits diophantine 3-tuple with property $D(9n^2)$.

Taking (b, c_2) and employing the above procedure, it is seen that the triple (b, c_2, c_3)

where

$$c_3 = 196n^2 + 70n$$

exhibits diophantine 3-tuple with property $D(9n^2)$.

Taking (b, c_3) and employing the above procedure, it is seen that the triple (b, c_3, c_4)

where

$$c_4 = 324n^2 + 108n$$

exhibits diophantine 3-tuple with property $D(9n^2)$.

The repetition of the above process leads to the generation of sequence of diophantine

3-tuples whose general form is given by (b, c_s, c_{s+1}) where

$$c_s = (4s + 2)^2 n^2 + (4s^2 + 10s + 4)n, \quad s = 1, 2, 3, \dots$$

VIII.3A On Sequences of Diophantine 3-Tuples Generated through Euler Polynomials

Sequence: 1

Consider the Euler polynomials $E_1(x)$ and $E_2(x)$ given by

$$E_1(x) = x - \frac{1}{2}, \quad E_2(x) = x^2 - x$$

Let $a = 4(E_1(x))^2$, $b = E_2(x)$

It is observed that

$$ab + 3x^2 - 3x + 1 = (2x^2 - 2x + 1)^2$$

Therefore, the pair (a, b) represents diophantine 2-tuple with the property $D(3x^2 - 3x + 1)$.

Let c_1 be any non-zero polynomial in x such that

$$ac_1 + 3x^2 - 3x + 1 = p^2 \quad (8.25)$$

$$bc_1 + 3x^2 - 3x + 1 = q^2 \quad (8.26)$$

Eliminating c_1 between (8.25) and (8.26), we have

$$bp^2 - aq^2 = (b - a)(3x^2 - 3x + 1) \quad (8.27)$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + bT \quad (8.28)$$

in (8.27) and simplifying we get

$$X^2 = abT^2 + 3x^2 - 3x + 1$$

which is satisfied by $T = 1$, $X = 2x^2 - 2x + 1$

In view of (8.28) and (8.25), it is seen that

$$c_1 = 9E_2(x) + 3$$

Note that (a, b, c_1) represents diophantine 3-tuple with property $D(3x^2 - 3x + 1)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 25E_2(x) + 8$$

exhibits diophantine 3-tuple with property $D(3x^2 - 3x + 1)$.

The repetition of the above process leads to the generation of sequence of diophantine 3-tuples whose general form is given by (a, c_n, c_{n+1}) where

$$c_n = (2n+1)^2 E_2(x) + n^2 + 2n, \quad n = 1, 2, 3, \dots$$

Now, consider (b, c_1) and employing the above procedure, it is seen that the triple

(b, c_1, c_2) where

$$c_2 = 16E_2(x) + 5$$

exhibits diophantine 3-tuple with property $D(3x^2 - 3x + 1)$.

The repetition of the above process leads to the generation of sequence of diophantine 3-tuples whose general form is given by (b, c_n, c_{n+1}) where

$$c_n = (n+2)^2 E_2(x) + 2n+1, \quad n = 1, 2, 3, \dots$$

Sequence: 2

Consider the Euler polynomials $E_1(x)$ and $E_3(x)$ given by

$$E_1(x) = x - \frac{1}{2}, E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}$$

Let $a = 2E_1(x)$, $b = 4E_3(x)$

It is observed that

$$ab + x^4 - 2x^3 + 9x^2 - 8x + 2 = (3x^2 - 3x + 1)^2$$

Therefore, the pair (a, b) represents diophantine 2-tuple with the property

$$D(x^4 - 2x^3 + 9x^2 - 8x + 2).$$

Let c_1 be any non-zero polynomial in x such that

$$ac_1 + x^4 - 2x^3 + 9x^2 - 8x + 2 = p^2 \quad (8.29)$$

$$bc_1 + x^4 - 2x^3 + 9x^2 - 8x + 2 = q^2 \quad (8.30)$$

Eliminating c_1 between (8.29) and (8.30), we have

$$bp^2 - aq^2 = (b - a)(x^4 - 2x^3 + 9x^2 - 8x + 2) \quad (8.31)$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + bT \quad (8.32)$$

in (8.31) and simplifying we get

$$X^2 = abT^2 + x^4 - 2x^3 + 9x^2 - 8x + 2$$

which is satisfied by $T = 1, X = 3x^2 - 3x + 1$

In view of (8.32) and (8.29), it is seen that

$$c_1 = 4x^3 - 4x + 2$$

Note that (a, b, c_1) represents diophantine 3-tuple with property $D(x^4 - 2x^3 + 9x^2 - 8x + 2)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 4x^3 + 6x^2 - 4x + 1$$

exhibits diophantine 3-tuple with property $D(x^4 - 2x^3 + 9x^2 - 8x + 2)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 4x^3 + 12x^2 - 2$$

exhibits diophantine 3-tuple with property $D(x^4 - 2x^3 + 9x^2 - 8x + 2)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4)

where

$$c_4 = 4x^3 + 18x^2 + 8x - 7$$

exhibits diophantine 3-tuple with property $D(x^4 - 2x^3 + 9x^2 - 8x + 2)$.

The repetition of the above process leads to the generation of sequence of diophantine 3-tuples.

Now, consider (b, c_1) and employing the above procedure, it is seen that the triple

(b, c_1, c_2) where

$$c_2 = 16x^3 - 12x^2 - 10x + 7$$

exhibits diophantine 3-tuple with property $D(x^4 - 2x^3 + 9x^2 - 8x + 2)$.

Taking (b, c_2) and employing the above procedure, it is seen that the triple (b, c_2, c_3)

where

$$c_3 = 36x^3 - 36x^2 - 16x + 14$$

exhibits diophantine 3-tuple with property $D(x^4 - 2x^3 + 9x^2 - 8x + 2)$

Taking (b, c_3) and employing the above procedure, it is seen that the triple (b, c_3, c_4)

where

$$c_4 = 64x^3 - 72x^2 - 22x + 23$$

exhibits diophantine 3-tuple with property $D(x^4 - 2x^3 + 9x^2 - 8x + 2)$

The repetition of the above process leads to the generation of sequence of diophantine

3-tuples whose general form is given by (b, c_n, c_{n+1}) where

$$c_n = (n+2)^2 E_2(x) + 2n+1, \quad n=1, 2, 3, \dots$$

VIII.3B On Sequences of Diophantine 3-Tuples Generated through Bernoulli Polynomials

Sequence: 1

Consider the Bernoulli polynomials $B_0(x)$ and $B_2(x)$ given by

$$B_0(x) = 1, \quad B_2(x) = x^2 - x + \frac{1}{6}$$

Let $a = B_0(x) = 1$, $b = 6B_2(x) = 6x^2 - 6x + 1$

It is observed that

$$ab + 3x^2 = 9x^2 - 6x + 1 = (3x - 1)^2$$

Therefore, the pair (a, b) represents diophantine 2-tuple with the property $D(3x^2)$.

Let c_1 be any non-zero polynomial in x such that

$$ac_1 + 3x^2 = p^2 \tag{8.33}$$

$$bc_1 + 3x^2 = q^2 \tag{8.34}$$

Eliminating c_1 between (8.33) and (8.34), we have

$$bp^2 - aq^2 = (b - a)(3x^2) \tag{8.35}$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + bT \tag{8.36}$$

in (8.35) and simplifying we get

$$X^2 = abT^2 + 3x^2$$

which is satisfied by $T = 1$, $X = 3x - 1$

In view of (8.36) and (8.33), it is seen that

$$c_1 = 9x^2 - 3x^2 = 6x^2$$

Note that (a, b, c_1) represents diophantine 3-tuple with property $D(3x^2)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 6x^2 + 6x + 1$$

exhibits diophantine 3-tuple with property $D(3x^2)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 6x^2 + 12x + 4$$

exhibits diophantine 3-tuple with property $D(3x^2)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4)

where

$$c_4 = 6x^2 + 18x + 9$$

exhibits diophantine 3-tuple with property $D(3x^2)$.

Taking (a, c_4) and employing the above procedure, it is seen that the triple (a, c_3, c_4)

where

$$c_5 = 6x^2 + 24x + 16$$

The repetition of the above process leads to the generation of sequence of diophantine

3-tuples whose general form is given by (a, c_n, c_{n+1}) where

$$c_n = 6x^2 + 6(n-1)x + (n-1)^2, \quad n=1, 2, 3, \dots$$

Sequence: 2

Consider the Bernoulli polynomials $B_1(x)$ and $B_3(x)$ given by

$$B_1(x) = x - \frac{1}{2}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

Let $a = 2B_1(x) = 2x - 1$, $b = 2B_3(x) = 2x^3 - 3x^2 + x$

It is observed that

$$ab + 3x^2 - 3x + 1 = (2x^2 - 2x + 1)^2$$

Therefore, the pair (a, b) represents diophantine 2-tuple with the property $D(3x^2 - 3x + 1)$.

Let c_1 be any non-zero polynomial in x such that

$$ac_1 + 3x^2 - 3x + 1 = p^2 \quad (8.37)$$

$$bc_1 + 3x^2 - 3x + 1 = q^2 \quad (8.38)$$

Eliminating c_1 between (8.37) and (8.38), we have

$$bp^2 - aq^2 = (b - a)(3x^2 - 3x + 1) \quad (8.39)$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + bT \quad (8.40)$$

in (8.39) and simplifying we get

$$X^2 = abT^2 + 3x^2 - 3x + 1$$

which is satisfied by $T = 1$, $X = 2x^2 - 2x + 1$

In view of (8.40) and (8.37), it is seen that

$$c_1 = 2x^3 + x^2 - x + 1$$

Note that (a, b, c_1) represents diophantine 3-tuple with property $D(3x^2 - 3x + 1)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 2x^3 + 5x^2 + x$$

exhibits diophantine 3-tuple with property $D(3x^2 - 3x + 1)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 2x^3 + 9x^2 + 7x - 3$$

exhibits diophantine 3-tuple with property $D(3x^2 - 3x + 1)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4)

where

$$c_4 = 2x^3 + 13x^2 + 17x - 8$$

exhibits diophantine 3-tuple with property $D(3x^2 - 3x + 1)$.

Taking (a, c_4) and employing the above procedure, it is seen that the triple (a, c_4, c_5)

where

$$c_5 = 2x^3 + 17x^2 + 31x - 15$$

exhibits diophantine 3-tuple with property $D(3x^2 - 3x + 1)$.

The repetition of the above process leads to the generation of sequence of diophantine

3-tuples whose general form is given by (a, c_n, c_{n+1}) where

$$c_n = 2x^3 + (4n-3)x^2 + (2n^2 - 4n + 1)x + (2n - n^2), \quad n = 1, 2, 3, \dots$$

VIII.3C On Sequences of Diophantine 3-Tuples Generated through Euler and Bernoulli Polynomials

Consider the Euler polynomial $E_2(x)$ and Bernoulli polynomial $B_2(x)$ given by

$$E_2(x) = x^2 - x, \quad B_2(x) = x^2 - x + \frac{1}{6}$$

Let $a = E_2(x) = x^2 - x$, $b = 6B_2(x) = 6x^2 - 6x + 1$

It is observed that

$$ab + 3x^4 + 3x^2 - 3x + 1 = (3x^2 - 2x + 1)^2$$

Therefore, the pair (a, b) represents diophantine 2-tuple with the property

$$D(3x^4 + 3x^2 - 3x + 1).$$

Let c_1 be any non-zero polynomial in x such that

$$ac_1 + 3x^4 + 3x^2 - 3x + 1 = p^2 \quad (8.41)$$

$$bc_1 + 3x^4 + 3x^2 - 3x + 1 = q^2 \quad (8.42)$$

Eliminating c_1 between (8.41) and (8.42), we have

$$bp^2 - aq^2 = (b - a)(3x^4 + 3x^2 - 3x + 1) \quad (8.43)$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + bT \quad (8.44)$$

in (8.43) and simplifying we get

$$X^2 = abT^2 + 3x^4 + 3x^2 - 3x + 1$$

which is satisfied by $T = 1$, $X = 3x^2 - 2x + 1$

In view of (8.44) and (8.41), it is seen that

$$c_1 = 13x^2 - 11x + 3$$

Note that (a, b, c_1) represents diophantine 3-tuple with property $D(3x^4 + 3x^2 - 3x + 1)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 22x^2 - 18x + 5$$

exhibits diophantine 3-tuple with property $D(3x^4 + 3x^2 - 3x + 1)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 33x^2 - 27x + 7$$

exhibits diophantine 3-tuple with property $D(3x^4 + 3x^2 - 3x + 1)$

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4)

where

$$c_4 = 46x^2 - 38x + 9$$

exhibits diophantine 3-tuple with property $D(3x^4 + 3x^2 - 3x + 1)$.

The repetition of the above process leads to the generation of sequence of diophantine

3-tuples whose general form is given by (a, c_s, c_{s+1}) where

$$c_s = (s^2 + 6s + 6)x^2 - (s^2 + 4s + 6)x + 2s + 1, \quad s = 1, 2, 3, \dots$$

A few numerical examples are presented in Table 8.1 below:

Table 8.1: Numerical Examples

x	D	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)	(a, c_4, c_5)
2	55	(2,33,57)	(2,57,85)	(2,85,117)	(2,117,153)
3	262	(6,87,149)	(6,149,223)	(6,223,309)	(6,309,407)
4	805	(12,167,285)	(12,285,427)	(12,427,593)	(12,593,783)
5	1936	(20,273,465)	(20,465,697)	(20,697,969)	(20,969,1281)

Now, consider (b, c_1) and employing the above procedure, it is seen that the triple

(b, c_1, c_2) where

$$c_2 = 37x^2 - 33x + 8$$

exhibits diophantine 3-tuple with property $D(3x^4 + 3x^2 - 3x + 1)$.

Taking (b, c_2) and employing the above procedure, it is seen that the triple (b, c_2, c_3)

where

$$c_3 = 73x^2 - 67x + 15$$

exhibits diophantine 3-tuple with property $D(3x^4 + 3x^2 - 3x + 1)$.

Taking (b, c_3) and employing the above procedure, it is seen that the triple (b, c_3, c_4)

where

$$c_4 = 121x^2 - 113x + 24$$

exhibits diophantine 3-tuple with property $D(3x^4 + 3x^2 - 3x + 1)$.

The repetition of the above process leads to the generation of sequence of diophantine

3-tuples whose general form is given by (b, c_n, c_{n+1}) where

A few numerical examples are presented in Table 8.2 below:

Table 8.2: Numerical Examples

x	D	(b, c_1, c_2)	(b, c_2, c_3)	(b, c_3, c_4)	(b, c_4, c_5)
2	55	(13, 33, 90)	(13, 90, 173)	(13, 173, 282)	(13, 282, 417)
3	262	(37, 87, 242)	(37, 242, 471)	(37, 471, 774)	(37, 774, 1151)
4	805	(73, 167, 468)	(73, 468, 915)	(73, 915, 1508)	(73, 1508, 2247)
5	1936	(121, 273, 768)	(121, 768, 1505)	(121, 1505, 2484)	(121, 2484, 3705)

For simplicity and brevity, some more sequences of 3-tuples generated through Euler and Bernoulli polynomials are presented in Table 8.3 below.

Table 8.3: Sequences of 3-tuples

a	b	c_1	D	Sequences of 3-tuples
$2E_1(x)$	$2B_3(x)$	$2x^3 + x^2 - x + 1$	$3x^2 - 3x + 1$	$\{a, c_s, c_{s+1}\}_{s=1,2,3,\dots}, \left\{ \begin{array}{l} c_s = 2x^3 + (4s-3)x^2 + \\ (2s^2 - 4s + 1)x + (-s^2 + 2s) \end{array} \right\}, s = 1, 2, 3, \dots$ $\{b, c_s, c_{s+1}\}_{s=1,2,3,\dots}, \left\{ \begin{array}{l} c_s = 2s^2x^3 + (-3s^2 + 4s)x^2 + \\ (s^2 - 4s + 2)x + 2s - 1 \end{array} \right\}, s = 1, 2, 3, \dots$
$2E_1(x)$	$6B_2(x)$	$6x^2 - 2x + 8$	$-12x^3 + 19x^2 + 17$	$\{a, c_s, c_{s+1}\}_{s=1,2,3,\dots}, \left\{ \begin{array}{l} c_s = 6x^2 + (2s^2 + 2s - 6)x \\ + (-s^2 + 8s + 1) \end{array} \right\}, s = 1, 2, 3, \dots$ $\{b, c_s, c_{s+1}\}_{s=1,2,3,\dots}, \left\{ \begin{array}{l} c_s = 6s^2x^2 + (-6s^2 + 2s + 2)x \\ + s^2 + 8s - 1 \end{array} \right\}, s = 1, 2, 3, \dots$
$E_0(x)$	$6B_2(x)$	$6x^2 + 2 - 2n$	$3x^2 + (6 - 6n)x + n^2 - 1$	$\{a, c_s, c_{s+1}\}_{s=1,2,3,\dots}, \left\{ \begin{array}{l} c_s = 6x^2 + 6(s-1)x \\ + (s^2 - 2ns + 1) \end{array} \right\}, s = 1, 2, 3, \dots$ $\{b, c_s, c_{s+1}\}_{s=1,2,3,\dots}, \left\{ \begin{array}{l} c_s = 6s^2x^2 - (6s^2 - 6s)x \\ - 2sn + (s^2 + 1) \end{array} \right\}, s = 1, 2, 3, \dots$

a	b	c_1	D	Sequences of 3-tuples
$B_0(x)$	$E_2(x)$	$x^2 + x + 2n + 1$	$(2n+1)x + n^2$	$\{a, c_s, c_{s+1}\}_{s=1,2,3,\dots}, \left\{ \begin{array}{l} c_s = x^2 + (2s-1)x \\ + 2sn + s^2 \end{array} \right\}, s = 1, 2, 3, \dots$ $\{b, c_s, c_{s+1}\}_{s=1,2,3,\dots}, \left\{ \begin{array}{l} c_s = s^2 x^2 + (-s^2 + 2s)x \\ + (2sn + 1) \end{array} \right\}, s = 1, 2, 3, \dots$
$2B_1(x)$	$4E_3(x)$	$4x^3 - 4x - 2$	$x^4 - 2x^3 - 3x^2 + 4x + 2$	$\{a, c_s, c_{s+1}\}_{s=1,2,3,\dots}, \left\{ \begin{array}{l} c_s = 4x^3 + 6(s-1)x^2 \\ - (s^2 + 2s - 1) + (2s^2 - 6s)x \end{array} \right\}, s = 1, 2, 3, \dots$ $\{b, c_s, c_{s+1}\}_{s=1,2,3,\dots}, \left\{ \begin{array}{l} c_s = 4s^2 x^3 + (6s - 6s^2)x^2 \\ + (2 - 6s)x + (s^2 - 2s - 1) \end{array} \right\}, s = 1, 2, 3, \dots$

VIII.4 Generation of Diophantine 3-tuples Through Matrix Method

Initially, construct a diophantine 2-tuple with property $D(k^2 + 10k - 3)$ and then, extend it to diophantine 3-tuple.

Let 4, 7 be two distinct integers such that

$$4 \cdot 7 + k^2 + 10k - 3 = (k + 5)^2, \text{ a perfect square}$$

Therefore, the pair (4,7) exhibits diophantine double having property $D(k^2 + 10k - 3)$.

If c is the 3rd tuple, then it satisfies corresponding double equations

$$4c + k^2 + 10k - 3 = p^2 \quad (8.45)$$

$$7c + k^2 + 10k - 3 = q^2 \quad (8.46)$$

The eliminant of c in the above two equations leads to

$$7p^2 - 4q^2 = 3(k^2 + 10k - 3) \quad (8.47)$$

Taking

$$p = X + 4T, \quad q = X + 7T \quad (8.48)$$

in (8.47) and simplifying, we get

$$X^2 = 28T^2 + k^2 + 10k - 3$$

which is satisfied by $T = 1, X = k + 5$

From (8.48) and (8.45), observe

$$c = 2k + 21$$

Note that $(4, 7, 2k + 21)$ is a diophantine triple satisfying the property $D(k^2 + 10k - 3)$.

The process of obtaining other diophantine triples with property $D(k^2 + 10k - 3)$ is illustrated below:

Let M be a 3×3 square matrix given by

$$M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad (8.49)$$

Now,

$$(4, 7, 2k + 21)M = (4, 2k + 21, 4k + 43)$$

Note that

$$4 * (2k + 21) + k^2 + 10k - 3 = (k + 9)^2$$

$$4 * (4k + 43) + k^2 + 10k - 3 = (k + 13)^2$$

$$(2k + 21) * (4k + 43) + k^2 + 10k - 3 = (3k + 30)^2$$

$\therefore$ The triple $(4, 2k + 21, 4k + 43)$ is a diophantine triple having the property

$$D(k^2 + 10k - 3).$$

Performing the above analysis, the general form of diophantine triple $(4, c_{s-1}, c_s)$ is given by

$$(4, 4s^2 + (2k + 2)s - 2k + 1, 4(s + 1)^2 + (2k + 2)s + 3), s = 1, 2, 3, \dots$$

A few numerical illustrations are given in table 8.4 below:

Table 8.4: Numerical Illustrations

k	$(4, c_0, c_1)$	$(4, c_1, c_2)$	$(4, c_2, c_3)$	$D(k^2 + 10k - 3)$
0	(4, 7, 21)	(4, 21, 43)	(4, 43, 73)	$D(-3)$
-1	(4, 7, 19)	(4, 19, 39)	(4, 39, 67)	$D(-12)$
-2	(4, 7, 17)	(4, 17, 35)	(4, 35, 61)	$D(-19)$
-3	(4, 7, 15)	(4, 15, 31)	(4, 31, 55)	$D(-24)$
1	(4, 7, 23)	(4, 23, 47)	(4, 47, 79)	$D(8)$

It is noted that the triple $(c_{s-1}, c_s + 4, c_{s+1})$, $s = 1, 2, 3, \dots$ forms an arithmetic progression.

Note 1

It is obvious that $(7, 4, 2k + 21)$ is a Diophantine 3-tuples with property $D(k^2 + 10k - 3)$.

Following the procedure as above, the Diophantine triples obtained are $(7, 4, 2k + 21)$,

$(7, 2k + 21, 4k + 52)$, $(19, 4k + 52, 6k + 97)$, each with property $D(k^2 + 10k - 3)$

whose general form $(7, c_{s-1}, c_s)$ is

$$(7, 7s^2 + (2k - 4)s - 2k + 1, 7(s + 1)^2 + (2k - 4)s - 3), s = 1, 2, 3, \dots$$

Note that $(c_{s-1}, c_s + 7, c_{s+1})$ forms an Arithmetic Progression.

Note 2

In addition to (8.108), one may consider the linear transformation given by

$$p = X - 4T, \quad q = X - 7T$$

For this case, employing the procedure as above one obtains two sets of sequences of Diophantine 3-tuples in which, each triple has the property $D(k^2 + 10k - 3)$. For simplicity and brevity, the general form of the triple in the sequence of Diophantine 3-tuples is presented:

Set 1: $(4, \alpha_{s-1}, \alpha_s)$ where $\alpha_{s-1} = 4s^2 + (-2k - 18)s + 2k + 21$, $s = 1, 2, 3, \dots$

Note that $(\alpha_{s-1}, \alpha_s + 4, \alpha_{s+1})$ forms an Arithmetic Progression.

Set 2: $(7, \alpha_{s-1}, \alpha_s)$ where $\alpha_{s-1} = 7s^2 + (-2k - 24)s + 2k + 21$, $s = 1, 2, 3, \dots$

Note that $(\alpha_{s-1}, \alpha_s + 7, \alpha_{s+1})$ forms an Arithmetic Progression.

Remark

Instead of (8.109), suppose we have a third order square matrix N given by

$$N = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

Following the procedure presented above, one obtains 4 more sets of Diophantine triples, each with property $D(k^2 + 10k - 3)$.

To conclude, one may search for other choices of Matrices for the formulation of collections of Diophantine triples with suitable properties.

Chapter – IX

Dio 3-Tuples

CHAPTER – IX

DIO 3-TUPLES

An attempt has made in constructing sequences of dio 3- tuples (a, b, c) such that the product of any two elements of the set added with the sum or minus the sum of the same elements and increased by a polynomial with integer coefficient is a perfect square.

Formulation of Special Dio 3-Tuples generated through Polynomials with Suitable Property

Sequence: 1

Let $a = 3, b = 5k$

It is observed that

$$ab + a + b + 4k^2 - 8k + 6 = (2k + 3)^2$$

Therefore, the pair (a, b) represents dio 2-tuple with the property $D(4k^2 - 8k + 6)$.

Let c_1 be any non-zero polynomial in x such that

$$(a+1)c_1 + a + 4k^2 - 8k + 6 = p^2 \quad (9.1)$$

$$(b+1)c_1 + b + 4k^2 - 8k + 6 = q^2 \quad (9.2)$$

Eliminating c_1 between (9.1) and (9.2), we have

$$(b+1)p^2 - (a+1)q^2 = (a-b) + (b-a)(4k^2 - 8k + 6) \quad (9.3)$$

Introducing the linear transformations

$$p = X + (a+1)T, \quad q = X + (b+1)T \quad (9.4)$$

in (9.3) and simplifying we get

$$X^2 = (a+1)(b+1)T^2 + 4k^2 - 8k + 5$$

which is satisfied by $T = 1$, $X = 2k + 3$

In view of (9.4) and (9.1), it is seen that

$$c_1 = 9k + 10$$

Note that (a, b, c_1) represents dio 3-tuple with property $D(4k^2 - 8k + 6)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 13k + 28$$

exhibits dio 3-tuple with property $D(4k^2 - 8k + 6)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 17k + 54$$

exhibits dio 3-tuple with property $D(4k^2 - 8k + 6)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4)

where

$$c_4 = 21k + 88$$

exhibits dio 3-tuple with property $D(4k^2 - 8k + 6)$.

The repetition of the above process leads to the generation of sequence of dio 3-tuples

whose general form is given by (a, c_n, c_{n+1}) where

$$c_n = (5 + 4n)k + 2n(2n + 3), \quad n = 1, 2, 3, \dots$$

A few numerical examples are presented in Table 9.1 below:

Table 9.1: Numerical Examples

k	D	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	6	(3, 28, 54)	(3, 54, 88)	(3, 88, 130)
3	18	(3, 37, 67)	(3, 67, 105)	(3, 105, 151)
4	38	(3, 46, 80)	(3, 80, 122)	(3, 122, 172)
5	66	(3, 55, 93)	(3, 93, 139)	(3, 139, 193)

Sequence: 2

Let $a = 5$, $b = 2k + 6$

It is observed that

$$ab - a - b + k^2 - 3 = (k + 4)^2$$

Therefore, the pair (a, b) represents dio 2-tuple with the property $D(k^2 - 3)$.

Let c_1 be any non-zero polynomial in x such that

$$(a-1)c_1 - a + k^2 - 3 = p^2 \quad (9.5)$$

$$(b-1)c_1 - b + k^2 - 3 = q^2 \quad (9.6)$$

Eliminating c_1 between (9.5) and (9.6), we have

$$(b-1)p^2 - (a-1)q^2 = (a-b) + (b-a)(k^2 - 3) \quad (9.7)$$

Introducing the linear transformations

$$p = X + (a-1)T, \quad q = X + (b-1)T \quad (9.8)$$

in (9.7) and simplifying we get

$$X^2 = (a-1)(b-1)T^2 + k^2 - 4$$

which is satisfied by $T = 1$, $X = k + 4$

In view of (9.8) and (9.5), it is seen that

$$c_1 = 4k + 18$$

Note that (a, b, c_1) represents dio 3-tuple with property $D(k^2 - 3)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 6k + 38$$

exhibits dio 3-tuple with property $D(k^2 - 3)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 8k + 66$$

exhibits dio 3-tuple with property $D(k^2 - 3)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4)

where

$$c_4 = 10k + 102$$

exhibits dio 3-tuple with property $D(k^2 - 3)$.

The repetition of the above process leads to the generation of sequence of dio 3-tuples

whose general form is given by (a, c_n, c_{n+1}) where

$$c_n = (2n + 2)k + (4n^2 + 8n + 6), \quad n = 1, 2, 3, \dots$$

A few numerical examples are presented in Table 9.2 below:

Table 9.2: Numerical Examples

k	D	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	1	(5, 26, 50)	(5, 50, 82)	(5, 82, 122)
3	6	(5, 30, 56)	(5, 56, 90)	(5, 90, 132)
4	13	(5, 34, 62)	(5, 62, 98)	(5, 98, 142)
5	22	(5, 38, 68)	(5, 68, 106)	(5, 106, 152)

Sequence: 3

Let $a = 5$, $b = 2k + 6$

It is observed that

$$ab + a + b + k^2 - 5 = (k + 6)^2$$

Therefore, the pair (a, b) represents dio 2-tuple with the property $D(k^2 - 5)$.

Let c_1 be any non-zero polynomial in x such that

$$(a+1)c_1 + a + k^2 - 5 = p^2 \quad (9.9)$$

$$(b+1)c_1 + b + k^2 - 5 = q^2 \quad (9.10)$$

Eliminating c_1 between (9.9) and (9.10), we have

$$(b+1)p^2 - (a+1)q^2 = (a-b) + (b-a)(k^2 - 5) \quad (9.11)$$

Introducing the linear transformations

$$p = X + (a+1)T, \quad q = X + (b+1)T \quad (9.12)$$

in (9.11) and simplifying we get

$$X^2 = (a+1)(b+1)T^2 + k^2 - 6$$

which is satisfied by $T = 1$, $X = k + 6$

In view of (9.12) and (9.9), it is seen that

$$c_1 = 4k + 24$$

Note that (a, b, c_1) represents dio 3-tuple with property $D(k^2 - 5)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 6k + 54$$

exhibits dio 3-tuple with property $D(k^2 - 5)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3) where

$$c_3 = 8k + 96$$

exhibits dio 3-tuple with property $D(k^2 - 5)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4) where

$$c_4 = 10k + 150$$

exhibits dio 3-tuple with property $D(k^2 - 5)$.

The repetition of the above process leads to the generation of sequence of dio 3-tuples whose general form is given by (a, c_n, c_{n+1}) where

$$c_n = (2n + 2)k + 6(n + 1)^2, \quad n = 1, 2, 3, \dots$$

A few numerical examples are presented in Table 9.3 below:

Table 9.3: Numerical Examples

k	D	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	-1	(5, 32, 66)	(5, 66, 112)	(5, 112, 170)
3	4	(5, 36, 72)	(5, 72, 120)	(5, 120, 180)
4	11	(5, 40, 78)	(5, 78, 128)	(5, 128, 190)
5	20	(5, 44, 84)	(5, 84, 136)	(5, 136, 200)

Sequence: 4

Let $a = 5$, $b = 6k^2 + 8k$

It is observed that

$$ab + a + b + 11 = (6k + 4)^2$$

Therefore, the pair (a, b) represents dio 2-tuple with the property $D(11)$.

Let c_1 be any non-zero polynomial in x such that

$$(a+1)c_1 + a + 11 = p^2 \quad (9.13)$$

$$(b+1)c_1 + b + 11 = q^2 \quad (9.14)$$

Eliminating c_1 between (9.13) and (9.14), we have

$$(b+1)p^2 - (a+1)q^2 = (a-b) + (b-a)(11) \quad (9.15)$$

Introducing the linear transformations

$$p = X + (a+1)T, \quad q = X + (b+1)T \quad (9.16)$$

in (9.15) and simplifying we get

$$X^2 = (a+1)(b+1)T^2 + 10$$

which is satisfied by $T=1, X=6k+4$

In view of (9.16) and (9.13), it is seen that

$$c_1 = 6k^2 + 20k + 14$$

Note that (a, b, c_1) represents dio 3-tuple with property $D(11)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 6k^2 + 32k + 40$$

exhibits dio 3-tuple with property $D(11)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 6k^2 + 44k + 78$$

exhibits dio 3-tuple with property $D(11)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4) where

$$c_4 = 6k^2 + 56k + 128$$

exhibits dio 3-tuple with property $D(11)$.

The repetition of the above process leads to the generation of sequence of dio 3-tuples whose general form is given by (a, c_n, c_{n+1}) where

$$c_n = 6k^2 + 4(3n+2)k + 2n(3n+4), \quad n = 1, 2, 3, \dots$$

A few numerical examples are presented in Table 9.4 below:

Table 9.4: Numerical Examples

k	D	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	11	(5, 78, 128)	(5, 78, 190)	(5, 190, 264)
3	11	(5, 128, 190)	(5, 190, 264)	(5, 264, 350)
4	11	(5, 190, 264)	(5, 264, 350)	(5, 350, 448)
5	11	(5, 264, 350)	(5, 350, 448)	(5, 448, 558)

Sequence: 5

Let $a = 5$, $b = 6k^2 + 4k - 2$

It is observed that

$$ab + a + b + 11 = (6k + 2)^2$$

Therefore, the pair (a, b) represents dio 2-tuple with the property $D(11)$.

Let c_1 be any non-zero polynomial in x such that

$$(a+1)c_1 + a + 11 = p^2 \tag{9.17}$$

$$(b+1)c_1 + b + 11 = q^2 \tag{9.18}$$

Eliminating c_1 between (9.17) and (9.18), we have

$$(b+1)p^2 - (a+1)q^2 = (a-b) + (b-a)(11) \quad (9.19)$$

Introducing the linear transformations

$$p = X + (a+1)T, \quad q = X + (b+1)T \quad (9.20)$$

in (9.19) and simplifying we get

$$X^2 = (a+1)(b+1)T^2 + 10$$

which is satisfied by $T=1, X=6k+2$

In view of (9.20) and (9.17), it is seen that

$$c_1 = 6k^2 + 16k + 8$$

Note that (a, b, c_1) represents dio 3-tuple with property $D(11)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 6k^2 + 28k + 30$$

exhibits dio 3-tuple with property $D(11)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 6k^2 + 40k + 64$$

exhibits dio 3-tuple with property $D(11)$.

Taking (a, c_3) and employing the above procedure, it is seen that the triple (a, c_3, c_4)

where

$$c_4 = 6k^2 + 52k + 110$$

exhibits dio 3-tuple with property $D(11)$.

The repetition of the above process leads to the generation of sequence of dio 3-tuples whose general form is given by (a, c_n, c_{n+1}) where

$$c_n = 6k^2 + 4(3n+1)k + (6n^2 + 4n - 2), \quad n = 1, 2, 3, \dots$$

A few numerical examples are presented in Table 9.5 below:

Table 9.5: Numerical Examples

k	D	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	11	(5, 64, 110)	(5, 110, 168)	(5, 168, 238)
3	11	(5, 110, 168)	(5, 168, 238)	(5, 238, 320)
4	11	(5, 168, 238)	(5, 238, 320)	(5, 320, 414)
5	11	(5, 238, 320)	(5, 320, 414)	(5, 414, 520)

Sequence 6

Let $a = 1, \quad b = n$

It is observed that

$$ab + a + b + n^2 + s^2 + (2s - 2)n - 1 = (n + s)^2$$

Therefore, the pair (a, b) represents dio 2-tuple with the property $D(n^2 + s^2 + (2s - 2)n - 1)$.

Let c_1 be any non-zero polynomial in x such that

$$(a + 1)c_1 + a + n^2 + s^2 + (2s - 2)n - 1 = p^2 \quad (9.21)$$

$$(b + 1)c_1 + b + n^2 + s^2 + (2s - 2)n - 1 = q^2 \quad (9.22)$$

Eliminating c_1 between (9.21) and (9.22), we have

$$(b + 1)p^2 - (a + 1)q^2 = (a - b) + (b - a)(n^2 + s^2 + (2s - 2)n - 1) \quad (9.23)$$

Introducing the linear transformations

$$p = X + (a + 1)T, \quad q = X + (b + 1)T \quad (9.24)$$

in (9.23) and simplifying we get

$$X^2 = (a+1)(b+1)T^2 + n^2 + s^2 + (2s-2)n$$

which is satisfied by $T=1$, $X=n+s$

In view of (9.24) and (9.21), it is seen that

$$c_1 = 3n + 2s + 2$$

Note that (a, b, c_1) represents dio 3-tuple with property $D(n^2 + s^2 + (2s-2)n - 1)$.

Taking (a, c_1) and employing the above procedure, it is seen that the triple (a, c_1, c_2)

where

$$c_2 = 5n + 4s + 8$$

exhibits dio 3-tuple with property $D(n^2 + s^2 + (2s-2)n - 1)$.

Taking (a, c_2) and employing the above procedure, it is seen that the triple (a, c_2, c_3)

where

$$c_3 = 7n + 6s + 18$$

exhibits dio 3-tuple with property $D(n^2 + s^2 + (2s-2)n - 1)$.

The repetition of the above process leads to the generation of sequence of dio 3-tuples

whose general form is given by (a, c_n, c_{n+1}) where

$$c_n = (2S+1)n + 2Ss + 2S^2, \quad S=1, 2, 3, \dots$$

A few numerical examples are presented in Table 9.6 below:

Table 9.6: Numerical Examples

n	s	D	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	2	11	(1, 12, 26)	(1, 26, 44)	(1, 44, 66)
3	3	29	(1, 17, 35)	(1, 35, 57)	(1, 57, 83)
4	4	55	(1, 22, 44)	(1, 44, 70)	(1, 70, 100)
5	5	89	(1, 27, 53)	(1, 53, 83)	(1, 83, 117)

Chapter – X

Diophantine and Dio Quadruples

CHAPTER – X

DIOPHANTINE AND DIO QUADRUPLES

Chapter X has two sections X.1 and X.2

Section X.1 presents diophantine quadruples (a, b, c, d) generated from two given pronic numbers such that the product of any two members of the set increased by one is a perfect square.

Section X.2 has two subsections X.2A and X.2B

The Sub-section X.2A deals with the study of formulating special Dio-quadruples (a, b, c, d) generated through Euler polynomials such that the product of any two of the set minus their sum and increased by two is a perfect square. Sub-section X.2B concerns with constructing special Dio-quadruples (a, b, c, d) generated through Euler polynomials such that the product of any two of the set minus their sum and increased by five is a perfect square.

X.1 Sequences of $D(1)$ Diophantine Quadruples generated through

Pronic Numbers

Let $a = PR_{(n-2)} = n^2 - 3n + 2$ and $b = PR_n = n^2 + n$ be two pronic numbers such that $ab+1$ is a perfect square.

Let c_s be any non-zero polynomial such that

$$ac_s + 1 = p_s^2 \quad (10.1)$$

$$bc_s + 1 = q_s^2 \quad (10.2)$$

Eliminating c_s from (10.1) and (10.2), we get

$$bp_s^2 - aq_s^2 = (b-a) \quad (10.3)$$

Setting,

$$p_s = X_s + a T_s \quad (10.4)$$

$$q_s = X_s + b T_s \quad (10.5)$$

in (10.3), we have

$$X_s^2 = ab T_s^2 + 1 \quad (10.6)$$

whose initial solution is $T_0 = 1, X_0 = n^2 - n - 1$

The general solution of (10.6) is

$$X_s = \frac{1}{2} f_s, \quad T_s = \frac{1}{2\sqrt{ab}} g_s \quad (10.7)$$

where

$$f_s = \left(n^2 - n - 1 + \sqrt{ab}\right)^{s+1} + \left(n^2 - n - 1 - \sqrt{ab}\right)^{s+1}$$

$$g_s = \left(n^2 - n - 1 + \sqrt{ab}\right)^{s+1} - \left(n^2 - n - 1 - \sqrt{ab}\right)^{s+1}$$

Substitution of (10.7) in (10.4) gives

$$p_s = \frac{1}{2} f_s + a * \frac{1}{2\sqrt{ab}} g_s \quad (10.8)$$

From (10.8) and (10.1) we get

$$c_s = \frac{(p_s^2 - 1)}{a} \quad (10.9)$$

Substituting $s = 0$ in (10.9) we have,

$$c_0 = \frac{(p_0^2 - 1)}{a}$$

$$(i.e.), c_0 = 4n^2 - 4n$$

Note that, the tuple (a, b, c_0) is a dio-triple with property $D(1)$.

Again, substituting $s = 1, 2$ in (10.9) and simplifying we get,

$$c_1 = (4n^2 - 2)(4n^4 - 12n^3 + 6n^2 + 6n - 2)$$

$$c_2 = (8n^5 - 16n^4 - 4n^3 + 16n^2 - 2)(8n^5 - 24n^4 + 4n^3 + 28n^2 - 4n - 6)$$

It is seen that each of the quadruples (a, b, c_0, c_1) and (a, b, c_1, c_2) represents diophantine quadruples with property $D(1)$.

In general, it is observed that the quadruple (a, b, c_{s-1}, c_s) , $s = 1, 2, 3, \dots$ is a diophantine quadruple with property $D(1)$. Some numerical examples are given in Table 10.1 below:

Table 10.1: $D(1)$ Diophantine-quadruples

n	(a, b, c_0, c_1)	(a, b, c_1, c_2)	(a, b, c_2, c_3)
3	(6,20,48,23188)	(6,20,23188,11176620)	(6,20,11176620,5787107704)
4	(2,12,24,2380)	(2,12,2380,233244)	(2,12, 233244,22855560)
5	(12,30,80,115444)	(12,30,115444,166470252)	(12,30,166470252,240049988024)

X.2 Two special Dio-quadruples generated through Euler Polynomials

Subsection X.2A: Formulation of $D(2)$ Dio-quadruples

Consider the Euler polynomials $E_1(x)$ and $E_2(x)$ given by

$$E_1(x) = x - \frac{1}{2}, \quad E_2(x) = x^2 - x$$

Let $a = [2E_1(x)]^2 = 4x^2 - 4x + 1$ and $b = E_2(x) = x^2 - x$ be two polynomials such that $ab - (a + b) + 2$ is a perfect square.

Let $c_N(x)$ be any non-zero polynomial such that

$$(a-1)c_N(x) - a + 2 = p_N^2 \quad (10.10)$$

$$(b-1)c_N(x) - b + 2 = q_N^2 \quad (10.11)$$

Eliminating $c_N(x)$ from (10.10) and (10.11), we get

$$(b-1)p_N^2 - (a-1)q_N^2 = (b-a) \quad (10.12)$$

Setting

$$p_N = X_N + (a-1)T_N \quad (10.13)$$

$$q_N = X_N + (b-1)T_N \quad (10.14)$$

in (10.12), we have

$$X_N^2 = (b-1)(a-1)T_N^2 + 1 \quad (10.15)$$

whose initial solution is $T_0 = 1$, $X_0 = 2x^2 - 2x - 1$

The general solution of (10.15) is

$$X_N = \frac{1}{2}f_N, \quad T_N = \frac{1}{2\sqrt{(a-1)(b-1)}}g_N \quad (10.16)$$

where

$$f_N = \left(X_0 + \sqrt{(a-1)(b-1)T_0} \right)^{N+1} + \left(X_0 - \sqrt{(a-1)(b-1)T_0} \right)^{N+1}$$

$$g_N = \left(X_0 + \sqrt{(a-1)(b-1)T_0} \right)^{N+1} - \left(X_0 - \sqrt{(a-1)(b-1)T_0} \right)^{N+1}$$

Substitution of (10.16) in (10.13) gives

$$p_N = \frac{1}{2} f_N + (a-1) * \frac{1}{2\sqrt{(a-1)(b-1)}} g_N \quad (10.17)$$

From (10.17) and (10.10) we get

$$c_N(x) = 1 + \frac{(p_N^2 - 1)}{(a-1)} \quad (10.18)$$

Substituting $N = 0$ in (10.18) we have,

$$c_0(x) = 1 + \frac{(p_0^2 - 1)}{(a-1)}$$

$$(i.e.) \ c_0(x) = 9(x^2 - x) - 2$$

Note that the tuple $(a, b, c_0(x))$ is Dio-triple with property $D(2)$.

Again, substituting $N = 1, 2$ in (10.18) and simplifying we get,

$$c_1(x) = 144[E_2(x)]^3 - 192[E_2(x)]^2 + 76E_2(x) - 7$$

$$c_2(x) = 1 + \left\{ 48[E_2(x)]^2 - 56E_2(x) + 15 \right\} \left\{ 48[E_2(x)]^3 - 56[E_2(x)]^2 + 15E_2(x) - 1 \right\}$$

It is seen that $(a, b, c_0(x), c_1(x))$ and $(a, b, c_1(x), c_2(x))$ represent Dio-quadruples with property $D(2)$ respectively.

In general, it is observed that the quadruple $(a, b, c_{N-1}(x), c_N(x))$, $N = 1, 2, 3, \dots$ is a Dio-quadruple with property $D(2)$. Some numerical examples are given in Table 10.2 below:

Table 10.2: $D(2)$ Dio-quadruples

x	$(a, b, c_0(x), c_1(x))$	$(a, b, c_1(x), c_2(x))$	$(a, b, c_2(x), c_3(x))$
2	(9, 2, 16, 529)	(9, 2, 529, 17956)	(9, 2, 17956, 609961)
3	(25, 6, 52, 24641)	(25, 6, 24641, 11876488)	(25, 6, 11876488, 5724442153)

Subsection X.2B: Formulation of $D(5)$ Dio-quadruples

Consider the Euler polynomials $E_0(x)$ and $E_1(x)$ given by

$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2}$$

Let $a = E_0(x) = 1$ and $b = [2E_1(x)] = 2x - 1$ be two polynomials such that $ab - (a + b) + 5$ is a perfect square.

Let c be any polynomial. Observe that, $ac - (a + c) + 5$ is automatically a perfect square.

Now, consider

$$bc - (b + c) + 5 = p^2 \quad (10.19)$$

After some calculations, it is seen that (10.19) is satisfied when $c = 2x + 3$.

Note that the triple $(1, 2x - 1, 2x + 3)$ is a Dio-triple with property $D(5)$.

Let d be any non-zero polynomial such that

$$bd - (b + d) + 5 = \alpha^2 \quad (10.20)$$

$$cd - (c + d) + 5 = \beta^2 \quad (10.21)$$

Eliminating d between the above two equations we have,

$$(2x + 2)\alpha^2 - (2x - 2)\beta^2 = 16 \quad (10.22)$$

Taking

$$\alpha = X + (2x - 2)T \quad (10.23)$$

$$\beta = X + (2x + 2)T \quad (10.24)$$

in (10.22), we get

$$X^2 = (4x^2 - 4)T^2 + 4$$

which is satisfied by

$$T = 1, \quad X = 2x \quad (10.25)$$

Using (10.25) in (10.23) and in view of (10.20), one obtains

$$d = 8x + 1$$

Observe that $(1, 2x - 1, 2x + 3, 8x + 1)$ is a Dio-quadruple with property $D(5)$.

The repetition of the above process leads to the generation of Dio-quadruples given by

$$(1, 2x + 3, 8x + 1, 18x + 7), \quad (1, 8x + 1, 18x + 7, 50x + 11), \quad (1, 18x + 7, 50x + 11, 128x + 33)$$

and so on.

Chapter – XI

Special Family of 3-Tuples

CHAPTER – XI

SPECIAL FAMILY OF 3-TUPLES

Chapter XI deals with special family of 3-tuples in two sections XI.1 to XI.2

Section XI.1 concerns with the study of formulating 3-tuples consisting of polygonal and pyramidal numbers such that, in each three tuple, the sum of any two members is a perfect square.

Section XI.2 deals with the study of formulation of special family of 3-tuples (a, b, c) such that the product of any two elements of the set added with their sum is a perfect square.

XI.1 On a Graceful Family of 3-Tuples

Triple 1

Let $a = 2t_{3,2k} = 4k^2 + 2k$ and $b = 2k + 1$

$$a + b = (2k + 1)^2$$

Let c be any non-zero integer such that

$$a + c = \alpha^2$$

$$b + c = \beta^2$$

Using some algebra

we have

$$c = 24k P_{k-1}^3 - 2k$$

Here $(2t_{3,2k}, 2k + 1, 24k P_{k-1}^3 - 2k)$ is the required triple such that the sum of any two members is a perfect square.

Properties

- $c - a + 2b + 2$ is a perfect square
- $c + 3a - 2b + 6$ is a perfect square
- $2a - b + c + 1 = 8k CP_k^3$

Triple 2

Let $a = Ct_{10,2k} = 20k^2 + 10k + 1$ and $b = 5t_{10,2k} = 80k^2 - 30k$

$$a + b = (10k - 1)^2$$

Let c be any non-zero integer such that

$$a + c = \alpha^2$$

$$b + c = \beta^2$$

Using some algebra

we have

$$c = 100(t_{8,k})^2 - 5t_{10,2k}, \quad k > 1$$

Here $(Ct_{10,2k}, 5t_{10,2k}, 100(t_{8,k})^2 - 5t_{10,2k})$ is the required triple such that the sum of any two members is a perfect square.

Properties

- $4(a-1) - b \equiv 0 \pmod{70}$
- $3(a-1) + b \equiv 0 \pmod{140}$
- $c - 4b - 15a \equiv 0 \pmod{15}$

Triple 3

Let $a = 8t_{3,k} = 4k^2 + 4k, k > 1$ and $b = 1$

$$a + b = (2k + 1)^2$$

Let c be any non-zero integer such that

$$a + c = \alpha^2$$

$$b + c = \beta^2$$

Using some algebra

we have

$$c = 2k SO_k + 12CS_k^4 + 4t_{3,k-1} - 6k$$

Here $(8t_{3,k}, 1, 2k SO_k + 12CS_k^4 + 4t_{3,k-1} - 6k)$ is the required triple such that the sum of any two members is a perfect square.

Properties

- $c - 2ka = 8k CP_k^3 - t_{25,k} - 15k$
- $2k^2 a - c = 8k CP_k^3 - t_{10,k} + 5k$

For simplicity some more triples satisfying the required condition are given below:

Triple 4	$(t_{12,2n} + 2t_{3,2n} + 1, t_{8,2n} - 2n, 6n + 24nCP_{9,n})$
Triple 5	$(t_{34,n} + t_{42,n}, 11GNO_n - 10, 4(t_{20,n})^2 - 48nt_{20,n} - 68t_{3,n-1} + 142n^2)$
Triple 6	$(S_n, 6t_{12,n} + 18, 36nCP_{24,n} + 78n^2 + 6n)$
Triple 7	$(4PR_n, 1, 8nP_n^5 - 24CP_n^3 + 8n)$
Triple 8	$(7(6P_n^3 - 2P_n^5), 4t_{3,n} + 4, 36(t_{3,n})^2 - 28t_{3,n})$

XI.2 Formulation of Curious Family of 3-Tuples

Sequence 1

Let $a = 2k^2 + 6k + 4, c_0 = 8k^2 + 16k + 9$

It is observed that

$$ac_0 + a + c_0 = (4k^2 + 10k + 7)^2$$

Let c_1 be any integer such that

$$(a+1)c_1 + a = \alpha^2 \quad (11.1)$$

$$(c_0+1)c_1 + c_0 = \beta^2 \quad (11.2)$$

Eliminating c_1 between (11.1) and (11.2), we have

$$(c_0+1)\alpha^2 - (a+1)\beta^2 = (a-c_0) \quad (11.3)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T, \beta = X + (c_0+1)T \quad (11.4)$$

in (11.3) and simplifying we get

$$X^2 = (a+1)(c_0+1)T^2 - 1$$

which is satisfied by $T=1, X = 4k^2 + 10k + 7$

In view of (11.4) and (11.1), it is seen that

$$c_1 = 18k^2 + 42k + 28$$

Let c_2 be any integer such that

$$(a+1)c_2 + a = \alpha^2 \quad (11.5)$$

$$(c_1+1)c_2 + c_1 = \beta^2 \quad (11.6)$$

Eliminating c_2 between (11.5) and (11.6), we have

$$(c_1 + 1)\alpha^2 - (a + 1)\beta^2 = (a - c_1) \quad (11.7)$$

Introducing the linear transformations

$$\alpha = X + (a + 1)T, \quad \beta = X + (c_1 + 1)T \quad (11.8)$$

in (11.7) and simplifying we get

$$X^2 = (a + 1)(c_1 + 1)T^2 - 1$$

which is satisfied by $T = 1$, $X = 6k^2 + 16k + 12$

In view of (11.8) and (11.5), it is seen that

$$c_2 = 32k^2 + 80k + 57$$

Let c_3 be any integer such that

$$(a + 1)c_3 + a = \alpha^2 \quad (11.9)$$

$$(c_2 + 1)c_3 + c_2 = \beta^2 \quad (11.10)$$

Eliminating c_3 between (11.9) and (11.10), we have

$$(c_2 + 1)\alpha^2 - (a + 1)\beta^2 = (a - c_2) \quad (11.11)$$

Introducing the linear transformations

$$\alpha = X + (a + 1)T, \quad \beta = X + (c_2 + 1)T \quad (11.12)$$

in (11.11) and simplifying we get

$$X^2 = (a + 1)(c_2 + 1)T^2 - 1$$

which is satisfied by $T = 1$, $X = 8k^2 + 22k + 17$

In view of (11.12) and (11.9), it is seen that

$$c_3 = 50k^2 + 130k + 96$$

The repetition of the above process leads to the generation of sequence of 3-tuples whose general form is given by (a, c_{s-1}, c_s) where

$$c_{s-1} = (2s^2 + 4s + 2)k^2 + (6s^2 + 8s + 2)k + (5s^2 + 4s), \quad s = 1, 2, 3, \dots$$

A few numerical examples are presented in Table 11.1 below:

Table 11.1: Numerical Examples

k	(a, c_0, c_1)	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	(24, 73, 184)	(24, 184, 345)	(24, 345, 556)	(24, 556, 817)
3	(40, 129, 316)	(40, 316, 585)	(40, 585, 936)	(40, 936, 1369)
4	(60, 201, 484)	(60, 484, 889)	(60, 889, 1416)	(60, 1416, 2065)
5	(84, 289, 688)	(84, 688, 1257)	(84, 1257, 1996)	(84, 1996, 2905)

Sequence 2

Let $a = 1, c_0 = 2k^2 - 2k$

It is observed that

$$ac_0 + a + c_0 = (2k - 1)^2$$

Let c_1 be any integer such that

$$(a+1)c_1 + a = \alpha^2 \tag{11.13}$$

$$(c_0+1)c_1 + c_0 = \beta^2 \tag{11.14}$$

Eliminating c_1 between (11.13) and (11.14), we have

$$(c_0+1)\alpha^2 - (a+1)\beta^2 = (a - c_0) \tag{11.15}$$

Introducing the linear transformations

$$\alpha = X + (a+1)T, \quad \beta = X + (c_0+1)T \tag{11.16}$$

in (11.15) and simplifying we get

$$X^2 = (a+1)(c_0+1)T^2 - 1$$

which is satisfied by $T = 1, X = 2k - 1$

In view of (11.16) and (11.13), it is seen that

$$c_1 = 2k^2 + 2k$$

Let c_2 be any integer such that

$$(a+1)c_2 + a = \alpha^2 \quad (11.17)$$

$$(c_1+1)c_2 + c_1 = \beta^2 \quad (11.18)$$

Eliminating c_2 between (11.17) and (11.18), we have

$$(c_1+1)\alpha^2 - (a+1)\beta^2 = (a-c_1) \quad (11.19)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T, \beta = X + (c_1+1)T \quad (11.20)$$

in (11.19) and simplifying we get

$$X^2 = (a+1)(c_1+1)T^2 - 1$$

which is satisfied by $T = 1, X = 2k + 1$

In view of (11.20) and (11.17), it is seen that

$$c_2 = 2k^2 + 6k + 4$$

Let c_3 be any integer such that

$$(a+1)c_3 + a = \alpha^2 \quad (11.21)$$

$$(c_2+1)c_3 + c_2 = \beta^2 \quad (11.22)$$

Eliminating c_3 between (11.21) and (11.22), we have

$$(c_2+1)\alpha^2 - (a+1)\beta^2 = (a-c_2) \quad (11.23)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T, \quad \beta = X + (c_2+1)T \quad (11.24)$$

in (11.23) and simplifying we get

$$X^2 = (a+1)(c_2+1)T^2 - 1$$

which is satisfied by $T=1, X=2k+3$

In view of (11.24) and (11.21), it is seen that

$$c_3 = 2k^2 + 10k + 12$$

The repetition of the above process leads to the generation of sequence of 3-tuples whose general form is given by (a, c_{s-1}, c_s) where

$$c_{s-1} = 2k^2 + (4s-6)k + (2s^2 - 6s + 4), \quad s = 1, 2, 3, \dots$$

A few numerical examples are presented in Table 11.2 below:

Table 11.2: Numerical Examples

k	(a, c_0, c_1)	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	(1, 4, 12)	(1, 12, 24)	(1, 24, 40)	(1, 40, 60)
3	(1, 12, 24)	(1, 24, 40)	(1, 40, 60)	(1, 60, 84)
4	(1, 24, 40)	(1, 40, 60)	(1, 60, 84)	(1, 84, 112)
5	(1, 40, 60)	(1, 60, 84)	(1, 84, 112)	(1, 112, 144)

Sequence 3

Let $a=1, c_0 = 2k^2 + 2k$

It is observed that

$$ac_0 + a + c_0 = (2k+1)^2$$

Let c_1 be any integer such that

$$(a+1)c_1 + a = \alpha^2 \quad (11.25)$$

$$(c_0+1)c_1 + c_0 = \beta^2 \quad (11.26)$$

Eliminating c_1 between (11.25) and (11.26), we have

$$(c_0 + 1)\alpha^2 - (a + 1)\beta^2 = (a - c_0) \quad (11.27)$$

Introducing the linear transformations

$$\alpha = X + (a + 1)T, \quad \beta = X + (c_0 + 1)T \quad (11.28)$$

in (11.27) and simplifying we get

$$X^2 = (a + 1)(c_0 + 1)T^2 - 1$$

which is satisfied by $T = 1, X = 2k + 1$

In view of (11.28) and (11.25), it is seen that

$$c_1 = 2k^2 + 6k + 4$$

Let c_2 be any integer such that

$$(a + 1)c_2 + a = \alpha^2 \quad (11.29)$$

$$(c_1 + 1)c_2 + c_1 = \beta^2 \quad (11.30)$$

Eliminating c_2 between (11.29) and (11.30), we have

$$(c_1 + 1)\alpha^2 - (a + 1)\beta^2 = (a - c_1) \quad (11.31)$$

Introducing the linear transformations

$$\alpha = X + (a + 1)T, \quad \beta = X + (c_1 + 1)T \quad (11.32)$$

in (11.31) and simplifying we get

$$X^2 = (a + 1)(c_1 + 1)T^2 - 1$$

which is satisfied by $T = 1, X = 2k + 3$

In view of (11.32) and (11.29), it is seen that

$$c_2 = 2k^2 + 10k + 12$$

Let c_3 be any integer such that

$$(a+1)c_3 + a = \alpha^2 \quad (11.33)$$

$$(c_2+1)c_3 + c_2 = \beta^2 \quad (11.34)$$

Eliminating c_3 between (11.33) and (11.34), we have

$$(c_2+1)\alpha^2 - (a+1)\beta^2 = (a-c_2) \quad (11.35)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T, \quad \beta = X + (c_2+1)T \quad (11.36)$$

in (11.35) and simplifying we get

$$X^2 = (a+1)(c_2+1)T^2 - 1$$

which is satisfied by $T=1, X=2k+5$

In view of (11.36) and (11.33), it is seen that

$$c_3 = 2k^2 + 14k + 24$$

The repetition of the above process leads to the generation of sequence of 3-tuples

whose general form is given by (a, c_{s-1}, c_s) where

$$c_{s-1} = 2k^2 + (4s-2)k + (2s^2 - 2s), \quad s=1, 2, 3, \dots$$

A few numerical examples are presented in Table 11.3 below:

Table 11.3: Numerical Examples

k	(a, c_0, c_1)	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	(1,12,24)	(1,24,40)	(1, 40, 60)	(1, 60, 84)
3	(1,24,40)	(1,40,60)	(1, 60, 84)	(1, 84, 112)
4	(1,40,60)	(1,60,84)	(1, 84, 112)	(1,112,144)
5	(1,60,84)	(1,84,112)	(1,112, 144)	(1,144,180)

Sequence 4

Let $a = 4, c_0 = 5k^2 + 4k$

It is observed that

$$ac_0 + a + c_0 = (5k + 2)^2$$

Let c_1 be any integer such that

$$(a+1)c_1 + a = \alpha^2 \quad (11.37)$$

$$(c_0+1)c_1 + c_0 = \beta^2 \quad (11.38)$$

Eliminating c_1 between (11.37) and (11.38), we have

$$(c_0+1)\alpha^2 - (a+1)\beta^2 = (a - c_0) \quad (11.39)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T, \quad \beta = X + (c_0+1)T \quad (11.40)$$

in (11.39) and simplifying we get

$$X^2 = (a+1)(c_0+1)T^2 - 1$$

which is satisfied by $T = 1, X = 5k + 2$

In view of (11.40) and (11.37), it is seen that

$$c_1 = 5k^2 + 14k + 9$$

Let c_2 be any integer such that

$$(a+1)c_2 + a = \alpha^2 \quad (11.41)$$

$$(c_1+1)c_2 + c_1 = \beta^2 \quad (11.42)$$

Eliminating c_2 between (11.41) and (11.42), we have

$$(c_1+1)\alpha^2 - (a+1)\beta^2 = (a - c_1) \quad (11.43)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T, \beta = X + (c_1+1)T \quad (11.44)$$

in (11.43) and simplifying we get

$$X^2 = (a+1)(c_1+1)T^2 - 1$$

which is satisfied by $T=1, X=5k+7$

In view of (11.44) and (11.41), it is seen that

$$c_2 = 5k^2 + 24k + 28$$

Let c_3 be any integer such that

$$(a+1)c_3 + a = \alpha^2 \quad (11.45)$$

$$(c_2+1)c_3 + c_2 = \beta^2 \quad (11.46)$$

Eliminating c_3 between (11.45) and (11.46), we have

$$(c_2+1)\alpha^2 - (a+1)\beta^2 = (a-c_2) \quad (11.47)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T, \beta = X + (c_2+1)T \quad (11.48)$$

in (11.47) and simplifying we get

$$X^2 = (a+1)(c_2+1)T^2 - 1$$

which is satisfied by $T=1, X=5k+12$

In view of (11.48) and (11.45), it is seen that

$$c_3 = 5k^2 + 34k + 57$$

The repetition of the above process leads to the generation of sequence of 3-tuples

whose general form is given by (a, c_{s-1}, c_s) where

$$c_{s-1} = 5k^2 + (10s-6)k + (5s^2 - 6s + 1), \quad s = 1, 2, 3, \dots$$

A few numerical examples are presented in Table 11.4 below:

Table 11.4: Numerical Examples

k	(a, c_0, c_1)	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	(4,28,57)	(4, 57, 96)	(4,96,145)	(4,145,204)
3	(4,57,96)	(4,96, 145)	(4,145,204)	(4,204,273)
4	(4,96,145)	(4,145,204)	(4,204,273)	(4,273,352)
5	(4,145,204)	(4,204,273)	(4,273,352)	(4,352,441)

Sequence 5

Let $a = 4, c_0 = 5k^2 - 4k$

It is observed that

$$ac_0 + a + c_0 = (5k - 2)^2$$

Let c_1 be any integer such that

$$(a+1)c_1 + a = \alpha^2 \quad (11.49)$$

$$(c_0+1)c_1 + c_0 = \beta^2 \quad (11.50)$$

Eliminating c_1 between (11.49) and (11.50), we have

$$(c_0+1)\alpha^2 - (a+1)\beta^2 = (a - c_0) \quad (11.51)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T, \quad \beta = X + (c_0+1)T \quad (11.52)$$

in (11.51) and simplifying we get

$$X^2 = (a+1)(c_0+1)T^2 - 1$$

which is satisfied by $T = 1, X = 5k - 2$

In view of (11.52) and (11.49), it is seen that

$$c_1 = 5k^2 + 6k + 1$$

Let c_2 be any integer such that

$$(a+1)c_2 + a = \alpha^2 \quad (11.53)$$

$$(c_1+1)c_2 + c_1 = \beta^2 \quad (11.54)$$

Eliminating c_2 between (11.53) and (11.54), we have

$$(c_1+1)\alpha^2 - (a+1)\beta^2 = (a-c_1) \quad (11.55)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T, \quad \beta = X + (c_1+1)T \quad (11.56)$$

in (11.55) and simplifying we get

$$X^2 = (a+1)(c_1+1)T^2 - 1$$

which is satisfied by $T=1, X=5k+3$

In view of (11.56) and (11.53), it is seen that

$$c_2 = 5k^2 + 16k + 12$$

Let c_3 be any integer such that

$$(a+1)c_3 + a = \alpha^2 \quad (11.57)$$

$$(c_2+1)c_3 + c_2 = \beta^2 \quad (11.58)$$

Eliminating c_3 between (11.57) and (11.58), we have

$$(c_2+1)\alpha^2 - (a+1)\beta^2 = (a-c_2) \quad (11.59)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T, \quad \beta = X + (c_2+1)T \quad (11.60)$$

in (11.59) and simplifying we get

$$X^2 = (a+1)(c_2+1)T^2 - 1$$

which is satisfied by $T = 1$, $X = 5k + 8$

In view of (11.60) and (11.57), it is seen that

$$c_3 = 5k^2 + 26k + 33$$

The repetition of the above process leads to the generation of sequence of 3-tuples

whose general form is given by (a, c_{s-1}, c_s) where

$$c_{s-1} = 5k^2 + (10s - 14)k + (5s^2 - 14s + 9), \quad s = 1, 2, 3, \dots$$

A few numerical examples are presented in Table 11.5 below:

Table 11.5: Numerical Examples

k	(a, c_0, c_1)	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	(4, 12, 33)	(4, 33, 64)	(4, 64, 105)	(4, 105, 156)
3	(4, 33, 64)	(4, 64, 105)	(4, 105, 156)	(4, 156, 217)
4	(4, 64, 105)	(4, 105, 156)	(4, 156, 217)	(4, 217, 288)
5	(4, 105, 156)	(4, 156, 217)	(4, 217, 288)	(4, 288, 369)

Chapter – XII

Equality of Polygonal Numbers

CHAPTER – XII

EQUALITY OF POLYGONAL NUMBERS

Chapter XII focuses on Equality of Polygonal Numbers in two sections XII.1 to XII.2

Section XII.1 illustrates formulas for the ranks of Triangular numbers, Hexagonal numbers, star numbers satisfying the relation $t_{3,N} = t_{6,h} = S_n$.

Section XII.2 exhibits formulas for the ranks of Triangular numbers, Hexagonal numbers, Centered Hexagonal numbers, Centered Octagonal numbers, Centered Decagonal numbers and Centered Dodecagonal numbers satisfying the relations

$$t_{3,N} = t_{6,h} = ct_{6,H}, \quad t_{3,N} = t_{6,h} = ct_{8,M}, \quad t_{3,N} = t_{6,h} = ct_{10,M}, \quad t_{3,N} = t_{6,h} = ct_{12,D}.$$

XII.1 Triangular Numbers Simultaneously Equal to Hexagonal and Star Numbers

Let N, h, n be the ranks of Triangular, Hexagonal and Star numbers respectively.

The relation

$$t_{3,N} = t_{6,h}$$

leads to

$$N = 2h - 1 \quad (12.1)$$

The assumption $t_{6,h} = S_n$ gives

$$2h^2 - h = 6n^2 - 6n + 1$$

Treating the above equation as a quadratic in n and solving for n , one obtains

$$n = \frac{1}{6}(3 + R) \quad (12.2)$$

where

$$R^2 = 12h^2 - 6h + 3 \quad (12.3)$$

On completing the squares on R.H.S in (12.3), one obtains

$$4R^2 - 3X^2 = 9 \quad (12.4)$$

$$\text{where } X = 4h - 1 \quad (12.5)$$

To solve (12.4), the introduction of the transformations

$$X = P + 4Q, R = P + 3Q \quad (12.6)$$

lead to

$$P^2 = 12Q^2 + 9 \quad (12.7)$$

whose smallest positive integer solution (P_0, Q_0) is

$$P_0 = 21, Q_0 = 6$$

To obtain the other solutions of (12.7), consider the pell equation

$$P^2 = 12Q^2 + 1$$

whose general solution is given by

$$\tilde{P}_s = \frac{1}{2} f_s, \quad \tilde{Q}_s = \frac{1}{4\sqrt{3}} g_s$$

where

$$f_s = (7 + 4\sqrt{3})^{s+1} + (7 - 4\sqrt{3})^{s+1}, \quad g_s = (7 + 4\sqrt{3})^{s+1} - (7 - 4\sqrt{3})^{s+1}, \quad s = -1, 0, 1, \dots$$

Applying Brahmagupta Lemma between (P_0, Q_0) and $(\tilde{P}_s, \tilde{Q}_s)$, the other integer

solutions of (12.7) are given by

$$P_{s+1} = \frac{21}{2} f_s + 6\sqrt{3} g_s$$

$$Q_{s+1} = 3f_s + \frac{21}{4\sqrt{3}} g_s$$

Substituting the values of P_{s+1} , Q_{s+1} in (12.6), we have

$$X_{s+1} = \frac{45}{2} f_s + 13\sqrt{3} g_s, \quad R_{s+1} = \frac{39}{2} f_s + \frac{45\sqrt{3}}{4} g_s$$

In view of (12.5), (12.2) and (12.1), we get

$$h_{s+1} = \frac{1}{4} \left(\frac{45}{2} f_s + 13\sqrt{3} g_s + 1 \right)$$

$$N_{s+1} = \frac{45}{4} f_s + \frac{13}{2} \sqrt{3} g_s - \frac{1}{2}$$

$$n_{s+1} = \frac{1}{6} \left(\frac{39}{2} f_s + \frac{45\sqrt{3}}{4} g_s + 3 \right)$$

Note that the values of $t_{3, N_{s+1}} = t_{6, h_{s+1}} = S_{n_{s+1}}$, $s = 0, 2, 4, \dots$

A few numerical examples satisfying the relations are given in the Table 12.1 below:

Table 12.1 Numerical Examples

s	f_s	g_s	N_{s+1}	h_{s+1}	n_{s+1}	$t_{3,N_{s+1}} = t_{6,h_{s+1}} = S_{n_{s+1}}$
0	14	8	313	157	91	49141
2	2702	1560	60817	30409	17557	1849384153
4	524174	302632	11798281	5899141	3405871	6.95997E+13
6	101687054	58709048	2288805793	1144402897	660721321	2.61932E+18

To conclude, one may search for the ranks of triples of other special polygonal numbers with the same value.

XII.2 On Three Figurate Numbers With Same Values

1. Equality of $t_{3,N} = t_{6,h} = ct_{6,H}$

Let N, h, H be the ranks of Triangular, Hexagonal and Centered Hexagonal numbers respectively.

The relation

$$t_{3,N} = t_{6,h}$$

leads to

$$N = 2h - 1 \tag{12.8}$$

The assumption $t_{6,h} = ct_{6,H}$ gives

$$2h^2 - h = 3H^2 - 3H + 1$$

which is written as

$$Y^2 = 6X^2 + 3 \tag{12.9}$$

where

$$Y = 4h - 1, X = 2H - 1 \tag{12.10}$$

To obtain the other solutions of (12.9), consider the pell equation

$$Y^2 = 6X^2 + 1$$

whose general solution is given by

$$\tilde{Y}_n = \frac{1}{2} f_n, \quad \tilde{X}_n = \frac{1}{2\sqrt{6}} g_n$$

where

$$f_n = (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1}, \quad g_n = (5 + 2\sqrt{6})^{n+1} - (5 - 2\sqrt{6})^{n+1}, \quad n = -1, 0, 1, \dots$$

Applying Brahmagupta Lemma between (X_0, Y_0) and $(\tilde{X}_n, \tilde{Y}_n)$, the other integer

solutions of (12.9) are given by

$$X_{n+1} = \frac{1}{2} f_n + \frac{3}{2\sqrt{6}} g_n$$

$$Y_{n+1} = \frac{3}{2} f_n + \frac{3}{\sqrt{6}} g_n$$

In view of (12.8) and (12.10), we get

$$h_{n+1} = \frac{1}{4} \left(\frac{3}{2} f_n + \frac{3}{\sqrt{6}} g_n + 1 \right)$$

$$N_{n+1} = \frac{3}{4} f_n + \frac{3}{2\sqrt{6}} g_n + \frac{1}{2}$$

$$H_{n+1} = \frac{1}{2} \left(\frac{1}{2} f_n + \frac{3}{2\sqrt{6}} g_n + 1 \right)$$

Note that the values of $t_{3,N_{n+1}} = t_{6,h_{n+1}} = ct_{6,H_{n+1}}$, $n = 0, 1, 2, \dots$

A few numerical examples satisfying the relations are given in the Table 12.2 below:

Table 12.2: Numerical Examples

n	N_{n+1}	h_{n+1}	H_{n+1}	$t_{3,N_{n+1}} = t_{6,h_{n+1}} = ct_{6,H_{n+1}}$
0	13	7	6	91
1	133	67	55	8911
2	1321	661	540	873181
3	13081	6541	5341	85562821
4	129493	64747	52866	8384283271

2. Equality of $t_{3,N} = t_{6,h} = ct_{6,M}$

Let N , h , M be the ranks of Triangular, Hexagonal and Centered Octagonal numbers respectively.

The relation

$$t_{3,N} = t_{6,h}$$

leads to

$$N = 2h - 1 \tag{12.11}$$

The assumption $t_{6,h} = ct_{8,M}$ gives

$$2h^2 - h = 4M^2 - 4M + 1$$

which is written as

$$Y^2 = 8X^2 + 1 \quad (12.12)$$

where

$$Y = 4h - 1, \quad X = 2M - 1 \quad (12.13)$$

The general solution of (12.12) is given by

$$Y_n = \frac{1}{2} f_n, \quad X_n = \frac{1}{4\sqrt{2}} g_n$$

where

$$f_n = (3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1}, \quad g_n = (3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1}, \quad n = -1, 0, 1, \dots$$

In view of (12.11) and (12.13), we get

$$h_n = \frac{1}{8} (f_n + 2)$$

$$N_n = \frac{1}{4} f_n - \frac{1}{2}$$

$$M_n = \frac{1}{8} \left(4 + \frac{1}{\sqrt{2}} g_n \right)$$

Note that the values of $t_{3,N_n} = t_{6,h_n} = ct_{8,M_n}$, $n = 0, 2, 4, \dots$

A few numerical examples satisfying the relations are given in the Table 12.3 below:

Table 12.3: Numerical Examples

n	N_n	h_n	M_n	$t_{3,N_n} = t_{6,h_n} = ct_{8,M_n}$
0	1	1	1	1
2	49	25	18	1225
4	1681	841	595	1413721

3. Equality of $t_{3,N} = t_{6,h} = ct_{10,M}$

Let N, h, M be the ranks of Triangular, Hexagonal and Centered Decagonal numbers respectively.

The relation

$$t_{3,N} = t_{6,h}$$

leads to

$$N = 2h - 1 \quad (12.14)$$

The assumption $t_{6,h} = ct_{10,M}$ gives

$$2h^2 - h = 5M^2 - 5M + 1$$

which is written as

$$Y^2 = 10X^2 - 1 \quad (12.15)$$

where

$$Y = 4h - 1, X = 2M - 1 \quad (12.16)$$

To obtain the other solutions of (12.15), consider the pell equation

$$Y^2 = 10X^2 + 1$$

whose general solution is given by

$$\tilde{Y}_s = \frac{1}{2} f_s, \tilde{X}_s = \frac{1}{2\sqrt{10}} g_s$$

where

$$f_s = (19 + 6\sqrt{10})^{s+1} + (19 - 6\sqrt{10})^{s+1}, g_s = (19 + 6\sqrt{10})^{s+1} - (19 - 6\sqrt{10})^{s+1}, s = -1, 0, 1, \dots$$

Applying Brahmagupta Lemma between (X_0, Y_0) and $(\tilde{X}_s, \tilde{Y}_s)$, the other integer

solutions of (12.15) are given by

$$X_{s+1} = \frac{1}{2} f_s + \frac{3}{2\sqrt{10}} g_s$$

$$Y_{s+1} = \frac{3}{2} f_s + \frac{5}{\sqrt{10}} g_s$$

In view of (12.14) and (12.16), we get

$$h_{s+1} = \frac{1}{4} \left(\frac{3}{2} f_s + \frac{5}{\sqrt{10}} g_s + 1 \right)$$

$$N_{s+1} = \frac{3}{4} f_s + \frac{5}{2\sqrt{10}} g_s - \frac{1}{2}$$

$$M_{s+1} = \frac{1}{2} \left(\frac{1}{2} f_s + \frac{3}{2\sqrt{10}} g_s + 1 \right)$$

Note that the values of $t_{3,N_{s+1}} = t_{6,h_{s+1}} = ct_{10,M_{s+1}}$, $s = -1, 1, 3, \dots$

A few numerical examples satisfying the relations are given in the Table 12.4 below:

Table 12.4: Numerical Examples

s	N_{s+1}	h_{s+1}	M_{s+1}	$t_{3,N_{s+1}} = t_{6,h_{s+1}} = ct_{10,M_{s+1}}$
-1	1	1	1	1
1	2221	1111	703	2467531
3	3203401	1601701	1013005	5.13089E+12

4. Equality of $t_{3,N} = t_{6,h} = ct_{12,D}$

Let N, h, D be the ranks of Triangular, Hexagonal and Centered Do Decagonal numbers respectively.

The relation

$$t_{3,N} = t_{6,h}$$

leads to

$$N = 2h - 1 \tag{12.17}$$

The assumption $t_{6,h} = ct_{12,D}$ gives

$$2h^2 - h = 6D^2 - 6D + 1$$

which is written as

$$Y^2 = 12X^2 - 3 \quad (12.18)$$

where

$$Y = 4h - 1, X = 2D - 1 \quad (12.19)$$

To obtain the other solutions of (12.18), consider the pell equation

$$Y^2 = 12X^2 + 1$$

whose general solution is given by

$$\tilde{Y}_n = \frac{1}{2} f_n, \tilde{X}_n = \frac{1}{4\sqrt{3}} g_n$$

where

$$f_n = (7 + 4\sqrt{3})^{n+1} + (7 - 4\sqrt{3})^{n+1}, g_n = (7 + 4\sqrt{3})^{n+1} - (7 - 4\sqrt{3})^{n+1}, n = -1, 0, 1, \dots$$

Applying Brahmagupta Lemma between (X_0, Y_0) and $(\tilde{X}_n, \tilde{Y}_n)$, the other integer

solutions of (12.18) are given by

$$X_{n+1} = \frac{1}{2} f_n + \frac{\sqrt{3}}{4} g_n$$

$$Y_{n+1} = \frac{3}{2} f_n + \sqrt{3} g_n$$

In view of (12.17) and (12.19), we get

$$h_{n+1} = \frac{1}{4} \left(\frac{3}{2} f_n + \sqrt{3} g_n + 1 \right)$$

$$N_{n+1} = \frac{3}{4} f_n + \frac{\sqrt{3}}{2} g_n - \frac{1}{2}$$

$$D_{n+1} = \frac{1}{2} \left(\frac{1}{2} f_n + \frac{\sqrt{3}}{4} g_n + 1 \right)$$

Note that the values of $t_{3,N_{n+1}} = t_{6,h_{n+1}} = ct_{12,D_{n+1}}$, $n = -1, 1, 3, \dots$

A few numerical examples satisfying the relations are given in the Table 12.5 below:

Table 12.5: Numerical Examples

n	N_{n+1}	h_{n+1}	D_{n+1}	$t_{3,N_{n+1}} = t_{6,h_{n+1}} = ct_{12,D_{n+1}}$
-1	1	1	1	1
1	313	157	91	49141
3	60817	30409	17557	1849384153
5	11798281	5899141	3405871	6.95997E+13

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ON HOMOGENEOUS QUADRATIC WITH FIVE UNKNOWNNS

$$4w^2 - x^2 - y^2 + z^2 = 16t^2$$

A. VIJAYASANKAR, SHARADHA KUMAR AND M. A. GOPALAN

ABSTRACT

The homogeneous quadratic diophantine equation with five unknowns given by:

$4w^2 - x^2 - y^2 + z^2 = 16t^2$ Is analyzed for determining its non-zero distinct integer solutions through employing linear transformations.

Keywords: homogeneous quadratic, quadratic with five unknowns, integer solutions

INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous or non-homogeneous quadratic diophantine equations with two or more variables have been an interest to mathematicians since antiquity [1-4]. In this context, one may refer [5-11] for different choices of quadratic diophantine equations with four unknowns. In [12], the quadratic diophantine equation with five unknowns given by $10w^2 - x^2 - y^2 + z^2 = t^2$ is analysed for obtaining its non-zero distinct integer solutions.

This motivated me for finding integer solutions to other choices of quadratic equations with five unknowns. This paper deals with the problem of determining non-zero distinct integer solutions to the quadratic

Diophantine equation with five unknowns given by $4w^2 - x^2 - y^2 + z^2 = 16t^2$.

METHOD OF ANALYSIS

The second degree diophantine equation with five unknowns to be solved is

$$4w^2 - x^2 - y^2 + z^2 = 16t^2 \quad (1)$$

The process of obtaining different sets of non-zero distinct integer solutions to (1) is exhibited below:

Set 1:

The substitution of the linear transformations

$$x = 4P + 12Q, y = 8Y, z = 4(P - Q), w = 4(P + Q), t = 2T \quad (2)$$

in (1) leads to the space pythagorean equation

$$P^2 = Q^2 + Y^2 + T^2 \quad (3)$$

which is satisfied by

$$P = a^2 + b^2 + c^2, T = a^2 - b^2 - c^2, Q = 2ab, Y = 2ac \quad (4)$$

In view of (2), one has the integer solutions to (1) given by

$$x = 4(a^2 + b^2 + c^2 + 6ab),$$

$$y = 16ac,$$

$$z = 4(a^2 + b^2 + c^2 - 2ab),$$

$$w = 4(a^2 + b^2 + c^2 + 2ab),$$

$$t = 2(a^2 - b^2 - c^2)$$

Set 2:

Introducing the linear transformations

$$x = (8a^2 - 1)s, y = 4aY, z = s, w = 4a^2s, t = aT \quad (5)$$

in (1), it simplifies to the Pythagorean equation

$$s^2 = Y^2 + T^2 \quad (6)$$

whose solutions may be taken as

$$s = p^2 + q^2, T = p^2 - q^2, Y = 2pq \quad (7)$$

In view of (5), the integer solutions to (1) are given by

$$x = (8a^2 - 1)(p^2 + q^2), y = 8apq, z = (p^2 + q^2), w = 4a^2(p^2 + q^2), t = a(p^2 - q^2)$$

Note 1:

The solutions to (6) is also taken as

$$s = p^2 + q^2, Y = p^2 - q^2, T = 2pq$$

In this case, the integer solutions to (1) are given by

$$x = (8a^2 - 1)(p^2 + q^2), y = 4a(p^2 - q^2), z = (p^2 + q^2), w = 4a^2(p^2 + q^2), t = 2apq$$

Set 3:

Taking

$$x = 4(P + Q), y = 4(P - Q), w = 4P, z = 4Q \quad (8)$$

in (1), it reduces to

$$Q^2 + t^2 = 2P^2 \quad (9)$$

After some algebra, it is seen that (9) is satisfied by

$$t = a^2 - b^2 + 2ab,$$

$$Q = a^2 - b^2 - 2ab,$$

$$P = a^2 + b^2$$

In view of (8), it is seen that

$$x = 8a(a - b),$$

$$y = 8b(a + b),$$

$$z = 4(a^2 - b^2 - 2ab),$$

$$w = 4(a^2 + b^2)$$

Thus, the above values of x, y, z, w, t satisfies (1).

Note 2:

After performing a few calculations, (9) is also satisfied by

$$t = 2a^2 - b^2,$$

$$Q = 2a^2 + b^2 + 4ab,$$

$$P = 2a^2 + b^2 + 2ab$$

In view of (8), the corresponding values of x, y, z, w are found to be

$$x = 4(4a^2 + 2b^2 + 6ab),$$

$$y = -8ab,$$

$$z = 4(2a^2 + b^2 + 4ab),$$

$$w = 4(2a^2 + b^2 + 2ab)$$

Set 4:

The choice

$$z = x + 4t \quad (10)$$

in (1) leads to

$$y^2 - 4w^2 = 8xt \quad (11)$$

which is expressed as the system of double equations as shown in Table: 1 below:

Table: 1 System of double equations

System	1	2	3	4
$y + 2w$	$8x$	$4x$	$8t$	$2x$
$y - 2w$	t	$2t$	x	$4t$

Solving each of the above systems, one obtains the values of x, y, w, t . In view of (10), the corresponding value of z is obtained. For simplicity, the integer solutions to the corresponding system of equations are exhibited below:

Solutions to system 1:

$$x = s, y = 4s + 2k, z = s + 16k, w = 2s - k, t = 4k$$

Solutions to system 2:

$$x = s, y = 2s + 2k, z = s + 8k, w = s - k, t = 2k$$

Solutions to system 3:

$$x = 4s, y = 2s + 4k, z = 4s + 4k, w = 2k - s, t = k$$

Solutions to system 4:

$$x = 2s, y = 2s + 2k, z = 2s + 4k, w = s - k, t = k$$

CONCLUSION

In this paper, an attempt has been made to obtain non-zero distinct integer solutions to the quadratic diophantine equation with five unknowns given by $4w^2 - x^2 - y^2 + z^2 = 16t^2$.

The readers of this paper may search for finding integer solutions to other choices of quadratic diophantine equations with five or more unknowns.

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ON THE HOMOGENEOUS CONE

$$z^2 = (2k^2 - 2k + 22)x^2 + y^2$$

A. VIJAYASANKAR, SHARADHAKUMAR AND M. A. GOPALAN

ABSTRACT

The homogeneous ternary quadratic equation given by $z^2 = (2k^2 - 2k + 22)x^2 + y^2$ is analysed for its non-zero distinct integer solutions through different methods. A few interesting properties between the solutions are presented. Also, formulae for generating sequence of integer solutions based on the given solution are presented.

Keywords: Ternary quadratic, Integer solutions, Homogeneous cone.

INTRODUCTION

It is well known that the quadratic Diophantine equations with three unknowns (homogeneous or non-homogeneous) are rich in variety [1, 2]. In particular, the ternary quadratic Diophantine equations of the form $z^2 = Dx^2 + y^2$ are analysed for values of $D=29,41,43,47, 53, 55, 61, 63, 67$ in [3-11]. In [12], the homogeneous cone represented by the ternary quadratic equation $z^2 = 74x^2 + y^2$ has been studied. This result motivated us for determining integer solutions to the homogeneous cone $z^2 = Dx^2 + y^2$ when D takes even values. In this communication, yet another interesting homogeneous ternary quadratic Diophantine equation given by $z^2 = (2k^2 - 2k + 22)x^2 + y^2$ is analysed for its non-zero distinct integer solutions through different methods. A few interesting properties between the solutions are presented. Also, formulas for generating sequence of integer solutions based on the given solution are presented.

METHODS OF ANALYSIS

The ternary quadratic equation to be solved for its integer solutions is

$$z^2 = (2k^2 - 2k + 22)x^2 + y^2 \tag{1}$$

We present below different methods of solving (1):

METHOD: 1

(1) Is written in the form of ratio as

$$\frac{z+y}{(2k^2 - 2k + 22)x} = \frac{x}{z-y} = \frac{r}{s}, s \neq 0 \tag{2}$$

which is equivalent to the system of double equations

$$\begin{aligned} (2k^2 - 2k + 22)rx - sy - sz &= 0 \\ sx + ry - rz &= 0 \end{aligned}$$

Applying the method of cross-multiplication to the above system of equations,

$$\begin{aligned} x &= x(r, s) = 2rs \\ y &= y(r, s) = (2k^2 - 2k + 22)r^2 - s^2 \\ z &= z(r, s) = (2k^2 - 2k + 22)r^2 + s^2 \end{aligned}$$

which satisfy (1)

NOTE: 1

It is observed that (1) may also be represented in the form of ratio as below:

$$(i) \frac{z+y}{2x} = \frac{(k^2 - k + 11)x}{z-y} = \frac{r}{s}, s \neq 0$$

The corresponding solutions to (1) are given as:

$$x = 2rs, y = 2r^2 - (k^2 - k + 11)s^2, z = 2r^2 + (k^2 - k + 11)s^2$$

$$(ii) \quad \frac{z+y}{(k^2 - k + 11)x} = \frac{2x}{z-y} = \frac{r}{s}, s \neq 0$$

The corresponding solutions to (1) are given as:

$$x = 2rs, y = (k^2 - k + 11)r^2 - 2s^2, z = (k^2 - k + 11)r^2 + 2s^2$$

METHOD: 2

(1) Is written as the system of double equation in Table 1 as follows:

Table: 1 System of Double Equations

System	1	2	3	4
$z+y$	$2x$	$(k^2 - k + 11)x^2$	$(2k^2 - 2k + 22)x$	$(k^2 - k + 11)x$
$z-y$	$(k^2 - k + 11)x$	2	x	$2x$

Solving each of the above system of double equations, the value of x, y & z satisfying (1) are obtained. For simplicity and brevity, in what follows, the integer solutions thus obtained are exhibited.

Solutions for system: I

$$x=2s, y = -(k^2 - k + 9)s, z = (k^2 - k + 13)s$$

Solutions for system: II

$$x = 2s, y = 2s^2(k^2 - k + 11) - 1, z = 2s^2(k^2 - k + 11) + 1$$

Solution for system: III

$$x = 2s, y = (2k^2 - 2k + 21)s, z = (2k^2 - 2k + 23)s$$

Solution for system: IV

$$x = 2s, y = s(k^2 - k + 11) - 2s, z = s(k^2 - k + 11) + 2s$$

METHOD: 3

(1) Is written as

$$y^2 + (2k^2 - 2k + 22)x^2 = z^2 = z^2 * 1 \quad (3)$$

Assume z as

$$z = a^2 + (2k^2 - 2k + 22)b^2 \quad (4)$$

Write 1 as

$$1 = \frac{\left[(2k^2 - 2k + 22)r^2 - s^2 + i2rs\sqrt{2k^2 - 2k + 22} \right] * \left[(2k^2 - 2k + 22)r^2 - s^2 - i2rs\sqrt{2k^2 - 2k + 22} \right]}{\left((2k^2 - 2k + 22)r^2 + s^2 \right)^2} \quad (5)$$

Using (4) & (5) in (3) and employing the method of factorization, consider

$$y + i\sqrt{2k^2 - 2k + 22}x = \frac{(a + ib\sqrt{2k^2 - 2k + 22})^2 \left[(2k^2 - 2k + 22)r^2 - s^2 + i\sqrt{2k^2 - 2k + 22} 2rs \right]}{(2k^2 - 2k + 22)r^2 + s^2}$$

Equating real & imaginary parts, it is seen that

$$y = \frac{1}{(2k^2 - 2k + 22)r^2 + s^2} \left[\left\{ (2k^2 - 2k + 22)r^2 - s^2 \right\} \left\{ a^2 - (2k^2 - 2k + 22)b^2 \right\} - 4abrs \left\{ 2k^2 - 2k + 22 \right\} \right]$$

$$x = \frac{1}{(2k^2 - 2k + 22)r^2 + s^2} \left[2ab \left\{ (2k^2 - 2k + 22)r^2 - s^2 \right\} + 2rs \left\{ a^2 - (2k^2 - 2k + 22)b^2 \right\} \right] \quad (6)$$

Since our interest is to find the integer solutions, replacing a by $\left[(2k^2 - 2k + 22)r^2 + s^2 \right]A$ & b by $\left[(2k^2 - 2k + 22)r^2 + s^2 \right]B$ in (6) & (4), the corresponding integer solutions to (1) are given by

$$x = x(A, B) = \left((2k^2 - 2k + 22)r^2 + s^2 \right) \left[\left(A^2 - (2k^2 - 2k + 22)B^2 \right) 2rs + 2AB \left((2k^2 - 2k + 22)r^2 - s^2 \right) \right]$$

$$y = y(A, B) = \left((2k^2 - 2k + 22)r^2 + s^2 \right) \left[\left(A^2 - (2k^2 - 2k + 22)B^2 \right) \left[(2k^2 - 2k + 22)r^2 - s^2 \right] - 4ABrs(2k^2 - 2k + 22) \right]$$

$$z = z(A, B) = \left((2k^2 - 2k + 22)r^2 + s^2 \right)^2 \left(A^2 + (2k^2 - 2k + 22)B^2 \right)$$

Following the above procedure, one may obtain difference sets of integer solutions to (1).

METHOD: 4

(1) Is written as

$$z^2 - (2k^2 - 2k + 22)x^2 = y^2 = y^2 * 1 \quad (7)$$

Assume y as

$$y = a^2 - (2k^2 - 2k + 22)b^2 \quad (8)$$

Write 1 as

$$1 = \frac{\left((2k^2 - 2k + 22)r^2 + s^2 + \sqrt{2k^2 - 2k + 22} 2rs \right) \left((2k^2 - 2k + 22)r^2 + s^2 - \sqrt{2k^2 - 2k + 22} 2rs \right)}{\left((2k^2 - 2k + 22)r^2 - s^2 \right)^2} \quad (9)$$

Using (8) & (9) in (7) and employing the method of factorization, consider

$$z + \sqrt{2k^2 - 2k + 22} x = \frac{\left[(2k^2 - 2k + 22)r^2 + s^2 + 2rs\sqrt{2k^2 - 2k + 22} \right] * \left[a^2 + (2k^2 - 2k + 22)b^2 \right] + 2ab\sqrt{2k^2 - 2k + 22}}{(2k^2 - 2k + 22)r^2 - s^2}$$

Equating rational and irrational parts, it is seen that,

$$x = \frac{(a^2 + (2k^2 - 2k + 22)b^2) 2rs + 2ab((2k^2 - 2k + 22)r^2 + s^2)}{(2k^2 - 2k + 22)r^2 - s^2}$$

$$z = \frac{(a^2 + (2k^2 - 2k + 22)b^2)((2k^2 - 2k + 22)r^2 + s^2) + 4abrs(2k^2 - 2k + 22)}{(2k^2 - 2k + 22)r^2 - s^2} \quad (10)$$

Since our interest to find the integer solution, replacing a by $\left((2k^2 - 2k + 22)r^2 - s^2 \right)A$ & b by $\left((2k^2 - 2k + 22)r^2 - s^2 \right)B$ in (10) & (8), the corresponding integer solutions to (1) are given by

$$x = x(A, B) = \left((2k^2 - 2k + 22)r^2 - s^2 \right) \left[\left(A^2 + (2k^2 - 2k + 22)B^2 \right) 2rs + 2AB \left((2k^2 - 2k + 22)r^2 + s^2 \right) \right]$$

$$y = y(A, B) = \left((2k^2 - 2k + 22)r^2 - s^2 \right)^2 \left[A^2 - (2k^2 - 2k + 22)B^2 \right]$$

$$z = z(A, B) = \left((2k^2 - 2k + 22)r^2 - s^2 \right) \left[\left(A^2 + (2k^2 - 2k + 22)B^2 \right) \left((2k^2 - 2k + 22)r^2 + s^2 \right) + 4ABrs(2k^2 - 2k + 22) \right]$$

Following the above procedure, one may obtain difference sets of integer solutions to (1).

GENERATION OF SOLUTIONS

Different formulas for generating sequence of integer solutions based on the given solution are presented below:

Let (x_0, y_0, z_0) be any given solution to (1)

Formula: 1

Let (x_1, y_1, z_1) given by

$$x_1 = 3x_0, y_1 = 3y_0 + h, z_1 = 3z_0 + 2h \quad (11)$$

be the 2nd solution to (1). Using (11) in (1) and simplifying, one obtains

$$h = 2y_0 - 4z_0$$

In view of (11), the values of y_1 and z_1 are written in the matrix form as

$$(y_1, z_1)^t = M(y_0, z_0)^t$$

where

$$M = \begin{bmatrix} 5 & -4 \\ 4 & -5 \end{bmatrix}$$

and t is the transpose

The repetition of the above proses leads to the n^{th} solutions y_n, z_n givenby

$$(y_n, z_n)^t = M^n(y_0, z_0)^t$$

If α, β are the distinct eigen values of M , then

$$\alpha = 3, \beta = -3$$

We know that

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{(\beta - \alpha)}(M - \alpha I), \quad I = 2 \times 2 \text{ Identity matrix}$$

$$x_n = 3^n x_0$$

Thus, the general formulas for integer solutions to (1) are given by
$$\begin{pmatrix} y_n \\ z_n \end{pmatrix} = \frac{1}{3} \begin{bmatrix} 4\alpha^n - \beta^n & -2\alpha^n + 2\beta^n \\ 2\alpha^n - 2\beta^n & -\alpha^n + 4\beta^n \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

Formula: 2

Let (x_1, y_1, z_1) given by

$$x_1 = h - (2k^2 - 2k + 23)x_0, y_1 = h - (2k^2 - 2k + 23)y_0, z_1 = (2k^2 - 2k + 23)z_0 \quad (12)$$

be the 2nd solution to (1). Using (12) in (1) and simplifying, one obtains

$$h = (4k^2 - 4k + 44)x_0 + 2y_0$$

In view of (12), the values of x_1 and y_1 are written in the matrix form as

$$(x_1, y_1)^t = M(x_0, y_0)^t$$

$$\text{Where } M = \begin{bmatrix} 2k^2 - 2k + 21 & 2 \\ 4k^2 - 4k + 44 & -(2k^2 - 2k + 21) \end{bmatrix}$$

And t is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, y_n given by

$$(x_n, y_n)^t = M^n(x_0, y_0)^t$$

If α, β are the distinct eigen values of M , then

$$\alpha = 2k^2 - 2k + 23, \beta = -(2k^2 - 2k + 23)$$

Thus, the general formulas for integer solutions to (1) are given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \frac{1}{(2k^2 - 2k + 23)} \begin{bmatrix} (2k^2 - 2k + 22)\alpha^n + \beta^n & \alpha^n - \beta^n \\ (2k^2 - 2k + 22)(\alpha^n - \beta^n) & \alpha^n + (2k^2 - 2k + 22)\beta^n \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$z_n = (2k^2 - 2k + 23)^n z_0$$

Formula: 3

Let (x_1, y_1, z_1) given by

$$x_1 = h - (2k^2 - 2k + 21)x_0, y_1 = (2k^2 - 2k + 21)y_0, z_1 = (2k^2 - 2k + 21)z_0 + h \quad (13)$$

be the 2^{nd} solution to (1). Using (13) in (1) and simplifying, one obtains

$$h = 2z_0 + (4k^2 - 4k + 44)x_0$$

In view of (13), the values of x_1 and z_1 are written in the matrix form as

$$(x_1, z_1)^t = M(x_0, z_0)^t$$

$$\text{Where } M = \begin{bmatrix} 2k^2 - 2k + 23 & 2 \\ 4k^2 - 4k + 44 & 2k^2 - 2k + 23 \end{bmatrix}$$

and t is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, z_n given by

$$(x_n, z_n)^t = M^n(x_0, z_0)^t$$

If α, β are the distinct eigen values of M , then

$$\alpha = 2k^2 - 2k + 23 + 2\sqrt{2k^2 - 2k + 22},$$

$$\beta = 2k^2 - 2k + 23 - 2\sqrt{2k^2 - 2k + 22}$$

Thus, the general formulas for integer solutions to (1) are given by

$$y_n = (2k^2 - 2k + 21)^n y_0$$

$$\begin{pmatrix} x_n \\ z_n \end{pmatrix} = \frac{1}{2} \begin{bmatrix} \alpha^n + \beta^n & \frac{\alpha^n - \beta^n}{2\sqrt{2k^2 - 2k + 22}} \\ \sqrt{2k^2 - 2k + 22}(\alpha^n - \beta^n) & \alpha^n + \beta^n \end{bmatrix} \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}$$

CONCLUSION

In this paper, an attempt has been made to obtain non-zero distinct integer solutions to the ternary quadratic Diophantine equation $z^2 = (2k^2 - 2k + 22)x^2 + y^2$ representing homogeneous cone. As there are varieties of cones, the readers may search for other forms of cones to obtain integer solutions for the corresponding cones.

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