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A study on Darboux polynomials and their significance in determining other integrability quantifiers: A case study in third-order nonlinear ordinary differential equations

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Abstract. In this paper, we present a method for deriving quantifiers of the extended Prelle–Singer (PS) method using Darboux polynomials for third-order nonlinear ordinary differential equations. By knowing the Darboux polynomials and their co-factors, we extract the extended PS method's quantities without evaluating the PS method's determining equations. We consider three different cases of known Darboux polynomials. In the first case, we prove the integrability of the given third-order nonlinear equation by utilising the quantifiers of the PS method from the two known Darboux polynomials. If we know only one Darboux polynomial, then the integrability of the given equation will be dealt as Case 2. Likewise, Case 3 discusses the integrability of the given system where we have two Darboux polynomials and one set of PS method quantity. The established interconnection not only helps in deriving the integrable quantifiers without solving the underlying determining equations, but also provides a way to prove the complete integrability and helps us in deriving the general solution of the given equation. We demonstrate the utility of this procedure with three different examples.

Keywords. Integrability; nonlinear ordinary differential equations; extended Prelle–Singer procedure; Darboux polynomials; integrating factors.

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1. Introduction

Identifying the integrable systems is of great interest due to their importance and significance in the field of science and engineering. Integrability is the unique property of these integrable systems. Though there exist many methods exist in the literature to deal with the integrability of a given system, there is no unique method to study all the integrable systems. Each integrable system has to be studied and analysed separately. These integrable systems are governed by either nonlinear partial differential equations (PDEs) or nonlinear ordinary differential equations (ODEs). Several studies had already been reported in the literature [1–6] to analyse the integrability of the systems governed by nonlinear PDEs. For more details about the integrability of nonlinear PDEs, one can refer to refs [7–11] and references therein. Likewise, methods have been developed to integrate and find the solutions of nonlinear ODEs [12,13]. Some of the well-known mathematical methods that are being used in the recent literature to derive the solutions of nonlinear ODEs are: (i) Lie symmetry analysis [14–17], (ii) extended Prelle–Singer procedure [18–23], (iii) Darboux method [24–28], (iv) Jacobi last multiplier method [29–31], (v) λ-symmetry analysis [32–37], (vi) homogeneous balance method [38], its extended versions and so on [39–43]. Each method has its own benefits and limitations. As far as the limitations of these methods are concerned, one has to go for an ansatz to determine the Darboux polynomials (DPs) in the Darboux method. Suppose the system admits a trigonometric form of integral, it then becomes very difficult to obtain through the DP method. As far as the extended Prelle–Singer (PS) procedure is concerned, a proper ansatz should be made to determine the underlying integrable quantifiers, namely the null forms (P, Q) and integrating factors (R). It has been shown that certain equations are to be integrable even though they do not possess Lie point symmetries. In such circumstances, to construct the integrals, it is informative to search for

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more generalised symmetries. Determining those generalised symmetries is often a cumbersome task.

Recently, efforts have been made to interlink the analytical methods [34,44–46]. By interconnecting the methods, we can derive one integrability quantifier from the other. For example, suppose we know PS quantities, that are null forms and integrating factors, then the established interconnections will help in deriving the other integrability quantifiers, namely DPs, Lie symmetries, Jacobi last multipliers and integrating factors without recourse to the respective method.

In one of our earlier works, by considering second-order ODEs, we have shown a method to derive extended PS quantities from DPs [44]. We have also analysed the connection between the various analytical methods for third-order ODEs and nth-order ODEs in refs [45,46], respectively. From the analysis, we have observed that some of the interconnections remain the same for any order ODEs except their order. However, the rare connections are more complex than the corresponding lower-order connections. In this paper, we create an interlink between DPs and the extended PS quantities for third-order ODEs. We begin our work with DP and its co-factor at hand. Using this co-factor, we determine another function which in turn helps to identify second null form from the first null form. Finding this function plays a major role in this procedure. Determination of this function enables us to derive other integrability quantifiers algebraically. We explain the interconnection in three categories. In the first category, we consider a situation in which we know two sets of DPs and their co-factors. Here, we aim to determine three null forms, three integrating factors and three integrals for the considered system from the two known DPs and their co-factors. Interestingly, we also explore the third DP and its co-factor through the established interconnections without solving the DP determining equation. In the second category, we consider a situation in which we know only one DP and its co-factor for the given ODE. The method of deriving the other unknown quantifiers, namely two more DPs and their co-factors and three sets of PS quantities and their associated integrals from the known quantifier will be discussed in the second case. In the third category, we consider a case in which we know two DPs and their co-factors and also one set of PS quantities. In this case, we show that the other quantifiers and the complete integrability of the given ODE can be established by mere algebraic calculations. The main advantage of the proposed procedure is that instead of solving the cumbersome determining equations in the concerned method, one can derive a number of integrability quantifiers from the known quantifier.

This framework of the article is as follows: In §2, we recall the two analytical methods for third-order nonlinear ODEs, namely Darboux method and the extended PS procedure. In §3, we connect DPs and their co-factors with extended PS quantities. In §4, we demonstrate the method of deriving unknown quantifiers from known quantifiers for three different situations. In §5, we illustrate the procedure with suitable examples. Finally, we conclude our article in §6.

2. Analytical methods for the third-order ODEs

In this section, in order to be self-contained, we briefly recall the analytical methods, namely Darboux method and the extended PS procedure, which we intend to interconnect.

2.1 *Darboux method*

Let us consider a third-order nonlinear ODE of the form

$$\dddot{x} = \phi(t, x, \dot{x}, \ddot{x}), \tag{1}$$

where ϕ is a function of t, x, $\dot{x}$ and $\ddot{x}$ and overdot denotes differentiation with respect to t. Let us suppose that eq. (1) admits an integral of the form $I = l_1(t, x, \dot{x}, \ddot{x})/l_2(t, x, \dot{x}, \ddot{x})$ where l_1 and l_2 are functions of their arguments. Upon differentiating this integral with respect to t, we find

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{l_1}{l_2} = 0 \Rightarrow \dot{l_1} = h(t, x, \dot{x}, \ddot{x})l_1$$
$$\Rightarrow D[l_1] = h(t, x, \dot{x}, \ddot{x})l_1. \tag{2}$$

Equation (2) is the determining equation for the DP [24,25,28]. Here

$$D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \phi\frac{\partial}{\partial \ddot{x}}$$

is the total differential operator and $h(t, x, \dot{x}, \ddot{x})$ is the DP cofactor. Solving (2), we can determine DPs (l) and their co-factors (h) of eq. (1). The ratio of two DPs whose co-factors are the same provides an integral for eq. (1) [25]. Even though it is sufficient to know particular solutions of (2) to explore those solutions, one often assumes an ansatz for the functions l and h.

2.2 *Extended PS procedure*

In this procedure, we rewrite eq. (1) in the form $\mathrm{d}\ddot{x} - \phi\,\mathrm{d}t = 0$. Now adding the terms $Q(t, x, \dot{x}, \ddot{x})\ddot{x}\,\mathrm{d}t - Q(t, x, \dot{x}, \ddot{x})\mathrm{d}\dot{x}$ and $P(t, x, \dot{x}, \ddot{x})\dot{x}\,\mathrm{d}t - P(t, x, \dot{x}, \ddot{x})\,\mathrm{d}x$ in this equation, we get

$$(\phi + P\dot{x} + Q\ddot{x})\mathrm{d}t - P\mathrm{d}x - Q\mathrm{d}\dot{x} - \mathrm{d}\ddot{x} = 0. \tag{3}$$

Note that the terms which we appended above reshape the third-order ODE (1) into a more general 1-form. The functions P and Q are called null forms. Upon multiplying eq. (3) by an integrating factor ($R(t, x, \dot{x}, \ddot{x})$), it can be rewritten as a perfect differential function, that is

$$R(\phi+P\dot{x}+Q\ddot{x})\mathrm{d}t - RP\mathrm{d}x - RQ\mathrm{d}\dot{x} - R\mathrm{d}\ddot{x} = \mathrm{d}I = 0, \tag{4}$$

where I is an integral of eq. (1).

Let us recall that the total differential I of (1) can also be written as

$$\mathrm{d}I = \frac{\partial I}{\partial t}\mathrm{d}t + \frac{\partial I}{\partial x}\mathrm{d}x + \frac{\partial I}{\partial \dot{x}}\mathrm{d}\dot{x} + \frac{\partial I}{\partial \ddot{x}}\mathrm{d}\ddot{x} = 0.$$

Comparing this total differential with the one obtained in (4), we can identify the following relations:

$$\begin{aligned} &I_t = R(\phi + \dot{x}P + Q\ddot{x}),\ I_x = -RP,\\ &I_{\dot{x}} = -RQ,\ I_{\ddot{x}} = -R. \end{aligned} \tag{5}$$

Upon integrating the first-order ODEs (5), we find

$$\begin{aligned} I(t, x, \dot{x}, \ddot{x}) = p_1 - p_2 &- \int \left(RQ + \frac{\mathrm{d}}{\mathrm{d}\dot{x}}[p_1 - p_2]\right)\mathrm{d}\dot{x}\\ &- \int \left(R + \frac{\mathrm{d}}{\mathrm{d}\ddot{x}}\left[p_1 - p_2 - \int \left[RQ + \frac{\mathrm{d}}{\mathrm{d}\dot{x}}[p_1 - p_2]\right]\mathrm{d}\dot{x}\right]\right)\mathrm{d}\ddot{x}, \end{aligned} \tag{6}$$

where

$$p_1 = \int R(\phi + P\dot{x} + Q\ddot{x})\mathrm{d}t,$$

$$p_2 = \int \left(RP + \frac{\mathrm{d}}{\mathrm{d}x}\int p_1\right)\mathrm{d}x.$$

Substituting the expressions P, Q and R in (6) and integrating the resultant expressions, we can obtain the integrals of (1).

The null functions (P and Q) and the integrating factor (R) can be determined from the integrability conditions, $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{t\ddot{x}} = I_{\ddot{x}t}$, $I_{x\dot{x}} = I_{\dot{x}x}$, $I_{x\ddot{x}} = I_{\ddot{x}x}$ and $I_{\dot{x}\ddot{x}} = I_{\ddot{x}\dot{x}}$, where the first-order partial derivatives of the integral I are given in (5). These six integrability conditions provide six equations among which three of them turn out to be the determining equations for the unknowns (P, Q and R) and the other three turn out to be a set of constraints that have to be satisfied by these functions. The determining equations and the constraints read as

$$D[P] = -\phi_x + P\phi_{\ddot{x}} + QP, \tag{7}$$

$$D[Q] = -\phi_{\dot{x}} + Q\phi_{\ddot{x}} - P + Q^2, \tag{8}$$

$$D[R] = -R(Q + \phi_{\ddot{x}}), \tag{9}$$

$$R_x = R_{\ddot{x}}P + RP_{\ddot{x}}, \tag{10}$$

$$R_{\dot{x}}P = -P_{\dot{x}}R + R_xQ + RQ_x, \tag{11}$$

$$R_{\dot{x}} = R_{\ddot{x}}Q + RQ_{\ddot{x}}. \tag{12}$$

The method of solving these equations has already been discussed in ref. [20].

Let the three independent solutions of eqs (7)–(12) be P_i, Q_i, R_i, $i = 1, 2, 3$. For each set, we can construct an integral through (6). Let us designate these integrals as

$$I_1 = N_1(t, x, \dot{x}, \ddot{x}), \tag{13}$$

$$I_2 = N_2(t, x, \dot{x}, \ddot{x}), \tag{14}$$

$$I_3 = N_3(t, x, \dot{x}, \ddot{x}). \tag{15}$$

Thus, the complete integrability of eq. (1) can be established through the extended PS procedure by determining three sets of null forms (P and Q) and the integrating factors (R).

3. Extracting one integrability quantifier from another known quantifier

Now, we connect DPs with the PS quantities. By succeeding in this task, we can construct the integrability quantifiers of the second method from the integrability quantifiers of the first method itself. Rearranging expression (13) for $\ddot{x}$ and substituting it in (14), we find

$$I_2 = \tilde{N}_2(t, x, \dot{x}, I_1). \tag{16}$$

Differentiating eq. (16) with respect to x, $\dot{x}$ and $\ddot{x}$ separately, we obtain

$$I_{2x} = \tilde{N}_{2x} + \tilde{N}_{2I_1}I_{1x} = -p_2P_2, \tag{17}$$

$$I_{2\dot{x}} = \tilde{N}_{2\dot{x}} + \tilde{N}_{2I_1}I_{1\dot{x}} = -p_2Q_2, \tag{18}$$

$$I_{2\ddot{x}} = \tilde{N}_{2I_1}I_{1\ddot{x}} = p_2, \tag{19}$$

where we have considered the expressions given in (5) to obtain the right-hand side of eqs (17)–(19). Replacing the terms $I_{1\ddot{x}}$ and $I_{2\ddot{x}}$ by p_1 and p_2 respectively in eq. (19), we find

$$p_2 = l_1p_1, \qquad l_1 = \tilde{N}_{2I_1}. \tag{20}$$

Substituting the relations, $I_{1x} = -p_1P_1$ and $I_{2\ddot{x}} = \tilde{N}_{2I_1}I_{1\ddot{x}} = p_2$, given in (5) in (19) and simplifying the resultant equation, we arrive at the following expression for the second null form P_2:

$$P_2 = m_1 + P_1, \tag{21}$$

where the unknown function $m_1(=\tilde{N}_{2x}/\tilde{N}_{2I_1}I_{1\ddot{x}})$ is to be determined. Expression (21) connects the first null form P_1 with the second null form P_2.

Let us consider eq. (18) and replace the terms $I_{1\dot{x}}$ by $-p_1Q_1$ and p_2 by $-\tilde{N}_{2I_1}I_{1\ddot{x}}$. This action yields the following expression for Q_2:

$$Q_2 = c_1 + Q_1, \tag{22}$$

where the function $c_1(=\tilde{N}_{2\dot{x}}/\tilde{N}_{2I_1}I_{1\ddot{x}})$ is yet another function. Expression (22) connects the first null form Q_1 with the second null form Q_2. Relations (20)–(22) provide a way to determine the second integrating factor (p_2) and the second set of null forms (P_2 and Q_2) from the quantities (P_1, Q_1 and p_1) by determining the functions m_1, c_1 and l_1. The functions m_1, c_1 and l_1 are functions of t, x, $\dot{x}$ and $\ddot{x}$. Now, we lay out a procedure to determine these three functions.

Let us differentiate eq. (21) with respect to t and substitute it into (7). In this process, we replace the functions Q_2 and P_2 by expressions (21) and (22). Implementing this, we end up at

$$D[m_1] = m_1(\phi_{\ddot{x}} + c_1 + Q_1) + c_1P_1. \tag{23}$$

A similar equation can also be derived for the unknown function c_1 by differentiating eq. (22) with respect to t and substituting the resultant expression along with (21) and (22) in the PS method determining equation (8). Here, we find

$$D[c_1] = c_1^2 + c_1(\phi_{\ddot{x}} + 2Q_1) - m_1. \tag{24}$$

On the other hand, differentiating eq. (20) with respect to t and substituting the determining equation (9) into it, we get

$$D[l_1] = -c_1l_1. \tag{25}$$

Equation (25) is nothing other than the determining equation for DP (see eq. (2)) in which l_1 is the DP and $(-c_1 = h)$ is the co-factor. We note that the third set of quantifiers (P_3, Q_3, R_3) can be determined using the same expressions (20), (21) and (22) by considering P_2, Q_2 and p_2 as known quantities and P_3, Q_3 and R_3 as unknowns.

4. Methodology to derive the integrability of the third-order ODEs

4.1 *Case* 1: *Two DPs are known*

To begin, let us assume that we know two DPs (l_1 and l_2) and their associated co-factors (c_1 and c_2) of eq. (1). These two polynomials are solutions of eq. (25), that is $D[l_i] = c_il_i$, $i = 1, 2$. Now we determine the PS quantities, P_i, Q_i, R_i, $i = 1, 2, 3$, and their integrals from the two known DPs.

Substituting the known expression c_1 in eq. (24), we express the null form Q_1 in terms of the function m_1. Inserting this expression into (23), we get an equation which involves P_1 and m_1. Now plugging this expression into (7) and solving the resultant differential equation, we obtain an explicit form of m_1.

Substituting the determined function m_1 back in (24), we can obtain the form of Q_1. Inserting this null form Q_1 in (8) we can get the null form P_1. The integrating factor p_1 can be calculated from the last expression given in (9). With the known expressions P_1 and m_1, eq. (21) yields the expression for second null form P_2. Since c_1 and Q_1 are known, inserting them in (22) helps us in identifying Q_2. The second integrating factor p_2 can be obtained from expression (20). In this way, we can generate two sets of PS quantities, namely U_i, S_i and R_i, $i = 1, 2$, from the known DP l_1 and its co-factor c_1. We note that one can also reformulate the steps given above and alternatively derive the quantifiers U_i, S_i and R_i, $i = 1, 2$,. This essentially depends upon the difficulties one may come across while following the steps given before. However, in either of the methods one has to solve only one differential equation. The rest of the calculations involve only algebra.

To explore the third null form Q_3, we reconsider expression (22) in the form

$$Q_3 = c_2 + Q_2. \tag{26}$$

Here c_2 is the co-factor of the second DP (l_2) and Q_2 is the second null form. Since both are known, the third null form Q_3 can be identified from (26). To determine the other null form P_3, we consider eq. (21) in the form

$$P_3 = m_2 + P_2. \tag{27}$$

Following our earlier footsteps, the function m_2 can be determined from eq. (24) by considering it in the form $D[c_2] = c_2^2 + c_2(\phi_{\ddot{x}} + 2Q_2) - m_2$. Plugging the known functions c_2 and Q_2 in this equation and rewriting, we can obtain the expression for m_2. Since the functions m_2 and P_2 are known, we can fix the third null form P_3. The integrating factor R_3 can be fixed from the relation

$$R_3 = l_2p_2. \tag{28}$$

Substituting each set of the functions P_i, Q_i and R_i, $i = 1, 2, 3$, separately in (6) and evaluating the integrals, we can obtain the first integrals of eq. (1).

Thus, knowledge of the DPs and their co-factors help to identify the integrability quantifiers and the integrals for the given ODE. One may observe that the function Q_3 can be obtained from Q_2 and only the co-factor c_2 associated with the second DP is known. In some circumstances, we may know only one DP and its co-factor for the given equation. In this case, we adopt another way and determine the second DP (l_2) and its co-factor (c_2). We investigate this situation as a separate case in the following.

4.2 *Case* 2: *Only one DP is known*

Let us suppose that we know only one DP (l_1) and its associated co-factor (c_1). In this case, we need to determine the second DP (l_2) and its co-factor (c_2) to capture other integrability quantifiers. To succeed in this case, we proceed as follows.

We start the procedure from eq. (28). By substituting eq. (20) into it we find $R_3 = l_2 l_1 p_1$ from which we can obtain the relation

$$l_2 l_1 = \frac{R_3}{p_1}. \tag{29}$$

Since l_1 and l_2 are two DPs, their product ($l_1 l_2$) can also be considered as a DP [28]. Since the left-hand side in (29) is a polynomial, the right-hand side (R_3/p_1) may also be considered as a polynomial. Considering this fact, we identify the following relation:

$$l_2 = \frac{f}{l_1}, \tag{30}$$

where $f = R_3/p_1$. Since the function f is a DP, it should satisfy the equation

$$D[f] = gf, \tag{31}$$

where g is the associated co-factor.

Differentiating eq. (30) with respect to t and replacing the terms $D[l_2]$ by $-c_2 l_2$ and $D[f]$ by gf and simplifying the resultant equation, we find

$$c_2 = -(c_1 + g). \tag{32}$$

Substituting (32) in the determining equation for c_2 (refer eq. (24)), the latter equation becomes

$$\begin{aligned} D[c_1] + D[g] = &- \left(c_1^2 + g^2 + 2gc_1\right) \\ &+ (c_1 + g)(\phi_{\ddot{x}} + 2Q_2) + m_2. \end{aligned} \tag{33}$$

The unknown functions in eq. (33) are g and m_2. Considering eq. (23) for m_2 in the form $D[m_2] = m_2(\phi_{\ddot{x}} + c_2 + Q_2) + c_2 P_2$ and substituting eq. (32) into it and rearranging the resultant equation, we end up with

$$D[m_2] = m_2(\phi_{\ddot{x}} - c_1 - g + Q_2) - (c_1 + g)P_2. \tag{34}$$

Inserting the expression m_2 from (33) in eq. (34), the latter equation turns out to be the determining equation for g. Solving this equation, we can determine the explicit form of g, from which we can identify the function m_2 through (33). The DP (f) can be found by substituting the expression g back in eq. (31) and solving the resultant equation. In this way, we can identify not only the second DP (l_2) and its co-factor g but also the needed functions m_2 and c_2. The third DP, its co-factor and the extended PS procedure quantities can be derived in the same way as we did in Case 1.

4.3 *Case* 3: *Two DPs and first set of PS method quantities are known*

Suppose we know two DPs, their co-factors and the first set of PS method quantities, then the remaining quantifiers of the considered system can be determined algebraically. By substituting the co-factor c_1 and the null form Q_1 in (24) and simplifying the resultant expression, we can obtain the function m_1. Substituting the DP (l_1), its cofactor (c_1), the function m_1 and the first set of PS quantities in expressions (20)–(22) and simplifying the resultant equations, we can identify the second set of PS quantities (P_2, Q_2, p_2). Now repeating the same procedure with the second set of PS quantities (P_2, Q_2, p_2), DP (l_2) and its co-factor (c_2), we can derive the third set of PS method quantities. The function m_2 can be derived using the expression (vide eq. (24))

$$D[c_2] = c_2^2 + c_2(\phi_{\ddot{x}} + 2Q_2) - m_2. \tag{35}$$

The associated integrals can be constructed with the help of (6).

5. Illustrations for the procedure

In this section, we consider three examples and apply the theory developed in the previous section to these examples. In the first example, we consider the situation in which two DPs are known and in the second example we consider the situation where only one DP is known. In the third example, we consider a situation in which we have partial information on DP and PS quantifiers. With these partial information at hand, in the third example, we demonstrate that the complete integrability of the given equation can be established in an algebraic manner through the interconnections found in this work.

5.1 *Example* 1

We consider Chazy equation of the form [20]

$$\dddot{x} + 4x\ddot{x} + 3\dot{x}^2 + 6x^2\dot{x} + x^4 = 0. \tag{36}$$

Equation (36) admits two DPs of the form

$$l_1 = -\frac{t(x(tx-2) + t\dot{x})(x^3 + 3x\dot{x} + \ddot{x})^2}{\left(x^2 + \dot{x}\right)\left(-tx^3 - 3tx\dot{x} - t\ddot{x} + x^2 + \dot{x}\right)^2}, \tag{37}$$

$$l_2 = \frac{t^2(x^2+\dot{x}) - 3tx + 3}{3(x(tx-2)+t\dot{x})}. \quad (38)$$

The associated co-factors are given by

$$c_1 = \frac{2x(-tx^3 - 3tx\dot{x} - t\ddot{x} + x^2 + \dot{x})}{t\left(x^2+\dot{x}\right)(x(tx-2)+t\dot{x})}, \quad (39)$$

$$c_2 = -\frac{(tx-3)(tx^3 + 3tx\dot{x} + t\ddot{x} - x^2 - \dot{x})}{(x(tx-2)+t\dot{x})\left(t^2\left(x^2+\dot{x}\right) - 3tx + 3\right)}. \quad (40)$$

One can check that expressions (37)–(40) satisfy eq. (2). In the following, we derive the PS quantifiers and establish the integrability of (36).

Substituting the co-factor c_1 in (23), the latter equation becomes

$$D[m_1] - m_1\left(\frac{2x(-tx^3 - 3tx\dot{x} - t\ddot{x} + x^2 + \dot{x})}{t\left(\dot{x}+x^2\right)(t\dot{x}+x(tx-2))} - 4x + Q_1\right) - \left(\frac{2x(-tx^3 - 3tx\dot{x} - t\ddot{x} + x^2 + \dot{x})}{t\left(\dot{x}+x^2\right)(t\dot{x}+x(tx-2))}\right)P_1 = 0. \quad (41)$$

Rewriting eq. (41) for the null form Q_1, we find

$$Q_1 = \frac{1}{m_1}\left[D[m_1] - m_1\left(\frac{2x(-tx^3 - 3tx\dot{x} - t\ddot{x} + x^2 + \dot{x})}{t\left(\dot{x}+x^2\right)(t\dot{x}+x(tx-2))} - 4x\right) - \left(\frac{2x(-tx^3 - 3tx\dot{x} - t\ddot{x} + x^2 + \dot{x})}{t\left(\dot{x}+x^2\right)(t\dot{x}+x(tx-2))}\right)P_1\right]. \quad (42)$$

Substituting this expression into eq. (8) and rewriting it, we obtain an evolution equation for P_1 in terms m_1. Now plugging this expression into (7) we can obtain a determining equation for the unknown m_1. Since the resultant expression is lengthy, we do not reproduce the equation here. Equation (42) admits a particular solution of the form

$$m_1 = -\frac{2(x^2-\dot{x})(tx^3 + 3tx\dot{x} + t\ddot{x} - x^2 - \dot{x})}{t\left(x^2+\dot{x}\right)(x(tx-2)+t\dot{x})}. \quad (43)$$

To determine the null form Q_1, we substitute the functions m_1 and c_1 back in eq. (24) which in turn yields the following expression for the null form Q_1, that is

$$Q_1 = \frac{2x^3 - \ddot{x}}{x^2+\dot{x}}. \quad (44)$$

Substituting this expression in eq. (8) and simplifying it, we find

$$P_1 = -\frac{\ddot{x}}{x}. \quad (45)$$

To determine the integrating factor p_1, we use eq. (9). Upon solving this equation, we find

$$p_1 = \frac{x^2+\dot{x}}{\left(x^3 + 3x\dot{x} + \ddot{x}\right)^2}. \quad (46)$$

Thus, from the knowledge of one DP (l_1) and its associated co-factor (c_1) we can obtain the PS quantifiers (Q_1, P_1, p_1).

Now inserting the functions P_1, p_1 and Q_1 in (6), and evaluating the integrals, we find an integral of (36), in the from

$$I_1 = \frac{x^2+\dot{x}}{x^3 + 3x\dot{x} + \ddot{x}} - t. \quad (47)$$

Now we proceed to calculate the second set of null forms (P_2, Q_2) and the integrating factor (p_2) from the known expressions m_1, c_1, l_1, Q_1, P_1 and p_1 (eqs (21), (22) and (20)). Following the procedure given in §4.1 we obtain the following expression for P_2, Q_2, p_2, that is

$$P_2 = \frac{t^2x^4 - 2t^2x\ddot{x} + \dot{x}(3t^2\dot{x} - 2) - 4tx^3 + 2t\ddot{x} + 2x^2}{t(x(tx-2)+t\dot{x})}, \quad (48)$$

$$Q_2 = \frac{2x(tx(tx-3)+1) - t^2\ddot{x}}{t(x(tx-2)+t\dot{x})}, \quad (49)$$

$$p_2 = -\frac{t(x(tx-2)+t\dot{x})}{\left(-tx^3 - 3tx\dot{x} - t\ddot{x} + x^2 + \dot{x}\right)^2}. \quad (50)$$

Inserting expressions (48)–(50) in (6) and evaluating the integrals, we identify the second integral of eq. (36) in the form

$$I_2 = \frac{t^2(-(x^3 + 3x\dot{x} + \ddot{x})) + 2t(x^2+\dot{x}) - 2x}{-t\left(x^3 + 3x\dot{x} + \ddot{x}\right) + x^2 + \dot{x}}. \quad (51)$$

To obtain the third set of integrable quantifiers (P_3, Q_3, R_3), we need to determine the function m_2. To do so, we consider eq. (24) in the form

$$D[c_2] = c_2^2 + c_2(\phi_{\ddot{x}} + 2Q_2) - m_2. \quad (52)$$

Inserting the known null form Q_2 and the DP co-factor c_2 in (52) and rearranging the resultant equation for m_2, we find

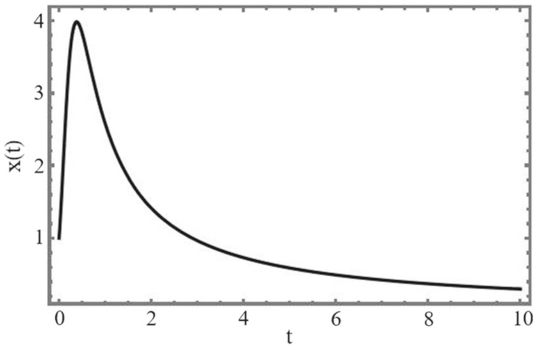

Figure 1. Solution plot for eq. (36) with the initial conditions $I_1 = 0.1,\ I_2 = 0.1,\ I_3 = 0.1$.

$$m_2 = -\frac{(t(x(tx-6)-t\dot{x})+6)(tx^3+3tx\dot{x}+t\ddot{x}-x^2-\dot{x})}{t(x(tx-2)+t\dot{x})\left(t^2\left(x^2+\dot{x}\right)-3tx+3\right)}. \tag{53}$$

With the help of the functions m_2 and c_2, the third set of null forms P_3, Q_3 and the integrating factor R_3 can be captured using expressions (27), (26) and (28). Our results show

$$P_3 = \frac{t^3x^4-2x(t^3\ddot{x}+3)-6t^2x^3+3t(\dot{x}(t^2\dot{x}+2)+t\ddot{x})+12tx^2}{t\left(t^2\left(x^2+\dot{x}\right)-3tx+3\right)}, \tag{54}$$

$$Q_3 = \frac{t^3(-\ddot{x})+tx(tx(2tx-9)+12)-3}{t\left(t^2\left(x^2+\dot{x}\right)-3tx+3\right)}, \tag{55}$$

$$R_3 = -\frac{t(t^2(x^2+\dot{x})-3tx+3)}{3\left(-tx^3-3tx\dot{x}-t\ddot{x}+x^2+\dot{x}\right)^2}. \tag{56}$$

This third set of null forms (P_3, Q_3) and the integrating factor (R_3) helps to construct the third integral of eq. (36) and the resultant form becomes

$$I_3 = \frac{t^3(-(x^3+3x\dot{x}+\ddot{x}))+3t^2(x^2+\dot{x})+6(1-tx)}{6\left(-t\left(x^3+3x\dot{x}+\ddot{x}\right)+x^2+\dot{x}\right)}. \tag{57}$$

The general solution of (36) can be derived using the integrals I_1, I_2 and I_3 and it is given by

$$x(t) = \frac{\frac{t^2}{2}+I_1t+I_1I_2}{\frac{t^3}{6}+I_1\frac{t^2}{2}+I_1I_2t+I_1I_3}. \tag{58}$$

This solution is plotted in figure 1.

5.2 *Example* 2

In the second example, we consider a situation in which we know only one DP and its co-factor. Now the task is to determine the remaining two sets of DPs and other integrability quantifiers used in the PS procedure. To demonstrate this, we consider an equation of the form [14]

$$\dddot{x} = \frac{6t\ddot{x}^3}{\dot{x}^2}+\frac{6\ddot{x}^2}{\dot{x}}. \tag{59}$$

The above equation admits a DP of the form

$$l_1 = \dot{x}, \tag{60}$$

with the co-factor

$$c_1 = -\frac{\ddot{x}}{\dot{x}}. \tag{61}$$

In the following, we adopt the procedure given in §4.1 and obtain the null forms and its integrating factors. Now, following the steps given in the previous example (see eqs (41)–(46)) we can deduce the first set of null form and the integrating factor in the form

$$P_1 = \frac{2\ddot{x}^2}{\dot{x}^2},\quad Q_1 = \frac{-6t\ddot{x}^2-2\dot{x}\ddot{x}}{\dot{x}^2},\quad p_1 = \frac{\dot{x}^2}{\ddot{x}^2}. \tag{62}$$

While deriving the above functions, we also come across the following expression for m_1:

$$m_1 = -\frac{2\ddot{x}^2}{\dot{x}^2}. \tag{63}$$

Using (62) and (63), we can obtain the second set of null forms (P_2, Q_2) and the integrating factor (p_2) with the help of eqs (21), (22) and (20). The resultant outcome shows that

$$P_2 = 0,\quad Q_2 = \frac{-6t\ddot{x}^2-3\dot{x}\ddot{x}}{\dot{x}^2},\quad p_2 = \frac{\dot{x}^3}{\ddot{x}^2}. \tag{64}$$

To obtain the third set of null forms (P_3, Q_3) and the integrating factor (R_3), we need to determine the second DP (l_2) and its co-factor (c_2) from the first DP (l_1).

Substituting the known expressions c_1 and Q_2 in eq. (33) and simplifying it, we obtain the following determining equation for g:

$$D[g]+g^2-2g\left(\frac{\ddot{x}(3t\ddot{x}+4\dot{x})}{\dot{x}^2}\right) - m_2 + 2\frac{\ddot{x}^2}{\dot{x}^3}(5\dot{x}+6t\ddot{x}) = 0. \tag{65}$$

Rewriting the above expression for m_2 and substituting this expression in (34) we obtain a differential equation in terms of the unknown function m_2. To proceed further, we choose a trivial solution for m_2, that is

$$m_2 = 0. \tag{66}$$

Substituting this trivial form in eq. (65) and solving the resultant expression, we identify the following particular solution for the function g,

$$g = \frac{2\ddot{x}}{\dot{x}}. \tag{67}$$

Plugging this expression in (31) and solving the latter equation, we find a particular solution for f as

$$f = \dot{x}^2. \tag{68}$$

Since the functions f and l_1 are now determined, the second DP (l_2) can be readily identified from $l_2 = f/l_1$, which in turn reads as

$$l_2 = \dot{x}. \tag{69}$$

We note here that the second DP is the same as the first DP (l_1). The associated co-factor turns out to be $c_2 = -\ddot{x}/\dot{x}$.

Equations (26)–(28) yield the third set of null forms and integrating factor in the form

$$P_3 = 0, \quad Q_3 = \frac{-6t\ddot{x}^2 - 4\dot{x}\ddot{x}}{\dot{x}^2}, \quad R_3 = \frac{\dot{x}^4}{\ddot{x}^2}. \tag{70}$$

The functions $P_i, Q_i, R_i, i = 1, 2, 3$, help in building the necessary integrals of (59) whose explicit forms are given by

$$I_1 = 6t\dot{x} - 2x + \frac{\dot{x}^2}{\ddot{x}}, \quad I_2 = 3t\dot{x}^2 + \frac{\dot{x}^3}{\ddot{x}},$$
$$I_3 = 2\left(2t\dot{x}^3 + \frac{\dot{x}^4}{\ddot{x}}\right). \tag{71}$$

Using the integrals I_1, I_2 and I_3 we can derive an implicit solution for eq. (59) in the form

$$I_2((2x + I_1)^2 - 12tI_2)^2 + 3t(9tI_3 + I_2(I_1 - 2x))^2$$
$$-(2x+I_1)((2x+I_1)^2-12tI_2)(9tI_3+I_2(2x+I_1)) = 0. \tag{72}$$

5.3 *Example* 3

Here, we consider a situation in which we have partial information on DPs and also have partial information on PS quantifiers. In this case, the procedure developed in this article supports to study the complete integrability of the considered equation in an algebraic manner, as shown below.

Let us consider a third-order nonlinear ODE, of the form [47]

$$\dddot{x} = \frac{\dot{x}\ddot{x}}{x} + \frac{\ddot{x}^2}{\dot{x}}. \tag{73}$$

Suppose that we know two DPs of this equation, say

$$l_1 = 2\dot{x} - \frac{2x\ddot{x}}{\dot{x}}, \tag{74}$$

$$l_2 = \frac{x\dot{x}(t\dot{x}^2 - x(t\ddot{x} + \dot{x}))}{4\sqrt{-x\ddot{x}\left(x\ddot{x} - 2\dot{x}^2\right)}\left(x\ddot{x} - \dot{x}^2\right)} \tag{75}$$

and their corresponding co-factors

$$c_1 = \frac{\dot{x}\ddot{x}}{\dot{x}^2 - x\ddot{x}}, \tag{76}$$

$$c_2 = -\frac{x\ddot{x}(x\ddot{x} - 2\dot{x}^2)}{\left(\dot{x}^2 - x\ddot{x}\right)\left(t\dot{x}^2 - x(t\ddot{x} + \dot{x})\right)}. \tag{77}$$

Let us also assume that we know the null forms (P_1, Q_1) and the integrating factor (p_1) of eq. (73), say

$$P_1 = -\frac{\ddot{x}}{x}, \quad Q_1 = -\frac{\ddot{x}}{\dot{x}}, \quad p_1 = -\frac{1}{x\dot{x}}. \tag{78}$$

Now we deduce the other quantifiers in an algebraic way as follows:

Since c_1 and Q_1 are known, m_1 can be fixed from (24). Here, we find

$$m_1 = \frac{\ddot{x}^2}{x\ddot{x} - \dot{x}^2}. \tag{79}$$

Substituting expression (78) in eqs (21), (22) and (20), we can obtain the following quantifiers:

$$P_2 = \frac{\dot{x}^2\ddot{x}}{x\left(x\ddot{x} - \dot{x}^2\right)},$$
$$Q_2 = \frac{x\ddot{x}^2}{\dot{x}^3 - x\dot{x}\ddot{x}},$$
$$p_2 = \frac{2\ddot{x}}{\dot{x}^2} - \frac{2}{x}. \tag{80}$$

To obtain the third set of integrable quantifiers P_3, Q_3 and R_3, we have to determine the function m_2. It can be derived from (52) through a direct path. The function m_2 turns out to be

$$m_2 = -\frac{x\ddot{x}(x\ddot{x} - 2\dot{x}^2)}{\left(\dot{x}^2 - x\ddot{x}\right)\left(t\dot{x}^2 - x(t\ddot{x} + \dot{x})\right)}. \tag{81}$$

We can capture the third set of integrable quantifiers with the help of eqs (26)–(28). The resultant expressions read as

$$P_3 = \frac{\dot{x}\ddot{x}(tx + \dot{x})}{x\left(x(\dot{x} + t\ddot{x}) - t\dot{x}^2\right)},$$
$$Q_3 = \frac{x\ddot{x}(t\ddot{x} + 2\dot{x})}{t\dot{x}^3 - x\dot{x}(t\ddot{x} + \dot{x})},$$
$$R_3 = \frac{t\dot{x}^2 - x(t\ddot{x} + \dot{x})}{2\dot{x}\sqrt{x\ddot{x}\left(2\dot{x}^2 - x\ddot{x}\right)}}. \tag{82}$$

The integrals can be constructed from eq. (6) by appropriately substituting the functions P_i, Q_i, R_i, i =

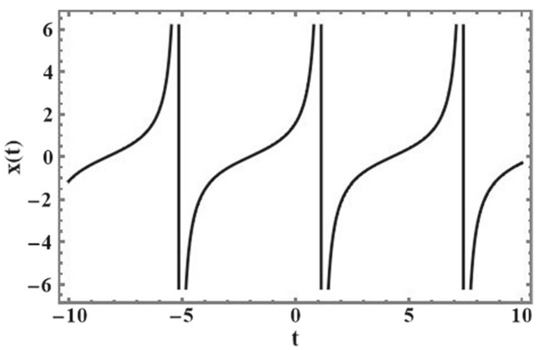

Figure 2. Solution plot for eq. (73) with the initial conditions $I_1 = 1,\ I_2 = 1,\ I_3 = 1$.

1, 2, 3, into it and the integrals are

$$I_1 = \frac{\ddot{x}}{x\dot{x}}, \quad I_2 = \ddot{x}\left(\frac{2}{x} - \frac{\ddot{x}}{\dot{x}^2}\right),$$
$$I_3 = -\frac{t}{2\dot{x}}\sqrt{\frac{\ddot{x}}{x}(2\dot{x}^2 - x\ddot{x})} + \tan^{-1}\sqrt{\frac{\ddot{x}}{x(2\dot{x}^2 - x\ddot{x})}}\,x. \tag{83}$$

We can write the general solution of (73) with the help of these three integrals and it takes the form

$$x(t) = \sqrt{\frac{I_2}{I_1}}\tan\left[\frac{1}{2}\left(\sqrt{I_1 I_2}t + 2I_3\right)\right]. \tag{84}$$

The solution plot for eq. (73) is displayed in figure 2.

It is clear from this demonstration that one can establish the integrability of the given equation in an algebraic manner with the help of the procedure developed here.

6. Conclusion

In this paper, we have presented a procedure to obtain the integrable quantifiers that are being used in the extended PS procedure from the DPs and their co-factors. In this procedure, one has to determine a function by solving a first-order differential equation. The rest of the procedure involves only algebraic calculations. Interestingly, any particular solution of the aforementioned first-order differential equation is sufficient to proceed further and identify the other quantifiers. The main advantage of this interconnection is that the underlying quantifiers can be derived without solving the determining equation in the respective method. Such determination helps in establishing the complete integrability as well as the general solution for the given nonlinear ODE. We have determined the usefulness of the interconnections by considering three different examples.

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