

SOME CONTRIBUTIONS TO DOMINATION THEORY IN GRAPHS

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It has been subjected to Plagiarism check in the Ouriginal software and the percentage of significance is **1%**. The report is enclosed in the thesis.

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“Earning is wealth; None could destroy it;
Nothing else could give genuine joy”

-Thirukural

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List of Publications

- 1 Joseline Manora J and Vignesh S, Inverse Majority Dominating Set of a Graph - **American International Journal of Research in Science, Technology, Engineering & Mathematics**, ISSN(O): 2328-3580, Feb 2019, pp 111-117.
- 2 Joseline Manora J and Vignesh S, Results on $\gamma_M^{-1}(G)$ and $\gamma_M^{-1}(\bar{G})$ - **Malaya Journal of Matematik**, Vol. S, No. 1, 358-362 (2020), ISSN(O):2321-5666.
- 3 Joseline Manora J and Vignesh S, Inverse Independent Majority Dominating Set of a Graph - **Malaya Journal of Matematik**, Vol. S, No. 1, 272-277(2021), ISSN(O): 2321-5666.
- 4 Joseline Manora J, and Vignesh S, Results On The Inverse Majority Domination and Majority Independence Number of a Graph - **Turkic World Mathematical Society Journal of Applied and Engineering Mathematics** V.11, Special Issue, 2021, pp. 103-111, ISSN(O): 2146-1147.
- 5 Joseline Manora J, Vignesh S and Paulraj Jayasimman I, Inverse Split Majority Dominating set of a graph - **Communications**

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- 6 Joseline Manora J and Vignesh S, Inverse Majority Domination Number on Subdivision Graphs - **Advances and Applications in Mathematical Science**, Volume 21, Issue 2, December 2021, Pages 805-817, ISSN(O):0974-6803.
- 7 Joseline Manora J and Vignesh S, Inverse Connected Majority Dominating Set of a Graph - Accepted to **Advances and Application in Discrete Mathematics**.
- 8 Joseline Manora J and Vignesh S, Inverse Non-Split Majority Dominating set of a Graph - Communicated to **Advances and Application in Mathematical Science**.

List of Notations

Notation	Meaning
$G = (V, E)$	Graph with vertex set V and edge set E
$\overline{G}$	Complement graph of G
$d(v)$	Degree of the vertex v
$\delta(G)$	Minimum degree of the graph G
$\Delta(G)$	Maximum degree of the graph G
$\lceil x \rceil$	Smallest integer greater than or equal to x
$\lfloor x \rfloor$	Largest integer less than or equal to x
$N(u)$	Open Neighborhood of a vertex u
$N[u]$	Closed Neighborhood of a vertex u
$pn[u, S]$	Private Neighbour of the vertex u with respect to the set S
$\kappa(G)$	Vertex Connectivity of G
$G_1 \circ G_2$	Corona graph of G_1 and G_2
$P_i \times P_j$	Grid graph
$D_n(m)$	(Dutch) Windmill graph
$S(G)$	Subdivision of a graph G

Notation	Meaning
$S_p(G)$	Splitting graph of a graph G
$\gamma(G)$	Domination Number of G
$\gamma_c(G)$	Connected Domination Number of G
$i(G)$	Independent Domination Number of G
$\gamma_s(G)$	Split Domination Number of G
$\gamma_{ns}(G)$	Non-Split Domination Number of G
$\gamma_M(G)$	Majority Domination Number of G
$i_M(G)$	Independent Majority Domination Number of G
$\beta_M(G)$	Majority Independence Number of G
$\gamma_{CM}(G)$	Connected Majority Domination Number of G
$\gamma_{SM}(G)$	Split Majority Domination Number of G
$\gamma_{NSM}(G)$	Non -Split Majority Domination Number of G
$\gamma^{-1}(G)$	Inverse Domination Number of G
$i^{-1}(G)$	Inverse Independent Domination Number of G
$\gamma_s^{-1}(G)$	Inverse Split Domination Number of G
$\gamma_{ns}^{-1}(G)$	Inverse Non-Split Domination Number of G
$\gamma_M^{-1}(G)$	Inverse Majority Domination Number of G
$i_M^{-1}(G)$	Inverse Independent Majority Domination Number of G
$\gamma_{CM}^{-1}(G)$	Inverse Connected Majority Domination Number of G
$\gamma_{SM}^{-1}(G)$	Inverse Split Majority Domination Number of G
$\gamma_{NSM}^{-1}(G)$	Inverse Non-Split Majority Domination Number of G

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Abstract

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The field of Mathematics plays a vital role in various fields. One of the important areas in mathematics is graph theory, which is used in structural models. This structural arrangements of various objects or technologies lead to new inventions and modifications in the existing environment for enhancement in those fields. Graphs are among the most ubiquitous models of both natural and human-made structures.

During the later part of the twentieth century, the areas of graph theory in computer engineering and operations research had an explosive growth. Graphs are used to represent networks of communication, data organization, computational devices, the flow of compu-

tation etc. Graphs have an intuitive and aesthetic appeal because of their diagrammatic representation.

One of the fastest growing areas in graph theory is the study of domination and related subset problems such as independence, covering, matching and inverse domination. In 1958, the concept ‘Domination’ was introduced in terms of stability number of a graph by Berge. In 1962, Oystein Ore [45] gave the formal mathematical name as “Dominating set” and “Domination number” of a graph. In 1977, Cockayne and Hedetniemi [9] made an interesting and extensive survey of the results known at that time about dominating sets in graphs through their survey paper. Domination has a wide range of applications in radio stations, modelling social networks, coding theory and nuclear power plants problems.

Acharya, Sampath Kumar and Walikar [55] are some Indian Mathematicians who have made substantial contributions to the study of domination in graphs. There are more than 75 models of dominating and relative types of sets in graphs, which have appeared in the research literature for the past 30 years. Many domination parameters are formed by combining domination with another graph theoretic property P . In an attempt to formalise the concept, Haynes

et al. defined the conditional domination number $\gamma(G : P)$ as the smallest cardinality of a dominating set $D \subseteq V(G)$ such that the induced subgraph $\langle D \rangle$ and $\langle V - D \rangle$ satisfy some property P. Some of the domination parameters are independent domination, connected domination total domination, paired domination, split domination, non-split domination and so on.

In 1991, Kulli and Sigarkanti considered the problem of selecting two disjoint sets of transmitting stations $D_1(D_2)$ so that every station not belonging to $D_1(D_2)$ has a link with atleast one station in $D_1(D_2)$, where $|D_1|$ and $|D_1 \cup D_2|$ are minimum among all points of disjoint transmitting stations. This leads then to define the inverse dominating sets in a graph. The concept “Inverse Domination Number in Graphs” was introduced and determined for many classes of graphs with interesting results by Kulli and Sigarkanti [30].

The idea of dominating atleast half of the vertex set is a crucial one and it gave an inspiration for defining majority dominating sets instead of functions to a graph structure. Incidentally, it also has a good application. In decision making process, sometimes it may not be necessary to have the consent of all but a majority opinion will do. In democratic institutions, majority rule is given importance.

In administration, it is enough if the executive body convinces a majority of the employees. In order to model this concept, Joseline Manora and Swaminathan [50] defined majority dominating sets and majority domination number $\gamma_M(G)$ on graphs in 2006.

Let $G = (V, E)$ be a simple, finite and undirected graph with p vertices and q edges and $V - D$ is the complement of a dominating set D of G then the conditional number $\gamma(G; C)$ is the minimum cardinality of a dominating set $D \subseteq V(G)$ such that the set $(V - D)$ satisfies condition C . Obviously $\gamma(G) \leq \gamma(G : C)$, for any condition C . The conditional domination property provides a method for finding many parameters by considering different conditions C on the complement of a various dominating sets of a graph G .

The researcher has been motivated by the above parameter to define a new graph theoretical parameter “Inverse Majority Dominating Sets in Graphs”. The researcher consider the parameters defined by imposing some constraints on the complement of a dominating set. It produces many interesting results, some beautiful inequalities and bounds in the field of domination theory on graphs.

This thesis entitled “**Some Contributions to Domination**

Theory in Graphs” consists of Seven chapters.

In the **first chapter**, the researcher presents the basic definitions and results on domination theory which are necessary to write this Thesis. Survey of the literature, the motivation and the scope of the thesis are also mentioned.

Chapter two consists of six sections. The new parameter an inverse majority dominating set of a graph has been defined with an illustrative example. Then an inverse majority domination number $\gamma_M^{-1}(G)$ is determined for various families of graphs. Also bounds of $\gamma_M^{-1}(G)$ in terms of p , characterization theorems for γ_M^{-1} -set and some interesting inequalities are established. In the next sections, results on the inverse majority domination and majority independence number of a graph are discussed. Particularly the relationship between $\gamma_M^{-1}(G)$ and $\beta_M(G)$ for cubic bipartite graphs are studied. An application for this Inverse Majority Dominating Set and an algorithm for finding a Minimal Inverse Majority Dominating Set and its number $\gamma_M^{-1}(G)$ for a given graph are established.

Chapter three deals with the exact values of $\gamma_M^{-1}(G)$ for some special graph structures. Inverse Majority Domination Number on subdivision graphs of some special graphs and splitting graph of a

graph are also discussed. In the subsequent sections, inverse majority domination number for a complementary graph $\bar{G}$ of a graph G and $\gamma_M^{-1}(G - u)$, for some vertex $u \in V(G)$ which is deleted from G are also discussed.

Chapter four discusses the new concept an “Inverse Independent Majority Dominating Sets of a Graph”. The inverse independent majority domination number $i_M^{-1}(G)$ for some standard graphs, bounds of $i_M^{-1}(G)$ and relationship between $i_M^{-1}(G)$, $i_M(G)$, $i^i(G)$ and $\gamma_M^{-1}(G)$ are also studied. The characterization theorem for a minimal inverse independent majority dominating set of G and the inverse independent majority domination number $i_M^{-1}(G)$ for disconnected graphs have been found. Nordhus –Gaddum type of results are also established for $i_M^{-1}(G)$.

In the **fifth chapter**, the parameter “Inverse Connected Majority Dominating Sets of a Graph” has been defined with an example. Then Inverse Connected Majority Domination number $\gamma_{CM}^{-1}(G)$ is determined for various families of graphs and also grid graphs. A minimal inverse connected majority dominating set of a graph is characterised. The algorithm and application to an inverse connected majority domination number of a graph G is discussed.

Chapter six includes the “Inverse Split Majority Dominating Sets of a Graph”. The Inverse Split Majority Domination Number $\gamma_{SM}^{-1}(G)$ for some classes of graphs and bounds of $\gamma_{SM}^{-1}(G)$ in terms of $\Delta(G)$ and $\Delta'(G)$ are determined. The necessary and sufficient condition for a Minimal Inverse Split Majority Dominating Set of G and the relationship between $\gamma_{SM}^{-1}(G)$, $\gamma_{SM}(G)$ and $\gamma_M^{-1}(G)$ have been produced.

Chapter seven deals with the concept of an “Inverse Non-Split Majority Dominating Sets and an inverse non split majority domination number $\gamma_{NSM}^{-1}(G)$ of a graph G ”. Lower and upper bounds of $\gamma_{NSM}^{-1}(G)$ and Characterisation theorem for a minimal inverse non-split majority dominating set have been established.

The **Conclusion** includes total summary of the Thesis highlighting all new findings developed using the newly coined parameter of various Inverse Majority Domination Parameters of a Graph. Also some references which are needed to the Thesis are given at the end.

Chapter 1

PROLEGOMENON

ABSTRACT

This chapter is introductory in nature which unlocks the fundamental theoretical background of the thesis. This chapter comprises the details of chronological survey of all literature, basic concepts of graph theory, domination theory and objectives of the study. The motivation, scope and organization of the thesis are also given at the end.

1.1 Introduction

Graph Theory is a branch of Mathematics which is most flourishing and interesting area for several reasons. In last three decades hundreds of research articles have been published in Graph Theory. There are many other areas of graph theory which have received good attention from mathematicians. Some of these areas are coloring of Graphs, Matching Theory, Domination Theory, Labeling of Graphs and areas related to Algebraic Graph Theory.

Domination in graphs has been an extensive research branch of graph theory. The theory of domination has been the nucleus of research activity in graph theory in recent times. One of the fastest growing area in graph theory is the study of domination and related subset problems such as independence, covering, matching and inverse domination, the reason being its many and varied applications in such fields as social sciences, communication networks, algorithm designs, computational complexity etc. Henda C. Swart [15] has rightly commented that the theory of domination in graphs is like a *growth industry*.

1.2 Survey of Literature

The rigorous study of dominating sets in graph theory began around 1960, even though the subject has historical roots dating back to 1862 when de Jaenisch studied the problems of determining the minimum number of queens which are necessary to cover or dominate an $n \times n$ chess board. The eminent mathematicians Berge [5] and Oystein Ore [45] gave the formal mathematical definition to the topic of domination. Berge called the domination as external stability and domination number as coefficient of external stability.

Ore [45], a well known lattice theorist, introduced the concept of domination in graphs in his famous book *Theory of Graphs*. It was only Claude Berge's [6] pioneering book *Graphs and Hyper Graphs* in 1973 which included a bound on the domination number of a graph and Vizing's bound on the size of a graph with a given order and domination number. Besides, this book also contains application to surveillance networks and game theory. Today, Ore's concept of domination in graphs has indeed become an independent theory. This concept lived almost in hibernation until 1975. When Cockayne and

Hedetniemi [9] published their paper *Towards a theory of Domination in Graphs* which appeared in *Networks* in 1977. This paper brought to light new ideas and potentiality of being applied in a variety of areas.

There are many origins to domination theory. The earliest idea of domination sets data back to the origin of game of chess in India. In 1979, Walikar et. al. [55] published a lecture notes volume titled *Recent Developments in the Theory of Domination in Graphs and its Applications*. In 1990 [24], Hedetniemi and Laskar published their *Bibliography on domination in graphs and some basic definitions of domination parameters*. This book contained at that time about 400 references. Towards the end of 1977, Haynes et. al brought out a comprehensive two volume text book namely, *Fundamentals of Domination in Graphs* [21] and *Domination in Graphs: Advanced Topics* [22], which contain more than 1200 bibliographical entries.

Then, few eminent indian mathematicians Walikar, Acharya and Sampathkumar [55], have made substantial contribution to the field of domination in graphs. More than fifty types of domination parameters such as independent domination, connected domination, total domination, paired domination etc... have been studied by different

graph theorists. The split domination number [31] and the non-split domination number of a graph [32] were introduced by Kulli and Janakiram.

In 1995 [8], Broere et. al., introduced the concept of majority dominating function in graphs. Tara S. Holm [26] found the majority domination numbers of certain families of graphs and showed that the decision problem corresponding to computing majority domination number of an arbitrary disjoint union of complete graphs is NP-complete. The idea of dominating half of the vertex set is a crucial one and it gave the inspiration for defining majority dominating sets instead of functions. Incidentally, it also has good application. In decision making process, sometimes it may not be necessary to have the consent of all; but a majority opinion will do.

In democratic institutions, majority rule is given importance. In administration it is enough if the executive body convinces a majority of the employees. In order to model this concept in graph theory. Joseline Manora and Swaminathan in 2006 [50], defined majority dominating sets and majority domination number on graphs. Further, this concept was developed into many area of domination and in 2010 [51], lot of interesting results on majority dominating sets

are introduced in their research article. In 2011 [35], they studied various parameters in this area such as majority domatic number, vertex and edge critical graphs on majority domination number. In 2015 [41], the construction of intersection graphs on majority dominating sets of a graph were introduced and analysed with their properties.

1.3 Basic Concepts in Graphs and Domination Theory

The following definitions are the preliminaries of a graph theory.

Definition 1.3.1. A **graph** $G = (V, E)$ consists of a set objects $V(G) = \{v_1, v_2, \dots\}$ called **vertices** and another set $E(G) = \{e_1, e_2, e_3, \dots\}$, whose elements are called **edges**, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. Then $e_i = v_j v_k, v_j$ and v_k are called **end vertices** of an edge e_i .

Definition 1.3.2. The cardinality of the vertex set $|V(G)| = p$ and the cardinality of the edge set $|E(G)| = q$ are respectively called the **order and size** of the graph G . A graph G with p vertices and q

edges is called a (p, q) -graph and it is denoted by $G = (V, E) = (p, q)$.

The $(1, 0) = K_1$ -graph is called a **trivial graph**.

Definition 1.3.3. If $e = \{v_1, v_2\} \in E(G)$, v_1 and v_2 are called **adjacent vertices**; Further, the vertices v_1 and v_2 are **incident** with an edge e . If two vertices are not joined by an edge, then they are **non-adjacent**. If two distinct edges e_1 and e_2 are incident with a common vertex, then it is said to be **adjacent edges**.

Definition 1.3.4. The **degree** of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $\deg_G(v)$ or $\deg(v)$. The **minimum and maximum degrees** of vertices of G are denoted by $\delta(G)$ and $\Delta(G)$ respectively.

Definition 1.3.5. A vertex of degree zero in G is called an **isolated vertex** and a vertex of degree one is called a **pendent vertex** or an end vertex of G . Any vertex which is adjacent to a pendent vertex is called a **support vertex**.

Definition 1.3.6. A graph H is called a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A **spanning subgraph** of G is a subgraph H with $V(H) = V(G)$. For any set S of vertices of G , the

induced subgraph $\langle S \rangle$ is the maximal subgraph of G with vertex set S such that two vertices of S are adjacent in $\langle S \rangle$ if and only if they are adjacent in G .

Definition 1.3.7. Let $v \in V(G)$. Then the graph $G - v$ is the subgraph of G obtained by the **removal of the vertex** v and edges incident with v . If $e \in E(G)$ then the spanning subgraph denoted by $G - e$ is the subgraph of G obtained by the **removal of the edge** e .

Definition 1.3.8. The **distance** $d(u, v)$ between two vertices u and v in a graph G is the length of a shortest $u - v$ path in G . The **diameter** of a connected graph G is the length of any longest path. The diameter of G is denoted by $\text{diam}(G)$.

Definition 1.3.9. A graph G is **connected** if every pair of distinct vertices of G are joined by a path. A **disconnected** graph has at least two components. If a graph G has no edges between the vertices then it is called **totally disconnected** or a **null graph**.

Definition 1.3.10. A connected graph without cycles is called a **tree**. A **spanning tree** of a graph G is a spanning subgraph of G that is a tree.

Definition 1.3.11. A vertex of a graph G is said to be a **full degree** or a **dominating vertex** if it is adjacent to all other vertices in G . That is, $d(v) = p - 1$.

Definition 1.3.12. A graph G is said to be **regular** of degree r if every vertex of G has degree r . Such graphs are called **r -regular graphs**. Any 3-regular graph is called a **cubic graph**.

Definition 1.3.13. A graph G is **complete** if every pair of its vertices are adjacent. A complete graph on p vertices is denoted by K_p . A **clique** of a graph G is a maximal complete subgraph of G .

Definition 1.3.14. The **complement** $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G .

Definition 1.3.15. A **bipartite graph** is a graph whose vertex set $V(G)$ can be partitioned into two subsets $V_1(G)$ and $V_2(G)$ such that every edge of G has one end in $V_1(G)$ and the other end in $V_2(G)$. If every vertex of $V_1(G)$ is joined to all the vertices of $V_2(G)$ and vice versa, then G is called a **complete bipartite graph**. If $|V_1(G)| = m$ and $|V_2(G)| = n$ then it is denoted by $K_{m,n}$, $m, n \geq 1$. The complete

bipartite graph with $(V_1(G), V_2(G)) = (1, p - 1)$ is called a **star** and it is denoted by $\mathbf{K}_{1,p-1}$.

Definition 1.3.16. A **walk** of G is a finite, alternating sequence of vertices and edges beginning and ending with vertices i.e.) $u_1e_1, u_2e_2, \dots, u_{n-1}e_{n-1}$. The number n is called the **length** of the walk. The walk is said to be **closed walk** if $u_1 = u_n$ and is an **open walk** otherwise.

Definition 1.3.17. A walk in which all the vertices are distinct is called a **path** and a path on p vertices is denoted by P_p . A closed walk $u_0, u_1, u_2, \dots, u_k$ in which the vertices $u_0, u_1, u_2, \dots, u_{k-1}$ are distinct is called a **cycle** and a cycle on p vertices is denoted by C_p .

Note 1.3.18. Let G_1 and G_2 be two graphs. If one of the graphs say $G_2 = K_1$ then the graph $G_1 \vee G_2$ is obtained by joining every vertex of G_1 with every vertex of G_2 and is denoted by $G_1 + K_1$. Hence the wheel graph $W_p = C_{p-1} + K_1$ and the fan graph $F_p = P_{p-1} + K_1$.

Definition 1.3.19. A **cut-vertex(cut-edge)** of a graph G is a vertex(edge) whose removal leaves the graph disconnected.

Definition 1.3.20. A **caterpillar** is a tree C for which the removal of all end vertices leaves a path, which is called the **spine** of C .

A **spider** is a tree which has at most one vertex of degree ≥ 3 . A **double star** is a graph obtained by taking two stars and joining the vertices of maximum degrees with an edge and is denoted by $D_{r,s}$.

Definition 1.3.21. A graph is called **acyclic**, if it has no cycles. A connected **acyclic** graph is called a tree. A tree is a **wounded spider** if the tree is $K_{1,r}$, $r \geq 0$, in which at most $r - 1$ of the edges are subdivided.

Next, in definitions of some special graphs structures are given below.

Definition 1.3.22. A **subdivision** of an edge $e = uv$ of a graph G is obtained by introducing a new vertex w and replacing the edge uv with edges uw and wv . The graph obtained from G by subdividing each edge of G exactly once is called the subdivision graph and is denoted by $S(G)$.

Definition 1.3.23. Let G_1 be an (n_1, m_1) -graph and let G_2 be an (n_2, m_2) -graph. Then the **corona** $G_1 \circ G_2$ is defined as the graph G

obtained by taking one copy of G_1 and n_1 copies of G_2 , and joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2 . The **corona** $H \circ K_1$ is denoted by H^+ , for any connected graph H .

Definition 1.3.24. Let G and H be any two connected graphs with the vertex set $(u_1, u_2, \dots, u_n)$ and $(v_1, v_2, \dots, v_n)$ respectively. The **(Cartesian) Product graph** $K = G \times H$ has $V(K) = V(G) \times V(H)$ and vertices (u_1, v_1) and (u_2, v_2) in $V(K)$ are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2$ in $E(H)$ or $v_1 = v_2$ and $u_1 u_2$ in $E(G)$. The **Grid** is $P_i \times P_j$; the **Cylinder** is $C_i \times P_j$, for $i \geq 3$ and $j \geq 3$; and the **Torus** $C_i \times C_j$, for $i \geq 3$ and $j \geq 3$.

Definition 1.3.25. For each $n \geq 3$ and $0 < k < n$, $P(n, k)$ denotes the **Generalized Petersen graph** with vertex set $V(G) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and the edge set $E(G) = \{u_i u_{i+1(mod\ n)}, u_i v_i, v_i v_{i+k(mod\ n)}\}, 1 \leq i \leq n$.

Definition 1.3.26. Let G be a graph and v be a vertex of G . $G - v$ is the **induced subgraph** $\langle V(G) - v \rangle$ of G and it is obtained from G by removing v and the edges incident with v . If $e \in E(G)$, then

$G - e$ is the spanning subgraph with edge set $E(G) - \{e\}$ and it is obtained from G by removing the edge e from G .

Definition 1.3.27. The **Dutch windmill graph** $D_n^{(m)}$, is the graph obtained by taking m copies of the cycle graph C_n with a vertex in common.

Definition 1.3.28. The **open neighborhood** $N(v)$ of a vertex v in a graph G is the set of all vertices adjacent to v in G . The **closed neighborhood** $N[v]$ of v is the set $N(v) \cup \{v\}$.

Definition 1.3.29. Let $G = (V, E)$ be a simple graph. Let a vertex $v \in V(G)$ and the set $S \subseteq V(G)$. A vertex v is a **private neighbor** of u (with respect to S) if $N[v] \cap S = \{v\}$. Further, the **private neighbor set** of u with respect to S is $pn[u, S] = \{v : N[v] \cap S = \{v\}\}$.

Definition 1.3.30. A set $S \subseteq V(G)$ is said to be **independent** if no two vertices in S are adjacent. The maximum number of vertices in an independent set of a graph G is called the **independence number** of G and is denoted by $\beta(G)$.

Definition 1.3.31. The **vertex connectivity** $\kappa = \kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph.

Definition 1.3.32. For any real number x , $\lceil x \rceil$ denotes the smallest integer greater than or equal to x , and $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .

Next, the following definitions are regarding domination theory and its domination parameters.

Definition 1.3.33. Let $G = (V, E)$ be a finite, undirected and simple graph. A subset S of $V(G)$ is said to be a **dominating set** of G if every vertex in $(V - S)$ is adjacent to at least one vertex in S . A dominating set is called minimal dominating set if no proper subset of S is a dominating set. The minimum cardinality of the minimal dominating set of G is called the **domination number** of G , denoted by $\gamma(G)$ and $\Gamma(G)$ denotes the cardinality of the largest minimal dominating sets of G and $\Gamma(G)$ is called the **upper domination number** of a graph G .

Definition 1.3.34. A subset S of $V(G)$ is called an **independent set** if no two vertices in S are adjacent. The minimum cardinality of

a maximal independent set is called the **independent domination number** of G and it is denoted by $i(G)$.

Definition 1.3.35. A dominating set S is said to be a **connected dominating set** if the subgraph $\langle S \rangle$ induced by S is connected in G . The minimum cardinality of a minimal connected dominating set of G is called the **connected domination number**, denoted by $\gamma_C(G)$.

Definition 1.3.36. A dominating set D of a graph $G = (V, E)$ is a **split dominating set** if the induced subgraph $\langle V - D \rangle$ is disconnected. The **split domination number** $\gamma_s(G)$ is the minimum cardinality of a minimal split dominating set of G .

Definition 1.3.37. A dominating set D of a graph $G = (V, E)$ is a **non-split dominating set** if the induced subgraph $\langle V - D \rangle$ is connected. The **non-split domination number** $\gamma_{ns}(G)$ is the minimum cardinality of a non-split dominating set of G .

Definition 1.3.38. A vertex v in a graph G is said to be a **majority dominating vertex** if $d(v) \geq \lceil \frac{p}{2} \rceil - 1$.

Definition 1.3.39. Let $G = (V, E)$ be a finite graph with p vertices and q edges. A subset $S \subseteq V(G)$ of vertices in a graph G is called a **majority dominating set** if at least half of the vertices of $V(G)$ are either in S or adjacent to vertices of S (i.e.) $|N[S]| \geq \lceil \frac{p}{2} \rceil$.

A majority dominating set D is said to be a **minimal majority dominating set** of G if no proper subset of D is a majority dominating set of G .

The minimum cardinality of a minimal majority dominating set is called the **majority domination number** and is denoted by $\gamma_M(G)$.

If $S \subseteq V(G)$ is a majority dominating set of minimum cardinality, then it is called a γ_M -set of G .

Definition 1.3.40. A set S of vertices of a graph G is said to be a **majority independent set** if it induces a totally disconnected subgraph with $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $|pn[v, S]| > |N[S]| - \lceil \frac{p}{2} \rceil$, for every $v \in G$. The minimum cardinality of a maximal majority independent set is called the lower majority independent set of G and it is also called **independent majority domination number** of G , denoted by $i_M(G)$. The independent majority domination number of a graph

G is the minimum cardinality of a minimal IMD set of G . The maximum cardinality of a maximal majority independent set is called the **majority independence number** and it is denoted by $\beta_M(G)$.

Definition 1.3.41. A set $S \subseteq V(G)$ is a **Connected Majority Dominating (CMD) set** if the set S is a majority dominating set and the subgraph $\langle S \rangle$ induced by S is connected in G .

The connected majority dominating set S is **minimal** if no proper subset of S is a connected majority dominating set. The minimum cardinality of a minimal connected majority dominating set is called the **Connected Majority Domination Number** and is denoted by $\gamma_{CM}(G)$. The maximum cardinality of a Minimal Connected Majority Dominating Set of G is called **Upper Connected Majority Domination** of G , denoted by $\Gamma_{CM}(G)$.

Definition 1.3.42. Let $G = (V, E)$ be a simple, finite and undirected graph with p vertices and q edges. A subset D of $V(G)$ is said to be a **split majority dominating set** of G if the set D satisfies the following conditions: (i) $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and (ii) The induced subgraph $\langle V - D \rangle$ is disconnected.

Definition 1.3.43. *The minimum cardinality of a minimal split majority dominating set is called the split majority domination number of a graph G and it is denoted by $\gamma_{SM}(G)$.*

Definition 1.3.44. *Let $G = (V, E)$ be a simple, finite and undirected graph on p vertices. A set $D \subseteq V(G)$ is said to be a **non-split majority dominating set** of G if the subset D satisfies the following conditions: (i) $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and (ii) The induced subgraph $\langle V - D \rangle$ is connected.*

Definition 1.3.45. *The minimum cardinality of a minimal non-split majority dominating set of the graph G is called the non-split majority domination number and it is denoted by $\gamma_{NSM}(G)$.*

Definition 1.3.46. *Let D be a minimum dominating set in a graph G . If the set $(V - D)$ contains a dominating set D' then D' is called an **inverse dominating set** of G with respect to D . The cardinality of a minimum inverse dominating set is defined as the **inverse domination number** of a graph G and it is denoted by $\gamma^{-1}(G)$.*

Definition 1.3.47. Let D be a minimum independent dominating set of G . If $(V - D)$ contains an independent dominating set D' then the set D' is called an **inverse independent dominating set** with respect to D . The **inverse independent domination number** $i^{-1}(G)$ is the minimum cardinality of a minimal inverse independent dominating set of G .

Definition 1.3.48. Let D be a minimum dominating set of G and D' be the minimum inverse dominating set of G with respect to D . Then D' is called an **inverse split dominating set** of G if the induced subgraph $\langle V - D' \rangle$ is disconnected. The **inverse split domination number** is denoted by $\gamma_S^{-1}(G)$ and it is the minimum cardinality taken over all the minimal inverse split dominating sets of G .

Definition 1.3.49. Let D be a minimum dominating set of G and D' be the minimum inverse dominating set of G with respect to D . Then D' is called an **inverse non-split dominating set** of G if the induced subgraph $\langle V - D' \rangle$ is connected. The **inverse non-split domination number** is denoted by $\gamma_{ns}^{-1}(G)$ and it is the minimum cardinality taken over all the minimal inverse non-split dominating sets of G .

1.4 Some Results on $\gamma_M(G)$ [50], $\beta_M(G)$ [33], $i_M(G)$ [34], $\gamma_{CM}(G)$ [37], $\gamma_{SM}(G)$ [39] and $\gamma_{NSM}(G)$ [40]

1. Let $G = C_p$, a cycle on p vertices. Then $\gamma_M(G) = \lceil \frac{p}{6} \rceil$.
2. Let G , be a caterpillar graph. Then $\gamma_M(G) = \lceil \frac{p}{8} \rceil$.
3. Let $G = K_{1,p-1}$. Then $\beta_M(G) = \lfloor \frac{p-1}{2} \rfloor$.
4. For any cubic bipartite graph G , Then $\beta_M(G) \leq \lceil \frac{p}{4} \rceil - 1$.
5. Let $G = W_p$. Then $\beta_M(G) = \lceil \frac{p-2}{6} \rceil$, if $5 \leq p \leq 18$ and $\beta_M(G) = \lceil \frac{p-3}{4} \rceil$, if $p \geq 19$.
6. Let G be a cycle of p vertices, $p \geq 3$. Then $\beta_M(G) = \lceil \frac{p}{6} \rceil$, if $3 \leq p \leq 16$ and $\beta_M(G) = \lceil \frac{p-4}{4} \rceil$, if $p \geq 17$.
7. For a path $G = P_p, p \geq 2$, $\beta_M(G) = \lceil \frac{p}{4} \rceil$, if $2 \leq p \leq 10$ and $\beta_M(G) = \lceil \frac{p-2}{6} \rceil$, if $p > 11$.
8. Let $G = F_p$. Then $\beta_M(G) = \lceil \frac{p}{6} \rceil$.
9. For any cubic graph, $\beta_M(G) = \lceil \frac{p}{8} \rceil$.
10. Let $G = K_{1,p-1}$. Then $\beta_M(G) = \lfloor \frac{p-1}{2} \rfloor$.
11. For a complete graph $G = K_p, \gamma_M(G) = \gamma_M^{-1}(G) = 1$.
12. Let $G = mK_2$. Then $i_M(G) = \lceil \frac{p}{4} \rceil$.
13. If $G = P_p$ be a path with $p \geq 2$ and C_p a cycle with $p \geq 3$ then $i_M(G) = \lceil \frac{p}{6} \rceil$.

14. Let $G = P_p$ be a path, $p \geq 3$. Then $\gamma_{CM}(G) = \lceil \frac{p}{2} \rceil - 2$.
15. For a grid graph $G = P_2 \times P_j, j \geq 3$

$$\gamma_{CM}(G) = \begin{cases} \lfloor \frac{p}{4} \rfloor, & \text{if } j \text{ is odd} \\ \lfloor \frac{p-1}{4} \rfloor, & \text{if } j \text{ is even} \end{cases}.$$
16. Let G be a uniform caterpillar with p vertices, $e \geq 2$ pendants and $k \geq 3$ central vertices. Then
$$\gamma_{CM}(G) = \begin{cases} \frac{p}{e+1} - \lfloor \frac{k}{2} \rfloor, & \text{if } k \text{ is odd} \\ \frac{p}{e+1} - \frac{k}{2}, & \text{if } k \text{ is even} \end{cases}.$$
17. Let $G = W_p$ be a wheel graph of p vertices. Then $\gamma_{SM}(G) = 3$.
18. Let $G = P_p$ be a path on p vertices. Then $\gamma_{NSM}(G) = \lceil \frac{p}{4} \rceil$ if $p \leq 8$ and $\gamma_{NSM}(G) = \lceil \frac{p}{2} \rceil - 2$ if $p \geq 9$.
19. Let G be any graph with p vertices and $G_1 = G \circ K_1$ be the corona of G and K_1 . $\gamma_{NSM}(G_1) = \lceil \frac{p}{2} \rceil$
20. Let G be a caterpillar graph with p vertices. Then $\gamma_{NSM}(G) = \lceil \frac{p}{4} \rceil$

1.5 Motivation and Scope of the Thesis

✠ In 2006 [50], Swaminathan and Joseline Manora introduced the concept “Majority Domination” with respect to sets with the idea of dominating atleast half of the vertices of a graph. Further the concept inverse domination in graphs is introduced by Kulli and Sigarkanti [30]. The researcher is motivated by these concepts to introduce new graph theoretical parameter “Inverse Majority Dominating set of a graph”.

- ✠ In 1998, Haynes et. al. [21], introduced the independent dominating set of a graph and the same concept was extended to inverse independent dominating set of a graph by Kulli et. al. [29]. In 2014 [33], Joseline Manora and John introduced majority independent dominating set of a graph and they developed this concept into independent majority dominating set of a graph [34]. The researcher was motivated by this idea and extended to the concept of “Inverse Independent Majority Dominating set of a Graph”.
- ✠ In 1979, The concept “Connected Domination in Graphs” was introduced by Sampath Kumar and Walikar [48]. Then it is extended to “Connected Majority Dominating set of a Graph” and this idea was studied by Joseline Manora and Muthukani Vairavel in 2017 [37]. This inspired the researchers to enter into the area of “Inverse Connected Majority Dominating set of a Graph.
- ✠ In 1997, Kulli and Janakiram [31] introduced split dominating set of a graph. Another work in 2015, Joseline Manora and Veeramaniandan [39] studied the extension of the idea and defined the parameter “The Split Majority Domination Number of a Graph”. In 2008 [2], the inverse split dominating set of a graph G was introduced by Ameenul Bibi and Selvakumar. These eminent mathematicians discussed the parameter with another domination parameters and produced many results on the inverse split domination number of G , denoted by $\gamma_S^{-1}(G)$. Here the researcher has developed this concept to “Inverse Split

Majority Dominating Sets and its number of a graph”.

- ✠ In 2000, and Janakiram [32] defined a Non-Split dominating set of a graph. This concept was extended to the non-split majority dominating set of a graph by Joseline Manora and Veeramanikadan [40] in 2015. Then Ameen Bibi and Selvakumar defined an inverse nonsplit dominating set of a graph G and they determined bounds of the inverse nonsplit domination number $\gamma_{ns}^{-1}(G)$ and some theorems were produced by these mathematicians. Now, the researcher has studied the concept “Inverse Non-Split Majority Dominating set of a Graph” and developed this notion to some extent.

1.6 Objectives of the Thesis

The objectives of this research work are:

- ✠ To introduce a new parameter inverse majority dominating set of a graph G and inverse majority domination number, denoted by $\gamma_M^{-1}(G)$.
- ✠ To find the exact values of inverse majority domination number for various families of graphs and to obtain the bounds of $\gamma_M^{-1}(G)$.
- ✠ To study the comparison of $\gamma_M^{-1}(G)$ with the majority independence number $\beta_M(G)$.
- ✠ To find the particular values of $\gamma_M^{-1}(G)$ for some special graph structures.

- ✠ To establish an algorithm to find an inverse majority dominating set in a given graph G .
- ✠ To define another new parameter in the complement of a independent majority dominating set namely, an inverse independent majority domination.
- ✠ To determine the inverse independent majority domination number $i_M^{-1}(G)$ for various classes of graphs and bounds of $i_M^{-1}(G)$.
- ✠ To characterise the value of $i_M^{-1}(G)$ with another parameters such as $i^{-1}(G)$, $i_M(G)$ and $\gamma_M^{-1}(G)$.
- ✠ To study the Nordhaus – Gaddum type results for $i_M^{-1}(G)$.
- ✠ To study another new concept “Inverse connected majority dominating sets in graphs” and its number $\gamma_{CM}^{-1}(G)$ for various classes of graphs.
- ✠ To determine the exact values of $\gamma_{CM}^{-1}(G)$ for product graphs and lower and upper bounds of $\gamma_{CM}^{-1}(G)$.
- ✠ To determine the characterisation theorem for a minimal inverse connected majority dominating set of a graph.
- ✠ To check the existence of an inverse connected majority dominating set for a disconnected graph.
- ✠ To introduce a new inverse parameter in the domination theory with respect to the complement of the majority dominating set of a graph G namely inverse split majority domination and inverse non-split majority domination.

- ✠ To investigate some properties of graph G with respect to the inverse split majority domination and inverse non-split majority domination.
- ✠ To determine the exact values of $\gamma_{SM}^{-1}(G)$ and $\gamma_{NSM}^{-1}(G)$ for various families of graphs.
- ✠ To obtain some relationships among the domination parameter such as $\gamma_M^{-1}(G)$, $\gamma_{SM}^{-1}(G)$, $\gamma_{SM}(G)$, $\gamma_{NSM}(G)$ and $\gamma_{NSM}^{-1}(G)$.
- ✠ To establish algorithms and applications for the above said inverse parameters.

1.7 Organisation of the Thesis

This thesis entitled “**Some Contributions to Domination Theory in Graphs**” comprises of seven chapters with conclusion and bibliography. The organization of the thesis follows the pattern given below.

1. Prolegomenon.
2. Inverse Majority Dominating Sets in Graph.
3. $\gamma_M^{-1}(G)$ for Some Special Graph Structures.
4. Inverse Independent Majority Dominating Sets of a Graph.
5. Inverse Connected Majority Dominating Sets of a Graph.
6. Inverse Split Majority Domination in Graphs.

7. Inverse Non- Split Majority Domination in Graphs.

Conclusion.

Bibliography.

In the first chapter, the researcher presents the basic definitions and results on domination theory which are necessary to write this thesis. Survey of the literature, the motivation and the scope of the thesis are also mentioned.

Chapter two consists of six sections .The new parameter inverse majority dominating set of a graph has been defined with an example. Then inverse majority domination number $\gamma_M^{-1}(G)$ is determined for various families of graphs. Also bounds of $\gamma_M^{-1}(G)$ in terms of p , characterization theorem for $\gamma_M^{-1}(G)$ - set and some interesting inequalities are established. The content of this section are published in **American International Journal of Research in Science, Technology, Engineering & Mathematics**.

In the next sections, Results on the inverse majority domination and majority independence number of a graph are discussed. In particular, relationship between $\gamma_M^{-1}(G)$ and $\beta_M(G)$ for cubic bipartite graphs are studied. This concept is published in **TWMS Journal of Applied and Engineering Mathematics**. An application of Inverse Majority Dominating Set and an algorithm for finding a minimal Inverse Majority Dominating set and its number $\gamma_M^{-1}(G)$ for a given graph are established.

Chapter three deals with the exact values of $\gamma_M^{-1}(G)$ for some special graph structures in six sections. Inverse Majority Domination

Number on subdivision graphs of special graphs and splitting graph of a graph are also discussed. This work is published in **Advances and Application in Mathematical Science**. In the subsequent section, inverse majority domination number for a complementary graph $\bar{G}$ and $\gamma_M^{-1}(G-u)$, for some vertex $u \in V(G)$ are also discussed. This content is published in **Malaya Journal of Matematik**.

Chapter four discusses the new concept “Inverse Independent Majority Dominating Sets of a Graph”. The inverse independent majority domination number $i_M^{-1}(G)$ for some standard graphs, bounds of $i_M^{-1}(G)$ and relationship between $i_M^{-1}(G)$ and $\gamma_M^{-1}(G)$ are also studied. The characterization theorem for a minimal inverse independent majority dominating set of G and the inverse independent majority domination number $i_M^{-1}(G)$ for disconnected graphs have been found. Nordhus–Gaddum type of results are also established. This work is published in **Malaya Journal of Matematik**.

In the fifth chapter, the parameter “Inverse Connected Majority Dominating Sets of a Graph” has been defined with an example. Then, Inverse Connected Majority Domination Number $\gamma_{CM}^{-1}(G)$ is determined for some classes of graphs and also grid graphs. A minimal inverse connected majority dominating set of a graph is characterised. The algorithm and application to find an inverse connected majority domination number of a graph G is discussed. This work is accepted to **Advances and Applications in Discrete Mathematics**.

Chapter six includes the “Inverse Split Majority Dominating Sets of a Graph”. The Inverse Split Majority Domination Number $\gamma_{SM}^{-1}(G)$

for some families of graphs and bounds of $\gamma_{SM}^{-1}(G)$ in terms of $\Delta(G)$ and $\Delta'(G)$ are determined. The necessary and sufficient theorem for an Inverse Split Majority Dominating Set of G and the relationship between $\gamma_{SM}^{-1}(G)$ and $\gamma_M^{-1}(G)$ have been produced. The concept of this Chapter is published in **Communications in Mathematics and Applications**.

Chapter seven deals with the concept of “Inverse Non-Split Majority Dominating Sets and inverse non split majority domination number $\gamma_{NSM}^{-1}(G)$ of a graph G ”. Lower and upper bounds of $\gamma_{NSM}^{-1}(G)$ and Characterisation theorem for a minimal inverse non-split majority dominating set have been established. This work is communicated to **Advances and Application in Mathematical Science**.

Conclusion includes the total summary of the thesis highlighting all new findings developed using the newly coined concepts of various Inverse Majority Domination Parameters of a Graph.

Chapter 2

Inverse Majority Dominating Sets in Graphs

Abstract

This chapter introduces a new notion an **inverse majority domination** of a graph G . For any graph G , the inverse majority dominating set with respect to a minimum majority dominating set of G is defined and its number **inverse majority domination number** denoted by $\gamma_M^{-1}(G)$ is determined for various classes of graphs. Bounds of $\gamma_M^{-1}(G)$ is found interms of ‘p’ vertices and a maximum degree $\Delta(G)$. Then the relationship of $\gamma_M^{-1}(G)$ with other domination parameters $\gamma_M(G)$, $\gamma^{-1}(G)$ and $\beta_M(G)$ are also discussed in detail.

The contents of this chapter are published in

1. American International Journal of Research in Science, Technology, Engineering & Mathematics, ISSN (O) 2328-3580, Feb 2019, pp 111-117.
2. TWMS J.App. and Eng. Math. V.11, Special Issue, 2021, pp. 103-111.

2.1 Introduction

The study on domination in graphs was initiated by Ore [45] and Berge [5]. In 1977, Cockayne and Hedetniemi [9] made a fascinating and broad overview of the outcomes known around with regards to dominating sets in graphs. The first paper on the inverse domination in graphs was contributed by Kulli and Sigarkanti [30] in 1991. In their article, they produced the exact values of $\gamma^{-1}(G)$ for certain classes of graphs. Kulli and Sigarkanti gave an upper bound involving the independence number and the inverse domination number and a lower bound in terms of the number of vertices p and edges q . Additionally, they fostered this area by determining few inequalities for $\gamma^{-1}(G)$ and interesting results.

In 2004, Domke et.al. [11], concentrated on this concept exhaustively and characterized the connected graphs with minimum degree at least one and two for which $\gamma(G) + \gamma^{-1}(G) = p$. Then they determined some lower bound and upper bound of $\gamma^{-1}(T)$ for any tree T . Recently this idea “Inverse Domination” is considered by several graph theorists and numerous new domination parameters are described in the books of Kulli.

The concept of majority dominating set in graphs was introduced and presented by Swaminathan and Joseline Manora [50]. It has great applications in real life circumstances. In any equitable arrangement, any of the findings are taken by the endorsement of the majority of the individuals of the set up. Accordingly majority rule is given significance in any decision making process. Enlivened by the two notions specifically inverse domination and majority domination, the new parameter “Inverse Majority Dominating Sets in Graphs” is introduced and its inverse majority domination number of G is studied in detailed way by producing many intriguing results and propositions.

The organization of this chapter is as follows. Section 2.1 is the introducing part of the chapter and it contains the motivation of defining the parameter inverse majority domination in graphs. Section 2.2 defines the concept of inverse majority domination for a graph G and illustrates the defined idea with an example. The inverse majority domination number $\gamma_M^{-1}(G)$ is determined for various classes of graphs in Section 2.3. In Section 2.4, bounds of $\gamma_M^{-1}(G)$ are established with sharpness and in Section 2.5, relationship between $\gamma_M^{-1}(G)$

and an independence number $\beta_M(G)$ is examined. Also algorithm for $\gamma_M^{-1}(G)$ and its real life application are likewise given in Section 2.6.

2.2 Inverse Majority Dominating Sets in Graphs

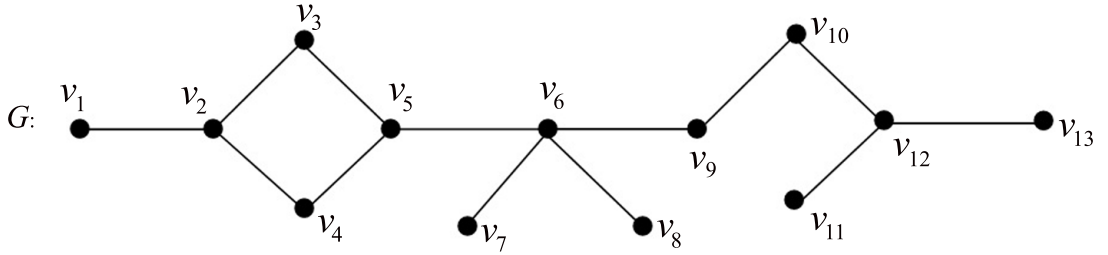
In this section, the definitions of inverse majority dominating set, minimal inverse majority dominating set, inverse majority domination number of a graph G are stated. An example illustrating these definitions is also shown.

Definition 2.2.1: Let G be a simple, finite graph and undirected graph with p vertices and q edges and D be a minimum majority dominating set of G . If the set $(V - D)$ contains a majority dominating set say D' then D' is called an inverse majority dominating set with respect to D .

Definition 2.2.2: An inverse majority dominating set D' is minimal if there exists no proper subset D'_1 of D' such that D'_1 is an inverse majority dominating set of G with respect to D .

Definition 2.2.3: The minimal inverse majority domination number, say $\gamma_M^{-1}(G)$ of a graph G is the minimum cardinality of all minimal inverse majority dominating sets of G and the maximum cardinality of all minimal inverse majority dominating sets of G is called upper inverse majority domination number of G , denoted by $\Gamma_M^{-1}(G)$.

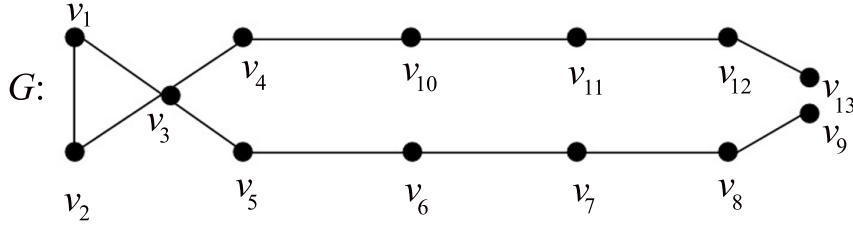
Example 2.2.4: Consider the following graph G with $p = 13$ vertices.



Let $V(G) = \{v_1, v_2, \dots, v_{13}\}$ with $|V(G)| = p = 13$. The following are some of the minimal majority dominating sets of G . $D_1 = \{v_1, v_5, v_9\}$, $D_2 = \{v_2, v_6\}$, $D_3 = \{v_3, v_{12}\}$, $D_4 = \{v_4, v_{10}, v_{13}\}$, $D_5 = \{v_1, v_7, v_8, v_{13}\}$. Clearly D_2 and D_3 are the minimum majority dominating sets for G . Consequently $\gamma_M(G) = 2$ and $\Gamma_M(G) = 4$. The inverse majority dominating sets with respect to D_2 and D_3 are $D'_2 = \{v_5, v_{10}\}$ and $D'_3 = \{v_6, v_1\}$. Infers that $\gamma_M^{-1}(G) = 2$. Also, the maximum cardinality of $D'_2 = \{v_1, v_7, v_8, v_{12}\} \subseteq (V - D_2)$ is

a minimal inverse majority dominating set with respect to D_2 and $\Gamma_M^{-1}(G) = 4$.

Example 2.2.5: Consider the following graph with $p = 13$ vertices.



Let $|V(G)| = p = 13$. Let $D = \{v_3, v_7, v_9, v_{11}, v_{13}\}$ and $D' = \{v_1, v_6, v_8, v_{10}, v_{12}\} \subseteq (V - D)$ be the dominating set and inverse dominating set with respect to D . Then $\gamma(G) = \gamma^{-1}(G) = 5$. The following are some of the minimal majority dominating sets of G . $D_1 = \{v_3, v_8\}$, $D_2 = \{v_9, v_{10}, v_{13}\}$, $D_3 = \{v_2, v_5, v_8\}$. Hence D_1 is a minimum majority dominating set of G . The inverse majority dominating set with respect to D_1 is $D'_1 = \{v_6, v_{10}, v_{13}\} \subseteq (V - D_1)$. Accordingly, $\gamma_M(G) = |D_1| = 2$ and $\gamma_M^{-1}(G) = |D'_1| = 3$. Then $\gamma_M(G) < \gamma_M^{-1}(G)$ and $\gamma_M^{-1}(G) < \gamma^{-1}(G)$.

Proposition 2.2.6:

- (i) For any graph G , $\gamma_M(G) \leq \gamma_M^{-1}(G)$.
- (ii) For any graph G , $\gamma_M^{-1}(G) \leq \gamma^{-1}(G)$.

(iii) Let G be any graph. Then $\gamma_M(G) \leq \gamma_M^{-1}(G) \leq \gamma^{-1}(G)$.

Proof: Since each inverse majority dominating set of G is a majority dominating set of G , $\gamma_M(G) \leq \gamma_M^{-1}(G)$. Also, since every inverse dominating set of G is an inverse majority dominating set for G , $\gamma_M^{-1}(G) \leq \gamma^{-1}(G)$. Therefore we get the inequality (iii). ■

Example 2.2.7:

1. Let $G = S(K_{1,10})$ be a subdivision graph of a star $K_{1,10}$.

Then $\gamma^{-1}(G) = 11$, $\gamma_M^{-1}(G) = 5$ and $\gamma_M(G) = 1$. It implies that

$$\gamma_M(G) < \gamma_M^{-1}(G) < \gamma^{-1}(G).$$

2. For $G = K_p$, a complete graph,

$$\gamma_M(G) = \gamma_M^{-1}(G) = \gamma^{-1}(G) = 1.$$

2.3 Inverse Majority Domination Number for Some Classes of Graphs

The inverse majority domination number $\gamma_M^{-1}(G)$ for some classes of graphs is determined in the following results and propositions.

2.3.1 Results on $\gamma_M^{-1}(G)$

- (1) For a Complete graph $G = K_p$, $\gamma_M^{-1}(K_p) = 1$.

(2) For a Path $G = P_p$, $\gamma_M^{-1}(P_p) = \lceil \frac{p}{6} \rceil, p \geq 2$.

(3) Let $G = K_{m,n}, m, n \geq 2$ be a Complete bipartite graph. Then

$$\gamma_M^{-1}(G) = 1.$$

(4) For a Fan graph $G = F_p, p \geq 4$, $\gamma_M^{-1}(G) = \lceil \frac{p-1}{6} \rceil$.

(5) For a Star $G = K_{1,p-1}, p \geq 2$, $\gamma_M^{-1}(G) = \lfloor \frac{p-1}{2} \rfloor$.

(6) Let $G = \bar{K}_p$ be a totally disconnected graph with p vertices.

Then $\gamma_M^{-1}(G) = \frac{p}{2}$, if p is even and $\gamma_M^{-1}(G)$ does not exist for p is odd.

(7) If $G = mK_2, m \geq 1$ then $\gamma_M^{-1}(G) = \lceil \frac{p}{4} \rceil$.

(8) For a Generalized Petersen graph $G = P(n, k), \gamma_M^{-1}(G) = \lceil \frac{p}{8} \rceil$.

Proposition 2.3.1: For any cycle C_p with $p \geq 3$, $\gamma_M^{-1}(C_p) = \lceil \frac{p}{6} \rceil$.

Proof: By the result (1.4) for a cycle $C_p, \gamma_M(C_p) = \lceil \frac{p}{6} \rceil$. Let $D \subseteq V(G)$ be a minimum majority dominating set of G with $|D| = \lceil \frac{p}{6} \rceil$ then $D' = \{u_{i+1}/u_i \in D\} \subseteq V - D$ with $|D'| = \lceil \frac{p}{6} \rceil$ is an inverse majority dominating set of G with respect to D with $|D'| = |D|$ and $D' \subseteq V - D$. Hence $\gamma_M^{-1}(G) = \lceil \frac{p}{6} \rceil$. ■

Proposition 2.3.2: Let $G = S(K_{1,p-1})$ be a subdivision graph of a star $K_{1,p-1}$ with p vertices. Then $\gamma_M^{-1}(G) = \lceil \frac{p-2}{4} \rceil$.

Proof: Let $V(G) = \{u, v_1, v_2, \dots, v_{\frac{p}{2}}, u_1, u_2, \dots, u_{\frac{p}{2}}\}$ where u is a central vertex, $u_1, u_2, \dots, u_{\frac{p}{2}}$ are pendants and $v_1, v_2, \dots, v_{\frac{p}{2}}$ are the middle vertex of each edge of G . Then $D = \{u\}$ is the minimum majority dominating set of G and $\gamma_M(G) = 1$. Let $D' = \{v_1, v_2, \dots, v_{\lceil \frac{p-2}{4} \rceil}\}$ such that $N[v_i] \cap N[v_j] = \{u\}$ for $i \neq j$. Then $|N[D']| = \sum_{i=1}^t d(v_i) + 1 = 2t + 1$, where $t = \lceil \frac{p-2}{4} \rceil$ and $|N[D']| = 2\lceil \frac{p-2}{4} \rceil + 1$. When $p = 4r, 4r+1, 4r+2, 4r+3$, $|N[D']| = \frac{p}{2}$. Therefore D' is a majority dominating set which is a subset of $(V - D)$ which is also minimum. It is inferred that D' is an inverse majority dominating set of G with respect to D of G . Hence $\gamma_M^{-1}(G) \leq |D'| = \lceil \frac{p-2}{4} \rceil$.

Let $S' = \{u_1, \dots, u_{\gamma_M^{-1}(G)}\}$ be an inverse majority dominating set of G . Then $|N[S']| \geq \lceil \frac{p}{2} \rceil$ and $S' \subseteq (V - D)$. Now $|N[S']| \leq \sum_{i=1}^{\gamma_M^{-1}(G)} d(u_i) + 1 = 2\gamma_M^{-1}(G) + 1$. Accordingly $\lceil \frac{p}{2} \rceil \leq |N[S']| \leq 2\gamma_M^{-1}(G) + 1$. Therefore, $\gamma_M^{-1}(G) \geq \frac{1}{2} \lceil \frac{p}{2} \rceil - \lceil \frac{1}{2} \rceil, \gamma_M^{-1}(G) \geq \lceil \frac{p-2}{4} \rceil$.

Hence $\gamma_M^{-1}(S(K_{1,p-1})) = \lceil \frac{p-2}{4} \rceil$. ■

Proposition 2.3.3: For the Double star $G = D_{r,s}$, $r < s$ and $r, s \geq 2$,

$$\gamma_M^{-1}(G) = \begin{cases} 1, & \text{if } s = r, r+1, r+2 \\ |e_i| + 1, & \text{if } s \geq r+3, \text{ where } |e_i| = \lceil \frac{p}{2} \rceil - |N[u]|. \end{cases}$$

Proof: Let $V(G) = \{u, v, u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_s\}$ with $p = r + s + 2$. Let u and v be the two central vertices of G with $(u_1, u_2, \dots, u_r)$ and $(v_1, v_2, \dots, v_s)$ pendants attached to u and v respectively and $\lceil \frac{p}{2} \rceil = \lceil \frac{r+s+2}{2} \rceil = \lceil \frac{r+s}{2} \rceil + 1$.

Case (i). When $s = r, r + 1, r + 2$.

Let $D = \{v\}$ be a minimum majority dominating set of G such that $|N[v]| = s + 2$. When $s = r, r + 1, r + 2$, $|N[v]| \geq r + 2 \geq \lceil \frac{p}{2} \rceil$. Choose another central vertex $\{u\} \subseteq V - D$ dominates $|N[u]| = r + 2 = \lceil \frac{p}{2} \rceil$ vertices. It implies that $D' = \{u\} \subseteq V - D$ is an inverse majority dominating set of G with respect to D . Hence $\gamma_M^{-1}(G) = 1$, if $s = r, r + 1, r + 2$.

Case (ii). When $s \geq r + 3$ and $r < s$.

Let $D = \{v\}$ is a minimum majority dominating set of G . In the graph G , there are $(r + s) = E_1 \cup E_2 =$ pendants such that $|E_1| = r$ and $|E_2| = s$. Select another centre vertex $\{u\} \subseteq V - D$ which dominates $|N[u]| = r + 2 < \lceil \frac{p}{2} \rceil$, if $s \geq r + 3$.

$$\text{Let } |e_i| = \left\lceil \frac{p}{2} \right\rceil - |N[u]|, \text{ for } e_i \in E_2. \quad (2.1)$$

Let $D' = (\{u\} \cup \{e_i\}) \subseteq V - D$ where $e_i \in E_2$. Then $|N[D']| = |e_i| + d(u) + 1$. By the condition (2.1), $|N[D']| = |e_i| + r + 2 =$

$\lceil \frac{p}{2} \rceil - (r + 2) + (r + 2), |N[D']| = \lceil \frac{p}{2} \rceil$. It indicates that D' is an inverse majority dominating set of G with respect to D and

$$\gamma_M^{-1}(G) \leq |D'| = |e_i| + 1, e_i \in E_2. \quad (2.2)$$

Suppose $|D_1| = |D'| - 1 \subseteq V - D$. Then $|N[D_1]| < \lceil \frac{p}{2} \rceil$ and clearly D_1 is not an inverse majority dominating set of G . Hence

$$\gamma_M^{-1}(G) \geq |D'| = |e_i| + 1. \quad (2.3)$$

From (2.2) and (2.3), We obtain $\gamma_M^{-1}(G) = |e_i| + 1$, where $e_i \in E_2$. ■

Proposition 2.3.4: Let $G = W_p$ be the wheel graph of p vertices with $p \geq 5$. Then $\gamma_M^{-1}(G) = \lceil \frac{p-2}{6} \rceil$.

Proof: Let $G = W_p = K_1 + C_{p-1}$ and $V(G) = \{v_1, v_2, \dots, v_{p-1}, v_p\}$, where v_p is the centre of wheel graph. The only minimum majority dominating set of G is $D = \{v_p\}$ and $\gamma_M(G) = 1$. Now, consider the inverse majority dominating set D' with respect to D in the set $(V - D) = V(C_{p-1})$. Let $D' = \{v_2, v_5, v_8, \dots, v_{\lceil \frac{p-2}{6} \rceil}\} \subseteq (V - D)$ with $d(v_i, v_j) \geq 3$ for $i \neq j$, $|D'| = t = \lceil \frac{p-2}{6} \rceil$. Then $|N[D']| = \sum_{i=1}^t d(v_i) + t - (t - 1)$ and $|N[D']| = 3t + 1 = 3\lceil \frac{p-2}{6} \rceil + 1$. When $p = 6r, 6r + 1, \dots, 6r + 5$, $|N[D']| = \lceil \frac{p}{2} \rceil$ and $D' \subseteq (V - D)$. It implies

that D' is an inverse majority dominating set of G with respect to D of G . Hence $\gamma_M^{-1}(G) \leq |D'| = \lceil \frac{p-2}{6} \rceil$.

Let $S' = \{v_1, \dots, v_{\gamma_M^{-1}(G)}\}$ be an inverse majority dominating set of G with respect to D and $|S'| = \gamma_M^{-1}(G)$. Then $|N[S']| \geq \lceil \frac{p}{2} \rceil$ and $S' \subseteq (V - D)$. Now, $|N[S']| = \sum_{i=1}^{\gamma_M^{-1}} d(v_i) + 1$, for $v_i \in V(C_{p-1})$. Now $\lceil \frac{p}{2} \rceil \leq |N[S']| \leq 3\gamma_M^{-1}(G) + 1$ and $\gamma_M^{-1}(G) \geq \lceil \frac{p}{2} \rceil - \frac{1}{3} = \lceil \frac{p-2}{6} \rceil$.

Therefore, $\gamma_M^{-1}(G) \geq \lceil \frac{p-2}{6} \rceil$. Thence $\gamma_M^{-1}(G) = \lceil \frac{p-2}{6} \rceil$. ■

Proposition 2.3.5: Let G be a caterpillar such that each vertex is attached with exactly one pendant. Then $\gamma_M^{-1}(G) = \lceil \frac{p}{8} \rceil$.

Proof: By the result (1.4), $\gamma_M(G) = \lceil \frac{p}{8} \rceil$. Then $D = \{v_2, v_5, \dots, v_{\lceil \frac{p}{8} \rceil}\}$ is a minimum majority dominating set of G such that $N[v_i] \cap N[v_j] = \emptyset$ for $i \neq j$ and $|D| = \lceil \frac{p}{8} \rceil$. Choose $D' = \{v_{i+1}/v_i \in D\} \subseteq (V - D)$ with $|D'| = |D|$. It is inferred that D' is an inverse majority dominating set of G with respect to D of G . Hence $\gamma_M^{-1}(G) = \lceil \frac{p}{8} \rceil$. ■

The next theorem is the characterization of a minimal inverse majority dominating set of G .

Theorem 2.3.6: Let G be an any graph with p vertices and D be a minimum majority dominating set of G . Then the set D' is a minimal

inverse majority dominating set of G with respect to D if and only if for each $u \in D'$, either the following condition (i) or (ii) holds.

$$(i) \ D' \subseteq V - D \text{ such that } |N[D']| > \lceil \frac{p}{2} \rceil \text{ and } |pn(u, D')| > |N[D']| - \lceil \frac{p}{2} \rceil.$$

$$(ii) \ D' \subseteq V - D \text{ such that } |N[D']| = \lceil \frac{p}{2} \rceil \text{ and either } u \text{ is an isolate of } D' \text{ or } pn(u, D') \cap (V - D') \neq \emptyset.$$

Proof: Let D be a γ_M -set of G . Let D' be a minimal inverse majority dominating set with respect to G . Then $D' \subseteq V - D$ such that $|N[D']| > \lceil \frac{p}{2} \rceil$. Let $u \in D'$ since D' is minimal, the set $D' - \{u\}$ is not an inverse majority dominating set of G . Then

$$|N[D' - \{u\}]| < \lceil \frac{p}{2} \rceil. \quad (2.4)$$

Since $|pn[u, D']| = |N[D']| - |N[D' - \{u\}]|$. Subsequently $|N[D' - \{u\}]| = |N[D']| - |pn[u, D']|$. By the condition (2.4), $|N[D']| - |pn[u, D']| < \lceil \frac{p}{2} \rceil$. Therefore, $|pn[u, D']| > |N[D']| - \lceil \frac{p}{2} \rceil$ and $D' \subseteq V - D$ such that $|N[D']| > \lceil \frac{p}{2} \rceil$. Thus, the condition (i) holds.

Assume that $|N[D']| = \lceil \frac{p}{2} \rceil$. Suppose the vertex u is neither an isolate of D' nor u has a private neighbor in the induced subgraph $\langle V - D' \rangle$. That is, $|pn[u, D']| = \emptyset$. Then $|N[D' - \{u\}]| = |N[D']| -$

$|pn[u, D']|$ implies that $|N[D' - \{u\}]| = |N[D']| = \lceil \frac{p}{2} \rceil$. Hence, $|N[D' - \{u\}]| = \lceil \frac{p}{2} \rceil$ and $D' \subseteq V - D$, which implies that $D' - \{u\}$ is an inverse majority dominating set with respect to D , which is a contradiction to the assumption. Therefore condition (ii) holds.

Conversely, Let D' be an inverse majority dominating set and suppose the conditions (i) and (ii) hold. To prove that D' is minimal. Suppose D' is not minimal inverse majority dominating set of G . Then $(D' - \{u\})$ is an inverse majority dominating set of G and

$$|N[D' - \{u\}]| > \left\lceil \frac{p}{2} \right\rceil, \text{ for some } u \in D' \quad (2.5)$$

Since $|N[D' - \{u\}]| = |N[D']| - |pn[u, D']|$ and by (2.5), $|pn[u, D']| < |N[D']| - \lceil \frac{p}{2} \rceil$, which is a contradiction to condition (i). Thence D' is a minimal inverse majority dominating set of G .

Let the condition (ii) be true. Then the subset $D' \subseteq V - D$ in such a way that $|N[D']| = \lceil \frac{p}{2} \rceil$. Suppose D' is not minimal, then $D' - \{u\}$ is an inverse majority dominating set and

$$|N[D' - \{u\}]| = \left\lceil \frac{p}{2} \right\rceil, \text{ for some vertex } u \in D'. \quad (2.6)$$

Since $|N[D' - \{u\}]| = |N[D']| - |pn[u, D']|$ and by (2.6), $|N[D']| - |pn[u, D']| = \lceil \frac{p}{2} \rceil$. Therefore $|N[D']| = \lceil \frac{p}{2} \rceil + |pn[u, D']|$ implies that

$|pn[u, D']| = \emptyset$. Then $pn[u, D'] \cap (V - D') = \emptyset$. Also, for some vertex $u \in D', N(u) \not\subseteq V - D'$. It implies that u is not an isolate of D' which is a contradiction to the condition (ii). Hence D' is a minimal inverse majority dominating set for G . ■

2.4 Bounds of $\gamma_M^{-1}(G)$

In this section, bounds of the inverse majority domination number $\gamma_M^{-1}(G)$ are determined for a connected and disconnected graphs in terms of p and $\Delta'(G)$, which is a maximum degree in $(V - D)$.

Proposition 2.4.1: For a connected graph G , $1 \leq \gamma_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil - 1$.

Proof: Suppose that the graph is minimally connected with p vertices. Then the graph G could have $(p-1)$ pendants and a full degree vertex ' u '. Therefore $D = \{u\}$ is a majority dominating set of G . Now choose $D' = \{u_1, u_2, \dots, u_t\} \subseteq (V - D)$ with $|D'| = t = \lceil \frac{p}{2} \rceil - 1$ and all are pendants. So, $|N[D']| = t + 1 = \lceil \frac{p}{2} \rceil$. It implies that D' is an inverse majority dominating set G with respect to D . Thus, $\gamma_M^{-1}(G) \leq |D'| = \lceil \frac{p}{2} \rceil - 1$.

Suppose the connected graph G has less than $(p-1)$ pendants, then $\gamma_M^{-1}(G) < \lceil \frac{p}{2} \rceil - 1$. Suppose G has atleast 2 pendants, then $G = P_p$

and $\gamma_M^{-1}(G) = \lceil \frac{p}{6} \rceil < \lceil \frac{p}{2} \rceil - 1$. If the graph G has no pendants then $\gamma_M^{-1}(G) < \lceil \frac{p}{6} \rceil < \lceil \frac{p}{2} \rceil - 1$ and if G is not a complete graph then $\gamma_M^{-1}(G) \geq 1$. Hence $1 \leq \gamma_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil - 1$. This upper bound is sharp if $G = K_{1,p-1}$, a star and the lower bound is sharp if $G = K_p$, a complete graph. ■

Proposition 2.4.2: For a tree T with p vertices, $\lceil \frac{p}{6} \rceil \leq \gamma_M^{-1}(T) \leq \lceil \frac{p}{2} \rceil - 1$. These bounds are sharp.

Proof: This result is proved by induction on the number of pendants ‘ e ’. Since every tree T has atleast two pendants, if $e = 2$ then a tree $T = P_p$, a path of p vertices. By the results (2.3.1)(2), $\gamma_M^{-1}(P_p) = \lceil \frac{p}{6} \rceil$. If $e = 3$ then G is any connected graph with three pendants or $T = K_{1,3}$ or $D_{1,2}$. By the results (2.3.1)(5) and proposition (2.3.3), We get $\gamma_M^{-1}(T) = \lfloor \frac{p-1}{2} \rfloor = \lceil \frac{p}{6} \rceil$. This is true for $e = 2, 3, 4, \dots, (p-2)$ pendants. When $e = p-1$ then $T = K_{1,p-1}$, a star. By the results (2.3.1)(5), $\gamma_M^{-1}(K_{1,p-1}) = \lfloor \frac{p-1}{2} \rfloor = \lceil \frac{p}{2} \rceil - 1$. Hence, $\lceil \frac{p}{6} \rceil \leq \gamma_M^{-1}(T) \leq \lceil \frac{p}{2} \rceil - 1$, for any tree T . ■

Corollary 2.4.3:

- (i) For any graph G with isolates, $\gamma_M^{-1}(G) \leq \binom{p}{2}$. The bound is sharp if $G = \bar{K}_p$, p is even.

(ii) For a disconnected graph G without isolates, $\gamma_M^{-1}(G) \leq \lceil \frac{p}{4} \rceil$.

This bound is sharp for $G = mK_2, m \geq 1$ and for $G = mP_5$ or $mC_7, \gamma_M^{-1}(G) < \lceil \frac{p}{4} \rceil$.

Theorem 2.4.4: Let G be a disconnected graph with p vertices, l -isolates and maximum degree $\Delta'(G)$ in $(V - D)$. Let D be a γ_M -set of G . Then $\gamma_M^{-1}(G) < \lceil \frac{p}{4} \rceil$, if $\Delta'(G) \geq 3$.

Proof: To prove $\gamma_M^{-1}(G) < \lceil \frac{p}{4} \rceil$, if $\Delta'(G) \geq 3$.

Let D be a γ_M -set of G and $\Delta'(G)$ be a maximum degree in $(V - D)$. Let $g_1, g_2, \dots, g_k$ be ' k ' components of G with g_i 's are components and g_j 's are isolates such that $(g_i \cup g_j) = g_k$.

Case (i): When $|l| < \lceil \frac{p}{2} \rceil$ and $\Delta'(G) \geq 3$. Let $G = g_1 \cup g_2 \cup \dots \cup g_k$ such that $|N[g_i]| \geq \lceil \frac{p}{2} \rceil$ and $|N[g_i]| < \lceil \frac{p}{2} \rceil$ where $|g_j| = |l|$. Let $D' = \{u_1, u_2, \dots, u_t\} \subseteq (V - D)$. Since $|N[g_i]| \geq \lceil \frac{p}{2} \rceil$, D' contains no isolates such that each $u_i \in V(g_i)$. Since $\Delta'(G) \geq 3$ in $(V - D)$, $d(u_i) \geq 3$ and $|N[D']| \geq \lceil \frac{p}{2} \rceil$ with $|D'| = t < \lceil \frac{p}{4} \rceil$. If g_i 's are all complete then $u_i \in V(g_i)$, for every $g_i, i = 1, 2, \dots, t$ and $|D'| = t < \lceil \frac{p}{4} \rceil$. It implies that D' is an inverse majority dominating set of G with respect to D and $\gamma_M^{-1}(G) < |D'| < \lceil \frac{p}{4} \rceil$. If g_i 's are not complete and $\Delta'(G) \geq 3$ then each $u_i \in D'$ dominates atleast 4 vertices and

$|N[D']| \geq \lceil \frac{p}{2} \rceil$ with $|D'| < \lceil \frac{p}{4} \rceil$. Therefore D' is an inverse majority dominating set of G and $\gamma_M^{-1}(G) \leq |D'| < \lceil \frac{p}{4} \rceil$. Hence $\gamma_M^{-1}(G) < \lceil \frac{p}{4} \rceil$, if $\Delta'(G) \geq 3$.

Case (ii): when $|l| > \lceil \frac{p}{2} \rceil$ and $\Delta'(G) \geq 3$. Since G contains more than $\lceil \frac{p}{2} \rceil$ isolates, the remaining components g_i such that $|N[g_i]| < \lceil \frac{p}{2} \rceil$. Since $\Delta'(G) \geq 3$ in $(V - D)$, each $u_i \in V(g_i)$ dominates nearly $\frac{p}{2}$ vertices. Then $D' = \{u_1, \dots, u_i, u_j\} \subseteq (V - D)$ such that $d(u_i) \geq 3$ and $d(u_j) = 0$. Therefore $|N[D']| = \lceil \frac{p}{2} \rceil$ and $|D'| = t = |i + j| < \lceil \frac{p}{4} \rceil$. Since $D' \subseteq (V - D)$, D' is an inverse majority dominating set of G with respect to D and $\gamma_M^{-1}(G) < |D'| < \lceil \frac{p}{4} \rceil$. Hence $\gamma_M^{-1}(G) < \lceil \frac{p}{4} \rceil$, if $\Delta'(G) \geq 3$. ■

Theorem 2.4.5: Let G be a disconnected graph with p vertices and l -isolates with a γ_M - set D of G and $\Delta'(G)$ is a maximum degree in $(V - D)$. Then $\gamma_M^{-1}(G) > \lceil \frac{p}{4} \rceil$, if $\Delta'(G) \leq 2$.

Proof: Let $G = (g_1 \cup g_2 \cup \dots \cup g_k)$ be a disconnected graph with $g_i \cup g_j = k$ components and l - isolates. When $\Delta'(G) = 0$, then $G = \bar{K}_p$. If p is even then $\gamma_M^{-1}(G) = \lceil \frac{p}{2} \rceil > \lceil \frac{p}{4} \rceil$ and $\gamma_M^{-1}(G)$ does not exist if p is odd. When $\Delta'(G) = 1$, then $(V - D)$ contains some pendants with l -isolates. If $|l| > \lceil \frac{p}{2} \rceil$ then $D' = \{u_1, \dots, u_i, u_j\} \subseteq V - D$ where

$u_i \in V(g_i)$ and u_j 's are all isolates. Since $\Delta'(G) = 1, d(u_i) = 1$, for every $u_i \in D'$ and $d(u_j) = 0$. Then $|N[D']| = \lceil \frac{p}{2} \rceil$ and $|D'| = t = (i + j) > \lceil \frac{p}{4} \rceil$. It implies that D' is an inverse majority dominating set of G and $\gamma_M^{-1}(G) > \lceil \frac{p}{4} \rceil$.

If $|l| < \lceil \frac{p}{2} \rceil$ and $\Delta'(G) = 1$, then the graph G consists of pendants attached to a Δ -degree vertex of G . In this case, $D' = \{u_1, \dots, u_i\} \subseteq V - D$ such that each $u_i \in V(g_i)$, where g_i 's connected components. Then $|N[D']| = \lceil \frac{p}{2} \rceil$ and since $\Delta'(G) = 1, |D'| = t > \lceil \frac{p}{4} \rceil$. Hence D' is an inverse majority dominating set of G and $\gamma_M^{-1}(G) > \lceil \frac{p}{4} \rceil$. When $\Delta'(G) = 2$, the $(V - D)$ contains some vertices of degree ≤ 2 and isolates. Apply the above argument for $|l| > \lceil \frac{p}{2} \rceil$ and $|l| < \lceil \frac{p}{2} \rceil, D'$ is an inverse majority dominating set of G and $\gamma_M^{-1}(G) > \lceil \frac{p}{4} \rceil$, if $\Delta'(G) \leq 2$. ■

Corollary 2.4.6: For a disconnected graph G with l isolates and $\Delta'(G) = 2$,

(i) $\gamma_M^{-1}(G) = \lfloor \frac{p}{4} \rfloor$, if $|l| < \lceil \frac{p}{2} \rceil$

(ii) $\gamma_M^{-1}(G) = \lceil \frac{p}{4} \rceil$, if $|l| > \lceil \frac{p}{2} \rceil$, where $\Delta'(G)$ is a maximum degree in $(V - D)$ and D is a γ_M -set of G .

Example 2.4.7:

1. Let $G = 2K_4 \cup 7K_1$. Then $\gamma_M^{-1}(G) = 2 < \lfloor \frac{p}{4} \rfloor = 3, |l| < \lceil \frac{p}{2} \rceil, \Delta'(G) \geq 3$.
2. Let $G = K_6 \cup 9K_1$. Then $\gamma_M^{-1}(G) = 3 < \lceil \frac{p}{4} \rceil = 4$, if $|l| > \lceil \frac{p}{2} \rceil, \Delta'(G) \geq 3$.
3. Let $G = K_{1,7} \cup 7K_1$. Then $\gamma_M^{-1}(G) = 7 > \lfloor \frac{p}{4} \rfloor$, if $|l| < \lceil \frac{p}{2} \rceil$ and $\Delta'(G) \leq 1$.
4. If $G = K_{1,5} \cup 9K_1$ then $\gamma_M^{-1}(G) = 7 > \lceil \frac{p}{4} \rceil$, if $\Delta'(G) \leq 2$.
5. If $G = 2K_3 \cup 9K_1$ then $\gamma_M^{-1}(G) = 4 = \lceil \frac{p}{4} \rceil$, if $|l| > \lceil \frac{p}{2} \rceil$, $\Delta'(G) = 2$.
6. If $G = K_3 \cup K_{1,4} \cup 7K_1$ then $\gamma_M(G) = 2$ and $\gamma_M^{-1}(G) = 5 > \lfloor \frac{p}{4} \rfloor$, if $\Delta'(G) = 2$.

Proposition 2.4.8: If G is any r -regular graph with p vertices then

$$\gamma_M^{-1}(G) = \gamma_M(G).$$

Proof: Let G be a r -regular graph with p vertices such that $d(u_i) = r$, for all $u_i \in V(G)$. Let D be a minimum majority dominating set of G and $|D| = \gamma_M(G)$.

Choose $D' = \{u_{i+1}/u_i \in D\}$ and $D' \subseteq V - D$ such that $|D'| = |D|$.

Then $|N[D']| \leq \sum_{i=1}^{|D|} d(u_{i+1}) + |D| \leq (r+1)|D|$. Since $|D| = \gamma_M(G)$, $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and also $|N[D']| \geq \lceil \frac{p}{2} \rceil$. It implies that D' is a majority dominating set in $V - D$ and therefore D' is an inverse majority dominating set of G with $|D'| = |D|$. Hereafter, $\gamma_M^{-1}(G) \leq |D'| = |D| = \gamma_M(G)$. Then by Proposition (2.2.6) (ii), for any graph G , $\gamma_M(G) \leq \gamma_M^{-1}(G)$. Hence we obtain $\gamma_M^{-1}(G) = \gamma_M(G)$. ■

Example 2.4.9: For a Cycle C_p , mK_2 , K_p a Complete graph, Petersen graph and Tutte graph, $\gamma_M^{-1}(G) = \gamma_M(G)$ and all are r - regular graphs.

Next result gives the comparison for $\gamma_M^{-1}(G)$ with an inverse domination number $\gamma^{-1}(G)$.

Theorem 2.4.10: Let G be any connected graph with ' p ' vertices. Then $\gamma_M^{-1}(G) \leq \lceil \frac{\gamma^{-1}(G)}{2} \rceil$, where $\gamma^{-1}(G)$ is the inverse domination number of G .

Proof: Let D be a minimum dominating set of G . Let $D' \subseteq V - D$ be an inverse dominating set and D' is a γ^{-1} - set of G with respect to D . Then $|N[D']| = |V(G)|$. Let $D' = D_1 \cup D_2$, where $|D_1| = \lceil \frac{\gamma^{-1}(G)}{2} \rceil$ and $|D_2| = \lfloor \frac{\gamma^{-1}(G)}{2} \rfloor$. Therefore $|N[D_1] \cup N[D_2]| = |N[D']| = |V(G)|$.

Since the dominating set D is also a majority dominating set of G and $D' \subseteq V - D, D_1 \subseteq V - D$ and $D_2 \subseteq V - D$. Then either $|N[D_1]| \geq \lceil \frac{V(G)}{2} \rceil$ or $|N[D_2]| \geq \lfloor \frac{V(G)}{2} \rfloor$. It implies that either D_1 is an inverse majority dominating set or D_2 is an inverse majority dominating set of G with respect to D . Therefore $\gamma_M^{-1}(G) \leq |D_1| = \lceil \frac{\gamma^{-1}(G)}{2} \rceil$ or $\gamma_M^{-1}(G) \leq |D_2| = \lfloor \frac{\gamma^{-1}(G)}{2} \rfloor$. Hence $\gamma_M^{-1}(G) \leq \lceil \frac{\gamma^{-1}(G)}{2} \rceil$. ■

Example 2.4.11: Let G be the graph obtained from $(K_{1,10})$ by dividing each edge exactly once and $G = S(K_{1,10})$ with $p = 21$ and $q = 20$. Then $\gamma(G) = 10$, $\gamma^{-1}(G) = 11$, $\gamma_M(G) = 1$ and $\gamma_M^{-1}(G) = 5$. Therefore $\gamma_M^{-1}(G) = 5 < \lceil \frac{11}{2} \rceil = \left\lceil \frac{\gamma^{-1}(G)}{2} \right\rceil = 6$.

Example 2.4.12: Let $D_{3,10}$ be a double star with $p = 15$. Then $\gamma(G) = 2, \gamma^{-1}(G) = 13$ and $\gamma_M(G) = 1$, By Proposition (2.3.3), $\gamma_M^{-1}(G) = |e_i| + 1 = \lceil \frac{p}{2} \rceil - |N[u]| + 1 = 4$. Hence $\gamma_M^{-1}(G) = 4 < \left\lceil \frac{\gamma^{-1}(G)}{2} \right\rceil = 7$.

Proposition 2.4.13: In a graph G , all the vertices of G are majority dominating vertex if and only if $\gamma_M^{-1}(G) = 1$.

Proof: Since each vertex $v_i \in V(G)$ is a majority dominating vertex, v_i satisfies $d(v_i) \geq \lceil \frac{p}{2} \rceil - 1$. Any single vertex is a minimum majority

dominating set D of G then we could find another majority dominating vertex D' in $(V - D)$ of G . Hence $\gamma_M^{-1}(G) = \gamma_M(G) = 1$. ■

Corollary 2.4.14:

(i) If the graph G has exactly one majority dominating vertex then

$$\gamma_M^{-1}(G) \geq 2.$$

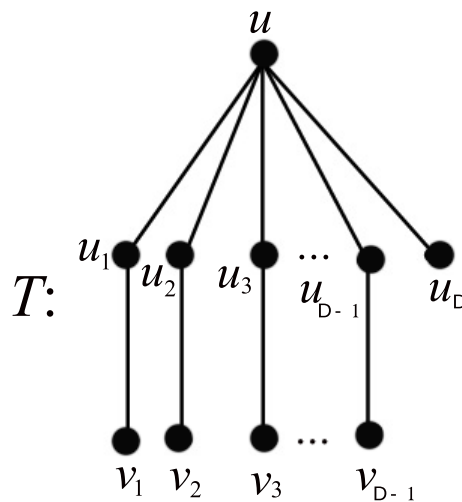
(ii) Let D be a minimum majority dominating set of a graph G . If

graph G has atleast one majority dominating vertex in $(V - D)$

then $\gamma_M^{-1}(G) = 1$.

Proposition 2.4.15: If a tree has a Δ -supports and Δ -pendants

then $\gamma_M^{-1}(T) = \left\lfloor \frac{\Delta(T)}{2} \right\rfloor$.



Proof: Since T has a Δ -supports and Δ -pendants, the tree has a wounded spider with a exactly one edge is not divided.

Let $V(G) = \{u, u_1, u_2, \dots, u_\Delta, v_1, v_2, \dots, v_{\Delta-1}\}$ and $|V(G)| = p = 2\Delta = \text{even}$. Let u be a support vertex of the pendant vertex u_Δ . Then $D = \{u\}$ is a majority dominating set of T . Let $D' = \{u_1, u_2, \dots, u_t\} \subseteq V - D$ with $|D'| = t = \lfloor \frac{\Delta}{2} \rfloor$. Since all the vertices of D' are supports of a tree T , $|N[D']| = 2\lfloor \frac{\Delta}{2} \rfloor + 1$. As p is even, $|N[D']| = \Delta + 1 > \frac{p}{2}$ and $D' \subseteq V - D$. It implies that D' is an inverse majority dominating set of T and $\gamma_M^{-1}(T) \leq |D'| = \lfloor \frac{\Delta}{2} \rfloor$. Supposing take $|D_1| < |D'|$ and $D_1 \subseteq V - D$. Therefore $|N[D_1]| = 2(\lfloor \frac{\Delta}{2} \rfloor - 1) = \Delta - 2 < \frac{p}{2}$. Then D_1 is not an inverse majority dominating set of T . $\gamma_M^{-1}(T) > |D_1|$ and $\gamma_M^{-1}(T) \geq |D'| = \lfloor \frac{\Delta}{2} \rfloor$. Hence $\gamma_M^{-1}(T) = \lfloor \frac{\Delta(T)}{2} \rfloor$. ■

Proposition 2.4.16: Let T be a tree with Δ -supports, Δ -pendants and a majority dominating vertex. Then $\gamma_M^{-1}(T) = \lceil \frac{\Delta}{2} \rceil$ if and only if $T = S(K_{1,p-1})$.

Proof: Let u be a majority dominating vertex of a tree T with $d(u) \geq \lceil \frac{p}{2} \rceil - 1$. The tree T has Δ -supports $S = \{v_1, v_2, \dots, v_s\}$ with $d(v_i) \geq 2$ and Δ -pendants $P = \{w_1, w_2, \dots, w_p\}$ with $d(w_i) = 1$. Then $d(u) = \Delta(T) \geq \lceil \frac{p}{2} \rceil - 1$ and $|V(G)| = 2\Delta + 1$. Since $|N[u]| = \lceil \frac{p}{2} \rceil$, $D = \{u\}$ is a majority dominating set of T . If $d(v_i) > 2$ then the number of pendants $\Delta(T)$ will be increased. Therefore $d(v_i) = 2$, for

every $v_i \in S$. Let $D' = \{v_1, v_2, \dots, v_t\} \subseteq S$ such that $|D'| = t = \lceil \frac{\Delta}{2} \rceil$.

$$\text{Then } |N[D']| = 2t + 1 = 2\lceil \frac{\Delta}{2} \rceil + 1 = \begin{cases} \Delta + 1, & \text{if } \Delta \text{ is even} \\ \Delta + 2, & \text{if } \Delta \text{ is odd} \end{cases}.$$

In both cases, $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $D' \subseteq V - D$. It implies that D' is an inverse majority dominating set of T and $\gamma_M^{-1}(T) \leq |D'| = \lceil \frac{\Delta}{2} \rceil$. Suppose $|D_1| < |D'|$ then $|N[D_1]| < \lceil \frac{p}{2} \rceil$ and $\gamma_M^{-1}(T) \geq |D'| = \lceil \frac{\Delta}{2} \rceil$. Hence $\gamma_M^{-1}(T) = \lceil \frac{\Delta}{2} \rceil$.

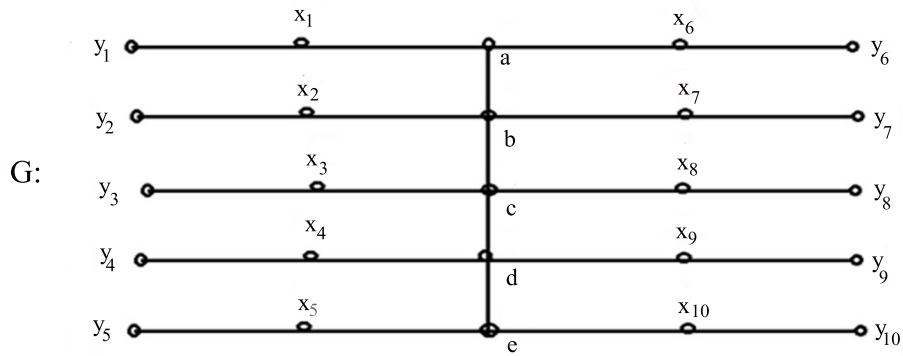
Conversely, assume that $\gamma_M^{-1}(T) = \lceil \frac{\Delta}{2} \rceil$. Since T has Δ -supports, Δ -pendants and a majority dominating vertex ' u ', $p = 2\Delta + 1$, odd. Let $D = \{u\}$ be a majority dominating set of T and $|N[D]| \geq \lceil \frac{p}{2} \rceil$. Then the set $(V - D)$ contains Δ -supports and Δ -pendants. By assumption, let $D' \subseteq (V - D)$ be an inverse majority dominating set of T with $|D'| = \lceil \frac{\Delta}{2} \rceil$ with respect to D . If the set $D' = \{v_1, \dots, v_t\}$ such that $|D'| = t = \lceil \frac{\Delta}{2} \rceil$ and all vertices v_i are only pendants meet at u , then $|N[D']| = t + 1 = \lceil \frac{\Delta}{2} \rceil + 1 = \frac{1}{2}\lceil \frac{p-1}{2} \rceil + 1 < \lceil \frac{p}{2} \rceil$. It implies that D' is not an inverse majority dominating set of T with respect to D . Therefore by proposition (2.3.1)(5), if $T = K_{1,p-1}$ then $\gamma_M^{-1}(T) = \lfloor \frac{\Delta}{2} \rfloor$ and hence the corresponding tree $T \neq K_{1,p-1}$.

Since T has Δ -supports, take $D' = \{u_1, u_2, \dots, u_t\} \subseteq V - D$ with $|D'| = t = \lceil \frac{\Delta}{2} \rceil$ and u_i 's are all support such that $d(u_i) \geq 2$. If

$d(u_i) > 2$ then the number of pendants increase beyond Δ -pendants of T . Hence $d(u_i) = 2$ and $D' \subseteq V - D$ then $|N[D']| = 2t + 1$. Therefore $|N[D']| = 2\lceil \frac{\Delta}{2} \rceil + 1 \geq \lceil \frac{p}{2} \rceil$. It implies that $\Delta(T) = \lfloor \frac{p}{2} \rfloor$ and the corresponding tree T is a subdivision of $(K_{1,p-1})$. Thus $G = S(K_{1,p-1})$. ■

2.5 Results on $\gamma_M^{-1}(G)$ and $\beta_M(G)$

Example 2.5.1: Consider the following graph $G = T_{5k}$, $k = 5$ with $p = 25$ vertices. The graph G contains five P_5 paths which is connected in the middle vertex of each path P_5 . The vertex set is labeled as $\{y_1, \dots, y_{10}\}$ are pendants, $\{x_1, \dots, x_{10}\}$ are two degree vertices and (a, b, c, d, e) are middle vertices of each P_5 .



In G , the set $D_1 = \{a, b, c, d\}$ is a majority dominating set and $\gamma_M(G) = |D_1| = 4$. Following the set $D_2 = \{x_1, x_2, x_3, e\}$ is a inverse majority dominating set of G with respect D_1 and $\gamma_M^{-1}(G) = |D_2| = 4$.

Next, the set $D_3 = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$ is a majority independent set of G and $\beta_M(G) = |D_3| = 7$. Hence $\gamma_M^{-1}(G) < \beta_M(G)$.

Proposition 2.5.2: For any graph G , the following inequalities are true. (i) $\gamma_M^{-1}(G) \leq \beta_M(G)$ and (ii) $\gamma_M(G) \leq \gamma_M^{-1}(G) \leq \beta_M(G)$.

Proof:

(i) Let D be a minimum majority dominating set and $D' \subseteq V - D$ be an inverse majority dominating set with respect to D of a graph G . Let S be any maximal majority independent set of G . Meanwhile $S \subseteq (V - D)$ any majority independent set S is an inverse majority dominating set of G . Henceforth $\gamma_M^{-1}(G) \leq \beta_M(G)$.

(ii) By Proposition (2.2.6)(i), $\gamma_M(G) \leq \gamma_M^{-1}(G)$ and by above outcome, we have $\gamma_M(G) \leq \gamma_M^{-1}(G) \leq \beta_M(G)$. ■

Proposition 2.5.3: If a graph G has a full degree vertex and others are pendants then $\gamma_M^{-1}(G) = \beta_M(G)$.

Proof: Let G be a graph with p vertices in which the only vertex u is a full degree and other $(p - 1)$ are pendants. Therefore $D = \{u\}$ is a γ_M -set of G . Then $D' = \{u_1, u_2, \dots, u_{\lceil \frac{p}{2} \rceil - 1}\} \subseteq (V - D)$ is an inverse

majority dominating set with $|D'| = \lceil \frac{p}{2} \rceil - 1$. Since all vertices of D' are pendants, $|N[D']| = \lceil \frac{p}{2} \rceil$. It is inferred that $\gamma_M^{-1}(G) = |D'| = \lceil \frac{p}{2} \rceil - 1$. By the result (1.4), $\beta_M(G) = \lfloor \frac{p-1}{2} \rfloor, p \geq 2$. When p is odd and even, $\lceil \frac{p}{2} \rceil - 1 = \lfloor \frac{p-1}{2} \rfloor$. Hence D' is also a majority independent set of G and $\beta_M(G) = |D'| = \lceil \frac{p}{2} \rceil - 1$. Thus $\gamma_M^{-1}(G) = \beta_M(G)$. ■

Corollary 2.5.4:

- (i) Let G be a totally disconnected graph with even number of vertices. Then $\gamma_M^{-1}(G) = \beta_M(G) = \frac{p}{2}$.
- (ii) If p is odd then $\gamma_M^{-1}(G)$ does not exist and $\beta_M(G) = \lceil \frac{p}{2} \rceil$.

Theorem 2.5.5: Let G be a disconnected graph with k components.

Then $\gamma_M^{-1}(G) \leq \beta_M(G)$.

Proof: Let G be a disconnected graph with ‘ k ’ components $g_1, g_2, \dots, g_k$ (say).

Case (i). If all ‘ k ’ components are regular then all vertices are of equal degree. Therefore γ_M – set D and inverse majority dominating set $D' \subseteq V - D$ both are equal that is $|D| = |D'|$. Also β_M -set S is equal to D' . Hence $|S| = |D'|$ implies that $\beta_M(G) = \gamma_M^{-1}(G)$.

Case (ii). If ‘ k ’ components are not regular but G has no isolates. Then the γ_M – set D contains Δ - degree vertices and the inverse

majority dominating set $D' \subseteq (V - D)$ contains some next maximum degree vertices with $|D| \leq |D'|$. It implies that $\gamma_M(G) \leq \gamma_M^{-1}(G)$. But the maximal majority independent set S consists of vertices u_i which satisfies the condition $|pn[u_i, s]| > |N[s]| - \lceil \frac{p}{2} \rceil$. It implies that $|S| \geq |D'|$ and $\gamma_M^{-1}(G) \leq \beta_M(G)$.

Case (iii). If the disconnected graph G with isolates then $\gamma_M(G) \leq \gamma_M^{-1}(G) = |D'|$. But the maximal majority independent set S contains isolates to get maximum size. Therefore $|S| > |D'|$ and $\gamma_M^{-1}(G) < \beta_M(G)$. Hence $\gamma_M^{-1}(G) \leq \beta_M(G)$. ■

Corollary 2.5.6: If the components of a disconnected graph G are regular then $\gamma_M^{-1}(G) = \beta_M(G)$.

Next, the relationship between the two parameters namely $\gamma_M^{-1}(G)$ and $\beta_M(G)$ for Cubic Bipartite Graphs is deliberated in detail.

Let G be a cubic bipartite graph with a partition of the vertex set $V_1(G)$ and $V_2(G)$ such that $|V_1| + |V_2| = p$. A cubic bipartite graph G with minimum number of vertices is $K_{3,3}$. Also graphs that have an odd number of vertices cannot be a cubic bipartite graph. We presently focus on the cubic bipartite graphs when $p = 6, 8, 10, 12, 14, 16, \dots$

Theorem 2.5.7: For all cubic and cubic bipartite graph G , $\gamma_M^{-1}(G) = \lceil \frac{p}{8} \rceil$.

Proof: Let $V(G) = \{v_1, v_2, \dots, v_p\}$ be the vertex set of the given graph G . Let D be a majority dominating set of G and $D' = \{v_1, v_2, \dots, v_t\}$ be the inverse majority dominating set of G with respect to D with $|D'| = t = \gamma_M^{-1}(G)$. Then, we have

$$|N[D']| \geq \left\lceil \frac{p}{2} \right\rceil \text{ and } D' \subseteq (V - D) \quad (2.7)$$

Now, $|N[D']| \leq \sum_{i=1}^t d(v_i) + t = 4t = 4\gamma_M^{-1}(G)$. By Condition (2.7), $\lceil \frac{p}{2} \rceil \leq 4\gamma_M^{-1}(G)$. Therefore, $\gamma_M^{-1}(G) \geq \lceil \frac{p}{8} \rceil$.

Suppose $D' = \{u_1, u_2, \dots, u_{\lceil \frac{p}{8} \rceil}\}$ is a subset of vertices in $(V - D)$ such that $N[u_i] \cap N[u_j] = \emptyset$, for $i \neq j$ and $|D'| = \lceil \frac{p}{8} \rceil$. Then $|N[D']| = 4\lceil \frac{p}{8} \rceil$. In all cases of $p = r(\text{mod } 8)$, when $0 \leq r \leq 7$, $|N[D']| = 4\lceil \frac{p}{8} \rceil \geq \lceil \frac{p}{2} \rceil$ and $D' \subseteq V - D$. Therefore D' is an inverse majority dominating set of G and $\gamma_M^{-1}(G) \leq |D'| = \lceil \frac{p}{8} \rceil$. Hence $\gamma_M^{-1}(G) = \lceil \frac{p}{8} \rceil$. ■

The following results are the characterization theorem of an inverse majority dominating sets and majority independent set of a graph G .

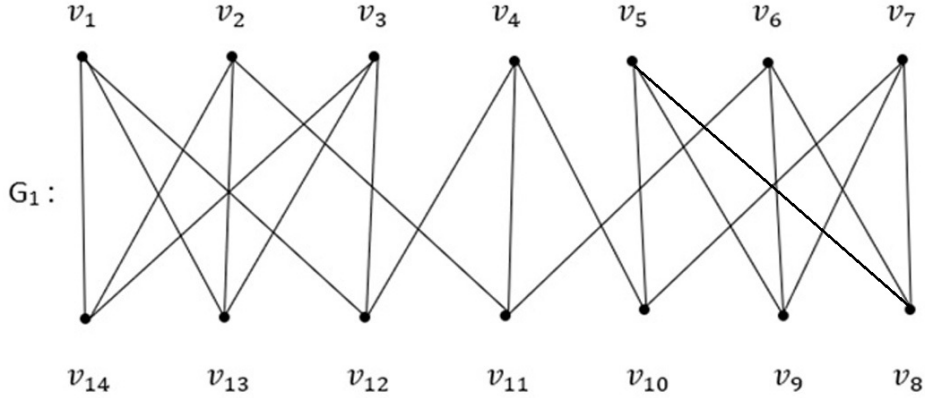
Proposition 2.5.8: $\gamma_M(G) = \gamma_M^{-1}(G) = \beta_M(G) = 1$ if and only if the cubic bipartite graph G has all vertices $u_i \in V(G)$ with $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$.

Proof: Let G be a cubic bipartite graph with p vertices. If the given graph G has vertices of degree $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$, for all $u_i \in V(G)$, then every vertex is a majority dominating vertex of G . It is inferred that $D = \{u_1\}$ is a minimal majority dominating set of G and $D' = \{u_2\} \subseteq V - D$ is a minimal inverse majority dominating set of G . Similarly any one vertex of G forms a majority independent set of G . Henceforth $\gamma_M(G) = \gamma_M^{-1}(G) = \beta_M(G) = 1$.

Conversely, suppose $D_1 = \{u_1\}, D_2 = \{u_2\} \subseteq (V - D_1)$ and $D_3 = \{u_3\}$ are a majority dominating set an inverse majority dominating set and a majority independent set of G respectively. Then $|N[D_i]| \geq \lceil \frac{p}{2} \rceil$, for $i = 1, 2, 3$ and each vertex u_i of degree $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$, for $u_i \in V(G)$. Hence the result. ■

Observation 2.5.9: Let G be any cubic bipartite graph with p vertices. Then the exact values of $\gamma_M(G), \gamma_M^{-1}(G)$ and $\beta_M(G)$ are given in the following way for the graph G .

1. If $p = 6$ or 8 , then $\gamma_M(G) = \gamma_M^{-1}(G) = \beta_M(G) = 1$.
2. If $p = 10$ or 12 , then $\gamma_M(G) = \gamma_M^{-1}(G) = \beta_M(G) = 2$.
3. The following graph G_1 is a cubic bipartite with $p = 14$.



In G_1 , $D_1 = \{v_1, v_4\}$ is a majority dominating set of G_1 and $D_2 = \{v_2, v_6\}$ is an inverse majority dominating set of G with respect to D_1 . Then $\gamma_M(G_1) = \gamma_M^{-1}(G_1) = 2$. Similarly $S = \{v_1, v_2, v_3\}$ is a majority independent set of G_1 and $\beta_M(G) = |S| = 3$.

4. For $p = 16$, $\gamma_M(G) = \gamma_M^{-1}(G) = 2$ and $\beta_M(G) = 3$.
5. For $p = 18, 20, 22$ $\gamma_M(G) = \gamma_M^{-1}(G) = 3$ and $\beta_M(G) = 4$.
6. For $p = 24$, $\gamma_M(G) = \gamma_M^{-1}(G) = 3$ and $\beta_M(G) = 5$.

and so on...

Next the result $\gamma_M^{-1}(G) < \beta_M(G)$, $p \geq 14$ is characterised in the following theorem.

Theorem 2.5.10: Let G be an cubic bipartite graph with p vertices.

The subsets D , D' and S of $V(G)$ are the minimum majority dominating set, an inverse majority dominating set with respect to D and a majority independent set of G respectively. Then $\gamma_M^{-1}(G) < \beta_M(G)$ when $p \geq 14$ if and only if

$$(i) \quad |pn[v_i, D']| \geq 2, \text{ for every } v_i \in D' \text{ and}$$

$$(ii) \quad |pn[v_i, S]| \leq 2, \text{ for every } v_i \in S.$$

Proof: Let D be the majority dominating set of G . Let $\gamma_M^{-1}(G) < \beta_M(G)$. Then the inverse majority domination number $\gamma_M^{-1}(G) = |D'|$ and the majority independence number $\beta_M(G) = |S|$. Also, D' is the minimum inverse majority dominating set with respect to D of G and S is the maximum majority independent set of a cubic bipartite graph G . Let $V_1(G) = \{v_1, v_2, \dots, v_{p_1}\}$ and $V_2(G) = \{u_1, u_2, \dots, u_{p_2}\}$ be the bipartition of $V(G)$ with $p = (p_1 + p_2)$. By the theorem (2.5.7), $\gamma_M^{-1}(G) \leq \lceil \frac{p}{8} \rceil$.

Let $D' = \{v_1, v_2, \dots, v_{\lceil \frac{p}{8} \rceil}\} \subseteq V_1(G)$ such that $d(v_i, v_j) \geq 2$ for any v_i and $v_j \in D'$ and $i \neq j$. Since each vertex of degree $d(v_i) = 3$, $N(v_i) = \{u_i, u_j, u_k\} \subseteq V_2(G)$, for each vertex $v_i \in D'$. Then $|N(v_i) \cap N(v_j)| \neq \emptyset$, for any $v_i, v_j \in D'$ and $i \neq j$. Therefore there exists atmost two vertex 'u' such that $N(v_i) \cap N(v_j) = \{u\}$ and $|N(v_i) \cap N(v_j)| \leq 2$. The private neighbour of each vertex $v_i \in D'$ is $\{v_i, u_i\}$ or $\{v_i, u_i, u_j\}$ or $\{v_i, u_i, u_j, u_k\}$. It implies that $|pn[v_i, D']| \geq 2$, for each $v_i \in D'$. Hence the condition (i) is true.

By the result (1.4), For any cubic bipartite graph G , $\beta_M(G) \leq \lceil \frac{p}{4} \rceil - 1$. Let $S = \{v_1, v_2, \dots, v_t\} \subseteq V_1(G)$ be a majority independent set of G where $t = \lceil \frac{p}{4} \rceil - 1$ such that $d(v_i, v_j) = 2$ for any i, j and $i \neq j$. For every vertex $v_i \in S$, $|N(v_i) \cap N(v_j)| \neq \emptyset$ and there exists atleast one vertex such that $|N(v_i) \cap N(v_j)| \geq 1$, for any $v_i, v_j \in S$. Since $d(v_i) = 3$, $|pn[v_i, S]| = 2$ or 1 and $|pn[v, S]| \leq 2$, for every $v_i \in S$. Thence the condition (ii) holds.

Conversely, assume that the conditions (i) and (ii) are true.

Let $D' = \{v_1, v_2, \dots, v_t\} \subseteq V-D$ be an inverse majority dominating set of G with respect to D with $|D'| = t$. Since $|pn[v_i, D']| \geq 2$, for every $v_i \in D'$, $|N(v_i) \cap N(v_j)| \leq 2$, for any $v_i, v_j \in D'$. It implies that $2 \leq |N[v_i]| \leq 4$, for every $v_i \in D'$ and each vertex $v_i \in D'$ dominates

atmost four vertices. Hence D' is an inverse majority dominating set of G with minimum cardinality. Therefore the inverse majority domination number is $\gamma_M^{-1}(G) = |D'| = \lceil \frac{p}{8} \rceil$.

Let $S = \{v_1, v_2, \dots, v_r\} \subseteq V(G)$ be a majority independent set of a cubic bipartite graph of G with $|S| = r$. Since $|pn[v_i, S]| \leq 2$, $|N(v_i) \cap N(v_j)| \geq 1$, for any $v_i, v_j \in S$ and $i \neq j$ and it satisfies $|pn[v_i, S]| > |N[S]| - \lceil \frac{p}{2} \rceil$, for every $v_i \in S$. Thus we obtain a majority independent set with maximum cardinality for G and majority independence number of G is $\beta_M(G) = |S| = \lceil \frac{p}{4} \rceil - 1$. Since $\gamma_M^{-1}(G) \leq \lceil \frac{p}{8} \rceil$ and $\beta_M(G) \leq \lceil \frac{p}{4} \rceil - 1$, we get $\gamma_M^{-1}(G) \leq \beta_M(G)$. ■

Corollary 2.5.11: Let G be a cubic bipartite graph with $p \leq 13$ vertices and D be an majority dominating set of G . If $|pn[v, D']| \geq 3$, for every $v \in D'$ and $|pn[v, S]| = 3$, for atleast one vertex $v \in S$ then $\gamma_M^{-1}(G) = \beta_M(G)$, where D' and S are the inverse majority dominating set with respect to D and majority independent set of G .

Proof: Let D be a minimum majority dominating set and S be a maximal majority independent set of G . Since for atleast one vertex $v \in S$, $|pn[v, S]| = 3$, $|N(v_i) \cap N(v_j)| = 1$, for any v_i and $v_j \in S$. Then $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $|pn[v, S]| > |N[S]| - \lceil \frac{p}{2} \rceil$, for every $v \in S$.

It implies that S is also a minimal majority dominating set with minimum cardinality which is in $(V - D)$. Hence $\gamma_M^{-1}(G) \geq \beta_M(G)$. By proposition (2.5.2)(i), for any graph G , $\gamma_M^{-1}(G) \leq \beta_M(G)$. Thus, $\gamma_M^{-1}(G) = \beta_M(G)$. ■

The succeeding results are the relation between the inverse majority domination number $\gamma_M^{-1}(G)$ and majority independence number $\beta_M(G)$ for some families of graphs.

Proposition 2.5.12: If $G = K_p$ is a complete graph with p vertices, $\gamma_M(G) = \gamma_M^{-1}(G) = \beta_M(G) = 1$.

Proof: Since the graph G is complete, it is a regular graph of degree $(p - 1)$. Each vertex of G is a full degree vertex. The majority dominating set, the inverse majority dominating set and also majority independent set are all equal to any one vertex $\{v\}$ of G . Hence the result is attained. ■

Proposition 2.5.13: [33] For a wheel graph $W_p, p \geq 5$, $\beta_M(G) = \lceil \frac{p-2}{6} \rceil$, if $5 \leq p \leq 18$ and $\beta_M(G) = \lceil \frac{p-3}{4} \rceil$, if $p \geq 19$.

Theorem 2.5.14: Let $G = W_p$ be a wheel of $p \geq 5$ vertices. Then

(i) $\gamma_M^{-1}(G) = \beta_M(G) = \lceil \frac{p-2}{6} \rceil$, if $5 \leq p \leq 18$ and

(ii) $\gamma_M^{-1}(G) < \beta_M(G)$, if $p \geq 19$.

Proof: By the Proposition (2.3.4), $\gamma_M^{-1}(G) = \lceil \frac{p-2}{6} \rceil$. Then by the above result, $\gamma_M^{-1}(G) = \beta_M(G) = \lceil \frac{p-2}{6} \rceil$, if $5 \leq p \leq 18$.

Similarly when $p \geq 19$, $\gamma_M^{-1}(G) = \lceil \frac{p-2}{6} \rceil$ and by the above result we obtain, $\gamma_M^{-1}(G) < \beta_M(G)$. ■

Proposition 2.5.15: [33] For a cycle $C_p, p \geq 3$, (i) $\beta_M(G) = \lceil \frac{p}{6} \rceil$, if $3 \leq p \leq 16$ and (ii) $\beta_M(G) = \lceil \frac{p-4}{4} \rceil$, if $p \geq 17$.

Theorem 2.5.16: Let $G = C_p$, be a cycle $p \geq 3$ vertices. Then

(i) $\gamma_M^{-1}(G) = \beta_M(G) = \lceil \frac{p}{6} \rceil$, if $3 \leq p \leq 16$ and

(ii) $\gamma_M^{-1}(G) < \beta_M(G) = \lceil \frac{p-4}{4} \rceil$, if $p \geq 17$.

Proof: By the Proposition (2.3.1) and the above result, $\gamma_M^{-1}(G) = \beta_M(G) = \lceil \frac{p}{6} \rceil$, if $3 \leq p \leq 16$. Also, when $p \geq 17$, $\gamma_M^{-1}(G) = \lceil \frac{p}{6} \rceil$ and $\beta_M(G) = \lceil \frac{p-4}{4} \rceil$. Hence $\gamma_M^{-1}(G) < \beta_M(G)$. ■

Proposition 2.5.17: [33] For a path $G = P_p, p \geq 2$, $\beta_M(G) = \lceil \frac{p}{4} \rceil$, if $2 \leq p \leq 10$ and $\beta_M(G) = \lceil \frac{p-2}{6} \rceil$, if $p > 11$.

Theorem 2.5.18: Let G be a path of $p \geq 2$ vertices then $\gamma_M^{-1}(G) = \beta_M(G) = \lceil \frac{p}{6} \rceil$, if $2 \leq p \leq 10$ and $\gamma_M^{-1}(G) < \beta_M(G) = \lceil \frac{p-2}{6} \rceil$, if $p \geq 11$.

Proof: By the results (2.3.1)(2) and the above proposition, $\gamma_M^{-1}(G) = \beta_M(G) = \lceil \frac{p}{6} \rceil$, if $2 \leq p \leq 10$ and when $p \geq 11$, $\gamma_M^{-1}(G) = \lceil \frac{p}{6} \rceil < \beta_M(G) = \lceil \frac{p-2}{6} \rceil$. ■

Proposition 2.5.19: [33] Let F_p , be a fan with $p \geq 4$ vertices. Then

(i) When $p \equiv 1 \pmod{6}$, $\gamma_M^{-1}(G) < \beta_M(G)$ and

(ii) When $p \not\equiv 1 \pmod{6}$, $\gamma_M^{-1}(G) = \beta_M(G)$.

Proof: By the results (2.3.1)(4) and the above proposition, $\gamma_M^{-1}(G) = \lceil \frac{p-1}{6} \rceil$ and $\beta_M(G) = \lceil \frac{p}{6} \rceil$. When $p \equiv 1 \pmod{6}$, $\lceil \frac{p-1}{6} \rceil < \lceil \frac{p}{6} \rceil$. Then $\gamma_M^{-1}(G) < \beta_M(G)$. When $p \equiv 0, 2, 3, 4, 5 \pmod{6}$, $\lceil \frac{p-1}{6} \rceil = \lceil \frac{p}{6} \rceil$.

Thus $\gamma_M^{-1}(G) = \beta_M(G)$, if $p \not\equiv 1 \pmod{6}$. ■

Proposition 2.5.20: If the cubic graph G is a Generalized Petersen $P(n, k)$ graph then $\gamma_M^{-1}(G) = \beta_M(G)$.

Proof: Since G is a 3-regular graph and by the result (1.4), we have $\beta_M(G) = \lceil \frac{p}{8} \rceil$. Also by the results (2.3.1)(8), for a Generalized Petersen graph $P(n, k)$, $\gamma_M^{-1}(G) = \lceil \frac{p}{8} \rceil = \beta_M(G)$. ■

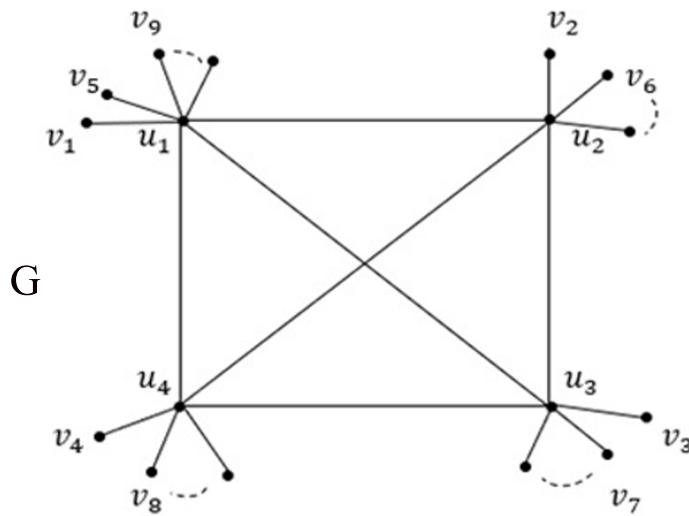
Proposition 2.5.21: Let $G = K_{1,p-1}$ be a star with $p \geq 2$ vertices.

Then $\gamma_M^{-1}(G) = \beta_M(G)$.

Proof: By the results (2.3.1)(5) and [1.4], $\gamma_M^{-1}(G) = \lfloor \frac{p-1}{2} \rfloor = \beta_M(G)$. ■

Results 2.5.22: There exists a graph G for which $\gamma_M(G) = \gamma_M^{-1}(G) = 2$ and $\beta_M(G) = 2t = \lceil \frac{p}{2} \rceil - \gamma_M^{-1}(G)$ where $t \geq 3$.

Proof: The graph G is obtained by adding one pendant at each vertex of a complete graph and then add a pendant each time at each vertex of K_4 . Finally we acquire a new structure with $p = 4 + 4t$, where t is the number of pendants at each time on each vertex of K_4 .



Let $|V(G)| = p = 4 + 4t$ when $t = 1$ then the vertex set $V(G) = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ where $(u_1, u_2, u_3, u_4) \subseteq V(K_4)$ and other vertices are pendants with $p = 8$.

Then $D = \{u_1\}$ and $D' = \{u_2\}$ and $S = \{v_1, v_2\}$ are majority dominating set, inverse majority dominating set and majority independent set of G . It implies that $\gamma_M(G) = \gamma_M^{-1}(G) = 1$ and $\beta_M(G) = 2$ where v_1 and v_2 are pendants which are adjacent to u_1 and u_2 in G . When $t = 2, p = 4 + 8 = 12$ then $\gamma_M(G) = \gamma_M^{-1}(G) = 1$ and $\beta_M(G) = |\{v_1, v_2, v_5, v_6\}| = 4 = 2t$. When $t = 3, p = 4 + (4 \times 3) = 16$.

Now, in the graph G , there are 3 pendants at each vertex of K_4 . Let $D = \{u_1, u_2\}$ be a majority dominating set of G and $D' = \{u_3, u_4\} \subseteq V - D$ is an inverse majority dominating set of G . Therefore $\gamma_M(G) = \gamma_M^{-1}(G) = 2$. The set $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}\}$ such that $|N[S]| = 8$ and since $|pn[v, S]| = 1$, S satisfies the condition $|pn[v, S]| > |N[S]| - \lceil \frac{p}{2} \rceil$ for every $u_i \in S$ and $\beta_M(G) = |S| = 6 = 2t$, if $t = 3$ and so on. Therefore $\lceil \frac{p}{2} \rceil - \gamma_M^{-1}(G) = 8 - 2 = 6 = \beta_M(G)$ if $t = 3$. Thus, $\beta_M(G) = 2t = \lceil \frac{p}{2} \rceil - \gamma_M^{-1}(G)$.

In general, in this structure has the difference between $\gamma_M^{-1}(G)$ and $\beta_M(G)$ is very large. Hence $\gamma_M^{-1}(G) < \beta_M(G)$ and the difference between these two numbers is very large when p is large and $t \geq 3$. ■

2.6 Algorithm and Application for an Inverse Majority Dominating Set

2.6.1 Algorithm for an Inverse Majority Dominating Set of a graph G

To find an Inverse Majority Dominating set for the given simple, connected and undirected graph G with p vertices and q edges.

Step 1: Find the adjacency matrix $[A(G)]_{p \times p}$ for the given graph G .

Step 2: Change all diagonal entries of $A(G)$ with the value one

$$\begin{array}{cccccc}
 & u_1 & u_2 & \dots & u_p & |N[u_i]| \\
 \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ \cdot \\ \cdot \\ u_p \end{array} & \begin{bmatrix} 1 & 1 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ 0 & 1 & \dots & 1 \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ 0 & 1 & \dots & 1 \end{bmatrix} & \Delta(G) & \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}
 \end{array}$$

Step 3: Let D be a minimum majority dominating set of G such

that $|N[D]| \geq \lceil \frac{p}{2} \rceil$.

Step 4: Initially choose the set $D' \subseteq V - D$ and $D' = \emptyset$.

Step 5: Next, choose the maximum degree vertex $u_i \in (V - D)$ in the column $|N[u_i]|$ of $A(G)$. If $|N[u_i]| \geq \lceil \frac{p}{2} \rceil$ then $D' = \{u_i\}$ is an inverse majority dominating set of G with respect to D . If $|N[u_i]| < \lceil \frac{p}{2} \rceil$ then go to step 6.

Step 6: Choose next maximum degree vertex u_j in the zero entry of row of the vertex u_i such that $d(u_i, u_j) \geq 3$. Now consider the set $D' = \{u_i, u_j\} \subseteq V - D$.

Step 7: If $|N[D']| \geq \lceil \frac{p}{2} \rceil$ then D' is an inverse majority dominating set of G with respect to D . If not, go to step (6) and apply the procedure to get an inverse majority dominating set D' of G with respect to the majority dominating set D of G .

2.6.2 Algorithm for $\gamma_M^{-1}(G)$

Using algorithm (2.6.1), find out all the inverse majority dominating sets for the given graph G .

Step 1: Let $\mathcal{D}' = \{D'_1, D'_2, \dots, D'_t\}$ be the set of all inverse majority dominating set of G with respect to D .

Step 2: Verify that the proper subset D''_i of each D'_i , for $D'_i \in \mathcal{D}'$ is an inverse majority dominating set of G .

Step 3: If the proper subset D''_i in any D'_i is an inverse majority dominating set of G with respect to D then the set D'_i is not a minimal inverse majority dominating set of G .

Step 4: Suppose there exists no such inverse majority dominating subset D''_i in the set D'_i then D'_i is a minimal inverse majority dominating set of G .

Step 5: Repeat the process to every $D'_i \in \mathcal{D}'$ and collect all the minimal inverse majority dominating sets for G with respect to D .

Step 6: Let $\mathcal{D}' = \{D'_1, D'_2, \dots, D'_r\}$, $r \leq t$ be the set of all minimal inverse majority dominating set of G with respect to the majority dominating set D of G .

Step 7: Find the cardinality of each set $D'_i \in \mathcal{D}'$, for $i = 1, 2, \dots, r$. Pick up the minimum and maximum cardinality of $D'_i \in \mathcal{D}'$ among all $D'_i, i = 1, 2, \dots, r$.

Step 8: The minimum cardinality of $D'_i = \gamma_M^{-1}(G)$ = inverse majority domination number of G and the maximum cardinality of $D'_i = \Gamma_M^{-1}(G)$ = upper inverse majority domination number of G .

2.6.3 Application for $\gamma_M^{-1}(G)$

Let D be a set of ministers of administering party and $(V - D)$ be a set of rest of administering party and adversary party. Presently, the elements in opponent party will attempt to take advantage of the circumstances by drawing in or convincing few individual members from the rest of ruling party who might be denied the Portfolio (or) Ministership. In the event that a few individuals move from dis-appointed or denied administering party to the adversary party on conspiracy, the opponent party may gain simple majority and they will move for no confidence motion and thereby defeating the ruling party. Then the rival party with the support of disagreeing and existing individuals from ruling party may prove majority and form a new government which may be called an inverse majority dominating set D' in $(V - D)$.

Chapter 3

$\gamma_M^{-1}(G)$ for Some Special Graphs

Abstract

This chapter includes majority domination number and inverse majority domination number for some special classes of graphs and its subdivision $S(G)$ graphs. Some results on the subdivision graphs $S(G)$ are discussed. The results on $\gamma_M^{-1}(G - u)$, the deletion of a vertex u from G are studied. The relationship between majority domination number and inverse majority domination number for splitting graph namely $\gamma_M(S_p(G))$ and $\gamma_M^{-1}(S_p(G))$ is determined. Also, inverse majority domination number for a complement $\bar{G}$ namely $\gamma_M^{-1}(\bar{G})$ of G are found for some classes of graphs.

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3.1 Introduction

Any graph model of a system provides a powerful means to understand the topology of any structure. In this chapter, some graph structures from a given graph G such as subdivision $S(G)$ of a graph G , deletion of a vertex u from G , splitting graph $S_p(G)$ of G and the complement $\bar{G}$ of a graph G are discussed.

An edge subdivision of an edge $e = uv$ in graph G is the graph obtained from G by replacing e by a path $\langle u, w, v \rangle$ where w is a new vertex of degree two. A refinement of G is a graph obtained from G by a finite number of subdivisions. This property of this graph structure is studied by many mathematicians and many results are produced in terms of graph theoretic parameters.

A vertex or an edge is critical with respect to an invariant if the removal of a vertex or an edge changes the value of the invariant. In 1979, Walikar and Acharya [54] studied this invariant in terms of domination and some results were produced in their research article. In 1985, Sampath Kumar and Neeralagi [47] have studied the relationship among such types of elements where $t = \gamma$. Since t can decrease by at most one when a vertex v is deleted, it follows that v is t -critical if and only if $t(G-v) = t(G) - 1$.

In 1980, Splitting graph $S_p(G)$ was introduced by Sampath Kumar and Walikar [49]. For each vertex v of a graph G , take a new vertex v' and join v' to all vertices of G adjacent to v . Based on some applications, many results were determined using this graph on domination parameter. This graph structure motivated the researcher to find its majority domination number and an inverse majority domination number.

In 2015, an inverse complementary domination graph was introduced by Pethanachi Selvam and Padmashiri [46] for any graph G , one could find a complement graph $\bar{G}$. Thence finally inverse majority domination number $\gamma_M^{-1}(\bar{G})$ is determined for standard graphs.

The organization of this Chapter as follows. Section 3.1 is the introductory part for various special classes of graph and Section 3.2 defines an inverse majority domination number on subdivision graphs of some special graph structures. In Section 3.3 some results on $\gamma_M(G')$ and $\gamma_M^{-1}(G')$ are established where G' is the subdivision graph of G , $\gamma_M^{-1}(G - u)$, for some vertex $u \in V(G)$ is studied in Section 3.4. Majority domination number and an inverse majority domination number for some families of splitting graph are also discussed in

Section 3.5. Finally, an inverse majority domination number for a complement graph $\bar{G}$ of a graph is determined.

3.2 Inverse Majority Domination Number on Subdivision Graphs

In this section, subdivision graphs $S(G) = G'$ of some special graph structures G are considered and an inverse majority dominating set for such structures G' are determined. Then majority domination number $\gamma_M(G)$, Inverse majority domination number $\gamma_M^{-1}(G)$, majority domination number for the subdivision graph G' of $G = \gamma_M(G')$ and Inverse Majority Domination number of $G' = \gamma_M^{-1}(G')$ are all established in the following propositions.

Proposition 3.2.1: Let G be the Dodecahedron graph and $G' = S(G)$ be the subdivision graph. Then $\gamma_M(G) = 3 = \gamma_M^{-1}(G)$ and $\gamma_M(G') = 7 = \gamma_M^{-1}(G')$.

Proof: Let G be the platonic solid dodecahedron and G is a 3-regular graph with $p = 20$ and $q = 30$. Let $D = \{v_1, v_4, v_6\}$ such that $d(v_i, v_j) = 3$. Then $|N[D]| = 11 > \lceil \frac{p}{2} \rceil$. Hence D is a majority dominating set of G and $\gamma_M(G) = 3$. Let $D' = \{v_2, v_5, v_8\} \subseteq V - D$,

such that $d(v_i, v_j) = 3$. By the above similar argument, Dx' is an inverse majority dominating set of G and $\gamma_M^{-1}(G) = 3$.

Let G' be a subdivision graph of dodecahedron with $p' = 50$ and $V(G') = \{v_1, \dots, v_{20}, u_1, u_2, \dots, u_q\}$ with $d(v_i) = 3, i = 1, 2, \dots, 20$ and $d(u_i) = 2, i = 1, 2, \dots, 30$. Let $S = \{v_1, v_3, v_6, v_7, v_{11}, v_{13}, v_{17}\}$ such that $d(v_i, v_j) = 4$, for every i, j and $i \neq j$. Then $|N[S]| = 28 > \lceil \frac{p}{2} \rceil$. It implies that S is a majority dominating set of G' and $\gamma_M(G') = |S| = 7$. Next choose $S' = \{v_2, v_4, v_5, v_7, v_8, v_{15}, v_{18}\}$ such that $d(v_i, v_j) = 4$ and $S' \subseteq V - S$. By the above similar argument, S' is an inverse majority dominating set of G and $\gamma_M^{-1}(G') = 7$. Hence $\gamma_M(G') = 7 = \gamma_M^{-1}(G')$. ■

Proposition 3.2.2: Let G be the Tetrahedron graph and Octahedron graph. Then $\gamma_M(G) = 1 = \gamma_M^{-1}(G)$ and $\gamma_M(G') = 2 = \gamma_M^{-1}(G')$, where G' is the subdivision of G .

Proof: The proof is obvious. ■

Proposition 3.2.3: Let G be an icosahedral graph and $S(G)$ be the subdivision graph of G . Let $G' = S(G)$. Then $\gamma_M(G) = 1 = \gamma_M^{-1}(G)$ and $\gamma_M(G') = 4 = \gamma_M^{-1}(G')$.

Proof: Let G be an icosahedral graph with p vertices and it is a 5-regular graph. Since each vertex dominates six vertices, $\gamma_M(G) = 1 = \gamma_M^{-1}(G)$.

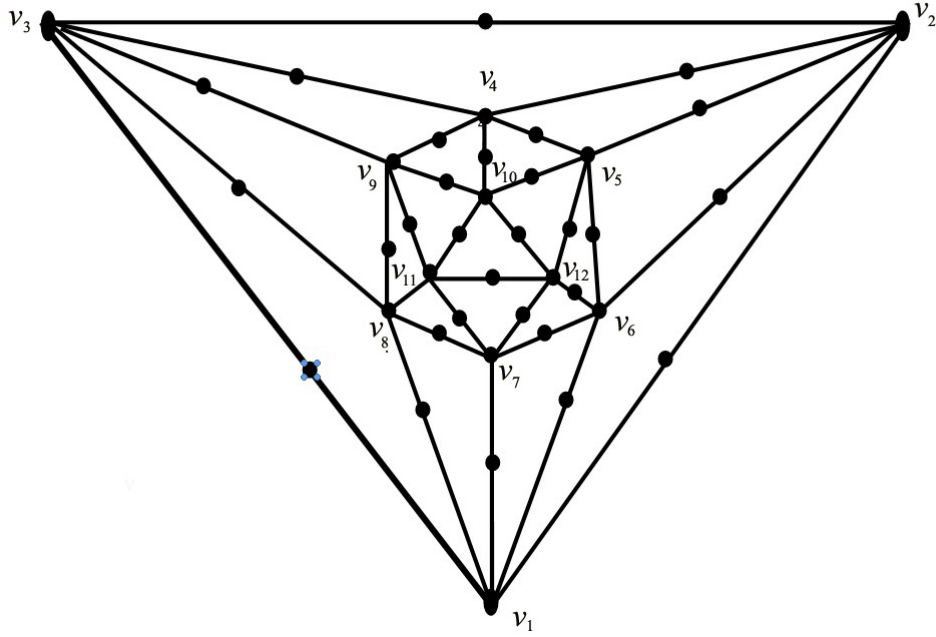


Figure 3.1: Subdivision of $G - S(G)$

Let G' be the subdivision graph of icosahedral with $p' = 42$. and $V(G') = \{v_1, v_2, \dots, v_{12}, u_1, u_2, \dots, u_{30}\}$ Now G' is not a regular graph with $d(v_i) = 5$, for all $i = 1, 2, \dots, 12$ and $d(u_i) = 2$, for all $i = 1, 2, \dots, 30$. Let $S = \{v_1, v_4, v_{11}, v_{12}\}$ such that $d(v_i, v_j) = 4$. Then $|N[S]| = \sum_{i=1}^S d(v_i) + |S| = [(3 \times 5) + 4 + 4] > \lceil \frac{p}{2} \rceil$. It implies that S is a majority dominating set of G' and $\gamma_M(G') = 4$. Choose $S' = \{v_2, v_7, v_9, v_{10}\} \subseteq V - S$. By the similar argument, $\gamma_M^{-1}(G') = 4$. ■

Proposition 3.2.4: Let G be a Frucht graph and $G' = S(G)$. Then $\gamma_M(G) = 2 = \gamma_M^{-1}(G)$ and $\gamma_M(G') = 4 = \gamma_M^{-1}(G')$.

Proof: The proof is obvious. ■

Proposition 3.2.5: Let G be a Doyle graph. Then $\gamma_M(G) = 3 = \gamma_M^{-1}(G)$. If $G' = S(G)$ is the subdivision of G then $\gamma_M(G') = 8 = \gamma_M^{-1}(G')$.

Proof: Let G be a Doyle graph and G is a 4-regular graph. The vertex set $V(G)$ can be partitioned into three vertex sets V_1, V_2 and V_3 each comprising of 9 vertices with degree 4 and V_1, V_2 and V_3 are vertices of a outer cycle C_1 , inner circle C_2 , innermost cycle C_3 respectively. Let $D = \{v_1, v_4, v_7\}$ and $D' = \{v_2, v_5, v_8\} \subseteq V - D$. Since G is a 4-regular graph, $|N[D']| = \sum d(v_i) + 3 = 15 > \lceil \frac{p}{2} \rceil$. Hence the sets D and D' are the majority dominating set and the inverse majority dominating set of G with respect to D respectively and $\gamma_M(G) = \gamma_M^{-1}(G) = 3$.

Let G' be the subdivision graph of a Doyle graph with $p' = 72$ vertices. Let $V(G') = \{v_1, v_2, \dots, v_{27}, u_1, u_2, \dots, u_{45}\}$ where $d(v_i) = 4$, for $i = 1$ to 27 and $d(u_j) = 2$, for $j = 1$ to 45 and $V_1(G') =$

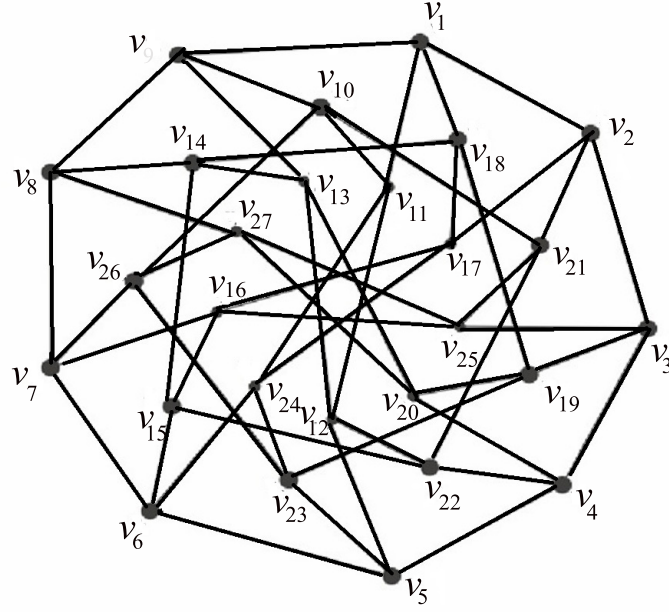


Figure 3.2: G:Doyle graph

$\{v_1, \dots, v_9\}$, $V_2(G') = \{v_{10}, \dots, v_{18}\}$, $V_3(G') = \{v_{19}, \dots, v_{27}\}$ and $U(G') = \{u_1, \dots, u_{45}\}$. Let $S = \{v_1, v_4, v_7, v_{13}, v_{17}, v_{18}, u_{20}, u_{25}\}$ such that $d(v_i, v_j) = 3$, for $i \neq j$ and $v_i, v_j \in S$. Then $|N[S]| = \sum_{v_i \in S} d(v_i) + |S| = (8 \times 4) + 8 = 40 > \lceil \frac{p}{2} \rceil$. Hence S is a majority dominating set of G' and $\gamma_M(G') = |S| = 8$. In $V - S$, choose $S' = \{v_2, v_5, v_8, v_{10}, v_{14}, v_{15}, v_{22}, v_{26}\}$ and $|N[S']| = 40 > \lceil \frac{p}{2} \rceil$. It implies that S' is an inverse majority dominating set of G' and $\gamma_M^{-1}(G') = |S'| = 8$. Hence $\gamma_M(G') = \gamma_M^{-1}(G') = 8$. ■

Proposition 3.2.6: (i) Let G be a Folkman graph. Then $\gamma_M(G) = 2 = \gamma_M^{-1}(G)$, $\gamma_M(G') = 6$ and $\gamma_M^{-1}(G') = 8$, where G' is the subdivision graph of G .

Proof: Let G be a Folkman graph and it is a bipartite, 4-regular, Hamiltonian graph and a four edge connected perfect graph. Let $D = \{u_1, v_1\}$ and $D' = \{u_3, v_3\} \subseteq V - D$ such that $d(u_i, v_j) \geq 4$. Since each vertex dominates 5 vertices, $|N[D]| = |N[D']| = 10 = \lceil \frac{p}{2} \rceil$ and $\gamma_M(G) = \gamma_M^{-1}(G) = 2$.

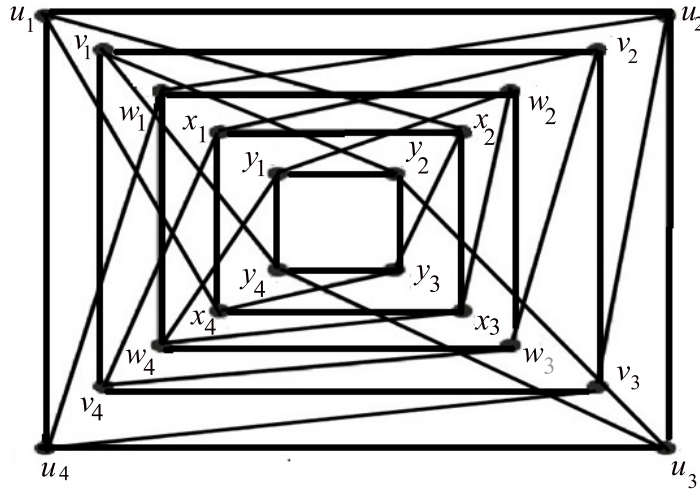


Figure 3.3: G :Folkman graph

Let G' be a subdivision graph of a Folkman graph with the vertex set $V(G') = \{u_1, \dots, u_4, v_1, \dots, v_4, w_1, \dots, w_4, x_1, \dots, x_4, y_1, \dots, y_4, z_1, z_2, \dots, z_{40}\}$ and $|V(G')| = 60$, where the vertices u_i, v_i, w_i, x_i and y_i are in the outer square to inner square of totally 5 squares in G' and $z_i, i = 1, \dots, 40$ denotes the newly added vertices in G . Let $S = \{u_1, v_1, w_1, x_1, y_1, y_3\}$ in which all are non-adjacent vertices in G' and $|N[S]| = \sum d(v_i) + 6 = 30 = \lceil \frac{p}{2} \rceil$. Hence S is a majority dominating set of G' and $\gamma_M(G') = |S| = 6$. Now choose $S' =$

$\{u_3, v_3, w_3, x_3, u_4, v_4, w_4, x_4\} \subseteq V - S$ and $|N[S']| = \sum_{v_i \in S'} d(v) + 8 = 32 > \lceil \frac{p}{2} \rceil$. It implies that S' is an inverse majority dominating set of G' and $\gamma_M^{-1}(G') = |S'| = 8$. Hence $\gamma_M^{-1}(G) < \gamma_M^{-1}(G')$. ■

Proposition 3.2.7: Let G be a Levi graph and $G' = S(G)$. Then $\gamma_M(G) = 4 = \gamma_M^{-1}(G)$ and $\gamma_M(G') = 9 = \gamma_M^{-1}(G')$.

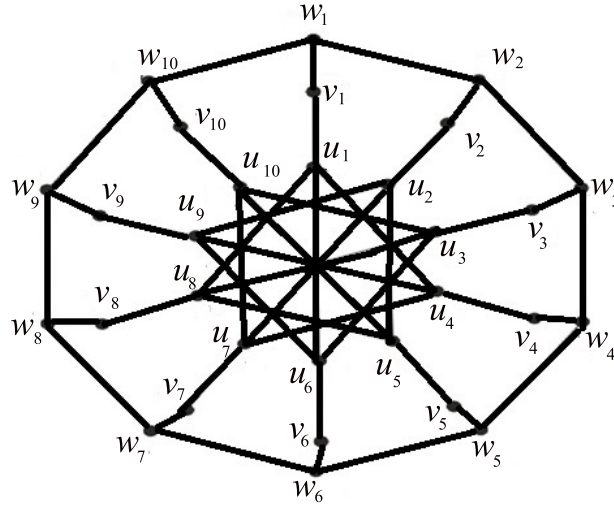


Figure 3.4: G: Levi graph

Proof: Let G be a Levi graph with $p = 30$, $q = 45$ and it is not a regular graph. Let $V(G) = \{w_1, w_2, \dots, w_{10}, v_1, \dots, v_{10}, u_1, \dots, u_{10}\}$ in which the vertices w_i, v_i and u_i , for $i = 1, \dots, 10$ from a outer circle to inner circle and $d(w_i) = 3, d(v_i) = 2, d(u_i) = 4, i = 1, \dots, 10$. Let $D = \{u_1, u_3, w_2, w_5\}$ and $|N[D]| = \sum d(u_i) + \sum d(w_i) + 4 = 18 > \lceil \frac{p}{2} \rceil$. Hence D is a majority dominating set of G and $\gamma_M(G) = |D| = 4$.

Next, choose $D' = \{u_5, u_7, w_5, w_8\} \subseteq V - D$. Then $|N[D']| = 18 > \lceil \frac{p}{2} \rceil$ and D' is a inverse majority dominating set of G . Hence $\gamma_M^{-1}(G) = 4 = \gamma_M(G)$.

Let G' be the subdivision graph of a Levi graph G with $p' = 75$. $V(G') = \{w_1, \dots, w_{10}, v_1, \dots, v_{10}, u_1, \dots, u_{10}, x_1, x_2, \dots, x_{45}\}$ where $d(w_i) = 3, d(v_i) = 2, d(u_i) = 4$ for $i = 1$ to 10 and $d(x_j) = 2, j = 1$ to 45 . Let $S = \{u_1, u_2, u_3, u_9, w_1, w_3, w_5, w_7, w_9\} \subseteq V(G')$ and $|N[S]| = \sum_i d(u_i) + \sum_i d(w_i) + |S| = 40 > \lceil \frac{p'}{2} \rceil$. It implies that $\gamma_M(G') = |S| = 9$. Next, choose $S' = \{u_4, u_5, u_6, u_{10}, w_2, w_4, w_6, w_8, w_{10}\} \subseteq V - S$. By the above calculations, $|N[S']| = 40 > \lceil \frac{p'}{2} \rceil$. Hence S' is an inverse majority dominating set of G' and $\gamma_M^{-1}(G') = |S'| = 9$. Thus, $\gamma_M(G') = 9 = \gamma_M^{-1}(G')$. ■

Proposition 3.2.8: Let G be Platonic Solid Cube and $G' = S(G)$ be the subdivision of G . Then $\gamma_M(G) = 1 = \gamma_M^{-1}(G)$ and $\gamma_M(G') = 3 = \gamma_M^{-1}(G')$.

Proof: The proof is obvious. ■

Proposition 3.2.9: Let $G = S(C_p)$ be a subdivision graph of a cycle with p vertices. Then $\gamma_M(G) = \gamma_M^{-1}(G) = \lceil \frac{p}{3} \rceil$.

Proof: Let $G = S(C_p)$ and $V(G) = \{u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_p\}$ and $|V(G)| = 2p$. Then $S(C_p)$ is also a cycle with $2p$ vertices. By the result (1.4), Proposition (2.3.1), $\gamma_M(C_p)$ and $\gamma_M^{-1}(C_p) = \lceil \frac{p}{6} \rceil$. Since $S(C_p)$ is a cycle, $\gamma_M^{-1}(G) = \lceil \frac{2p}{6} \rceil = \lceil \frac{p}{3} \rceil$. Thus $\gamma_M(G) = \gamma_M^{-1}(G) = \lceil \frac{p}{3} \rceil$. ■

Proposition 3.2.10: If G is a subdivision of a complete graph K_p then $\gamma_M(G) = 2$ and $\gamma_M^{-1}(G) = 2$ where $G = S(K_p)$.

Proof: Since K_p is complete graph, there are p vertices and $\frac{p(p-1)}{2} = q$ edges. Let $G = S(K_p)$ with $V(G) = \{u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_q\}$ and $|V(G)| = p' = (p + q)$, where $d(u_i) = p - 1$, for every $u_i, \dots, u_p$ and $d(v_i) = 2$, for every $v_i, \dots, v_q$ and $p' = p + \frac{p(p-1)}{2} = \frac{p(p+1)}{2}$ and $\lceil \frac{p'}{2} \rceil = \lceil \frac{p(p+1)}{4} \rceil$. Let $D = \{u_1, u_3\}$ such that $d(v_i, v_j) = 4$. Then $|N[D]| = (p - 1) + (p - 2) + 2 = 2p - 1 > \lceil \frac{p'}{2} \rceil$. It implies that D is a majority dominating set of G and $\gamma_M(G) = |D| = 2$. Let $D' = \{u_2, u_4\} \subseteq V - D$ such that $d(v_2, v_4) = 4$. Then $|N[D']| = 2p - 1 > \lceil \frac{p'}{2} \rceil$ and $\gamma_M^{-1}(G) = |D'| = 2$. Hence $\gamma_M(G) = 2 = \gamma_M^{-1}(G)$. ■

Proposition 3.2.11: Let $G = S(P_p)$ be a subdivision graph of a path with $p \geq 2$ vertices. Then $\gamma_M^{-1}(G) = \lceil \frac{p}{3} \rceil$.

Proof: Since the subdivision graph of path P_p is also a path, the result is obvious. ■

Proposition 3.2.12: Let $G = S(W_p)$ be a subdivision graph of a wheel. Then $\gamma_M^{-1}(G) = \gamma_M(G) = \lceil \frac{p'}{8} \rceil$, where $p' = (p + q)$.

Proof: Let $G = S(W_p)$ and $V(G) = \{u, u_1, u_2, \dots, u_{p-1}, v_1, v_2, \dots, v_q\}$ and $|V(G)| = p + q = p'$, where u is a central vertex $d(v_i) = 2$, for every $v_i, i = 1, \dots, q$ and $d(u_i) = 3$, for every $u_i, i = 1, 2, \dots, p - 1$. Let $D = \{u_1, u_3, \dots, u_t\}$ such that $t = \lceil \frac{p'}{8} \rceil$ and $d(u_i, u_j) \geq 4$, for $i \neq j$. Then $|N[D]| = \sum_{i=1}^t d(v_i) + t = 4t = 4\lceil \frac{p'}{8} \rceil \geq \lceil \frac{p'}{2} \rceil$. It implies that D is a majority dominating set of G and $\gamma_M(G) = |D| = \lceil \frac{p'}{8} \rceil$.

Now, choose the set $D' = \{u_2, u_5, \dots, u_t\} \subseteq V - D$ such that $t = \lceil \frac{p'}{8} \rceil$ and $d(u_i, u_j) \geq 4$, for $i \neq j$. By the above argument, the set D' is an inverse majority dominating set of G and $\gamma_M^{-1}(G) \leq |D'| = \lceil \frac{p'}{8} \rceil$. Suppose $D_1 \subseteq V - D$ be a set with $|D_1| < |D'| = \lceil \frac{p'}{8} \rceil$. Then $|N[D_1]| < \lceil \frac{p'}{2} \rceil$. It implies that D_1 is not an inverse majority dominating set of G and $\gamma_M^{-1}(G) \geq \lceil \frac{p'}{8} \rceil$. Hence $\gamma_M^{-1}(G) = \lceil \frac{p'}{8} \rceil$. ■

Proposition 3.2.13: If $G = S(D_{r,s})$ is a subdivision graph of a double star then $\gamma_M(G) = 2$ and

$$\gamma_M^{-1}(G) = \begin{cases} r+1, & \text{if } r = s \\ r+t, & \begin{cases} \text{if } r < s \text{ and } s = r+i, \\ t=1 \text{ and } i=(0,1), t=2 \text{ and } i=(2,3), \dots \end{cases} \end{cases}$$

Proof: Let $G = S(D_{r,s})$ and $V(G) = \{u, v, u_1, u_2, \dots, u_q, v_1, v_2, \dots, v_{r+s}\}$ and $|V(G)| = p' = p + q = 2(r + s) + 3$. Let u and v be the two central vertices with $(u_1, u_2, \dots, u_q)$ of $\deg(u_i) = 2$ and $(v_1, v_2, \dots, v_{r+s})$ are pedants in G .

Case (i): When $r = s$.

Since the graph $D_{r,s}$ has exactly two majority dominating vertices u and v the subdivision graph of $D_{r,s}$ has the same two vertices u and v with $d(u) = d(v) = r + 1$ but u and v not majority dominating vertices in $S(D_{r,s})$.

Let $D = \{u, v\}$ with $d(u, v) = 2$. Then $|N[D]| = d(u) + d(v) + 1 = r + s + 3 = 2r + 3 \geq \lceil \frac{p'}{2} \rceil$, if $r = s$. Therefore $D = \{u, v\}$ is the minimum majority dominating set of G and $\gamma_M(G) = 2$. Now, choose $D' = \{u_1, \dots, u_t\} \subseteq V - D$ with $|D'| = t = r + 1$ and $d(u_i) = 2$, for every u_i . Then $|N[D']| = \sum_{i=1}^t d(u_i) + 1 = 2t + 1 = 2(r + 1) + 1 =$

$2r + 3 \geq \lceil \frac{p'}{2} \rceil$. It implies that D' is an inverse majority dominating set of G with respect to D and $\gamma_M^{-1}(G) \leq |D'| = r + 1$. Suppose $D_1 < D'$ then $|N[D_1]| < \lceil \frac{p'}{2} \rceil$ and $\gamma_M^{-1}(G) > D_1 \geq |D'| = r + 1$ and $D_1 \subseteq V - D$. Hence $\gamma_M^{-1}(G) = r + 1$, if $r = s$.

Case (ii): When $r < s$ and $s = r + 1, r + 2, \dots$

By case (i), $\gamma_M(G) = 2$. Let $D' = \{u_{r+1}, u_{r+2}, \dots, u_{r+t}\} \subseteq V - D$ with $|D'| = t$. Since $d(u_i) = 2, |N[D']| = 2(r + t) + 1$.

Subcases:

a. If $s = r + i$ and $i = 0$ and 1 then $t = 1$ and $p' = 2p - 1$. Now,

$|N[D']| = 2r + 3 = \lceil \frac{p'}{2} \rceil$, for $i = 0 \& 1$ and $t = 1$. Hence D' is an inverse majority dominating set of G .

b. If $s = r + i, i = 2$ and 3 then $t = 2$ and $p' = 4r + 7$ i.e.,

$[p' = 2(r + r + 2 + 2) - 1]$. Therefore $|N[D']| = 2(r + 2) + 1 = 2r + 5 > \lceil \frac{p'}{2} \rceil$ and so on...

Hence D' is an inverse majority dominating set of G if $r < s$ and $s = r + i$, where $i = (0, 1), (2, 3), \dots$ and when $t = 1, 2, \dots$

It implies that $\gamma_M^{-1}(G) = |D'| = r + t$ if $t = 1 \& i = (0, 1), t = 2 \& i = (2, 3), \dots$ ■

3.3 Results on $\gamma_M(G')$ and $\gamma_M^{-1}(G')$

The following results provide some properties of a subdivision graph $G' = S(G)$ of G and the relationship between the numbers $\gamma_M(G')$ and $\gamma_M^{-1}(G')$ is also discussed.

Observation 3.3.1:

- 1) A full degree vertex of a graph becomes a majority dominating vertex in the subdivision graph $S(G)$ of G .
- 2) The degree of each vertex v_i of G will never change in the subdivision graph $S(G)$ and the degree of a newly added vertex is always two in $S(G)$.
- 3) The regular graph G is not a regular in $S(G)$ except $G = C_p$.
- 4) Any pendant edge becomes a path K_2 in $S(G)$.

Proposition 3.3.2: Let G and G' be the disconnected graph and its Subdivision graph $S(G)$ with p and p' vertices respectively. Then $\gamma_M(G') = \gamma_M^{-1}(G')$ where $G' = S(G)$.

For Example: Let $G = 5K_2$. Since $\gamma_M(G) = \lceil \frac{p}{4} \rceil$, $\gamma_M(G) = 3 = \gamma_M^{-1}(G) = 3$. Let the subdivision graph $S(G) = G' = 5P_3$ with $p' = 15$. Now, the set $D = \{u_1, u_2, u_3\}$, where $d_{G'}(u_i) = 2$, with

$|N[D]| > \lceil \frac{p'}{2} \rceil$ and D is a majority dominating set of G' and $\gamma_M(G') =$

3. $D' = \{u_4, u_5, v_1\} \subseteq V - D$ is an inverse majority dominating set of G' and $\gamma_M^{-1}(G') = 3 = \gamma_M(G')$.

(ii) Let $G = 5K_3$ with $p = 15$ and $V(G) = \{v'_1, v'_2, v'_3, \dots, v'_1, v'_2, v'_3\}$.

Then $D = \{v_1^1, v_1^2, v_1^3\}$ and $D' = \{v_2^1, v_2^2, v_2^3\} \subset (V - D)$ are the majority dominating set and an inverse majority dominating set of

G and $\gamma_M^{-1}(G) = \gamma_M(G) = 3$. Let $G' = 5C_6$ be the subdivision graph of G with $p' = 30$. Then $\{u_1^1, u_1^2, u_1^3, \dots, u_1^5, u_2^5, u_3^5\}$ is the newly

added vertex set of G . Let $D = \{u_1^1, u_1^2, u_1^3, u_1^4, u_1^5\}$ with each u_1^i belongs to each C_6 and $1 \leq i \leq 5$. Then $|N[D]| = \lceil \frac{p'}{2} \rceil$ and D is

a majority dominating set of G' , $\gamma_M(G') = |D| = 5$. Now the set

$D' = \{u_2^1, u_2^2, u_2^3, u_2^4, u_2^5\} \subseteq V - D$ is an inverse majority dominating set of G' and $\gamma_M^{-1}(G') = |D'| = 5$. Hence $\gamma_M(G') = \gamma_M^{-1}(G')$.

Proposition 3.3.3: If the graph G is regular then $\gamma_M(G') = \gamma_M^{-1}(G')$

where $G' = S(G)$ is the subdivision graph of G .

Proof: Since G is a regular graph and by the Proposition (2.4.8),

$\gamma_M(G) = \gamma_M^{-1}(G)$. Let $G' = S(G)$ be the subdivision graph of G .

Then G' is not a regular graph except $G = C_p$, a cycle. Since the

degree of the vertices of G is equal to the degree of the vertices of

G' except the newly added vertex u_i with $d(u_i) = 2, i = 1, 2, \dots, q$. To get the minimality select the vertices v_i of G with the distance $d(v_i, v_j) \geq 4$ in G' then it will form a minimum majority dominating set and minimum inverse majority dominating set in G' with the same cardinality. Hence $\gamma_M(G') = \gamma_M^{-1}(G')$. ■

Theorem 3.3.4: If a graph G with $p > 8$ vertices has a full degree vertex and others vertices v_i are of degree $d(v_i) \leq 3$ then $\gamma_M(G') < \gamma_M^{-1}(G')$ where G' is the subdivision graph of G .

Proof: Since any full degree vertex v of G becomes a majority dominating vertex v of G' , the set $D = \{v\}$ is a majority dominating set of G' and $\gamma_M(G') = 1$. Since $d_{G'}(v_i) \leq 3$, for $i = 1, 2, \dots, p-1$ and the newly added vertex u_i such that $d_{G'}(v_i) = 2, i = 1, 2, \dots, q$, the vertices of $v_i \subseteq V - D$ will form an inverse majority dominating set D' with $|D'| > 1$ of G' with respect to D . It implies that $\gamma_M^{-1}(G') > 1$. Hence $\gamma_M(G') < \gamma_M^{-1}(G')$. ■

Corollary 3.3.5: If the graph G contains a full degree vertex and others are pendants then $\gamma_M(G') < \gamma_M^{-1}(G')$ where $G' = S(G)$.

Proof: By the observation (3.3.1), since any full degree vertex v of G becomes a majority dominating vertex v of G' , $D = \{v\}$ is a majority dominating set of G' and $\gamma_M(G') = 1$. Since G has pendant vertices, $d(u_i) = 2$, for $i = 1, 2, \dots, q$ then the vertices u_i of $G' \subseteq V - D$ will form an inverse majority dominating set D' with $|D'| > 1$ of G' with respect to D . It implies that $\gamma_M^{-1}(G') > 1$ and $\gamma_M(G') < \gamma_M^{-1}(G')$. ■

Proposition 3.3.6: If the graph G has exactly one majority dominating vertex and other are pendants then $\gamma_M(G') > 1$ and $\gamma_M(G') < \gamma_M^{-1}(G')$ where G' is the subdivision graph of G .

Proof: Let $|V(G)| = p$ and $|V(G')| = (p + q) = p'$. Since the majority dominating vertex v of degree $d_G(v) \geq \lceil \frac{p}{2} \rceil - 1$, $d_{G'}(v) < \lceil \frac{p'}{2} \rceil - 1$. Then the majority dominating set D will contain atleast two vertices of G' and $\gamma_M(G') = |D| > 1$. Since other vertices v_i are pendants and the new vertices $d(u_i) = 2$, the inverse majority dominating set D' contains the vertices of $u_i \subseteq V - D$ of G' . It implies that $\gamma_M^{-1}(G') > 2$ and $\gamma_M(G') < \gamma_M^{-1}(G')$. ■

3.4 Inverse Majority Domination Number in Vertex Deletion

In this section, the removal of a vertex u from the vertex set $V(G)$ is discussed and the inverse majority domination number $\gamma_M^{-1}(G - u)$ for the graph $G - \{u\}$ is determined. Then how this removal of a single vertex u can affect the inverse majority domination number $\gamma_M^{-1}(G - u)$ at various level is surveyed.

Theorem 3.4.1: If a graph G has atleast a full degree vertex u without pendants then $\gamma_M^{-1}(G - u) = \gamma_M(G - u)$.

Proof: Let u be a full degree vertex of G . Since G has no pendants, the graph $G - u$ is connected. Let $G' = G - \{u\}$ with $p' = p - 1$ vertices. Then the degree of each vertex u_i in G' is $d(u_i) \geq 1$. Also, $1 \leq d(u_i) \leq p - 2$, for every $u_i \in V(G')$.

Case (i): When $d(u_i) = p - 2$, for every $u_i \in V(G')$.

If $d(u_i) = p - 2$ then $|N[u_i]| > \lceil \frac{p}{2} \rceil$, for every $u_i \in V(G')$. It implies that each vertex of G' form a majority dominating set and an inverse majority dominating set of G' . Therefore $\gamma_M(G') = 1 = \gamma_M^{-1}(G')$.

Case (ii): When $d(u_i) = 1$, for every $u_i \in V(G')$. Then G has a structure of some triangles meet at a full degree vertex u and $(G -$

u) consists of only m -edges i.e., mK_2 . By the results (2.3.1)(7), $\gamma_M(G') = \lceil \frac{p'}{4} \rceil$ and $\gamma_M^{-1}(G') = \lceil \frac{p'}{4} \rceil$ where $|V(G')| = p'$, Hence $\gamma_M(G - u) = \gamma_M^{-1}(G - u)$.

Case(iii): When $d(u_i) \geq 2$, for every $u_i \in V(G')$.

Then G may be a Wheel or a Fan graphs or other graph structure with u . Then $G - u$ is a Cycle or a Path or any other structure contains at most equal degree vertices. By the results (2.3.1)(2), We have $\gamma_M(G') = \lceil \frac{p}{6} \rceil = \gamma_M^{-1}(G')$ or $\gamma_M(G') > \lceil \frac{p}{6} \rceil$ and $\gamma_M^{-1}(G') > \lceil \frac{p}{6} \rceil$. Hence in all cases, $\gamma_M(G - u) = \gamma_M^{-1}(G - u)$. ■

Corollary 3.4.2: If a tree T with a full degree vertex u and others are pendants then $\gamma_M(G - u) \neq \gamma_M^{-1}(G - u)$, if p is odd and $\gamma_M(G - u) = \gamma_M^{-1}(G - u)$, if p is even.

Proposition 3.4.3: For a tree with a full degree vertex ' u ' and others are pendants, $\gamma_M^{-1}(G) = \gamma_M(G - u)$, if p is odd and $\gamma_M^{-1}(G) < \gamma_M(G - u)$, if p is even. Also $\gamma_M^{-1}(G - u)$ does not exist.

For Example: Let $G = K_{1,p-1}$, p is odd and $p - 1$ is even. Then $\gamma_M(G) = 1$, $\gamma_M(G - u) = \frac{p-1}{2}$, where u is a central vertex. Also, $\gamma_M^{-1}(G) = \left(\frac{p-1}{2}\right) = \gamma_M(G - u)$, if p is odd. When p is even and

$(p-1)$ is odd, $\gamma_M(G) = 1, \gamma_M^{-1}(G) = \lfloor \frac{p-1}{2} \rfloor$. If u be a central vertex of G then $\gamma_M(G-u) = \lceil \frac{p-1}{2} \rceil$ and $\gamma_M^{-1}(G-u)$ does not exist. Hence $\gamma_M^{-1}(G) < \gamma_M(G-u)$, if p is even.

Proposition 3.4.4: Let $G = S(F_n)$ be a subdivision of a Fan graph F_n . Then $\gamma_M^{-1}(G) = \lceil \frac{p-1}{8} \rceil = \gamma_M^{-1}(G-u)$.

Proof: Let $F_n = P_{n-1} \vee K_1$ and $G = S(F_n)$ with $|V(G)| = p = 2n+1$. Then $V(G) = \{u, v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$ where u is a majority dominating vertex of G with $d(u) = \lceil \frac{p}{2} \rceil - 1, d(v_i) = 2$ where v_i 's are newly added vertices, $d(u_i) = 2$, for u_1, u_n and $d(u_i) = 3, i = u_2, \dots, u_{n-1}$. It implies that $\gamma_M(G) = |D| = |\{u\}| = 1$. Let $D' = \{u_2, u_5, \dots, u_t\}$ such that $|D'| = t = \lceil \frac{p-1}{8} \rceil$ and $D' \subseteq V - D$ with $d(u_i, u_j) \geq 3, u_i \neq u_j$. Then $|N[D']| = \sum_{i=1}^t d(v_i) + t = 4\lceil \frac{p-1}{8} \rceil \geq \lceil \frac{p-1}{2} \rceil$, therefore D' is an inverse majority dominating set of G and $\gamma_M^{-1}(G) \leq |D'| = \lceil \frac{p-1}{8} \rceil$.

Suppose $D_1 \subseteq V - D$ be a set with $|D_1| < |D| = \lceil \frac{p-1}{8} \rceil$. Then $|N[D_1]| < \lceil \frac{p}{2} \rceil$ and D_1 is not an inverse majority dominating set of G with respect to D . Therefore $\gamma_M^{-1}(G) > |D_1|$ and $\gamma_M^{-1}(G) \geq |D'| = \lceil \frac{p-1}{8} \rceil$. Hence $\gamma_M^{-1}(G) = \lceil \frac{p-1}{8} \rceil$.

Let $G' = G - u$ and D_1 be a majority dominating set of $(G - u)$.

Since the set $D' \subseteq V(G - u)$, we could find $D'_1 = \{u_3, u_6, \dots, u_t\} \subseteq$

$(V - D_1)$ with $|D'_1| = |D'|$. Hence D'_1 is an inverse majority dominating set of G' with respect to D_1 and $\gamma_M^{-1}(G - u) = |D'_1| = \lceil \frac{p-1}{8} \rceil$. Hence $\gamma_M^{-1}(G) = \gamma_M^{-1}(G - u) = \lceil \frac{p-1}{8} \rceil$. ■

Proposition 3.4.5: If a graph G has exactly one vertex u such that $d(u) = \lceil \frac{p}{2} \rceil - 1$ and other vertices $d(u_i) \leq 3$, for every $u_i \in V(G)$ then $\gamma_M^{-1}(G) = \gamma_M(G - u)$, if p is odd.

Proof: Let $V(G) = \{u, u_1, \dots, u_r, v_1, \dots, v_s\}$ with $p = r + s + 1$. Let u be a vertex of G with degree $d(u) = \lceil \frac{p}{2} \rceil - 1$. Then $D = \{u\}$ is a majority dominating set of G . Since the remaining vertices of G in $(V - D)$ are of degree $< \lceil \frac{p}{2} \rceil - 1$, the inverse majority dominating set of G contains atleast two vertices.

Case(i): when $d(u_i) \leq 2$ and $u_i \neq u$, for every $u_i \in V(G)$.

Let $D' = \{u_1, u_2, \dots, u_t\} \subseteq V - D$ with $|t| = \lceil \frac{p-1}{4} \rceil$ such that $d(u_i) = 2$ for every $u_i \in D'$. Then $|N_G[D']| = 2\lceil \frac{p-1}{4} \rceil + 1 \geq \lceil \frac{p}{2} \rceil$ and D' is an inverse majority dominating set of G and

$$\gamma_M^{-1}(G) = |D'| = \left\lceil \frac{p-1}{4} \right\rceil, \text{ if } p \text{ is odd} \quad (3.1)$$

Let $G' = (G - u)$ and $|V(G')| = p - 1 = \text{even}$. In G' , since every vertex $u_i \in V - D$ of degree $d(u_i) = 1$, the induced subgraph $\langle G' \rangle = mK_2$

is disconnected, where $m = \left(\frac{p-1}{2}\right)$. Choose $D_1 = \{u_1, u_2, \dots, u_t\}$ with $|D_1| = |t| = \lceil \frac{p-1}{4} \rceil$. Then $|N_G[D_1]| = 2\lceil \frac{p-1}{4} \rceil \geq \left(\frac{p}{2}\right)$ and D_1 is a majority dominating set of G' . Hence $\gamma_M(G') = |D_1| = \lceil \frac{p-1}{4} \rceil$.

$$\text{Therefore } \gamma_M(G - u) = \left\lceil \frac{p-1}{4} \right\rceil. \quad (3.2)$$

Thus, $\gamma_M^{-1}(G) = \gamma_M(G - u) = \lceil \frac{p-1}{4} \rceil$, if p is odd.

Case(ii): when $d(v_i) \leq 3, u_i \neq u$ and for every $u_i \in V(G)$.

Let D be a majority dominating set of G with $|D| = \lceil \frac{p}{8} \rceil$. Then Choose $D' = \{v_2, v_5, \dots, v_t\} \subseteq V - D$ with $|D'| = |t| = \lceil \frac{p}{8} \rceil$ such that $d(v_i) = 3$, for every $v_i \in V - D$ and $|N_G[D']| = 3\lceil \frac{p}{8} \rceil + \lceil \frac{p}{8} \rceil = 4\lceil \frac{p}{8} \rceil \geq \lceil \frac{p}{2} \rceil$. It implies that D' is an inverse majority dominating set of G with respect to D and

$$\gamma_M^{-1}(G) = |D'| = \left\lceil \frac{p}{8} \right\rceil, \text{ if } p \text{ is odd.} \quad (3.3)$$

Since $d(u_i) \leq 3$, for every $u_i \in V(G')$ the induced subgraph $\langle G' \rangle$ is connected and it has a caterpillar structure or $(P_{p-1} \circ K_1)$ where $G' = G - u$.

Choose $D_1 = \{v_2, v_5, \dots, v_t\} \subseteq V(G')$ and $D'_1 = \{v_3, v_6, \dots, v_{t+1}\} \subseteq (V - D_1)$ with $|D'_1| = |D_1| = \lceil \frac{p}{8} \rceil$. Then $|N[D'_1]| = |N[D_1]| \geq \lceil \frac{p}{2} \rceil$.

It implies that D_1 is a majority dominating set of G' and D'_1 is an inverse majority dominating set of G' with respect to D_1 .

$$\gamma_M(G') = \gamma_M^{-1}(G') = \left\lceil \frac{p}{8} \right\rceil \text{ and } \gamma_M(G - u) = \left\lceil \frac{p}{8} \right\rceil. \quad (3.4)$$

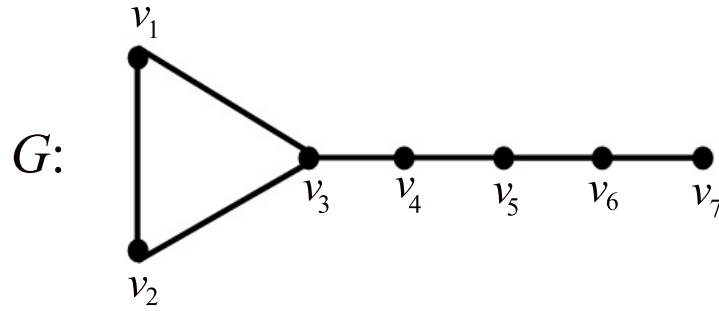
From (3.3) and (3.4), $\gamma_M^{-1}(G) = \gamma_M(G - u)$, if p is odd. ■

3.5 $\gamma_M^{-1}(G)$ for a Splitting graph of G

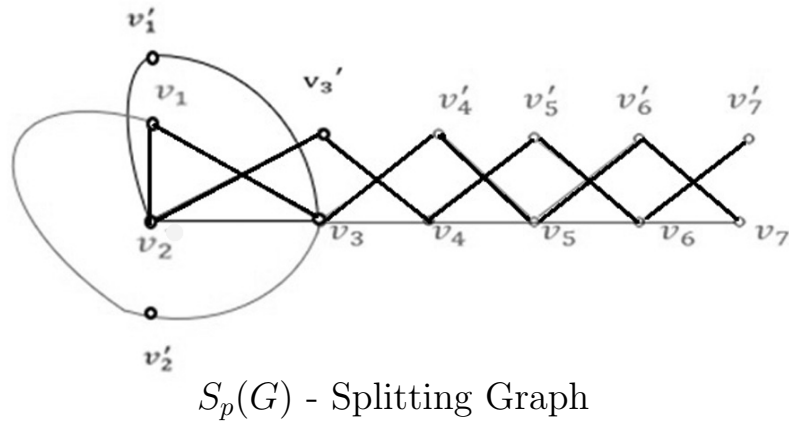
An interesting structure is splitting graph $S_p(G)$ of any graph G by adding new vertex set and joining these vertices according to some rules. In this section, splitting graph $S_p(G)$ for some classes of graphs are determined. Then the numbers $\gamma_M(S_p(G))$ and $\gamma_M^{-1}(S_p(G))$ are found and the relationship between these numbers is also studied.

Definition 3.5.1: [49] For each vertex v of a graph G , take a new vertex v' . Join v' to all the vertices of G adjacent to v . The graph $S(G)$ thus obtained is called splitting graph of G .

Example 3.5.2: Consider the following graph G and its splitting graph with $p = |V(G)| = 7$ and $p' = |V(S_p(G))| = 14$.



For a graph G , the degree of each vertex v_i of G is $d(v_i) = t$. Then $\gamma_M(G) = 1, \gamma_M^{-1}(G) = 2$.



In $S_p(G)$, the degree of a vertex v_i is $d(v_i) = 2t$ and $d(v'_i) = t = d(v_i)$, where v'_i 's are newly added vertices then $\gamma_M(S_p(G)) = 1$. Let $D' = \{v_5, v_6\} \subseteq V - D$. Such that $|N[D']| = 8 > \lceil \frac{p}{2} \rceil$. Therefore $\gamma_M^{-1}(S_p(G)) = |D'| = 2$.

Observation 3.5.3:

- (i) If $S_p(G)$ is a splitting graph of G then all the degree of the vertices of $S_p(G)$ is twice of $d(v_i)$, i.e.) if $d_G(v_i) = t$ then $d_s(v_i) = 2t$.

(ii) The degree of the newly added vertices v'_i of $S_p(G)$ will get the same degree of G . i.e.) if $d_s(v'_i) = t = d_G(v_i)$.

(iii) $\gamma_M(S_p(G)) \leq \gamma_M^{-1}(S_p(G))$, for the splitting graph $S_p(G)$ of any graph G .

(iv) $\gamma_M^{-1}(S_p(G)) \leq \gamma_M^{-1}(G)$, for any G and its splitting graph $S_p(G)$.

Proposition 3.5.4: Let $G = S_p(P_p)$ be a splitting graph of a path, $p \geq 2$. Then $\gamma_M(G) = \gamma_M^{-1}(G) = \lceil \frac{p}{5} \rceil$.

Proof: Let $|V(G)| = \{v_1, \dots, v_p, v'_1, \dots, v'_p\} = 2p$ and $d_s(v_1, v_p) = 2$ and $d_s(v_2, \dots, v_{p-1}) = 4$ also, $d_s(v'_1, v'_p) = 1$ and $d_s(v'_2, \dots, v'_{p-1}) = 2$. Choose $D = \{v_2, v_5, \dots, v_t\} \subseteq V(G)$ such that $d(v_i, v_j) \geq 3$ with $|D| = t = \lceil \frac{p}{5} \rceil$. Then $|N[D]| = \sum_{i=1}^t d(v_i) + t = 4t + t = 5t = 5\lceil \frac{p}{5} \rceil \geq p$. It implies that D is a majority dominating set of G and $\gamma_M(G) = \lceil \frac{p}{5} \rceil$. Also, choose $D' = \{v_3, v_6, \dots, v_t\} \subseteq V - D$ such that $d(v_i, v_j) \geq 3$ with $|D'| = t = \lceil \frac{p}{5} \rceil$. By the same argument, $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and D' is an inverse majority dominating set and $\gamma_M^{-1}(G) = |D'| = \lceil \frac{p}{5} \rceil$. Hence $\gamma_M(G) = \gamma_M^{-1}(G) = \lceil \frac{p}{5} \rceil$. ■

Corollary 3.5.5: If $G = S_p(C_p)$ is a cycle with $p \geq 3$ then $\gamma_M(G) = \lceil \frac{p}{5} \rceil = \gamma_M^{-1}(G)$.

Proposition 3.5.6: Let $G = S_p(K_p)$ be a splitting graph of a complete graph K_p . Then $\gamma_M(S_p(G)) = \gamma_M^{-1}(S_p(G)) = 1$.

Proof: Let $V(G) = 2p$. In K_p every full degree vertex $v_i \in V(K_p)$ becomes a vertex of degree $d_s(v_i) = 2p - 2$ in $S_p(G)$. Then any one vertex $D = \{v_i\} \in G$ is a majority dominating set of $S_p(G)$ and any other vertex $v_j \in V(G) - \{v_i\}$ and $v_j \subseteq V - D$, is an inverse majority dominating set of $S_p(G)$. Hence $\gamma_M(S_p(G)) = \gamma_M^{-1}(S_p(G)) = 1$. ■

Proposition 3.5.7: If a graph G is a Caterpillar with p vertices, then $\gamma_M(S_p(G)) = \gamma_M^{-1}(S_p(G)) = \lceil \frac{p}{7} \rceil$.

Proof: Let $S_p(G)$ be the splitting graph of a caterpillar graph with $|S_p(G)| = 2p$. Let $V(G) = \{v_1, v_2, \dots, v_{\frac{p}{2}}, u_1, \dots, u_{\frac{p}{2}}\}$ where v_i 's are lying in a path such that $d(v_i) = 3$ and u_i 's are pendants of G . Also, the degree of the vertices in $S_p(G)$ are $d_s(v_i) = 6, i = 2, \dots, (\frac{p}{2} - 1), d_s(v_i) = 4$, for $i = 1, (\frac{p}{2})$ and for the new vertices, $d_s(v_i') = 3$ or $2, d_s(u_i) = 2, i = (\frac{p}{2} + 1), \dots, p$ and $d_s(u_i') = 1$. Choose $D = \{v_2, v_5, \dots, v_t\} \subseteq V(G)$ with $d(v_i, v_j) \geq 3$ and $|D| = t = \lceil \frac{p}{7} \rceil$. Then $|N[D]| = \sum_{v_i \in D} d(v_i) + t = 7t = 7\lceil \frac{p}{7} \rceil \geq p$. Therefore, D is a majority dominating set of $S_p(G)$ and $\gamma_M(S_p(G)) \leq \lceil \frac{p}{7} \rceil$.

Let $D' = \{v_3, v_6, \dots, v_t\} \subseteq V - D$ with the same conditions as in the set D with $|D'| = \lceil \frac{p}{7} \rceil$. By the same argument, $\gamma_M^{-1}(S_p(G)) \leq \lceil \frac{p}{7} \rceil$. Suppose choose a set D_1 with $|D_1| = t - 1$ then $|N[D_1]| < p$ implies that D_1 is not a majority dominating set of $S_p(G)$. Hence $\gamma_M(S_p(G)) > |D_1|$ and $\gamma_M(S_p(G)) \geq |D| = \lceil \frac{p}{7} \rceil$. Also, by the same argument for an inverse majority dominating set of $S_p(G)$, we get, $\gamma_M^{-1}(S_p(G)) \geq \lceil \frac{p}{7} \rceil$. Hence $\gamma_M(S_p(G)) = \gamma_M^{-1}(S_p(G)) = \lceil \frac{p}{7} \rceil$. ■

Results 3.5.8:

1. Let $G = S_p(F_p)$. Then $\gamma_M(G) = \gamma_M^{-1}(G) = 1$, where F_p is a fan with p vertices.
2. If $G = S_p(D_{r,s})$ is a double star, $\gamma_M(G) = \gamma_M^{-1}(G) = 1$.
3. Let $G = S_p(W_p)$, where W_p is a wheel with p vertices. Then $\gamma_M(G) = \gamma_M^{-1}(G) = 1$.
4. If $G = S_p(K_{1,p-1})$ then $\gamma_M(G) = 1 = \gamma_M^{-1}(G)$.
5. If $G = S_p(K_{m,n})$ then $\gamma_M(G) = 1 = \gamma_M^{-1}(G)$.

Observation 3.5.9:

1. If a graph $G(p, q)$ has all the vertices of degree $d(v_i) = p-1$ then in the splitting graph $S_p(G)$, the vertex has degree $d_s(v_i) = 2p-2$ and $\gamma_M(S_p(G)) = \gamma_M^{-1}(S_p(G)) = 1$.

2. If a graph $G(p, q)$ has exactly one vertex v_i of degree $d(v_i) = p-1$ then the degree of vertex $d_s(v_i) = 2p-2$ and $d_s(v'_1) = p-1$, where v'_1 is a newly added vertex in $S_p(G)$ and $\gamma_M(S_p(G)) = \gamma_M^{-1}(S_p(G)) = 1$.
3. If a graph G has vertices of degree $d(v_i) = \lceil \frac{p}{2} \rceil$ or $\lceil \frac{p}{2} \rceil - 1$ then $d_s(v_i) = p$ or $p-1$ and $\gamma_M(S_p(G)) = \gamma_M^{-1}(S_p(G)) = 1$.
4. If a graph G contains atleast one majority dominating vertex 'v' such that $d(v) \geq \lceil \frac{p}{2} \rceil - 1$ then $\gamma_M(S_p(G)) = \gamma_M^{-1}(S_p(G)) = 1$.
5. If graph G contains all vertices v_i of degree $d(v_i) < \lceil \frac{p}{2} \rceil - 1$ then $\gamma_M(S_p(G)) \geq 2$ and $\gamma_M^{-1}(S_p(G)) \geq 2$.
6. If $\gamma_M(G) < \gamma_M^{-1}(G)$ then $\gamma_M(S_p(G)) < \gamma_M^{-1}(S_p(G))$, for any graph G .

Theorem 3.5.10: If a graph G contains atleast one vertex 'v' of degree $d(v) \geq \lceil \frac{p}{2} \rceil - 1$ if and only if $\gamma_M(S_p(G)) = \gamma_M^{-1}(S_p(G)) = 1$.

Proof: By Observation (3.5.9)(4), we get the result. ■

Proposition 3.5.11: If a graph G contains atleast half of the vertices of $V(G)$ have the same degree then $\gamma_M(S_p(G)) = \gamma_M^{-1}(S_p(G))$.

Proof: Let D be a majority dominating set of $S_p(G)$. Since the graph G contains more than half of the vertices have the same degree, we could choose an inverse majority dominating set $D' \subseteq (V - D)$ in $S_p(G)$ with $|D'| = |D|$. Hence $\gamma_M(S_p(G)) = \gamma_M^{-1}(S_p(G))$. ■

Corollary 3.5.12: If a graph G is regular then $\gamma_M(S_p(G)) = \gamma_M^{-1}(S_p(G))$.

Theorem 3.5.13: Let $\gamma_M^{-1}(G)$ and $\gamma_M^{-1}(S_p(G))$ be the inverse majority domination number of G and the splitting graph $S_p(G)$ respectively. Then $\gamma_M^{-1}(S_p(G)) \leq \gamma_M^{-1}(G)$.

Proof: Since every degree of a vertex ' v ' of G , $d_G(v) = t$ is increased twice in $S_p(G)$ i.e., $d_s(v) = 2t$, the cardinality of a minimal inverse majority dominating set of $S_p(G)$ is minimum that of the cardinality of a minimal inverse majority dominating set of G . Hence $\gamma_M^{-1}(S_p(G)) \leq \gamma_M^{-1}(G)$. ■

Proposition 3.5.14: If a graph G has exactly one vertex ' v ' such that $d(v) \geq \lceil \frac{p}{2} \rceil - 1$ and others are $d(v_i) < \lceil \frac{p}{2} \rceil - 1$ then $\gamma_M(S_p(G)) < \gamma_M^{-1}(S_p(G))$.

Proof: Let $v \in V(G)$ such that $d_G(v) \geq \lceil \frac{p}{2} \rceil - 1$ and $|N[v]| \geq \lceil \frac{p}{2} \rceil$.

In $S_p(G)$, the degree of the vertex v is $d_s(v) \geq p - 1$ and $|N_s[v]| \geq p$.

Hence $D = \{v\}$ is a majority dominating set of $S_p(G)$.

Since other vertices v_i are of degree $d_G(v_i) < \lceil \frac{p}{2} \rceil - 1$, $d_s(v_i) < p - 1$ and there is no majority dominating vertex in $(V - D)$ of $S_p(G)$.

Therefore, choose a set $D' \subseteq V - D$ such that $|D'| \geq 2$. Then

$|N_s[D']| \geq \lceil \frac{p}{2} \rceil$ and D' is an inverse majority dominating set of $S_p(G)$.

Hence $\gamma_M^{-1}(S_p(G)) = |D'| \geq 2$. Since $\gamma_M(S_p(G)) = 1$, $\gamma_M(S_p(G)) < \gamma_M^{-1}(S_p(G))$. ■

3.6 Inverse Majority Domination Number for Complement $\bar{G}$ of G

In this section, the majority domination number and an inverse majority domination number for the complement $\bar{G}$ of a graph G are determined for some standard graphs. Then few results regarding various domination numbers $\gamma_M(\bar{G})$ and $\gamma_M^{-1}(\bar{G})$ are produced in the case of complement $\bar{G}$ of G .

Proposition 3.6.1: For the complement of $G = K_{m,n}$, $m \leq n$, with $m, n \geq 2$, $\gamma_M(\bar{K}_{m,n}) = \gamma_M^{-1}(\bar{K}_{m,n}) = 1$.

Proof: Let $G = K_{m,n}$ be a complete bipartite graph with $|V_1(G)| = m$ and $|V_2(G)| = n$, $m, n \geq 2$. Then $\bar{G} = K_m \cup K_n$ where two complete graphs K_m and K_n with $m \leq n$.

Case (i): when $n = m, m + 1, m + 2$.

Since $\bar{G}$ contains complete graph, each vertex of V_1 or V_2 is a majority dominating set and an inverse majority dominating set of $\bar{G}$ and $\gamma_M(\bar{K}_{m,n}) = \gamma_M^{-1}(\bar{K}_{m,n}) = 1$.

Case (ii): when $n \geq m + 3$. Let $\bar{G} = K_m \cup K_n$ with $m < n$. Let $u_i \in V(K_n)$ and $v_i \in V(K_m)$. Then the sets $D = \{u_1\}$ and $D' = \{u_2\}$ are majority dominating and inverse majority dominating sets of $\bar{G}$ and $\gamma_M(\bar{K}_{m,n}) = \gamma_M^{-1}(\bar{K}_{m,n}) = 1$. ■

Proposition 3.6.2: Let $G = D_{r,s}$, $r \leq s$ be a tree with $p = r + s + 2$ vertices. Then

- (i) $\gamma_M(\bar{G}) = \gamma_M^{-1}(\bar{G}) = 1$, if $s = r, r + 1, r + 2$ and
- (ii) $\gamma_M(\bar{G}) = \gamma_M^{-1}(\bar{G})$, if $s \geq r + 3$.

Proof: Let $G = D_{r,s}$, $r \leq s$ be a tree with p vertices and $V(G) = \{u, v, u_1, u_2, u_3, \dots, u_r, v_1, v_2, \dots, v_s\}$ where u_i and v_j are pendants, $i = 1, \dots, r$ and $j = 1, \dots, s$ at u and v respectively,

Case (i): When $s = r, r + 1, r + 2$.

The graph G has two majority dominating vertex u and v . It gives that $\gamma_M(G) = \gamma_M^{-1}(G) = 1$. In the complement $\bar{G}$ of G , each pendant vertex u_i or v_i of G becomes a majority dominating vertex of $\bar{G}$ (i.e.,) $d(u_i) = p - 2 = d(v_i)$. It implies that $\gamma_M(\bar{G}) = \gamma_M^{-1}(\bar{G}) = 1$.

Case (ii): When $s \geq r + 3$ and $r < s$.

Then the graph G has exactly one majority dominating vertex ' v ' and $\gamma_M(G) = |D| = |\{v\}| = 1$. Suppose $s = r + 3$. Choose $D' = \{u, v_1\} \subseteq V - D$ and $|N[D']| = |N[u]| + |N[v_1]| = \lceil \frac{p}{2} \rceil - 1 + 1 = \lceil \frac{p}{2} \rceil$. It implies that D' is an inverse majority dominating set of G . Hence $\gamma_M^{-1}(G) = |D'| = 2$. Thus, if $s \geq r + 3, \gamma_M^{-1}(G) \geq 2$. Hence $\gamma_M(G) < \gamma_M^{-1}(G)$. By the same argument as in case (i), we obtain $\gamma_M(\bar{G}) = \gamma_M^{-1}(\bar{G}) = 1$. ■

Proposition 3.6.3: Let $\bar{G}$ be the complement of a graph $G = P_p$ or C_p with $p \geq 3$. Then $\gamma_M^{-1}(\bar{P}p) = \gamma_M^{-1}(\bar{C}p) = 1$.

Proof: Let $\bar{G} = \bar{P}p, p \geq 2$. Since the graph $G = P_p$ contains exactly two pendent vertices u_1 and $u_p, \bar{d}(u_i) = p - 2$, for $u_i \in V(\bar{P}p), i = 1, p$ and all other vertices v_i of G such that $d_G(v_i) = 2$ becomes $\bar{d}(v_i) = p - 3$ in $\bar{G}, i = 2, \dots, p - 1$. Then the two vertices u_i and u_p are majority dominating vertices of $\bar{G}$ and $\gamma_M^{-1}(\bar{P}p) = 1 = \gamma_M(\bar{P}p)$.

Let $\bar{G} = \bar{C}p, p \geq 3$ and $\delta(G) = \Delta(G) = 2$. Then all the degree of the vertices of G becomes $\bar{d}(v_i) = p - 3$ in $\bar{G}$. Therefore any single vertex would form a majority dominating set and an inverse majority dominating set for $\bar{G}$, $\gamma_M^{-1}(\bar{C}p) = 1$. ■

Proposition 3.6.4: There exists a graph G which is disconnected with p vertices such that $\gamma_M^{-1}(G) - \gamma_M^{-1}(\bar{G}) = 1$.

Proof: There exists a disconnected graph G with p vertices such that $G = K_{\lfloor \frac{p}{2} \rfloor} \cup \bar{K}_{\lceil \frac{p}{2} \rceil}$. Let $V(G) = \{v_1, \dots, v_{\lfloor \frac{p}{2} \rfloor}, u_{\lceil \frac{p}{2} \rceil}, \dots, u_p\}$ where $(u_{\lceil \frac{p}{2} \rceil}, \dots, u_p)$ are isolates and $v_i \in V(K_{\lfloor \frac{p}{2} \rfloor})$. Let $D = \{v_1, u_{\lceil \frac{p}{2} \rceil}\}$. Then $|N[D]| = |N[v_1]| + |N[u_{\lceil \frac{p}{2} \rceil}]| = \lfloor \frac{p}{2} \rfloor + 1 = \lceil \frac{p}{2} \rceil$. Therefore D is a majority dominating set of G and $\gamma_M(G) = |D| = 2$. By the similar argument, choose the set $D' = \{v_2, u_{\lceil \frac{p}{2} \rceil+1}\} \subseteq V - D$, such that $|N[D']| = \lceil \frac{p}{2} \rceil$. Therefore D' is an inverse majority dominating set of G with respect to D . Hence

$$\gamma_M^{-1}(G) = |D'| = 2. \quad (3.5)$$

Let $\bar{G} = \bar{K}_{\lfloor \frac{p}{2} \rfloor} \cup K_{\lceil \frac{p}{2} \rceil}$ be a complement of G and $V(\bar{G}) = \{x_1, \dots, x_{\lfloor \frac{p}{2} \rfloor}, y_{\lceil \frac{p}{2} \rceil}, \dots, y_p\}$ where x_i 's are isolates in $\bar{G}$ and $d_{\bar{G}}(y_j) = p - 1$, for $j = 1, \dots, \lceil \frac{p}{2} \rceil$. From these vertices of $\bar{G}$, any single vertex $y_j, y_j \in$

$V(K_{\lceil \frac{p}{2} \rceil})$ would form a majority dominating set $\bar{D}$ of $\bar{G}$ an inverse majority dominating set $\bar{D}'$ of $\bar{G}$. Hence

$$\gamma_M(G) = \gamma_M^{-1}(\bar{G}) = 1. \quad (3.6)$$

From (3.5) and (3.6) we obtain, $\gamma_M^{-1}(G) - \gamma_M^{-1}(\bar{G}) = 1$. ■

Proposition 3.6.5: If a graph G contains only full degree vertices with $p = 2n$ then $\gamma_M(\bar{G}) = \gamma_M^{-1}(\bar{G}) = \frac{p}{2}$.

Proof: Let the graph G has all vertices are of degree $d(v_i) = p - 1$, for every $v_i \in V(G)$ with p is even. The complement of G is a totally disconnected graph. Then $\gamma_M(\bar{G}) = \frac{p}{2} = \gamma_M^{-1}(\bar{G})$, since p is even. ■

Corollary 3.6.6: If a graph G contains only full degree vertices with $p = (2n + 1)$ then the inverse majority domination number of $\bar{G}$, $\gamma_M^{-1}(\bar{G})$ does not exist.

Proof: Let the graph G be a complete graph and p is odd. Then $G = K_p, p$ is odd. Therefore $\gamma_M(G) = \gamma_M^{-1}(G) = 1$. The complement of G is $\bar{G} = (\bar{K}_p), p$ is odd. Then the majority dominating set $|D| = \gamma_M(\bar{G}) = \lceil \frac{p}{2} \rceil$ and the remaining vertices $\lfloor \frac{p}{2} \rfloor$ in $(V - D)$ could not

form an inverse majority dominating set for $\bar{G}$. Hence the inverse majority domination number $\gamma_M^{-1}(\bar{G})$ does not exist. ■

Proposition 3.6.7: If a graph G has all vertices of degree $d(u) = \lceil \frac{p}{2} \rceil - 1$ and $\lceil \frac{p}{2} \rceil$ then $\gamma_M(G) = \gamma_M^{-1}(G) = \gamma_M(\bar{G}) = \gamma_M^{-1}(\bar{G}) = 1$.

Proof: Since the graph G has all vertices of degree $d(u) = \lceil \frac{p}{2} \rceil - 1$ and $\lceil \frac{p}{2} \rceil$, then G has only majority dominating vertices and $|N[u]| = \lceil \frac{p}{2} \rceil$. It implies that each vertex forms an inverse majority dominating set of G and $\gamma_M(G) = \gamma_M^{-1}(G) = 1$.

Then the complement of $\bar{G}$ of this graph G is having vertices of degree $\bar{d}(u) = \lfloor \frac{p}{2} \rfloor + 2$ and $\lfloor \frac{p}{2} \rfloor + 1$. Therefore $|N_{\bar{G}}[u]| = \lceil \frac{p}{2} \rceil$. From these vertices, any single vertex would form a majority dominating set and an inverse majority dominating set in $\bar{G}$ and $\gamma_M(\bar{G}) = \gamma_M^{-1}(\bar{G}) = 1$. ■

Theorem 3.6.8: For any tree T with $p \geq 7$, inverse majority domination number $\gamma_M^{-1}(T)$ is reduced to $\gamma_M^{-1}(\bar{T}) = 1$ in the complement $\bar{T}$ of T .

Proof: Let T be a tree with $p \geq 7$. Let $\gamma_M(T)$ and $\gamma_M^{-1}(T)$ be the majority domination number and an inverse majority domination number of a tree T .

Case (i): Since T has atleast two pendant vertices u_1 and u_2 , then $T = P_p$. By the results (2.3.1)(2), when $p \geq 7$, we have $\gamma_M^{-1}(T) = \lceil \frac{p}{6} \rceil$ and hence $\gamma_M^{-1}(T) \geq 2$.

In $\bar{T}$, there are two vertices with degree $\bar{d}(u_i) = p - 2$ and $|N_{\bar{G}}[u_i]| = p - 1, i = 1, 2$. It implies that $D = \{u_1\}$ and $D' = \{u_2\}$ are the majority dominating set and inverse majority dominating set of $\bar{T}$. Hence $\gamma_M(\bar{T}) = \gamma_M^{-1}(\bar{T}) = 1$.

Thus, in the case of the complement $\bar{T}$ of a tree T , $\gamma_M^{-1}(T)$ is reduced to $\gamma_M^{-1}(\bar{T}) = 1$.

Case (ii): Suppose T has atmost $(p - 1)$ pendants. Then $T = K_{1,p-1}$. By the results (2.3.1)(5), $\gamma_M^{-1}(T) = \lfloor \frac{p-1}{2} \rfloor$ and $\gamma_M^{-1}(T) \geq 2$. In $\bar{T}$, there are $p - 1$ vertices with degree $\bar{d}(u_i) = p - 2, i = 1, 2, \dots, p - 1$. It implies that any single vertex u_i would form a majority dominating set and an inverse majority dominating set of $\bar{T}$ and $\gamma_M(T) = \gamma_M^{-1}(\bar{T}) = 1$. Since $\gamma_M^{-1}(T) \geq 2, \gamma_M^{-1}(T)$ will take any integer value for a tree T but $\gamma_M^{-1}(T)$ is reduced to $\gamma_M^{-1}(\bar{T}) = 1$ in the complement $\bar{T}$. Thus the result. ■

Chapter 4

Inverse Independent Majority Dominating Set of a Graph

Abstract

This chapter introduces an **inverse independent majority dominating set** of a graph G with respect to an independent majority dominating set of a graph G . An **inverse independent majority domination number** $i_M^{-1}(G)$ is determined for some families of graphs. Some Characterisation theorems on $i_M^{-1}(G)$ are discussed. Bounds of an inverse independent majority domination number for a connected and disconnected graphs are found and its relationship with other domination parameters namely $i^{-1}(G)$, $\gamma_M^{-1}(G)$ and $i_M(G)$ are also established.

4.1 Introduction

In 1991, Kulli and Sigarkanti [30] initially introduced the concept inverse domination and produced many results in their research paper. In 2006, majority dominating set in graphs was studied by Swaminathan and Joseline Manora [50]. The concept of an independent majority dominating sets in graphs was introduced and propelled by Joseline Manora and John in 2014 [34]. In this work, they have defined an independent majority domination number $i_M(G)$ of a graph. Then bounds of an independent majority domination number and many interesting results with inequality are determined. Motivated by these concepts, an inverse independent majority dominating set of G and an inverse independent majority domination number $i_M^{-1}(G)$ of a graph G are introduced in this chapter.

The chapter includes the following sections. The introductory part is given in the first Section and Section 4.2 defines the concept of an inverse independent majority dominating set for a graph G and illustrates the defined concept with an example. The inverse independent majority domination number $i_M^{-1}(G)$ is determined for various classes of graphs in Section 4.3. Section 4.4 contains some

characterisation theorems on $i_M^{-1}(G)$ of a graph. In the next two Sections 4.5 and 4.6, bounds of $i_M^{-1}(G)$ are determined with sharpness and comparative study of the parameters $i^{-1}(G)$, $i_M^{-1}(G)$, $\gamma_M^{-1}(G)$ and $i_M(G)$ is discussed. Section 4.7 deals with the bounds of $i_M^{-1}(G)$ for disconnected graphs. Finally Nordhus-Gaddum type results are studied in Section 4.8.

4.2 Inverse Independent Majority Dominating Set

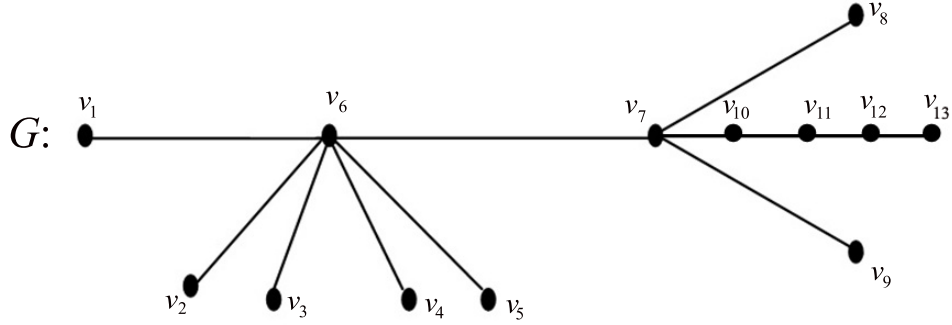
In this section, the definitions of an inverse independent majority dominating set with respect to a minimum independent majority dominating set of a graph G , an inverse independent majority domination number of a graph G are given. An example illustrating these definitions is also shown.

Definition 4.2.1: Let $G = (V, E)$ be a simple graph with p vertices and q edges. Let $D \subseteq V(G)$ be a minimum independent majority dominating set of a graph G . If $V - D$ contains an independent majority dominating set D' then the set D' is called an Inverse Independent Majority Dominating set of G with respect to D .

An inverse independent majority dominating set D' is minimal if there exists no proper subset D'_1 of D' such that D'_1 is an inverse independent majority dominating set of G with respect to D .

Definition 4.2.2: The Inverse Independent Majority Domination number, denoted by $i_M^{-1}(G)$ of a graph G is the minimum cardinality taken over all the minimal inverse independent majority dominating sets of a graph G .

Example 4.2.3: Consider the graph G as given below with $p = 13$.

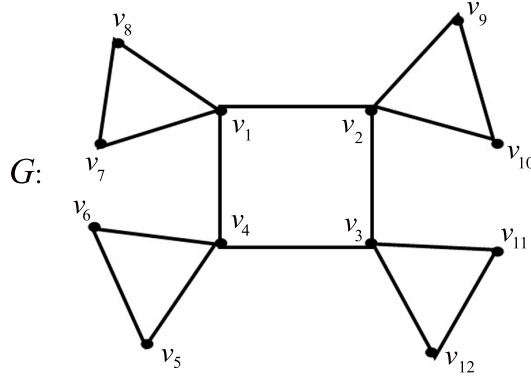


In G , $|V(G)| = p = 13$. $D = \{v_6\}$ is a minimum independent majority dominating set of G and $i_M(G) = |D| = 1$. An independent domination number $i(G) = |\{v_6, v_8, v_9, v_{11}, v_{13}\}| = 5$. Choose $D' = \{v_7, v_{12}\}$ such that $|N[D']| = 7$ and $D' \subseteq V - D$ such that $\langle D' \rangle$ has no edges. Therefore D' is an Inverse Independent Majority Dominating set of G with respect to D and $i_M^{-1}(G) = |D'| = 2$. Then inverse inde-

pendent domination number $i^{-1}(G) = |\{v_1, v_2, v_3, v_4, v_5, v_7, v_{12}\}| = 7$.

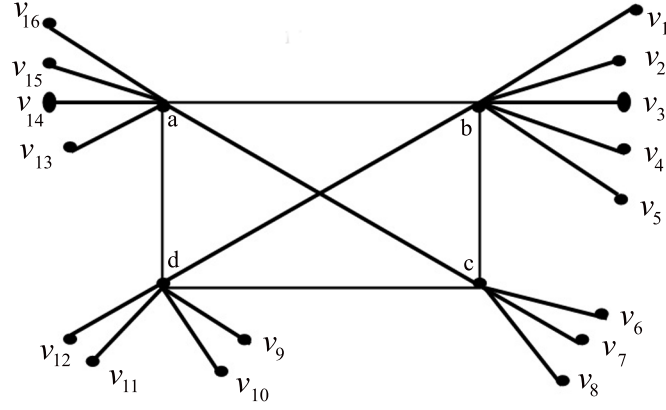
Hence $i_M^{-1}(G) < i^{-1}(G)$ and $i_M(G) < i_M^{-1}(G)$.

Example 4.2.4: Let $G = C_4 \circ K_2$ be a corona graph with $p = 12$.



For the graph G , a majority dominating set and an independent majority dominating set $D = \{v_1, v_9\}$ and their numbers $\gamma_M(G) = i_M(G) = |D| = 2$. Next, an inverse majority dominating set and an inverse independent majority dominating set $D' = \{v_2, v_{11}\} \subseteq V - D$ and their numbers $\gamma_M^{-1}(G) = i_M^{-1}(G) = |D'| = 2$. Also an independent set $S = \{v_6, v_8, v_{10}, v_{12}\}$ and the independence number is $i(G) = |S| = 4$. An inverse independent set $S' = \{v_5, v_7, v_9, v_{11}\} \subseteq V - S$ and the inverse independence number $i^{-1}(G) = |S'| = 4$. Thus $i_M^{-1}(G) < i^{-1}(G)$ and $\gamma_M^{-1}(G) = i_M^{-1}(G)$.

Example 4.2.5: Consider the following graph G with $p = 20$.



Let $D = \{a, b\}$ be a majority dominating set and $D' = \{c, d\}$ is an inverse majority dominating set with respect to D such that $D' \subseteq V - D$. Then $\gamma_M(G) = 2$ and $\gamma_M^{-1}(G) = 2$. Let $D_i = \{b, v_6\}$ be an independent majority dominating set and $D'_i = \{d, v_1, v_2\} \subseteq V - D_i$ is an inverse independent majority dominating set with respect to D_i . Then $i_M(G) = |D_i| = 2$ and $i_M^{-1}(G) = |D'_i| = 3$. Hence $\gamma_M^{-1}(G) < i_M^{-1}(G)$ and $\gamma_M(G) = i_M(G)$.

Example 4.2.6: Consider the graph G with $p = 20$ and G has an induced subgraph K_4 with 4 pendants at each vertex (a, b, c, d) .

Let $D = \{a, b\}$ be a majority dominating set and $D' = \{c, d\}$ is an inverse majority dominating set with respect to D such that $D' \subseteq V - D$. Then $\gamma_M(G) = 2$ and $\gamma_M^{-1}(G) = 2$. Let $D_i = \{a, v_1, v_2\}$ be an independent majority dominating set and $D'_i = \{b, v_5, v_6\} \subseteq V - D_i$ is

an inverse independent majority dominating set with respect to D_i .

Then $i_M(G) = 3 = i_M^{-1}(G)$. It gives the inequality, $\gamma_M(G) < i_M(G)$ and $\gamma_M^{-1}(G) < i_M^{-1}(G)$.

Proposition 4.2.7: (i) For any graph G , $i_M(G) \leq i_M^{-1}(G)$.

Proof: Since every inverse independent majority dominating set of G is an independent majority dominating set of G , the independent majority domination number $i_M(G)$ and inverse independent majority domination number $i_M^{-1}(G)$ satisfies $i_M(G) \leq i_M^{-1}(G)$. ■

(ii) For any graph G , $i_M^{-1}(G) \leq i^{-1}(G)$

Proof: Since every inverse independent dominating set of G is an inverse independent majority dominating set of G , $i_M^{-1}(G) \leq i^{-1}(G)$. ■

(iii) For any graph G , $\gamma_M^{-1}(G) \leq i_M^{-1}(G)$.

Proof: Since every inverse independent majority dominating set of G is an inverse majority dominating set of G , $\gamma_M^{-1}(G) \leq i_M^{-1}(G)$. ■

Observation 4.2.8: For every graph G , an inverse independent majority dominating set exists whereas an inverse independent dominating set does not exist for every graph without isolated vertices.

4.3 Inverse Independent Majority Domination Number for Some Classes of Graphs

The inverse independent majority domination number $i_M^{-1}(G)$ for various classes of graphs is established in the following results.

Results 4.3.1:

1. Let $G = K_p, p \geq 2$ be a complete graph. Then $i_M^{-1}(G) = 1$.
2. Let $G = K_{1,p-1}$ be a star with $p \geq 2$. Then $i_M^{-1}(G) = \lfloor \frac{p-1}{2} \rfloor$.
3. If $G = P_p$ a path with $p \geq 2$ and C_p a cycle with $p \geq 3$,
 $i_M^{-1}(G) = \lceil \frac{p}{6} \rceil$.
4. If $G = W_p$ is a wheel with $p \geq 5$, $i_M^{-1}(G) = \lceil \frac{p-2}{6} \rceil$.
5. If $G = F_p$ is a fan graph with $p \geq 4$, $i_M^{-1}(G) = \lceil \frac{p-2}{6} \rceil$.
6. Let $G = K_{m,n}, m, n \geq 2$ be a complete bipartite graph. Then
 $i_M^{-1}(G) = 1$.
7. Let $G = mK_2, m \geq 1$. Then $i_M^{-1}(G) = \lceil \frac{p}{4} \rceil$.

Proposition 4.3.2: Let $G = D_3(C_t)$ be a windmill graph with only three cycles of different size $t \geq 4$. Then (i) $i_M(G) = \lfloor \frac{p-2}{6} \rfloor$ and $i_M^{-1}(G) = \lceil \frac{p}{6} \rceil$ (ii) $i_M(G) < i_M^{-1}(G)$.

Proof: Let $G = D_3(C_t)$ be a windmill graph with only three cycles and each cycle is of size $t \geq 4$. Then $V(G) = \{u, c_{11}, c_{12}, \dots, c_{21}, c_{22}, \dots, c_{31}, c_{32}, \dots\}$ with $|V(G)| = p = 3t - 2$. All these three cycles C_1, C_2, C_3 meet at a vertex u . When $t = 4$ and $G = D_3(C_4)$. Then $V(G) = \{u, c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}, c_{31}, c_{32}, c_{33}\}$ and $p = 10$. Here $i_M(G) = |\{u\}| = 1$ and $i_M^{-1}(G) = |\{c_{12}, c_{22}\}| = 2$. When $t = 5$ and $G = D_3(C_5)$. Then $|V(G)| = p = 13$. Here, $i_M(G) = |\{u\}| = 1$ and $i_M^{-1}(G) = |\{c_{12}, c_{22}, c_{32}\}| = 3$.

If $t = 6$ then $G = D_3(C_6)$ and $p = 16$. Choose a set $D = \{u, c_{13}\}$ and a set $D' = \{c_{12}, c_{22}, c_{32}\} \subseteq V - D$ such that $|N[D]| = |N[D']| \geq \lceil \frac{p}{2} \rceil$. It implies that $i_M(G) = 2$ and $i_M^{-1}(G) = 3$. In general, let $V(G) = p = 3t - 2$ and $\lceil \frac{p}{2} \rceil = \lceil \frac{3t}{2} \rceil - 1$. Now, select the set $D = \{u, c_{13}, c_{23}, \dots, c_{ij}\}$ with $|D| = \lfloor \frac{p-2}{6} \rfloor$ such that $d(x_i, x_j) \geq 3$, where the vertices $x_i, x_j \in D$. Then $|N[D]| = |N[u]| + 3(|D| - 1) = 7 + 3\lfloor \frac{p-2}{6} \rfloor - 3$. Therefore $|N[D]| = 3\lfloor \frac{p-2}{6} \rfloor + 4 \geq \lceil \frac{p}{2} \rceil$. Since $d(x_i, x_j) \geq 3$, the induced subgraph $\langle D \rangle$ has only isolates. Thus D is an independent majority dominating set of G and

$$i_M(G) \leq |D| = \left\lfloor \frac{p-2}{6} \right\rfloor. \quad (4.1)$$

Suppose $|D_1| < |D|$ and $|D_1| = |D| - 1$. Then $|N[D_1]| < \lceil \frac{p}{2} \rceil$ and the induced subgraph $\langle D_1 \rangle$ has only isolates. But D_1 is not an independent majority dominating set of G and

$$i_M(G) > |D_1| \geq |D| = \left\lfloor \frac{p-2}{6} \right\rfloor \quad (4.2)$$

Combining (4.1) and (4.2), we obtain $i_M(G) = \lfloor \frac{p-2}{6} \rfloor$.

Let $D' = \{C_{13}, C_{16}, \dots, C_{1i}, C_{23}, C_{26}, \dots, C_{2j}, C_{33}, C_{36}, \dots, C_{3k}\}$ such that $d(x_i, x_j) \geq 3$ with $|D'| = \lceil \frac{p}{6} \rceil$. Then $|N[D']| = \sum_{x_i \in D'} d(x_i) = 3|D'| = 3\lceil \frac{p}{6} \rceil \geq \lceil \frac{p}{2} \rceil$. Also $D' \subseteq V - D$ and the induced subgraph $\langle D' \rangle$ has only isolates. Therefore D' is an inverse independent majority dominating set with respect to D and

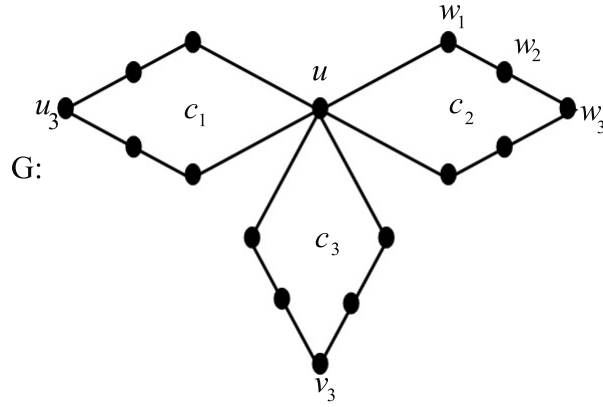
$$i_M^{-1}(G) \leq |D'| = \left\lceil \frac{p}{6} \right\rceil \quad (4.3)$$

Suppose $|D'_1| < |D'|$ and $|D'_1| = |D'| - 1$. Apply the above argument, D'_1 is not an inverse independent majority dominating set of G . Then

$$i_M^{-1}(G) > |D'_1| \text{ and } i_M^{-1}(G) \geq |D'| = \left\lceil \frac{p}{6} \right\rceil \quad (4.4)$$

Thus, by (4.3) and (4.4), $i_M^{-1}(G) = \lceil \frac{p}{6} \rceil$.

For Example; Consider the following graph $G = D_3(C_6)$.



When $t = 6$, in this graph $G = D_3(c_6)$ with 3 cycles c_6 meet at a vertex ‘ u ’ and $p = 16$. By the above result, $i_M(G) = |\{u, u_3\}| = 2$ and $i_M^{-1}(G) = |\{v_3, w_3, w_1\}| = 3$. ■

Theorem 4.3.3: Let $G = D_t(7)$ be a windmill graph with the number $t \geq 2$ of 7-cycles. Then

- (i) $i_M(G) = \begin{cases} \lceil \frac{p}{6} \rceil - \lfloor \frac{\Delta+1}{3} \rfloor, & \text{if } p \equiv 1, 2 \pmod{\left(\frac{\Delta+1}{3}\right)}. \\ \lceil \frac{p}{6} \rceil - \lfloor \frac{\Delta+1}{3} \rfloor + 1, & \text{if } p \equiv 0 \pmod{\left(\frac{\Delta+1}{3}\right)}. \end{cases}$
- (ii) $i_M^{-1}(G) = \lceil \frac{p}{6} \rceil$.
- (iii) $i_M(G) < i_M^{-1}(G)$

Proof: By the same argument of Proposition (4.3.2), the result (i) and (ii) are obtained and it implies that $i_M(G) < i_M^{-1}(G)$. ■

4.4 Characterization Theorems on Inverse Independent Majority Domination Number

This section deals with characterization theorems for $i_M^{-1}(G)$ and a necessary and sufficient theorem for a minimal inverse independent majority dominating set of a graph G .

Proposition 4.4.1: Let D be a i_M - set of a connected graph G . Then $i_M^{-1}(G) = 1$ if and only if the set $(V - D)$ contains atleast one majority dominating vertex of G .

Proof: Let $i_M^{-1}(G) = 1$. Then $D' = \{u\}$ is an inverse independent majority dominating set of G and $D' \subseteq (V - D)$. Then $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $|N[u]| \geq \lceil \frac{p}{2} \rceil$. It implies that the degree a vertex ' u ' such that $d(u) \geq \lceil \frac{p}{2} \rceil - 1$ and $u \in V - D$. Hence the vertex u is a majority dominating vertex of G in $(V - D)$. Suppose $(V - D)$ contains more than one majority dominating vertex of G , then one majority dominating vertex is enough to find an inverse independent majority dominating set for G . Hence $(V - D)$ contains atleast one majority dominating vertex of G . The converse is obvious. ■

Results 4.4.2: Given any positive integer $k > 1$, there exists a graph G such that $i_M^{-1}(G) = 1$.

Proof: Let $k \geq 2$ be any positive integer. Then there is a complete graph $G = K_k$. By the result (4.3.1)(1), $i_M^{-1}(G) = 1$, for any $k \geq 2$. ■

Theorem 4.4.3: Let D be a i_M -set of a connected graph G . Then $i_M^{-1}(G) \geq 2$ if and only if all the vertices u_i are of degree such that $d(u_i) < \lceil \frac{p}{2} \rceil - 1$, for every $u_i \in (V - D)$.

Proof: Let $i_M^{-1}(G) \geq 2$ and D' be a i_M^{-1} -set of G . Then the inverse independent majority dominating set D' which contains more than one vertex. Suppose $D' = \{u_1, u_2\}$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $D' \subseteq (V - D)$. Then $\lceil \frac{p}{2} \rceil \leq |N[D']| = \sum [d(u_1) + d(u_2)]$. It implies that either $d(u_1) < \lceil \frac{p}{2} \rceil - 1$ and $d(u_2) \geq 1$ or vice versa. Since $D' \subseteq (V - D)$, all the vertices $u_i \in D'$ with degree $d(u_i) < \lceil \frac{p}{2} \rceil - 1$. Conversely, if the set $(V - D)$ consists of the vertices with the degree $d(u_i) < \lceil \frac{p}{2} \rceil - 1$. Then any single vertex u_i could not form an inverse independent majority dominating set for G with respect to D . Hence $i_M^{-1}(G) \geq 2$. ■

Theorem 4.4.4: Let G be any connected graph with $p \geq 2$ vertices. Then $i_M^{-1}(G) = p - \kappa(G)$ if and only if G is a complete graph of order p , where $\kappa(G)$ is a vertex connectivity of G .

Proof:

$$\text{Let } i_M^{-1}(G) = p - \kappa(G). \quad (4.5)$$

Let D and D' be the independent majority dominating set and inverse independent majority dominating set of G respectively. Let u be a vertex of G such that $d(u) = \Delta(G) = |N(u)|$. Since $N[V(G) - N(u)] = V(G)$, $[V(G) - N(u)] = D'$ is an inverse independent majority dominating set of G . Hence $i_M^{-1}(G) \leq |D'| = |V(G)| - |N(u)| = p - \Delta(G)$. By assumption, $p - \kappa(G) = i_M^{-1}(G) \leq p - \Delta(G)$. It implies that $\kappa(G) \geq \Delta(G)$. By the known inequality “for any graph G , $\kappa(G) \leq \delta(G) \leq \Delta(G)$ ”. Thus, we obtain $\kappa(G) = \delta(G) = \Delta(G)$ and G is a regular graph, say κ -regular graph.

Let u be any vertex of κ -regular graph G . Then the vertex ‘ u ’ dominates $(\kappa+1)$ vertices of G . If $(\kappa+1) < \lceil \frac{p}{2} \rceil$ then $D_1 = [N[u] \cup S]$ is an inverse independent majority dominating set of G with respect to D where S is a set of $[\lceil \frac{p}{2} \rceil - (\kappa+1)]$ vertices disjoint from the vertices

of $N[u]$. Therefore $i_M^{-1}(G) \leq 1 + \lceil \frac{p}{2} \rceil - (\kappa + 1) = \lceil \frac{p}{2} \rceil - \kappa$. Then by the condition (4.5), $p - \kappa(G) \leq \lceil \frac{p}{2} \rceil - \kappa(G)$, which is impossible. Hence $(\kappa + 1) \geq \lceil \frac{p}{2} \rceil$ and $(\kappa + 1)$ vertices are dominated by one and only one vertex 'u' and $i_M^{-1}(G) = 1$. By assumption, $p - \kappa(G) = 1$. It implies that $\kappa(G) = p - 1$, vertex connectivity of G . Thus, the corresponding graph G is a complete graph of order p .

Conversely, let $G = K_p$ then the vertex connectivity of G with $\kappa(G) = p - 1$. The independent majority dominating set D and inverse independent majority dominating set D' set of a complete graph G is $D = \{u_1\}$ and $D' = \{u_2\}$ respectively, for $u_1, u_2 \in V(G)$ and $u_2 \subseteq V - D$. It implies that $i_M^{-1}(G) = |D'| = 1$ and $i_M^{-1}(G) = 1 = p - (p - 1)$. Therefore, by the result $\kappa(G) = p - 1$, $i_M^{-1}(G) = p - \kappa(G)$. Hence $i_M^{-1}(G) = p - \kappa(G)$ if and only if $G = K_p$. ■

The following theorem provides a necessary and sufficient condition for the two sets $i_M(G)$ and $i_M^{-1}(G)$ which are equal.

Theorem 4.4.5: Let G be a connected graph with p vertices. Let D and D' be the i_M -set and i_M^{-1} -set of G respectively. Then $i_M(G) = i_M^{-1}(G)$ if and only if one of the following conditions holds.

- (i) The graph G contains atleast two vertices with degree $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$.
- (ii) The graph G is regular.
- (iii) The graph G contains atleast $\lceil \frac{p}{2} \rceil$ vertices with equal degree r and other vertices u_i such that $d(u_i) < r$.
- (iv) The graph G contains an induced complete subgraph with equal degree and $\delta(G) \leq 2$.

Proof: Assume that $i_M(G) = i_M^{-1}(G)$. Since the sets D and D' are an independent majority dominating set and an inverse independent majority dominating set of G , $|D| = |D'|$ and $|N[D]| = |N[D']| \geq \lceil \frac{p}{2} \rceil$. Then $D' \subseteq (V - D)$ and an induced subgraphs $\langle D \rangle$ and $\langle D' \rangle$ both have no edges.

Case (i): Suppose $|D| = |D'| = 1$. Then $D = \{u_1\}$ and $D' = \{u_2\} \subseteq (V - D)$ such that $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$, for $i = 1, 2$. It implies that the graph G has atleast two vertices with $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$ and others are $d(u_i) < \lceil \frac{p}{2} \rceil - 1$. Hence the condition (i) holds.

If all the vertices of G may have the degree $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$, for all $u_i \in V(G)$ then all vertices u_i with same degree such that

either $d(u_i) = p - 1$ or $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$. Suppose $d(u_i) = p - 1$, for every $u_i \in V(G)$, then $G = K_p$, a complete graph with p vertices. Suppose $d(u_i) = \frac{p}{2}$, for every $u_i \in V(G)$ then $G = K_{m,n}$, a complete bipartite graph. If all the vertices u_i are having equal degree such that $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$ then the graph G must be regular. Hence the condition (ii) holds.

Case(ii): Suppose $|D| = |D'| \geq 2$. Then the vertices u_i of G are of degree $d(u_i) < \lceil \frac{p}{2} \rceil - 1$. By the assumption, the independent majority dominating set D and the inverse independent majority dominating set D' both have the same cardinality. Therefore i_M -set is $D = \{u_2, u_5, u_8, \dots, u_t\}$ and i_M^{-1} -set is $D' = \{u_3, u_6, u_9, \dots, u_t\} \subseteq (V - D)$ such that $d(u_i, u_j) \geq 3$, for $u_i, u_j \in D$, $u_i, u_j \in D'$ and $|N[D]| = |N[D']|$. Since the set D and D' are having vertices such that each vertex of D is adjacent to the vertices of D' , either all the vertices $u_i \in V(G)$ are having equal degree say ' r ' or at least $\lceil \frac{p}{2} \rceil$ vertices with equal degree r . Hence the graph G must be r -regular thus the condition (ii) holds. If the graph G has at least $\lceil \frac{p}{2} \rceil$ vertices with equal degree r then other remaining vertices u_i are having degree $d(u_i) < r$. Thus the condition (iii) holds.

Case(iii): Suppose $|D| = |D'| \geq 2$. It is proved by induction on $|D|$ and $|D'|$. When $|D| = |D'| = 2$. Then $D = \{u_1, v_1\}$ and $D' = \{u_2, v_2\} \subseteq (V - D)$ where $d(v_1) = d(v_2) \leq 2$ and $d(u_1) = d(u_2) < \lceil \frac{p}{2} \rceil - 1$. It implies that the equal degree vertices u_1 and u_2 such that u_1 and u_2 are adjacent it to a complete subgraph ‘ g ’ with degree $r = \lceil \frac{p}{2} \rceil - 2$ and other vertices v_i are of degree either $d(v_1) = 1$ or $d(v_2) = 2$, for $i = 1, 2$.

When $|D| = |D'| = 3$. Then $D = \{u_1, v_1, v_2\}$ and $D' = \{u_2, v_3, v_4\} \subseteq (V - D)$ such that $d(x_i, x_j) \geq 2$, for any $x_i, x_j \in D$ and D' and $N(u_1) = u_2$ and $N(u_2) = u_1$ with equal degree $r = \lceil \frac{p}{2} \rceil - 3$. These vertices u_1 and u_2 belong to a complete subgraph g with $d(u_1) = d(u_2)$ and other vertices v_i in D and D' are of degree either $d(v_i) = 1$ or 2 for $i = 1, 2, 3, 4$. This result is true for $|D| = |D'| = 1, 2, 3, \dots, (t-1)$.

When $|D| = |D'| = t$. Then $D = \{u_1, v_1, v_2, \dots, v_{t-1}\}$ and $D' = \{u_2, v_i, v_j, \dots, v_{t-1}\} \subseteq (V - D)$ such that $d(x_i, x_j) \geq 2$, for any $x_i, x_j \in D, D'$. Also, $u_1 = N(u_2)$, $N(u_1) = u_2$ with equal degree $r = \lceil \frac{p}{2} \rceil - t$ and $d(v_i) \leq 2$ for all v_i . It implies that $|N[D]| \geq d(u_1) + \sum_i d(v_i) \geq \lceil \frac{p}{2} \rceil - t + (t-1) + 1$ and $|N[D]| \geq \lceil \frac{p}{2} \rceil$. In a similar way, we obtain $|N[D']| \geq \lceil \frac{p}{2} \rceil$. Hence u_i ’s are adjacent in an induced subgraph of G and its forms an induced complete subgraph ‘ g ’ of G with equal

degree r . All other vertices v_i are of degree $d(v_i) \leq 2$ and hence $\delta(G) \leq 2$. Thus the condition (iv) holds. The converse part is obvious. ■

Theorem 4.4.6: Let D and D' be the i_M -set and i_M^{-1} -set of a graph G with p vertices. Then $i_M(G) < i_M^{-1}(G)$ if and only if one of the following conditions holds.

- (i) The graph G contains exactly one vertex u with $d(u) \geq \lceil \frac{p}{2} \rceil - 1$ and other vertices u_i such that $d(u_i) < \lceil \frac{p}{2} \rceil - 1$.
- (ii) The graph G contains atleast one vertex u with $d(u) < \lceil \frac{p}{2} \rceil - 1$ and other vertices v_i such that $d(v_i) < d(u)$ with $\langle v_i, v_j \rangle = e_i$, if $d(v_i) = d(v_j)$, for every $v_i, v_j \in V - D$.
- (iii) The graph G contains an induced complete subgraph $K_k, k \geq 3$ with vertices u_i such that $d(u_i) \leq \lceil \frac{p}{2} \rceil - 2$ and other vertices x_i such that $d(x_i) \leq 2$.

Proof: Let D and D' be the i_M -set and i_M^{-1} -set of G . Assume that $i_M(G) < i_M^{-1}(G)$. Then $|D| < |D'|$ and $|D'| \geq |D| + 1$.

Case (i): Suppose $i_M(G) = |D| = 1$ and $i_M^{-1}(G) = |D'| = 2$. It implies that $D = \{u\}$ and G contains only one vertex ' u ' such that

$d(u) \geq \lceil \frac{p}{2} \rceil - 1$. Since $i_M^{-1}(G) = |D'| = 2$, all other vertices $(u_i) \subseteq V - D$, are of degree $d(u_i) < \lceil \frac{p}{2} \rceil - 1$. If $d(u_i) < \lceil \frac{p}{2} \rceil - 1$ then u_i 's may also be pendants and $u_i \in V - D$, for every u_i . In this case, $i_M^{-1}(G) \geq 2$. Hence G contains exactly one vertex u such that $d(u) \geq \lceil \frac{p}{2} \rceil - 1$ and other vertices u_i such that $d(u_i) < \lceil \frac{p}{2} \rceil - 1$.

Case (ii): Suppose $i_M(G) = |D| = 2$ and $i_M^{-1}(G) = |D'| \geq 3$. By assumption, $|D| < |D'|$. By case (i), i_M -set D must have two vertices and $D = \{u_1, u_2\}$ with $d(u_1, u_2) \geq 2$ such that $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and $\langle D \rangle$ has no edges. Then the vertices u_i of D satisfies $d(u_i) < \lceil \frac{p}{2} \rceil - 1, i = 1, 2$. Since $i_M^{-1}(G) = |D'| \geq 3$, $D' = \{v_1, v_2, v_3\} \subseteq V - D$ such that $d(v_i, v_j) \geq 2$, for every $v_i, v_j \in D'$ and $|N[D']| \geq \lceil \frac{p}{2} \rceil$ with $\langle D' \rangle$ has no edges. Then the vertices of D' such that $d(v_i) < d(u_i)$, for every $v_i \in D'$ and $u_i \in D$.

If any of these vertices $v_i \in V - D$ with equal degree, then they must be adjacent. Suppose $v_i, v_j \in V - D$ and both are not adjacent with $d(v_i) = d(v_j)$. Then the i_M^{-1} -set D' includes these vertices and $|D'| = |D|$. Therefore $i_M(G) = i_M^{-1}(G)$, which is a contradiction to our assumption. Hence the vertices $v_i \in V - D$ with equal degree such that $\langle v_i, v_j \rangle = e_i$.

Case (iii): Suppose $|D| \geq 2$ and $|D'| \geq 3$. Then the i_M - set D contains atleast 2 vertices u_1 and u_2 with degree $d(u_1) \leq \lceil \frac{p}{2} \rceil - 2$ and $d(u_2) \leq 2$. Also, the i_M^{-1} - set D' contains atleast 3 vertices namely $\{v_1, v_2, v_3\} \subseteq V - D$ with $2 \leq d(v_i) \leq \lceil \frac{p}{2} \rceil - 3, i = 1, 2, 3$ and $\langle D' \rangle$ has no edges. It implies that the vertices u_1 and v_1 are adjacent in G and they form a complete subgraph $K_k, k \geq 3$. All other vertices x_i such that $d(x_i) \leq 2, x_i \neq u_1, v_1$. Hence the graph G contains an induced complete subgraph $K_k, k \geq 3$ with $d(u_i) \leq \lceil \frac{p}{2} \rceil - 2$ and other vertices are $d(x_i) \leq 2$. ■

Next, relationships among three different inverse parameters $i^{-1}(G)$, $i_M^{-1}(G)$ and $\gamma_M^{-1}(G)$ are discussed below.

Theorem 4.4.7: Let G be a connected graph with p vertices. The $i_M^{-1}(G) = i^{-1}(G)$ if and only if the graph G has atleast two full degree vertices.

Proof: Assume that $i_M^{-1}(G) = i^{-1}(G)$. Let D and D' be an independent dominating set and an inverse independent set with respect to D of G . Suppose $\Delta(G) \neq p - 1$ then $i^{-1}(G) \neq 1$ and $i^{-1}(G) = |D'| \geq 2$ is an inverse independent domination number of G . Let $D' = D_1 \cup D_2$

where $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$ such that $D_1 \cap D_2 \neq \emptyset$. Since $|N[D']| \neq \emptyset$ and $D_1 \cap D_2 \neq \emptyset$, $|N[D_1]| \geq \lceil \frac{p}{2} \rceil$ or $|N[D_2]| \geq \lceil \frac{p}{2} \rceil$. Hence either the set D_1 or D_2 is an inverse independent majority dominating set of G and $i_M^{-1}(G) \leq |D_1|$ or $i_M^{-1}(G) \leq |D_2|$. Since $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$, $i_M^{-1}(G) < |D'| = i^{-1}(G)$, which is a contradiction to the assumption. Hence $\Delta(G) = p - 1$. It implies that the graph G contains atleast one full degree vertex ' u ' such that $d(u) = p - 1$. Suppose G has exactly one vertex u with $d(u) = p - 1$, then $i(G) = 1$ and $i_M(G) = 1$ but $i^{-1}(G) \geq 2$ and $i_M^{-1}(G) \geq 2$, which is a contradiction to the assumption. Hence the graph G has atleast two full degree vertices. The converse is obvious. ■

Theorem 4.4.8: Let D and D' be the independent majority dominating set and the inverse independent majority dominating set of G . Then $i_M(G) = i_M^{-1}(G) = 1$ if and only if the graph contains atleast two majority dominating vertices.

Proof: Let $i_M(G) = |D| = |D'| = i_M^{-1}(G) = 1$. Since $i_M(G) = 1$, $D = \{u\}$ such that $|N[u]| \geq \lceil \frac{p}{2} \rceil$, G contains a vertex ' u ' with $d(u) \geq \lceil \frac{p}{2} \rceil - 1$. Since $i_M^{-1}(G) = 1$, $D' = \{v\} \subseteq V - D$ and $(V - D)$

contains a vertex 'v' with $d(v) \geq \lceil \frac{p}{2} \rceil - 1$. It implies that the vertices u and v are the majority dominating vertices of G . Suppose G contains more than two majority dominating vertices then certainly $|D| = |D'| = 1$. Thus G contains atleast two majority dominating vertices. The converse is obvious. ■

Corollary 4.4.9: If the graph G contains exactly one majority dominating vertex then $i_M^{-1}(G) \geq 2$.

Proposition 4.4.10: The connected graph G satisfies $i_M^{-1}(G) = \gamma_M^{-1}(G) = 1$ if and only if the graph G contains atleast two vertices of degree $\geq \lceil \frac{p}{2} \rceil - 1$.

Proof: Let $i_M^{-1}(G) = \gamma_M^{-1}(G) = 1$. Then $i_M(G) = \gamma_M(G) = 1$. It implies that $D = \{u_1\}$ is a majority dominating set and also an independent majority dominating set of G . Then $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and vertex u_1 has degree $d(u_1) \geq \lceil \frac{p}{2} \rceil - 1$. By the hypothesis, $D' = \{u_2\}$ is an inverse independent majority dominating set and an inverse majority dominating set of G with $|N[D']| \geq \lceil \frac{p}{2} \rceil$. It implies that the vertex u_2 is of degree $d(u_2) \geq \lceil \frac{p}{2} \rceil - 1$. Thus if $i_M^{-1}(G) = \gamma_M^{-1}(G) = 1$ then there are two vertices u_1 and u_2 in G with $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1, i =$

1, 2. These two vertices are enough to obtain the value $i_M^{-1}(G) = 1 = \gamma_M^{-1}(G)$. Hence the graph G contains atleast two vertices of degree $\geq \lceil \frac{p}{2} \rceil - 1$. ■

4.5 Bounds of the Inverse Independent Majority Domination Number

In this section, lower and upper bounds for the inverse independent majority domination number with respect to the vertices p and maximum degree Δ of G .

Proposition 4.5.1: For any graph G , $1 \leq i_M^{-1}(G) \leq \frac{p}{2}$. The bound is sharp when $G = \overline{K_p}$, p is even and $G = K_p$, a complete graph.

Proof: If a graph G has all vertices are of full degree then each single vertex forms an independent majority dominating set and an inverse independent majority dominating set for G . Hence $i_M(G) = i_M^{-1}(G) = 1$. Suppose the graph G contains all vertices are of degree $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$, then each single vertex is a majority dominating vertex of G . Therefore each vertex forms an independent majority dominating set and an inverse independent majority dominating set for G . Hence $i_M(G) = i_M^{-1}(G) = 1$.

If the graph G is minimally connected graph then $\delta(G) \geq 1$ and $\Delta(G) \leq p - 1$. Then $i_M(G) = 1$ and $i_M^{-1}(G) \geq 2$. Suppose the graph G is disconnected without isolates, then $i_M(G) \geq 2$ and $i_M^{-1}(G) \geq 2$. If the graph G has components with isolates then $i_M(G) < \binom{p}{2}$ and $i_M^{-1}(G) < \binom{p}{2}$. The bound is sharp. Let $G = \overline{K_p}$, p is even then $i_M(G) = \frac{p}{2}$ and $i_M^{-1}(G) = \frac{p}{2}$ and $i_M^{-1}(G) = 1$, if $G = K_p$. Hence $1 \leq i_M^{-1}(G) \leq \binom{p}{2}$. ■

Proposition 4.5.2: For a tree T with p vertices, $\lceil \frac{p}{6} \rceil \leq i_M^{-1}(T) \leq \lceil \frac{p}{2} \rceil - 1$. These bounds are sharp.

Proof: This result is proved by induction on the number of pendants ‘ e ’. Since every tree has $e \geq 2$ pendants, if $e = 2$, then $T = P_p$ is a path of p vertices. By the result (4.3.1)(3), $i_M^{-1}(P_p) = \lceil \frac{p}{6} \rceil$. If $e = 3$ then G is any connected graph with three pendants. The graph G is either $T = K_{1,3}$, a star or $T = D_{1,2}$, a double star or a caterpillar with $e = 3$ pendants. Then $i_M^{-1}(G) = 1 = \binom{p}{2} - 1$, if $T = K_{1,3}$ and $i_M^{-1}(G) = 1 < \lceil \frac{p}{2} \rceil - 1$, if $T = D_{1,2}$ and $i_M^{-1}(G) = 2 < \lceil \frac{p}{2} \rceil - 1$, if T is a caterpillar. Suppose the tree T takes the caterpillar structure with $(p - e)$ intermediate vertices and $e = 3$ pendants. Then $i_M(G) = i_M^{-1}(G) = \lceil \frac{p}{6} \rceil$.

Hence this is true for $e = 2, 3, 4, \dots, (p-2)$ pendants. Then $i_M^{-1}(T) \geq \lceil \frac{p}{6} \rceil$ and $i_M^{-1}(T) \leq \lceil \frac{p}{2} \rceil - 1$. If $e = p-1$ pendants then the graph T becomes $T = K_{1,p-1}$, a star. By the result (4.3.1)(2), $i_M^{-1}(K_{1,p-1}) = \lfloor \frac{p-1}{2} \rfloor = \lceil \frac{p}{2} \rceil - 1$, If p is odd. Hence $\lceil \frac{p}{6} \rceil \leq i_M^{-1}(T) \leq \lceil \frac{p}{2} \rceil - 1$, for any tree T . ■

Proposition 4.5.3: For any connected graph G , $i_M^{-1}(G) \leq \lfloor \frac{p-1}{2} \rfloor$ and $i_M^{-1}(G) = \lfloor \frac{p-1}{2} \rfloor$ if and only if $G = K_{1,p-1}$.

Proof: Let D be an i_M^{-1} -set of G . The theorem is proved by induction on the number of pendants ‘ e ’. If $e = 1$ then the connected graph G with only one pendant with atleast one cycle C and $\Delta(G) \geq 3$. Suppose G contains exactly one vertex ‘ u ’ such that $d(u) = \Delta(G) = 3$ and other vertices $d(u_i) < 3$. Then $i_M(G) \leq \lceil \frac{p}{6} \rceil$. Since $d(u_i) = 2$, by the result (4.3.1)(3), $i_M^{-1}(C_p) = \lceil \frac{p}{6} \rceil < \lfloor \frac{p-1}{2} \rfloor$. If $e = 2$ then $G = P_p$ is a path with two pendants and by the result (4.3.1)(3) and $i_M^{-1}(P_p) = \lceil \frac{p}{6} \rceil < \lfloor \frac{p-1}{2} \rfloor$.

This result is true for all pendants $e = 1, 2, 3, \dots, (p-3)(p-2)$ and we obtain $i_M^{-1}(G) < \lfloor \frac{p-1}{2} \rfloor$. Suppose $e = p-1$ then G is a star $K_{1,p-1}$ with $(p-1)$ pendants. By the result (4.3.1)(2), $i_M^{-1}(G) = \lfloor \frac{p-1}{2} \rfloor$. Hence for any connected graph G , $i_M^{-1}(G) \leq \lfloor \frac{p-1}{2} \rfloor$. Also equality

holds only for $G = K_{1,p-1}$. Thus $i_M^{-1}(G) = \lfloor \frac{p-1}{2} \rfloor$ if and only if $G = K_{1,p-1}$. ■

Theorem 4.5.4: For any graph G with p vertices, $i_M^{-1}(G) \geq \lceil \frac{p}{2(\Delta+1)} \rceil$.

The bound is sharp.

Proof: Let $D = \{v_2, v_4, \dots, v_{i_M}\}$ be a i_M -set of a graph G . Then $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and $\langle D \rangle$ has only isolates such that $d(v_i, v_j) \geq 2$, for any $v_i, v_j \in D, i \neq j$. Let $D' = \{v_3, v_5, \dots, v_t\} \subseteq V - D$ be a i_M^{-1} -set of G with $|D'| = t$. Then $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and D' contains independent vertices. Since each vertex of D' is adjacent to the vertex of D and D' contains the maximum degree vertices $\Delta(G)$, $|N[D']| = \sum_{v_i \in D'} d(v_i) + t$ and $\lceil \frac{p}{2} \rceil \leq |N[D']| = \sum_{v_i \in D'} \Delta(G) + t = t(\Delta(G) + 1)$ where $d(v_i) = \Delta(G)$. Therefore $\lceil \frac{p}{2} \rceil \leq |D'|(\Delta + 1)$. It implies that $\lceil \frac{p}{2} \rceil \leq i_M^{-1}(G)(\Delta + 1)$. Hence $i_M^{-1}(G) \geq \frac{\lceil \frac{p}{2} \rceil}{(\Delta + 1)} = \lceil \frac{p}{2(\Delta + 1)} \rceil$. The bound is sharp for $G = K_p$ and $P_p, p \geq 2$. By the result (4.3.1)(1), $i_M^{-1}(K_p) = 1 = \lceil \frac{p}{2(\Delta + 1)} \rceil$, when $\Delta = p - 1$ and $i_M^{-1}(P_p) = \lceil \frac{p}{6} \rceil = \lceil \frac{p}{2(\Delta + 1)} \rceil$, when $\Delta = 2$. ■

Corollary 4.5.5: For any connected graph G , $\lceil \frac{p}{2(\Delta + 1)} \rceil \leq i_M^{-1}(G) \leq \lfloor \frac{\Delta}{2} \rfloor$. This upper bound is sharp if $G = K_{1,p-1}$ when $(p - 1)$ is odd.

Proposition 4.5.6: Let $G = (V, E)$ be any graph with an inverse independent domination number $i^{-1}(G)$. Then $i_M^{-1}(G) \leq \lceil \frac{i^{-1}(G)}{2} \rceil$.

Proof: Let D_M and D be an independent majority dominating set and independent dominating set of G respectively. Suppose D' is an inverse independent dominating set of G . Then D' is a i^{-1} -set of G and $|N[D']| = |V(G)|$. Let $D' = D_1 \cup D_2$ where $|D_1| = \lfloor \frac{i^{-1}(G)}{2} \rfloor$ and $|D_2| = \lceil \frac{i^{-1}(G)}{2} \rceil$. Now $N[D] = |(N[D_1] - N[D_2]) \cup (N[D_2])|$ and $|N[D]| = |N[D_1] - N[D_2]| \cup |N[D_2]|$. It implies that either $|N[D_1] - N[D_2]| \geq \lceil \frac{p}{2} \rceil$ or $|N[D_2]| \geq \lceil \frac{p}{2} \rceil$. If $|N[D_2]| \geq \lceil \frac{p}{2} \rceil$ and $D_2 \subseteq V - D_M$ then D_2 is a i_M^{-1} -set of G where D_M is an independent majority dominating set of G . If $|N[D_1] - N[D_2]| \geq \lceil \frac{p}{2} \rceil$ and $D_2 \subseteq V - D_M$ then D_2 is an inverse independent majority dominating set of G when p is even. Hence $i_M^{-1}(G) \leq |D_2| = \lceil \frac{i^{-1}(G)}{2} \rceil$. Thus $i_M^{-1}(G) \leq \lceil \frac{i^{-1}(G)}{2} \rceil$.

For Example: Let cycle $G = C_{12}$. Then $i^{-1}(G) = 4$ and $i_M^{-1}(G) = 2 = \lceil \frac{i^{-1}(G)}{2} \rceil$. Let $G = S(K_{1,10})$ be a subdivision of a star by dividing each edge exactly once. Then $i^{-1}(G) = 11$ and $i_M^{-1}(G) = 5 < \lceil \frac{i^{-1}(G)}{2} \rceil$.

■

Theorem 4.5.7: Let D be a i_M -set of a connected graph with p vertices. Let $\Delta_1(G)$ and $\Delta_2(G)$ be the first maximum degree and second maximum degree of a graph $(V - D) \subseteq V(G)$ respectively. Then

(i) $i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil - \Delta_2(G)$, if $\Delta_2(G) < \lceil \frac{p}{2} \rceil - 1$. This bound is sharp if $G = P_7$.

(ii) $i_M^{-1}(G) \leq \lceil \frac{p - \Delta_2(G)}{2} \rceil$, if $\Delta_2(G) \geq \lceil \frac{p}{2} \rceil - 1$. This bound is sharp if $G = K_p$.

Proof:

Case (i): Let $\Delta_1(G) = \Delta_2(G) \geq \lceil \frac{p}{2} \rceil - 1$. If $\Delta_2(G) \geq \lceil \frac{p}{2} \rceil - 1$ then certainly $\Delta_1(G) \geq \lceil \frac{p}{2} \rceil - 1$. Therefore $D_1 = \{u_1\}$ and $D_2 = \{u_2\} \subseteq V - D_1$ are i_M -set and i_M^{-1} -set of G such that $|N[D_1]| \geq \lceil \frac{p}{2} \rceil$ and $|N[D_2]| \geq \lceil \frac{p}{2} \rceil$ with $d(u_1) = \Delta_1(G)$ and $d(u_2) = \Delta_2(G)$. Hence $i_M(G) = |D_1| = 1$ and $i_M^{-1}(G) = |D_2| = 1$.

Subcase (i): Let $\Delta_2(G) = \lceil \frac{p}{2} \rceil - 1$. Then $|N[u_2]| = \lceil \frac{p}{2} \rceil$ and $i_M^{-1}(G) = \left\lceil \frac{p - \lceil \frac{p}{2} \rceil + 1}{2} \right\rceil = \left\lceil \frac{\lfloor \frac{p}{2} \rfloor + 1}{2} \right\rceil = \lfloor \frac{p}{4} \rfloor + 1 < \lceil \frac{p - \Delta_2(G)}{2} \rceil$. Thus $i_M^{-1}(G) < \lceil \frac{p - \Delta_2(G)}{2} \rceil$, if $\Delta_2(G) \geq \lceil \frac{p}{2} \rceil - 1$.

Subcase (ii): Let $\Delta_1(G) = \Delta_2(G) = (p - 1)$. Then the graph G contains atleast two vertices u_1, u_2 and $D_1 = \{u_1\}$ such that $|N[D_1]| = p$ and $D_2 = \{u_2\} \subseteq V - D$ such that $|N[D_2]| = p$. Hence

$$i_M(G) = |D_1| = 1 \text{ and } i_M^{-1}(G) = |D_2| = 1 = \lceil \frac{p-(p-1)}{2} \rceil = \lceil \frac{p-\Delta_2(G)}{2} \rceil.$$

Thus $i_M^{-1}(G) = \lceil \frac{p-\Delta_2(G)}{2} \rceil$, if $\Delta_2(G) > \lceil \frac{p}{2} \rceil - 1$. This bound is sharp for a complete graph $G = K_p, p \geq 2$.

Case (ii): Let $\Delta_1(G) \geq \lceil \frac{p}{2} \rceil - 1$ and $\Delta_2(G) < \lceil \frac{p}{2} \rceil - 1$. Then the graph G contains exactly one majority dominating vertex u_1 and $D_1 = \{u_1\}$ such that $|N[D_1]| \geq \lceil \frac{p}{2} \rceil$. It implies that $i_M(G) = |D_1| = 1$.

Subcase (i): Suppose $\Delta_2(G) = 1 < \lceil \frac{p}{2} \rceil - 1$. Then choose the set $D_2 = \{u_2, \dots, u_t\} \subseteq V - D_1$ such that $|D_2| = t = \lceil \frac{p-1}{2} \rceil$ where $u_2, \dots, u_t$ are pendants and $|N[D_2]| = \lceil \frac{p-1}{2} \rceil + 1 = \lceil \frac{p}{2} \rceil$. Hence D_2 is a i_M^{-1} -set of G with respect to D_1 and $i_M^{-1}(G) = |D_2| \leq \lceil \frac{p-1}{2} \rceil \leq \lceil \frac{p}{2} \rceil - 1$. Thus $i_M^{-1}(G) = \lceil \frac{p}{2} \rceil - \Delta_2(G)$ if $\Delta_2(G) < \lceil \frac{p}{2} \rceil - 1$. This bound is sharp when $G = K_{1,p-1}$, if p is odd with $\Delta_2(G) = 1$.

Subcase (ii): Suppose $\Delta_2(G) = 2 < \lceil \frac{p}{2} \rceil - 1$, Then choose the set $D_2 = \{u_2, u_5, \dots, u_t\} \subseteq V - D_1$ with $|D_2| = |t| = \lceil \frac{p}{6} \rceil$ such that $d(u_i, u_j) \geq 3$ for $u_i, u_j \in D_2$. Therefore $|N[D_2]| \geq \sum_{i=1}^t d(u_i) + t \geq 3t = 3\lceil \frac{p}{6} \rceil \geq \lceil \frac{p}{2} \rceil$. Since $D_2 \subseteq V - D_1$ and $\langle D_2 \rangle$ has no edges, D_2 is a i_M^{-1} -set of G with respect to D_1 and $i_M^{-1}(G) \leq |D_2| = \lceil \frac{p}{6} \rceil \leq \lceil \frac{p}{2} \rceil - \Delta_2(G)$. This bound is sharp if $G = P_7$ with $\Delta_2(G) = 2$. By the result (4.3.1)(3), $i_M^{-1}(G) = 2 = \lceil \frac{p}{6} \rceil = \lceil \frac{p}{2} \rceil - \Delta_2(G)$. For $G = P_{19}$ with $\Delta_2(G) = 2$, $i_M^{-1}(P_p) = \lceil \frac{p}{6} \rceil = 4 < \lceil \frac{p}{2} \rceil - \Delta_2(G)$.

Subcase (iii): Suppose $3 \leq \Delta_2(G) < \lceil \frac{p}{2} \rceil - 1$. Then the set $D_2 = \{u_2, \dots, u_t\} \subseteq V - D_1$ such that $|D_2| = t = \lceil \frac{p}{8} \rceil$ and $|N[D_2]| \geq 4t = 4\lceil \frac{p}{8} \rceil = \lceil \frac{p}{2} \rceil$. Hence D_2 is a i_M^{-1} -set of G with respect to D_1 and $i_M^{-1}(G) = |D_2| = \lceil \frac{p}{8} \rceil < \lceil \frac{p}{2} \rceil - \Delta_2(G)$. The bound is sharp. If $G = W_{15}$, a wheel graph with $\Delta_2(G) = 3$ then $i_M^{-1}(G) = 3 < \lceil \frac{p}{2} \rceil - \Delta_2(G)$. Thus, in all cases of $\Delta_2(G) < \lceil \frac{p}{2} \rceil - 1$, $i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil - \Delta_2(G)$.

Case(iii): Let $\Delta_1(G) = \Delta_2(G) < \lceil \frac{p}{2} \rceil - 1$. Let D_1 and D_2 be the independent majority dominating set and inverse independent majority dominating set of G and $i_M(G) = i_M^{-1}(G)$. Since G is connected graph $\Delta_1(G) = \Delta_2(G) = 2$. Then the graph becomes $G = P_p$ or C_p . By the result (4.3.1)(3), $i_M(G) = i_M^{-1}(G) = \lceil \frac{p}{6} \rceil < \lceil \frac{p}{2} \rceil - \Delta_2(G)$, if $p \geq 9$ and $i_M^{-1}(G) = \lceil \frac{p}{6} \rceil = \lceil \frac{p}{2} \rceil - \Delta_2(G)$, if $G = P_7, P_8, C_7, C_8$. Suppose $\Delta_1(G) = \Delta_2(G) \geq 3$ then the graph G becomes a caterpillar structure. Hence $i_M^{-1}(G) \geq \lceil \frac{p}{8} \rceil$ and $i_M^{-1}(G) < \lceil \frac{p}{2} \rceil - \Delta_2(G)$. Thus in all these three cases, we find that $i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil - \Delta_2(G)$, if $\Delta_2(G) < \lceil \frac{p}{2} \rceil - 1$ and $i_M^{-1}(G) \leq \lceil \frac{p - \Delta_2(G)}{2} \rceil$, if $\Delta_2(G) \geq \lceil \frac{p}{2} \rceil - 1$. ■

4.6 Comparison of $i_M^{-1}(G)$, $\gamma_M^{-1}(G)$, $i^{-1}(G)$ and $i_M(G)$

The following results are concerned with the comparison of domination parameters $i_M^{-1}(G)$, $\gamma_M^{-1}(G)$, $i^{-1}(G)$ and $i_M(G)$ of a graph G .

Proposition 4.6.1: If a graph G is a path or a cycle for $p \geq 7$ with $i_M^{-1}(G) \geq 2$ and $\gamma_M^{-1}(G) \geq 2$ then $i_M^{-1}(G) = \gamma_M^{-1}(G)$.

Proof: Let $G = P_p$ or $G = C_p, p \geq 7$. Let the inverse independent majority domination number $i_M^{-1}(G) \geq 2$ and the inverse majority domination number $\gamma_M^{-1}(G) \geq 2$. Let D_1 and D_2 be the inverse majority dominating sets and majority dominating sets of G . By the Proposition [2.3.1](2), $\gamma_M^{-1}(G) = \lceil \frac{p}{6} \rceil$ and by the result (4.3.1)(3), $i_M^{-1}(G) = \lceil \frac{p}{6} \rceil$. Since $|D_1| \geq 2$, $D_1 = \{u_i, u_j\}$ such that $d(u_i, u_j) \geq 3$ and $D'_1 = \{u_{i+1}, u_{j+1}\} \subseteq V - D_1$ such that $d(u_{i+1}, u_{j+1}) \geq 3$ is an inverse independent majority dominating set of G . By the results (4.3.1)(3), $i_M^{-1}(G) = \lceil \frac{p}{6} \rceil, p \geq 7$. Similarly, since $|D_2| \geq 2$, D_2 and D'_2 are taken by the above argument in G . Then by the results (2.3.1)(2), $\gamma_M^{-1}(G) = \lceil \frac{p}{6} \rceil, p \geq 7$. Hence $i_M^{-1}(G) = \gamma_M^{-1}(G) = \lceil \frac{p}{6} \rceil, p \geq 7$. ■

Corollary 4.6.2: If a graph G contains atleast two full degree vertices then $i_M^{-1}(G) = \gamma_M^{-1}(G)$.

Proof: Since u_1 and u_2 are full degree vertices of G , $d(u_i) = p-1, i = 1, 2$. Then $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$ and u_i 's, are majority dominating vertices of G . By the Proposition [4.4.10], $i_M^{-1}(G) = \gamma_M^{-1}(G)$. ■

Proposition 4.6.3: If a graph G contains one majority dominating vertex and other vertices of degree $d(u_i) \leq 2$, then $i_M^{-1}(G) = \gamma_M^{-1}(G)$.

Proof: Let $V(G) = \{u, u_1, u_2, \dots, u_{p-1}\}$ where $d(u) \geq \lceil \frac{p}{2} \rceil - 1$ and $d(u_i) \leq 2$, for $i = 1, 2, \dots, p-1$. Then $D = \{u\}$ be a majority dominating set and an independent majority dominating set of G . Let $D' = \{u_1, u_2, \dots, u_t\} \subseteq (V - D)$ such that $d(u_i, u_j) \geq 2$ and $|N[D']| \geq \lceil \frac{p}{2} \rceil$ with $|D'| = t$. Then D' is an inverse majority dominating set of G and $\gamma_M^{-1}(G) = |D'| = t$. Since each $u_i, i = 1, 2, \dots, t$ is a pendant vertex or $d(u_i) = 2$ with $d(u_i, u_j) \geq 2, \langle D' \rangle$ has no edges and D' is also an inverse independent majority dominating set of G . Then $i_M^{-1}(G) = |D'| = t$. It implies that $i_M^{-1}(G) = \gamma_M^{-1}(G) = t$. ■

Corollary 4.6.4: If a graph G contains a complete subgraph $g = K_k, k \geq 4$ with $d(u_i) \leq \lceil \frac{p}{2} \rceil - 3$, for every $u_i \in V(g)$ and pendants at each vertex of the subgraph ‘ g ’ then $\gamma_M^{-1}(G) \leq i_M^{-1}(G)$.

Corollary 4.6.5: If the graph G contains one vertex u such that $d(u) = \lceil \frac{p}{2} \rceil - 2$ and other vertices u_i such that $d(u_i) < \lceil \frac{p}{2} \rceil - 2$ and G includes an induced complete subgraph K_k then $\gamma_M^{-1}(G) < i_M^{-1}(G)$.

Result 4.6.6: (i) Given any positive integer $k \geq 1$, there exists a graph G which is not complete with $p = 2k + 1$ such that $i_M^{-1}(G) = k$.

Proof: There exists a graph $G = K_{1,p-1}$, a star with p vertices and $i_M^{-1}(G) = \lfloor \frac{p-1}{2} \rfloor$. Then $\left(\frac{p-1}{2}\right) = k$, if $p = 2k + 1$. Let $G = K_{1,2k}$, $k \geq 1$ be the graph with $p = 2k + 1$ vertices. By the result (4.3.1)(2), $i_M^{-1}(G) = \frac{2k}{2} = k$. Hence for a given integer $k \geq 1$, there exists a graph $G = K_{1,2k}$ which is not complete such that $i_M^{-1}(G) = k$. ■

Results 4.6.7: Given any positive integer $k \geq 1$, there exists a graph G with $p = 2k + 1$ for which $i^{-1}(G) - i_M^{-1}(G) = k$.

Proof: Let $k \geq 1$ be any given integer. Let $G = K_{1,p-1}$, p is odd. By the result (4.3.1)(2), $i_M^{-1}(G) = \frac{p-1}{2}$ and $i^{-1}(G) = p - 1$. When $p = 2k + 1$, then $i_M^{-1}(G) = \frac{p-1}{2} = \frac{2k}{2} = k$ and $i^{-1}(G) = 2k$. Thus, we obtain $i^{-1}(G) - i_M^{-1}(G) = 2k - k = k$, $k \geq 1$. ■

Results 4.6.8: There exists a graph G with p vertices for which $i_M^{-1}(G) - \gamma_M^{-1}(G) = 1$.

Proof: There exists a graph structure $G = K_4^{(4)}$ with a complete subgraph K_4 and four pendants at each vertex of K_4 . Let $V(G) = \{v_1, v_2, v_3, v_4, v_{11}, \dots, v_{14}, v_{21}, \dots, v_{24}, v_{31}, \dots, v_{34}, v_{41}, \dots, v_{44}\}$ and $p = |V(G)| = 20$. The set $D = \{v_1, v_2\}$ and $D' = \{v_3, v_4\} \subseteq V - D$ are the majority dominating set and the inverse majority dominating set of G . Then we get $\gamma_M(G) = 2 = \gamma_M^{-1}(G)$.

Now, the set $S = \{v_1, v_{21}, v_{22}\}$ and $S' = \{v_3, v_{41}, v_{42}\} \subseteq V - S$ such that $|N[S]| = |N[S']| = \lceil \frac{p}{2} \rceil$ and the vertices of S and S' are independent. Then $i_M(G) = |S| = 3 = |S'| = i_M^{-1}(G)$. Hence $i_M^{-1}(G) = 3$ and $\gamma_M^{-1}(G) = 2$. This graph G always satisfies $i_M^{-1}(G) - \gamma_M^{-1}(G) = 1$. ■

The next result shows the property to satisfy the result $i_M(G) = i_M^{-1}(G)$.

Theorem 4.6.9: If a connected graph G contains an induced complete subgraph $\langle K_k \rangle, k \geq 3$ with equal degree and other vertices are pendants then $i_M(G) = i_M^{-1}(G)$.

Proof: Let G be a connected graph which contains an induced complete subgraph ' g ' = $\langle K_k \rangle, k \geq 3$. Let $V(G) = \{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_{p-k}\}$ and $V(g) = \{u_1, u_2, \dots, u_k\} \subseteq V(G)$ such that $d(u_i) = d(u_j)$, for every $u_i, u_j \in V(g)$ and the vertices $v_i \in V(G) - V(g)$ such that $d(v_i) = 1$. The theorem is proved by induction on $k \geq 3$.

When $k = 3, V(g) = \{u_1, u_2, u_3\}$ and the remaining $(p - 3)$ vertices are pendants in G . If $d(u_1) \geq \lceil \frac{p}{2} \rceil - 1$ then $D = \{u_1\}$ is an independent majority dominating set of G . Since $d(u_1) = d(u_2) = d(u_3), D' = \{u_2\}$ is an inverse independent majority dominating set of G with respect to D . Hence $i_M(G) = i_M^{-1}(G) = 1$. If $d(u_1) < \lceil \frac{p}{2} \rceil - 1$

then $D = \{u_1, v_1, v_2, \dots, v_t\}$ with $|D| = t + 1$ such that $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and it forms an independent majority dominating set for G where $(v_1, v_2, \dots, v_t)$ are pendants. Since the vertices $u_i, i = 1, 2, 3$ are equal degree, the set $D' = \{u_2, v_{t+1}, v_{t+2}, v_{2t}\} \subseteq V - D$ with $|D'| = t + 1$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and it forms an inverse independent majority dominating set for G with respect to D . Since v_i 's are pendants, the induced subgraphs $\langle D \rangle$ and $\langle D' \rangle$ both have no edges. Then $i_M(G) = |D| = t + 1$ and $i_M^{-1}(G) = |D'| = t + 1$. Hence $i_M(G) = i_M^{-1}(G)$.

When $k = 4, V(g) = \{u_1, u_2, u_3, u_4\}$ and the remaining $(p - 4)$ vertices are pendants in G . If $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1, i = 1, \dots, 4$ then $i_M(G) = i_M^{-1}(G) = 1$. Suppose $d(u_i) < \lceil \frac{p}{2} \rceil - 1$. Then $D = \{u_1, v_1, v_2, \dots, v_t\}$ with $|D| = t + 1$, where $d(v_i) = 1, i = 1, 2, \dots, t$ and $D' = \{u_2, v_{t+1}, v_{t+2}, \dots, v_{2t}\} \subseteq V - D$ with $|D'| = t + 1$, where $d(v_j) = 1$ for $j = t + 1, \dots, 2t$. Then $|N[D]| = |N[D']| \geq \lceil \frac{p}{2} \rceil$ and the induced subgraph $\langle D \rangle$ and $\langle D' \rangle$ both have no edges. It implies that D and D' are the independent majority dominating set and the inverse independent majority dominating set for G . Hence $i_M(G) = |D| = t + 1 = |D'| = i_M^{-1}(G)$. This result is true for $k = 3, 4, \dots, t$.

Suppose the graph G contains $(p - k)$ pendants and a complete subgraph $K_k, k \geq 3$. Then if $d(u_i) > \lceil \frac{p}{2} \rceil - 1$, for any $u_i \in V(g)$ and all u_i 's are equal degree then $i_M(G) = 1 = i_M^{-1}(G)$. Suppose $d(u_i) < \lceil \frac{p}{2} \rceil - 1$. Then the independent majority dominating set D contains atleast two vertices in which any one vertex $u_i \in V(g)$ and other vertices v_i 's are pendants. Therefore in general, the set $D = \{u_1, v_1, v_2, \dots, v_t\}$ such that $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and $|D| = t + 1$. Since ' g ' is a complete subgraph, all vertices of D are non-adjacent vertices and $i_M(G) = |D| = t + 1$.

Next the set $D' = \{u_2, v_{t+1}, \dots, v_{2t}\} \subseteq V - D$ with the same property $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $\langle D' \rangle$ has no edges. It implies that D' is an inverse independent majority dominating set of G with respect to D with $|D'| = t + 1$. Hence $i_M^{-1}(G) = |D'| = t + 1 = i_M(G)$. In all values of $k \geq 3, i_M(G) = i_M^{-1}(G)$. ■

Proposition 4.6.10: Let G be a disconnected graph with isolates and p vertices. If the graph G contains exactly one vertex u such that $d(u) \geq \lceil \frac{p}{2} \rceil - 2$ and $\delta(G) \leq 1$ then $i_M(G) < i_M^{-1}(G)$.

Proof: Let $g_1, g_2, \dots, g_m$ be the m-component of a disconnected graph G . If the component g_1 contains a vertex u such that $d(u) \geq \lceil \frac{p}{2} \rceil - 2$

then $D = \{u, u_1\}$ is an independent majority dominating set of G . Since $\delta(G) \leq 1$, $|N[D]| \geq \lceil \frac{p}{2} \rceil - 1 + 1 = \lceil \frac{p}{2} \rceil$ and $i_M(G) = |D| = 2$. Let $D' = \{u_2, \dots, u_{t+1}\} \subseteq (V - D)$ with $|D'| = t$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$. Since other vertices $u_i \in V(G)$ are all pendants and isolates, $|D'| = t \geq 3$ and D' is an inverse independent majority dominating set of G with respect to D . Hence $i_M^{-1}(G) = |D'| \geq 3$. Thus $i_M(G) < i_M^{-1}(G)$. ■

4.7 Bounds of $i_M^{-1}(G)$ for Disconnected Graphs

In this section, the bounds of an inverse independent majority domination number for a disconnected graph with isolates and without isolates are discussed in detail.

Proposition 4.7.1: If a disconnected graph G with isolates and p vertices then $i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil - 1$.

Proof: This result is proved by induction on the number of edges ‘ q ’. Let $q = 1$, then $G = K_2 \cup \overline{K_{p-2}}$ and $V(G) = \{u_1, u_2, u_i\}$ where u_i ’s are isolates for $i = 3, \dots, p$. Let $D = \{u_1, u_i\}$, where $u_1 \in V(K_2)$

and $|u_i| = \lceil \frac{p}{2} \rceil - 2$ such that $|D| = \lceil \frac{p}{2} \rceil - 1$ and $|N[D]| = |N[u_1]| + |u_i| = \lceil \frac{p}{2} \rceil$. Therefore D is an independent majority dominating set of G and $i_M(G) = \lceil \frac{p}{2} \rceil - 1$. Let $D' = \{u_2, u_i\}$ with $|D'| = \lceil \frac{p}{2} \rceil - 1$ where $u_2 \in V(K_2)$ and by the above argument, $|N[D']| = \lceil \frac{p}{2} \rceil$ and $D' \subseteq V - D$. Then D' is an inverse independent majority dominating set of G and $i_M^{-1}(G) = |D'| = \lceil \frac{p}{2} \rceil - 1$.

When $q = 2$ then $G = P_3 \cup \overline{K_{p-3}}$ and $V(G) = \{u_1, u_2, u_3, u_i\}$ where u_i 's are isolates. Let $D = \{u_2, u_i\}$ where $u_2 \in V(P_3)$ and $|D - \{u_2\}| = |u_i| = \lceil \frac{p}{2} \rceil - 3$ with $|D| = \lceil \frac{p}{2} \rceil - 2$. Then $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and $i_M(G) = |D| = \lceil \frac{p}{2} \rceil - 2$. Let $D' = \{u_1, u_i\}$ where $u_1 \in V(P_3)$ and $|D' - \{u_1\}| = |u_i| = \lceil \frac{p}{2} \rceil - 2$ isolates with $|D'| = \lceil \frac{p}{2} \rceil - 1$. Then $|N[D']| = \lceil \frac{p}{2} \rceil$ and $D' \subseteq (V - D)$. It implies that D' is an inverse independent majority dominating set of G with respect to D and $i_M^{-1}(G) = |D'| = \lceil \frac{p}{2} \rceil - 1$. Hence $i_M^{-1}(G) = \lceil \frac{p}{2} \rceil - 1$.

If $q = 1, 2, \dots, (q-2)$, then $i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil - 1$. Suppose $q = q-1$ and $u_i = 1$ or 2 isolates. If any component with $(q-1)$ edges then $i_M^{-1}(G) < \lceil \frac{p}{2} \rceil - 1$. If the graph G consists of a complete graph K_{p-1} with one isolate or K_{p-2} with two isolates then $i_M(G) = 1 = i_M^{-1}(G) < \lceil \frac{p}{2} \rceil - 1$. Hence $i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil - 1$. This bound is sharp if $G = K_2 \cup \overline{K_{p-2}}$. ■

Corollary 4.7.2: If a graph G is a totally disconnected graph with p vertices then $i_M^{-1}(G) = \frac{p}{2}$, when p is even.

Remark 4.7.3: For any disconnected graph G , the inverse independent majority domination number $i_M^{-1}(G)$ never reach the bound $\lceil \frac{p}{2} \rceil$. When p is odd, $i_M(G) = \lceil \frac{p}{2} \rceil$ but $i_M^{-1}(G)$ does not exist. This is true for, $G = \overline{K_p}$, p is odd.

Theorem 4.7.4: Let G be a disconnected graph without isolates and with p vertices. Then $i_M^{-1}(G) \leq \lceil \frac{p}{4} \rceil$ and $i_M^{-1}(G) \leq \lceil \frac{p}{4} \rceil$ if and only if $G = mK_2, m \geq 2$.

Proof: Let G be a disconnected graph with no isolates and p vertices. Then G has $g_1, g_2, \dots, g_m$ components.

Case (i): In a graph G , each component $g_i, i = 1, \dots, m$ are r -regular subgraphs.

Subcase (i): If the degree of each vertex in each components g_i is one then $G = mK_2$ and g_i 's are 1-regular. Since $\Delta(G) = 1$, By the result (1.4), $i_M(G) = \lceil \frac{p}{4} \rceil$. Therefore $D = \{u_1, u_2, \dots, u_{\lceil \frac{p}{4} \rceil}\}$ is a i_M - set of G . In a similar way, $D' = \{v_1, v_2, \dots, v_{\lceil \frac{p}{4} \rceil}\} \subseteq V - D$

and $\langle u_i, v_i \rangle = e_i = K_2$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $\langle D' \rangle$ no isolates.

Therefore D' is an inverse independent majority dominating set of G with respect to D . Hence $i_M^{-1}(G) = \lceil \frac{p}{4} \rceil$.

Subcase (ii): If the degree of each vertex in each components $g_i \geq 2$, then $G = mg_r$, each g_r is r -regular graph and $r \geq 3$ with degree $d(u_i) \geq 2$. When $r = 2$, then $G = mc_i$, all are Cycles. By the results (4.3.1)(3), $i_M^{-1}(G) = \lceil \frac{p}{6} \rceil < \lceil \frac{p}{4} \rceil$. When $r \geq 3$, let $V(G) = \{v_1, v_2, \dots, v_{g_1}, u_1, u_2, \dots, u_{g_2}, \dots, w_1, w_2, \dots, w_{g_3}, \dots\}$, such that $d(x_i) \geq 3$, for every $x_i \in V(G)$.

Choose the set $D = \{v_1, u_1, \dots, w_1\}$ with $|D| = t$ where each vertex belongs to each component $g_i, i = 1, 2, \dots, m$. Such that $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and $\langle D \rangle$ has isolates. Suppose $d(x) = r = 3$, then $|D| = \lceil \frac{p}{8} \rceil$ and $|N[D]| \geq \lceil \frac{p}{2} \rceil$. Therefore $i_M(G) \leq |D| = \lceil \frac{p}{8} \rceil$. Similarly, choose the set $D' = \{v_2, u_2, \dots, w_2\} \subseteq V - D$ with $|D'| = \lceil \frac{p}{8} \rceil$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$. Since $\langle D' \rangle$ has no edges, D' is an inverse independent majority dominating set of G with respect to D and $i_M^{-1}(G) \leq |D'| \leq \lceil \frac{p}{8} \rceil < \lceil \frac{p}{4} \rceil$.

The result is true for $r = 1, 2, \dots, (r - 1)$ and the degree of each vertex $(r - 1)$. Suppose $g_r, r \geq 2$ and the value of r increases, the

degree of each vertex in g_r will increase. It implies that the inverse independent majority domination number $i_M^{-1}(G) < \lceil \frac{p}{6} \rceil < \lceil \frac{p}{4} \rceil$. Hence $i_M(G) \leq \lceil \frac{p}{4} \rceil$, if the components g_i are r -regular.

Case (ii): In the graph G , each component $g_i, i = 1, 2, \dots, m$ are not regular.

Subcase (i): If the components g_i contains all the vertices of degree $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$, for all $i = 1, 2, \dots, m$. Then $i_M(G) = i_M^{-1}(G) = 1$. Hence $i_M^{-1}(G) < \lceil \frac{p}{4} \rceil$.

Subcase (ii): If the components g_i contains the vertices of degree $d(u_i) < \lceil \frac{p}{2} \rceil - 2$, for all $i = 1, 2, \dots, m$. Then $i_M(G) \geq 2$ and $i_M^{-1}(G) \geq 2$. Thus $i_M^{-1}(G) < \lceil \frac{p}{4} \rceil$.

Subcase (iii): In general, if the components g_i contains the vertices of degree $d(u_i) \leq 2$ then $i_M(G) = \lceil \frac{p}{6} \rceil$ and $i_M^{-1}(G) = \lceil \frac{p}{6} \rceil < \lceil \frac{p}{4} \rceil$. Hence in all cases, $i_M^{-1}(G) < \lceil \frac{p}{4} \rceil$. Thus for a disconnected graph G without isolates, $i_M^{-1}(G) \leq \lceil \frac{p}{4} \rceil$ and by the result (4.3.1)(7), $G = mK_2$, if and only if $i_M^{-1}(G) = \lceil \frac{p}{4} \rceil$. ■

4.8 Nordhus-Gaddum Type Results

The following theorem gives a famous property Nordhus-Gaddum type results with respect to $i_M(G)$ and $i_M^{-1}(G)$.

Theorem 4.8.1: If a connected graph G has i_M -set and i_M^{-1} -sets then

$$(i) \ 2 \leq i_M(G) + i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil \text{ and } (ii) \ 1 \leq i_M(G) \cdot i_M^{-1}(G) \leq \left(\lceil \frac{p}{6} \rceil \right)^2.$$

These bounds are sharp.

Proof: Case (i): Suppose all the vertices u_i of G are of full degree then $d(u_i) = p-1$, for every $u_i \in V(G)$. Let $D = \{u_1\}$ and $D' = \{u_2\}$ be a i_M -set and i_M^{-1} -set of G such that $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and $|N[D']| \geq \lceil \frac{p}{2} \rceil$ with $D' \subseteq V - D$ respectively. Then $i_M(G) = 1 = i_M^{-1}(G)$. Thus $i_M(G) + i_M^{-1}(G) = 2$. The lower bound is sharp if $G = K_p$, a complete graph $p \geq 2$. Also $i_M(G) \cdot i_M^{-1}(G) = 1$.

Case (ii): If the graph G contains exactly one full degree vertex and others are pendants then $d(u_1) = p-1$ and $d(u_i) = 1$, for every $u_i \in V(G), i = 2, \dots, p$. Then $D = \{u_1\}$ is a i_M -set of G and $i_M(G) = 1$. Since all other vertices are pendants, the graph structure is $K_{1,p-1}$, a star. By the results (4.3.1)(2), $i_M^{-1}(G) = \lfloor \frac{p-1}{2} \rfloor$. Therefore $i_M(G) + i_M^{-1}(G) = 1 + \lfloor \frac{p-1}{2} \rfloor = \lceil \frac{p}{2} \rceil, p \geq 2$. Hence $i_M(G) + i_M^{-1}(G) = \lceil \frac{p}{2} \rceil$. The upper bound is sharp if $G = K_{1,p-1}, p \geq 2$. Also, $i_M(G) \cdot i_M^{-1}(G) = \lfloor \frac{p-1}{2} \rfloor = \lceil \frac{p}{2} \rceil - 1$.

Case (iii): Suppose all vertices are majority dominating vertices then $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$, for every $u_i \in V(G)$. Then $D = \{u_1\}$

and $D' = \{u_2\} \subseteq V - D$ are the i_M -set and i_M^{-1} -set of G with respect to D respectively. In particular, suppose G contains atleast two majority dominating vertices, then $i_M(G) = i_M^{-1}(G) = 1$. Thus $i_M(G) + i_M^{-1}(G) = 2$ and $i_M(G) \cdot i_M^{-1}(G) = 1$.

Case (iv): Suppose G has exactly one majority dominating vertex u_1 and other vertices are of degree $< \lceil \frac{p}{2} \rceil - 1$. Then $d(u_1) \geq \lceil \frac{p}{2} \rceil - 1$ and $d(u_i) < \lceil \frac{p}{2} \rceil - 1$, for every $u_i \in V(G), i = 1, 2, \dots, p$. Now $D = \{u_1\}$ is a i_M -set of G and $i_M(G) = |D| = 1$. Let $D' = \{u_2, \dots, v_t\} \subseteq V - D$ such that $d(u_i, u_j) \geq 2$, for $i \neq j$ and $u_i, u_j \in D'$ and $|D'| = t \leq \lceil \frac{p}{6} \rceil$. Then $|N[D']| \geq (2t + t) = 3t = 3\lceil \frac{p}{6} \rceil > \lceil \frac{p}{2} \rceil$. Therefore D' is a i_M^{-1} -set of G and $i_M^{-1}(G) = |D'| \leq \lceil \frac{p}{6} \rceil$. Thence $i_M(G) + i_M^{-1}(G) \leq 1 + \lceil \frac{p}{6} \rceil < \lceil \frac{p}{2} \rceil$ and $i_M(G) \cdot i_M^{-1}(G) \leq \lceil \frac{p}{6} \rceil < \lceil \frac{p}{6} \rceil^2$.

Case (v): Suppose G contains all vertices of degree $d(u_i) < \lceil \frac{p}{2} \rceil - 1$, for every $u_i \in V(G)$.

Subcase (i): Suppose G is regular and $d(u_i) < \lceil \frac{p}{2} \rceil - 1$, for all $u_i \in V(G)$ then $i_M(G) = i_M^{-1}(G)$ and $i_M^{-1}(G) \geq 2$ and $i_M(G) \geq 2$. By case (iv), $i_M(G) = i_M^{-1}(G) \geq \lceil \frac{p}{6} \rceil$ and $i_M(G) + i_M^{-1}(G) \geq \lceil \frac{p}{6} \rceil + \lceil \frac{p}{6} \rceil = 2\lceil \frac{p}{6} \rceil \geq \lceil \frac{p}{3} \rceil$. Thus, $i_M(G) + i_M^{-1}(G) < \lceil \frac{p}{2} \rceil$ and $i_M(G) \cdot i_M^{-1}(G) = \lceil \frac{p}{6} \rceil^2$.

Subcase (ii): Suppose all the degree of u_i of G are not equal such that $d(u_i) < \lceil \frac{p}{2} \rceil - 1$, for every $u_i \in V(G)$. Then $i_M(G) \geq 2$ and $i_M^{-1}(G) \geq 3$. It implies that $i_M(G) \geq \lceil \frac{p}{6} \rceil$ and $i_M^{-1}(G) \geq \lceil \frac{p}{8} \rceil$. Hence $i_M(G) + i_M^{-1}(G) \geq \lceil \frac{p}{6} \rceil + \lceil \frac{p}{8} \rceil \leq \lceil \frac{p}{3} \rceil + 1$ and $i_M(G) + i_M^{-1}(G) < \lceil \frac{p}{2} \rceil$ and $i_M(G) \cdot i_M^{-1}(G) \leq \lceil \frac{p}{6} \rceil \cdot \lceil \frac{p}{8} \rceil < \left(\lceil \frac{p}{6} \rceil\right)^2$. Thence in all cases, we obtain $2 \leq i_M(G) + i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil$ and $i_M(G) \cdot i_M^{-1}(G) \leq \lceil \frac{p}{6} \rceil^2$.

Hence the result. ■

Theorem 4.8.2: Let G be a disconnected graph without isolates and with i_M -set and i_M^{-1} -set. Then $i_M(G) + i_M^{-1}(G) \leq \frac{p}{2} + 1$ and $i_M(G) \cdot i_M^{-1}(G) \leq \lceil \frac{p^2}{16} \rceil$. These bounds are sharp if $G = mK_2$.

Proof: Let G be a disconnected graph with p vertices and $g_1, g_2, \dots, g_m$ are components of G . Let D and D' be the i_M -set and i_M^{-1} -set of G respectively. The theorem is proved by induction on the degree $\Delta_1(G)$ and $\Delta_2(G)$ where Δ_1 is the maximum degree of G and Δ_2 is a maximum degree of $(V - D)$.

Case (i): When $\Delta_1(G) \geq \lceil \frac{p}{2} \rceil - 1$.

Subcase (i): Suppose $\Delta_1(G) = \Delta_2(G) \geq \lceil \frac{p}{2} \rceil - 1$. Then, $i_M(G) = 1 = i_M^{-1}(G)$. Hence $i_M(G) + i_M^{-1}(G) = 2 < (\frac{p}{2}) + 1$ and $i_M(G) \cdot i_M^{-1}(G) = 1 < \lceil \frac{p^2}{16} \rceil$.

Subcase (ii): Suppose $\Delta_1(G) \geq \lceil \frac{p}{2} \rceil - 1$ and $\Delta_2(G) < \lceil \frac{p}{2} \rceil - 1$. Then $i_M(G) = 1$ and $i_M^{-1}(G) \geq 2 = \lceil \frac{p}{6} \rceil$. Hence $i_M(G) + i_M^{-1}(G) \geq \lceil \frac{p}{6} \rceil + 1$ and $i_M(G) \cdot i_M^{-1}(G) \geq \lceil \frac{p}{6} \rceil$. Thence $i_M(G) + i_M^{-1}(G) \leq \frac{p}{2} + 1$ and $i_M(G) \cdot i_M^{-1}(G) < \lfloor \frac{p^2}{16} \rfloor$.

Case (ii): When $\Delta_1(G) < \lceil \frac{p}{2} \rceil - 1$.

Subcase (i): If the degree of the vertices u_i such that $d(u_i) = 1$, for every $u_i \in V(G)$ and $G = g_1 \cup \dots \cup g_m$. Then the graph G is $mK_2, m \geq 1$. By the results (1.4) and (4.3.1)(7), $i_M(G) = \lceil \frac{p}{4} \rceil = i_M^{-1}(G)$. Then $i_M(G) + i_M^{-1}(G) = \lceil \frac{p}{4} \rceil + \lceil \frac{p}{4} \rceil = 2\lceil \frac{p}{4} \rceil = \lceil \frac{p}{2} \rceil$. Hence $i_M(G) + i_M^{-1}(G) = \frac{p}{2} + 1$ and $i_M(G) \cdot i_M^{-1}(G) \leq \lceil \frac{p}{4} \rceil \lceil \frac{p}{4} \rceil \leq \lceil \frac{p^2}{16} \rceil$. This bound is sharp if $G = mK_2, m$ is even.

Subcase (ii): If the degree of vertices u_i such that $d(u_i) = 2$, for every $u_i \in V(G)$ and $G = g_1 \cup \dots \cup g_m$. Then the graph becomes $G = mC_p, p \geq 3$ or $G = mP_p$, except $d(u_i)$ and $d(u_p), p \geq 4$ and $m \geq 2$. By the result (1.4), $i_M(G) = \lceil \frac{p}{6} \rceil$ and $i_M^{-1}(G) = \lceil \frac{p}{6} \rceil$. Hence $i_M(G) + i_M^{-1}(G) = 2\lceil \frac{p}{6} \rceil = \lceil \frac{p}{3} \rceil + 1 < \frac{p}{2} + 1$ and $i_M(G) \cdot i_M^{-1}(G) < \lfloor \frac{p^2}{16} \rfloor$.

Subcase (iii): Suppose $d(u_i) \geq 3$, for every $u_i \in V(G_i)$ and $G_i = g_1, g_2, \dots, g_m$. Then $i_M(G) + i_M^{-1}(G) \geq \lceil \frac{p}{8} \rceil + \lceil \frac{p}{8} \rceil = \lceil \frac{p}{4} \rceil + 1$ and $i_M(G) \cdot i_M^{-1}(G) \geq \lceil \frac{p}{8} \rceil \cdot \lceil \frac{p}{8} \rceil = \lceil \frac{p^2}{64} \rceil + 3$. Hence $i_M(G) + i_M^{-1}(G) < (\frac{p}{2}) + 1$ and $i_M(G) \cdot i_M^{-1}(G) < \lceil \frac{p^2}{16} \rceil$. ■

Theorem 4.8.3: Let G be a disconnected graph with isolates and with i_M -set and i_M^{-1} -set then $2 \leq i_M(G) + i_M^{-1}(G) \leq p$ and $i_M(G) \cdot i_M^{-1}(G) \leq \frac{p^2}{4}$. The bound is sharp if $G = \overline{K_p}$, p is even.

Proof: This theorem is proved by induction on the number of isolates ‘ t ’. Let $G = (g_1 \cup g_2 \cup \dots \cup g_m)$ where $g_1, g_2, \dots, g_m$ are ‘ m ’ components of G with p vertices. When $t = 1$ and $g_m = \{u_p\}$. All other components are connected subgraph g_i of G .

Suppose $G = g_1 \cup g_2$, where $g_1 = K_{1,p-2}$ and $g_2 = \{u_p\}$ then $i_M(G) = 1$ and $i_M^{-1}(G) \leq \lfloor \frac{p-1}{2} \rfloor$. Hence $i_M(G) + i_M^{-1}(G) \leq 1 + \lfloor \frac{p-1}{2} \rfloor < p$ and $i_M(G) \cdot i_M^{-1}(G) \leq \lfloor \frac{p-1}{2} \rfloor < \frac{p^2}{4}$. Suppose $G = g_i \cup g_m, i \geq 2$ with $g_m = \{u_p\}$ and g_i ’s are any connected subgraphs of G . Since G contains only one $t = 1$ isolate, $i_M(G) \leq \lceil \frac{p}{4} \rceil$ and $i_M^{-1}(G) \leq \lceil \frac{p}{4} \rceil$, if $g_i = iK_2, i \geq 2$. Then $i_M(G) + i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil + 1 < p$ and $i_M(G) \cdot i_M^{-1}(G) \leq \lceil \frac{p}{4} \rceil^2 < \frac{p^2}{4}$.

Suppose $G = g_1 \cup t_r$ where $t_r \leq \frac{p}{2}$ isolates and the component g_1 . If the subgraph g_1 contains atleast two majority dominating vertices u_1 and u_2 such that $d(u_i) \geq \frac{p}{2} - 1, i = 1, 2$, then $i_M(G) = i_M^{-1}(G) = 1$. In this case, $i_M(G) + i_M^{-1}(G) = 2$, the lower bound exists and $i_M(G) + i_M^{-1}(G) = 1$.

Therefore this result is true for $t = 1, 2, \dots, \frac{p}{2}$. When $t_r = \lceil \frac{p}{2} \rceil$, then the graph G contains $\lceil \frac{p}{2} \rceil$ isolates and $G = g_i \cup t_r, i \geq 1$ and $i_M(G) = 2$ and $i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil - 1$. Therefore $i_M(G) + i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil + 1 < p$ and $i_M(G) \cdot i_M^{-1}(G) \leq 2 \left(\frac{p}{2} - 1 \right) + 1 = p + 1 < \frac{p^2}{4}$. When $t_r = \lceil \frac{p}{2} \rceil + 1$ then $i_M(G) \geq 3$ and $i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil - 2$. Hence $i_M(G) + i_M^{-1}(G) \leq \lceil \frac{p}{2} \rceil + 5 < p$ and $i_M(G) \cdot i_M^{-1}(G) \leq 3 \left(\frac{p}{2} + 1 \right) - 2 \leq \left(\frac{3p}{2} + 1 \right) < \frac{p^2}{4}$. This result is true for $t_r = \lceil \frac{p}{2} \rceil, \lceil \frac{p}{2} \rceil + 1, \dots, (p - 1)$ isolates. When $t_r = p$ isolates. Then $G = \overline{K_p}$, p is even and $i_M(G) = \left(\frac{p}{2} \right) = i_M^{-1}(G)$. Hence $i_M(G) + i_M^{-1}(G) = p$ and $i_M(G) \cdot i_M^{-1}(G) = \frac{p^2}{4}$. If $G = \overline{K_p}$, p is odd then $i_M(G) = \lceil \frac{p}{2} \rceil$ and $i_M^{-1}(G)$ does not exist. Hence in all cases of isolates $t_r = 1, 2, \dots, p$, a disconnected graph G with isolates satisfy $i_M(G) + i_M^{-1}(G) \leq p$ and $i_M(G) \cdot i_M^{-1}(G) \leq \frac{p^2}{4}$. The equality holds if and only if $G = \overline{K_p}$, p is even. Thus the result. ■

Chapter 5

Inverse Connected Majority Dominating Set of a Graph

Abstract

This chapter introduces an **inverse connected majority dominating set** with respect to minimum connected majority dominating set of a graph G and an **inverse connected majority domination number** $\gamma_{CM}^{-1}(G)$. Characterisation of a minimal inverse connected majority dominating set, exact values of $\gamma_{CM}^{-1}(G)$ for some families of graphs and bounds of $\gamma_{CM}^{-1}(G)$ are also determined. Algorithm and application for an inverse connected majority dominating set are also discussed.

5.1 Introduction

In 1979, Sampath Kumar and Walikar [48] introduced connected domination number of a graph. The concept of Cartesian product graph was initially defined by Faudree and Schelp in 1990 [14]. In 2006, Swaminathan and Joseline Manora [50] studied the concept Majority Dominating set of a graph and they elucidated the parameter in various levels by establishing many results. In 2017, Joseline Manora and Muthukani Vairavel [37] have studied connected majority dominating set of a graph. Bounds of a connected majority domination number $\gamma_{CM}(G)$ and many interesting results with inequalities were determined in their research articles. The concept of an inverse majority dominating set in graphs was introduced and propelled by Joseline Manora and Vignesh in 2019 with the help of the above concepts, the researcher has travelled in the next direction and defined the parameter namely an inverse connected majority dominating set of a graph.

The chapter aims at the study of a new concept called an inverse connected majority dominating set and its number. The organization of this chapter as follows: Section 5.1 is the introductory part

and Section 5.2 defines the concept of inverse connected majority domination for a graph G and illustrates the defined concept with an example. The inverse connected majority domination number $\gamma_{CM}^{-1}(G)$ is determined for various classes of graphs in section 5.3. Section 5.4 contains characterization of minimal inverse connected majority dominating set of a graph and also some necessary and sufficient condition for $\gamma_{CM}^{-1}(G)$ to satisfy some property. In section 5.5, bounds of $\gamma_{CM}^{-1}(G)$ are determined with sharpness. Final section deals with algorithm and application for an inverse connected majority dominating set of a graph G .

5.2 Inverse Connected Majority Dominating Set

Definition 5.2.1: Let G be a connected and simple graph with p vertices. Let $D \subseteq V(G)$ be a minimum connected majority dominating set of a graph G . A subset D' of $V(G)$ is an Inverse Connected Majority Dominating set of G (ICMD) if (i) $D' \subseteq V - D$ is a majority dominating set and (ii) The subgraph induced by the set D' is connected in G . The inverse connected majority dominating set D'

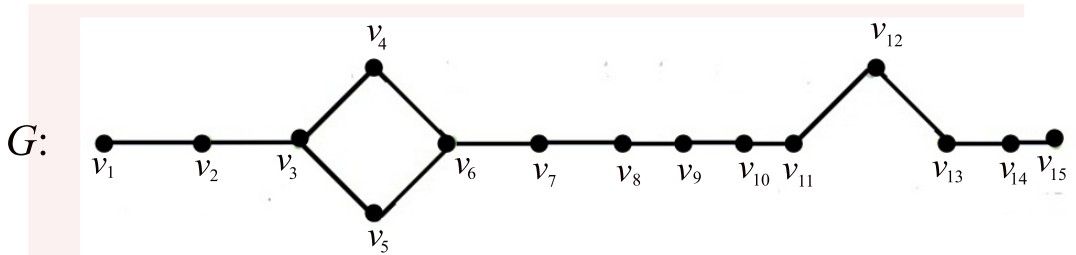
is minimal if no proper subset of D' is an inverse connected majority dominating set of G with respect to D of G .

5.2.1 Inverse Connected Majority Domination Number

The minimum and maximum cardinality of a minimum inverse connected majority dominating set of G is called an inverse connected majority domination number, denoted by $\gamma_{CM}^{-1}(G)$ and an upper inverse connected majority domination number, denoted by $\Gamma_{CM}^{-1}(G)$ of a graph G respectively.

5.2.2 Example for $\gamma_{CM}^{-1}(G)$ and $\Gamma_{CM}^{-1}(G)$

Consider the following connected graph with $p = 15$ vertices.



Let $V(G) = \{v_1, v_2, \dots, v_{15}\}$ with $p = 15$. The set $D = \{v_2, v_3, v_4, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$ is a connected dominating set of G implies $\gamma_C(G) = 12$. An inverse connected dominating set $D' \subseteq$

$V - D$ does not exist for the above graph G . The following are some of the minimal connected majority dominating sets of G . $D_1 = \{v_2, v_3, v_4, v_6, v_7\}$. It implies that $|D_1| = 5$, $D_2 = \{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$ and $|D_2| = 6$. But D_1 is a minimum connected majority dominating set of G . Therefore $\gamma_{CM}(G) = |D_1| = 5$ and $\Gamma_{CM}(G) = 6$. Then the inverse connected majority dominating set with respect to D_1 is $D'_1 = \{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\} \subseteq V - D_1$ and $\gamma_{CM}^{-1}(G) = |D'_1| = 6 = \Gamma_{CM}^{-1}(G)$.

Proposition 5.2.2: For any graph G , $\gamma_{CM}(G) \leq \gamma_{CM}^{-1}(G)$ where $\gamma_{CM}(G)$ is the connected majority domination number of G .

Proof: Since every inverse connected majority dominating set of G is a connected majority dominating set of G , the result is true. ■

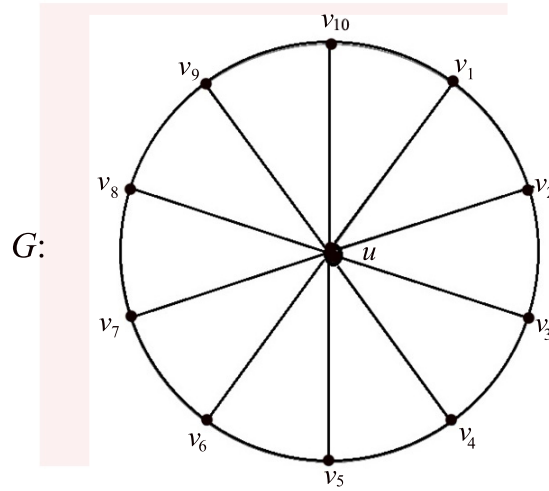
Proposition 5.2.3: For any connected graph G , $\gamma_{CM}^{-1}(G) \leq \gamma_C^{-1}(G)$ where $\gamma_C^{-1}(G)$ is an inverse connected domination number of G .

Proof: Since every inverse connected dominating set of G with respect to a connected dominating set is an inverse connected majority dominating set of G , we get the result. ■

Proposition 5.2.4: Let $\gamma_M^1(G)$ be an inverse majority dominating set of G . Then $\gamma_M^{-1}(G) \leq \gamma_{CM}^{-1}(G) \leq \gamma_C^{-1}(G)$ for any connected graph G .

Proof: Since an inverse connected majority dominating set of G is an inverse majority dominating set of G , $\gamma_M^{-1}(G) \leq \gamma_{CM}^{-1}(G)$. Then by proposition (5.2.3), we obtain this inequality. ■

Example 5.2.5: Consider the following Wheel graph $G = W_{11}$.



In this graph G , $|V(G)| = p = 11$. The set $D = \{u\}$ is a connected dominating set of G and $\gamma_C(G) = 1$. The inverse connected dominating set is $D' = \{v_2, v_5, v_8, v_{10}\} \subseteq V - D$ and $\gamma_C^{-1}(G) = 4$. The set $D_1 = \{u\}$ is a majority dominating set of G and $\gamma_M(G) = 1$. The inverse majority dominating set $D'_1 \subseteq V - D$, $|D'_1| = |\{v_2, v_5\}|$

and $\gamma_M^{-1}(G) = 2$. Let S_1 and S'_1 be a connected and inverse connected majority dominating sets of G . The set $S_1 = \{u\}$ and $S'_1 = \{v_2, v_3, v_4\} \subseteq V - S_1$, $|S'_1| = 3$ and $|S_1| = 1$. Hence $\gamma_{CM}^{-1}(G) = 3$ and $\gamma_{CM}(G) = 1$. Thus $\gamma_M^{-1}(G) < \gamma_{CM}^{-1}(G) < \gamma_C^{-1}(G)$.

Observation 5.2.6:

1. If $\gamma_{CM}^{-1}(G) = 1$ then $\gamma_{CM}(G) = 1$ but the converse need not be true. For example, in a comet graph, $\gamma_{CM}(G) = 1$ and $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 2$.
2. If the graph G contains only the vertices u_i such that $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$ then $\gamma_{CM}(G) = \gamma_{CM}^{-1}(G) = 1$.
3. If a graph G contains the vertices u_i such that $d(u_i) \leq 2$ then $\gamma_{CM}(G) = \gamma_{CM}^{-1}(G) \geq 2$.
4. Let G be a regular connected graph with degree $r \leq \lceil \frac{p}{2} \rceil - 1$. Then $\gamma_{CM}(G) = \gamma_{CM}^{-1}(G) \geq 1$.

5.3 $\gamma_{CM}^{-1}(G)$ for Various Classes of Graphs

In this section, particular values of $\gamma_{CM}^{-1}(G)$ is determined for some classes of graphs and product graphs.

Proposition 5.3.1: For a path $G = P_p$, $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 2$, if $p \geq 3$.

Proof: Let $G = P_p$ be a path on $p \geq 5$ vertices and $V(G) = \{u_1, u_2, u_3, \dots, u_{\lceil \frac{p}{2} \rceil - 1}, u_{\lceil \frac{p}{2} \rceil}, \dots, u_p\}$. Let D be a minimum connected majority dominating set of G and by the result (1.4), $\gamma_{CM}(P_p) = \lceil \frac{p}{2} \rceil - 2$. This proposition is proved by induction on ‘ p ’ vertices. When $p = 3, 4, 5, 6$ then $D = \{u_2\}$ is a γ_{CM} -set of G and $\gamma_{CM}(G) = 1$. Also, the inverse connected majority dominating set $D' = \{u_3\} \subseteq V - D$ and $\gamma_{CM}^{-1}(G) = 1 = \lceil \frac{p}{2} \rceil - 2$. When $p = 7, 8$ then $D = \{u_2, u_3\}$ is a γ_{CM} -set and $D' = \{u_4, u_5\} \subseteq V - D$ is an inverse connected majority dominating set of G with respect to D . It implies that $\gamma_{CM}^{-1}(G) = |D'| = 2$ and $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 2$.

This result is true for $p = 3, 4, 5, 6, 7, 8, \dots, (p - 1)$. For the graph G has p vertices. Let $D = \{u_2, u_3, u_4, \dots, u_{\lceil \frac{p}{2} \rceil - 1}\} \subseteq V(G)$ is a γ_{CM} -set of G . For p is odd, $D' = \{u_{\lceil \frac{p}{2} \rceil}, u_{\lceil \frac{p}{2} \rceil + 1}, \dots, u_{p-2}\} \subseteq V - D$ such that $d(u_i, u_j) = 1$, for $i \neq j$ and $|D'| = \lceil \frac{p}{2} \rceil - 2$. Then $|N[D']| = \sum_{i=\lceil \frac{p}{2} \rceil}^{p-2} d(u_i) + 2 = (p - 2) - (\lceil \frac{p}{2} \rceil - 1) + 2 = (\lfloor \frac{p}{2} \rfloor) + 1 = \lceil \frac{p}{2} \rceil$, if p is odd. For p is even, $D' = \{u_{\lceil \frac{p}{2} \rceil}, u_{\lceil \frac{p}{2} \rceil + 1}, \dots, u_{p-3}\} \subseteq V - D$ and $|N[D']| = \sum_{i=\lceil \frac{p}{2} \rceil}^{p-3} d(u_i) + 2 = (p - 3) - (\lceil \frac{p}{2} \rceil - 1) + 2 = \frac{p}{2}$, if p is even. Hence $|N[D']| = \lceil \frac{p}{2} \rceil$, $D' \subseteq V - D$ and since $d(u_i, u_j) = 1$,

the induced subgraph $\langle D' \rangle$ is connected. It implies that D' is an inverse connected majority dominating set of G with respect to D and $\gamma_{CM}^{-1}(G) \leq |D'| = \lceil \frac{p}{2} \rceil - 2$.

Suppose take the set D_1 such that $|D_1| < |D'| = \lceil \frac{p}{2} \rceil - 2$. Then $|N[D_1]| < \lceil \frac{p}{2} \rceil$, $D_1 \subseteq (V - D)$ and $\langle D_1 \rangle$ is connected. It leads to D_1 is not an inverse connected majority dominating set of G and $\gamma_{CM}^{-1}(G) > |D_1|$. Hence $\gamma_{CM}^{-1}(G) \geq |D'| = \lceil \frac{p}{2} \rceil - 2$. Thus $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 2$. ■

Proposition 5.3.2: Let $G = W_p$, be a wheel graph of p vertices with $p \geq 5$. Then (i) $\gamma_{CM}^{-1}(G) = 1$, when $p = 5, 6$ and (ii) $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 3$, when $p \geq 7$.

Proof: Let $G = W_p = C_{p-1} \cup K_1$ and $V(G) = \{v_1, v_2, \dots, v_{p-1}, v_p\}$ where v_p is the centre of the wheel graph with $d(v_p) = p - 1$. The only minimum connected majority dominating set of G is $D = \{v_p\}$ and $\gamma_{CM}(G) = 1$. Let $D' = \{v_1, v_2, \dots, v_t\} \subseteq V - D$ such that $|D'| = t = \lceil \frac{p}{2} \rceil - 3$ with $d(u_i, u_j) = 1$ for any i, j and $i \neq j$. Then $|N[D']| = \sum_{i=1}^t d(v_i) + 3 = |t| + 3 = \lceil \frac{p}{2} \rceil - 3 + 3 = \lceil \frac{p}{2} \rceil$. Since $d(v_i, v_j) = 1$, the induced subgraph $\langle D' \rangle$ is connected. Hence D' is an inverse

connected majority dominating set of G with respect to D . Therefore

$$\gamma_{CM}^{-1}(G) \leq |D'| = \lceil \frac{p}{2} \rceil - 3.$$

Suppose $D_1 \subseteq V - D$ with $|D_1| < t$. Then $|D_1| < \lceil \frac{p}{2} \rceil - 3$ and $|N[D_1]| \leq \lceil \frac{p}{2} \rceil$. It implies that D_1 is not an inverse connected majority dominating set of G with respect to D . $\gamma_{CM}^{-1}(G) \geq |D'| \geq \lceil \frac{p}{2} \rceil - 3$. Then combining these two inequalities, we obtain $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 3$. ■

Proposition 5.3.3: For a Fan graph F_p , $p \geq 5$ (i) $\gamma_{CM}^{-1}(G) = 1$, if $p \leq 8$ and (ii) $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 3$, if $p \geq 9$.

Proof: Let $G = F_p = P_{p-1} \cup K_1$ and $V(G) = \{v_1, v_2, \dots, v_{p-1}, v_p\}$.

Applying the same arguments as in Proposition (5.3.2), $D = \{v_p\}$ and $D' = \{v_2, \dots, v_t\} \subseteq V - D$ with $|D'| = \lceil \frac{p}{2} \rceil - 3$ are the connected majority dominating set and the inverse connected majority dominating set of G with respect to D respectively. Hence $\gamma_{CM}^{-1}(G) = |D'| = \lceil \frac{p}{2} \rceil - 3$. ■

Results 5.3.4:

1. Let $G = C_p$ be a cycle, $p \geq 3$. Then $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 2$.
2. For a star $K_{1,p-1}$, $p \geq 2$, $\gamma_{CM}^{-1}(G)$ does not exist.

3. Let $G = K_{m,n}$, $n \geq 2$ and $m \leq n$. Then $\gamma_{CM}^{-1}(G) = 1$.

4. For a complete graph $G = K_p$, $\gamma_{CM}^{-1}(G) = 1$.

5. For a double star $G = D_{r,s}$, $r \leq s$ and $r, s \geq 2$,

$$\gamma_{CM}^{-1}(G) = \begin{cases} 1, & \text{if } s = r, r+1, r+2 \\ \text{not exist,} & \text{if } s \geq r+3 \end{cases}.$$

6. For a Petersen graph G with $p = 10$ vertices and $q = 15$ edges

$$\gamma_{CM}^{-1}(G) = 2.$$

7. For a subdivision graph $G = S(K_{1,p-1})$, $\gamma_{CM}^{-1}(G)$ does not exist.

8. For a generalized Petersen graph $P(n, k) = P(9, 2)$,

$$\gamma_{CM}^{-1}(G) = 3.$$

Next, the product graph namely Grid graph $G = P_i \times P_j$, $i, j \geq 3$ is considered and the exact values of $\gamma_M^{-1}(G)$ and $\gamma_{CM}^{-1}(G)$ are determined. Also, the interesting inequality $\gamma_M^{-1}(G) < \gamma_{CM}^{-1}(G)$ is found in this grid graphs.

Theorem 5.3.5: For a grid graph $G = P_2 \times P_j$, $j \geq 3$, $\gamma_M(G) =$

$$\gamma_M^{-1}(G) = \lceil \frac{j}{8} \rceil \text{ and } \gamma_{CM}^{-1}(G) = \lfloor \frac{j-2}{4} \rfloor.$$

Proof: Consider the grid graph $G = P_2 \times P_j$ (Ladder graph) with vertex sets $(u_{11}, u_{12}, \dots, u_{1j})$ in the first row and $(u_{21}, u_{22}, \dots, u_{2j})$ in the second row respectively. Now, choose a set $D \subseteq V(G)$ such that $D = \{u_{12}, u_{15}, u_{18}, \dots, u_{1t}\}$ with $d(u_{1i}, u_{1j}) \geq 4$, for every $i \neq j$ and $|D| = t = \lceil \frac{p}{8} \rceil$. Then $|N[D]| \geq \sum_{i=1}^t d(u_{1i}, u_{1j}) = 4t \geq 4 \lceil \frac{p}{8} \rceil \geq \lceil \frac{p}{2} \rceil$. Therefore D is a majority dominating set of G . Hence $\gamma_M(G) \leq |D| = \lceil \frac{p}{8} \rceil$.

Suppose D is a majority dominating set then, $|N[D]| \leq \sum_{i=1}^t d(u_{1i}, u_{1j}) + \gamma_M(G)$. By the assumption, $\lceil \frac{p}{2} \rceil \leq 4 \cdot \gamma_M(G)$ and it implies that $\gamma_M(G) \geq \frac{1}{4} \lceil \frac{p}{2} \rceil \geq \lceil \frac{p}{8} \rceil$. Hence $\gamma_M(G) = \lceil \frac{p}{8} \rceil$. Let $D' = \{u_{13}, u_{16}, u_{19}, \dots, u_{1t}\} \subseteq V - D$ such that each vertex of D' is adjacent to the vertex of D . It leads to D' is an inverse majority dominating set of G with respect to D . Apply the similar argument in $(V - D)$, we get $\gamma_M^{-1}(G) = \lceil \frac{p}{8} \rceil$.

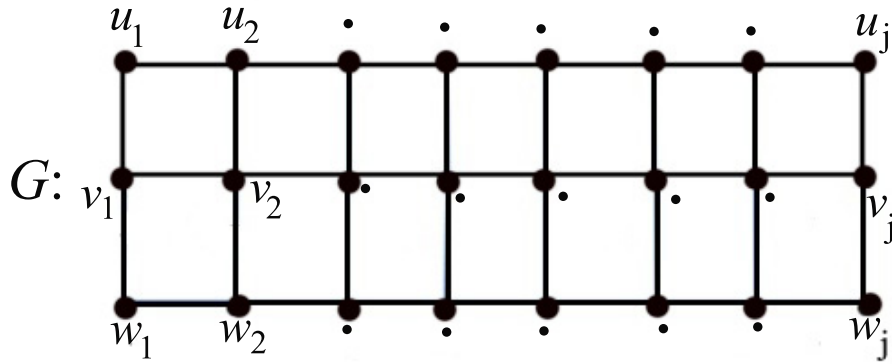
Let $S = \{u_{12}, u_{13}, \dots, u_{1(t-1)}\}$ be a connected majority dominating set of G . By the result (1.4), $\gamma_{CM}(G) = \begin{cases} \lfloor \frac{p}{4} \rfloor, & \text{if } j \text{ is odd} \\ \lfloor \frac{p-1}{4} \rfloor, & \text{if } j \text{ is even.} \end{cases}$ Choose the set $S' = \{u_{22}, u_{23}, \dots, u_{2(t-1)}\} \subseteq V - S$ with $|S'| = \lfloor \frac{p-2}{4} \rfloor = t$ and $d(u_{2i}, u_{2j}) = 1$, for $u_i, u_j \in S'$. Then $|N[S']| \leq \sum_{r=2}^t d(u_{2r}) - (|S'| - 2) = 3 \left(\frac{p-2}{4} \right) - |S'| + 2 = \frac{p}{2} + 1$. Hence $|N[S']| \geq \lceil \frac{p}{2} \rceil$

and $\langle S' \rangle$ is connected. That implies S' is an inverse connected majority dominating set of G with respect to S and $\gamma_{CM}^{-1}(G) \leq |S'| = \lfloor \frac{p-2}{4} \rfloor$.

Suppose $S' = S - \{v\}$ with $|S'| = \lfloor \frac{p-2}{4} \rfloor - 1$ then $|N[S']| \leq \sum_{r=2}^{t-2} d(u_{2r}) - (|S'| - 1) < \lceil \frac{p}{2} \rceil$ and S' would not be an inverse connected majority dominating set of G . Hence $\gamma_{CM}^{-1}(G) > |S'|$ and $\gamma_{CM}^{-1}(G) \geq |S| = \lfloor \frac{p-2}{4} \rfloor$. Therefore $\gamma_{CM}^{-1}(G) = \lfloor \frac{p-2}{4} \rfloor$. ■

Theorem 5.3.6: For a grid graph $G = P_3 \times P_j, j \geq 4$, $\gamma_M(G) = \gamma_M^{-1}(G) = \lceil \frac{p}{10} \rceil$ and $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{6} \rceil$.

Proof: Consider the grid graph $G = P_3 \times P_j, j \geq 4$. Let $V(G) = \{u_1, u_2, \dots, u_j, v_1, v_2, \dots, v_j, w_1, w_2, \dots, w_j\}$ and it forms I row, II row and III row respectively for G .



Let $D = \{u_{22}, u_{25}, \dots, u_{2t}\} \subseteq V(G)$ with $|D| = t = \lceil \frac{p}{10} \rceil$ such that $d(u_i, u_j) \geq 3$, for every $i \neq j$ and $u_i, u_j \in D$. Then $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and therefore D is a majority dominating set of G . Hence $\gamma_M(G) = \lceil \frac{p}{10} \rceil$.

Now choose a set $D' = \{u_{23}, u_{26}, \dots, u_{2t}\} \subseteq V - D$ such that each vertex D' is adjacent to the vertices of D with $|D'| = |D| = \lceil \frac{p}{10} \rceil$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$. It implies that D' is an inverse majority dominating set of G with respect to D and $\gamma_M^{-1}(G) = \lceil \frac{p}{10} \rceil$.

Let $S = \{v_1, v_2, \dots, v_t\}$ be a connected majority dominating set of G with $|S| = \lceil \frac{p}{6} \rceil$. Now, choose a set $S' = \{v_{t+1}, v_{t+2}, \dots, v_j, w_{t+1}\} \subseteq V - S$ such that $d(v_i, v_j) = 1$ with $|S'| = \lceil \frac{p}{6} \rceil$ and the induced subgraph $\langle S' \rangle$ is a connected. Since each vertex of S' in G dominates three vertices vertically and the first vertices of S' dominates 4 vertices then w_{t+1} dominates only one vertex of G , $|N[S']| = 3(j-t)+1+1 = |N[S']| = 3(|S'| - 1) + 3 = 3\lceil \frac{p}{6} \rceil \geq \lceil \frac{p}{2} \rceil$. Therefore $|N[S']| \geq \lceil \frac{p}{2} \rceil$ and $S' \subseteq V - S$. Hence S' is an inverse connected majority dominating set of G with respect to S and $\gamma_{CM}^{-1}(G) \leq |S'| = \lceil \frac{p}{6} \rceil$. Suppose the set $|S_1| = |S'| - 1$ with $|S_1| = \lceil \frac{p}{6} \rceil - 1$. Then $|N[S_1]| = \frac{p}{2} - 1$. It implies that S_1 would not be an inverse connected majority dominating set of G and $\gamma_{CM}^{-1}(G) \geq |S'| = \lceil \frac{p}{6} \rceil$. Hence $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{6} \rceil, j \geq 4$. ■

Proposition 5.3.7: Let $G = P_4 \times P_j$, be a grid graph with $j \geq 4$. Then $\gamma_M^{-1}(G) = \lceil \frac{p}{10} \rceil$ and $\gamma_{CM}^{-1}(G) = \lfloor \frac{p}{6} \rfloor$.

Proof: Apply the same argument as in Theorem 5.3.6. ■

Proposition 5.3.8: Let $G = P_i \times P_j$ be a grid graph with p vertices and $i, j \geq 2$. Then $\gamma_M^{-1}(G) \leq \gamma_{CM}^{-1}(G)$.

Proof: From the above results of Proposition (5.3.5) to (5.3.7), the result is obtained. ■

Theorem 5.3.9: Let G be a uniform caterpillar with p vertices, $e \geq 2$ pendants and $k \geq 3$ central vertices. Then

$$\gamma_{CM}^{-1}(G) = \begin{cases} \frac{p}{2(e+1)}, & \text{if } k \text{ is even} \\ \text{not exist,} & \text{if } k \text{ is odd} \end{cases}$$

Proof: Let $V(G) = \{v_1, v_2, \dots, v_k, e_{11}, e_{21}, e_{22}, \dots, e_{2i}, \dots, e_{k1}, e_{k2}, \dots, e_{ki}\}$ $i \geq 1$ and $|V(G)| = p = k(e+1)$. Let $D = \{v_2, v_3, \dots, v_{t_1}\}$ be a γ_{CM} -set of G . By the result (1.4),

$$\gamma_{CM}(G) = \begin{cases} \frac{p}{(e+1)} - \lfloor \frac{k}{2} \rfloor, & \text{if } k \text{ is odd} \\ \frac{p}{(e+1)} - \frac{k}{2}, & \text{if } k \text{ is even} \end{cases}$$

Case (i): Let k be even.

$$\begin{aligned} \text{Let } D' &= \left\{ v_{\frac{k}{2}+1}, \dots, v_k \right\} \subseteq (V - D) \text{ with } |D'| = t \text{ where } t = \\ \frac{1}{2} \left(\frac{p}{e+1} \right) \cdot |N[D']| &= \sum_{j=\frac{k}{2}+1}^k d(v_j) - |D'| + 2 = (|D'| - 1)d(v_j) + d(v_k) - \end{aligned}$$

$|D'| + 2$, where $(\frac{k}{2} + 1 \leq j \leq k - 1)$. Then $|N[D']| = (t - 1)d(v_j) + d(v_k) - t + 2 = (t - 1)(e + 2) + (e + 1) - t + 2 = t(e + 1) + 1$, where $t = \frac{p}{2(e+1)}$ and $|N[D']| = \frac{p}{2} + 1 \geq \lceil \frac{p}{2} \rceil$. Since $d(v_r, v_s) = 1$, for $v_r, v_s \in D'$, the induced subgraph $\langle D' \rangle$ is connected. Therefore D' is an inverse connected majority dominating set of G with respect to D and $\gamma_{CM}^{-1}(G) \leq |D'| = t = \left(\frac{p}{2(e+1)}\right)$, if k is even.

Case (ii): Let k be odd and D be a γ_{CM} -set of G . Since G is minimally connected, $\langle V - D \rangle$ is disconnected with components $g_i, i \geq 2$. In $(V - D)$, if each components g_i with $|N[g_i]| < \frac{p}{2}$ then there exists no connected majority dominating set D' in $(V - D)$ such that $D \cap D' = \emptyset$. Hence there is no inverse connected majority dominating set for a graph G with respect to D or any other γ_{CM} -set D . If k is odd. Thus $\gamma_{CM}^{-1}(G)$ does not exist. ■

5.4 Characterization Theorems

The following result gives a necessary and sufficient condition for a minimal inverse connected majority dominating set for a graph G .

Theorem 5.4.1: Let D be a minimum connected majority dominating set of a graph G . Let D' be an inverse connected majority

dominating set with respect to D . Then $D' \subseteq V - D$ is minimal if and only if for every vertex $v \in D'$ either the following condition (i) or condition (ii) holds.

(i) If $|N[D']| = \lceil \frac{p}{2} \rceil$ then either v is an enclave of D' or $pn[v, D'] \cap (V - D') \neq \emptyset$ and $\langle D' \rangle$ is connected.

(ii) If $|N[D']| > \lceil \frac{p}{2} \rceil$ then $|pn[v, D']| > |N[D']| - \lceil \frac{p}{2} \rceil$ and $\langle D' \rangle$ is connected.

Proof: Let $D' \subseteq V - D$ be a minimal connected majority dominating set of a graph G . Then $|N[D']| \geq \lceil \frac{p}{2} \rceil$ is connected in $(V - D)$.

Case (i): Let $|N[D']| = \lceil \frac{p}{2} \rceil$. Suppose the condition (i) is not true for any $v \in D'$. Then v is neither an enclave of D' nor v has a private neighbour in $(V - D')$. Therefore $pn[v, D'] \cap (V - D') = \emptyset$ and $\langle D' \rangle$ is not connected. We know that $pn[v, D'] = |N[D']| - |N[D' - v]|$, for every $v \in D'$. Since $pn[v, D'] = \emptyset$, $|N[D' - v]| = |N[D']| = \lceil \frac{p}{2} \rceil$. It implies that $D' - \{v\}$ is an inverse connected majority dominating set with respect to D , which contradicts that D' is minimal.

Suppose v is not an enclave of D' then $N[v] \not\subseteq D'$ and $N(v) \subseteq V - D'$, for atleast one vertex $w \in N(v)$. It implies that $pn[v, D'] \geq 1$ and $pn[v, D'] \subseteq V - D'$. Therefore, $pn[v, D'] \cap (V - D') \neq \emptyset$. Since

$pn[v, D'] = \emptyset, V - D' = \emptyset$ and $D' = V(G)$, which is a contradiction to $D' \subseteq V - D$. If $\langle D' \rangle$ is not connected then D' is not an inverse connected majority dominating set which is a contradiction to the result. Hence the condition (i) is true if $|N[D']| = \lceil \frac{p}{2} \rceil$.

Case (ii): Let $|N[D']| > \lceil \frac{p}{2} \rceil$. Since D' is a minimal inverse connected majority dominating set, $D' - \{v\}$ is not an inverse connected majority dominating set of G . It implies that $|N[D' - v]| < \lceil \frac{p}{2} \rceil$. But $|N[D']| - |pn[v, D']| < \lceil \frac{p}{2} \rceil$. Therefore $|pn[v, D']| > |N[D']| - \lceil \frac{p}{2} \rceil$, for every $v \in D'$. Also the set $D' \subseteq V - D$ consists of vertices with distance $d(v_i, v_j) = 1$ for every $v_i, v_j \in D'$ and $i \neq j$ and the induced subgraph $\langle D' \rangle$ is a connected subgraph of G . Thus the condition (ii) holds for every $v \in D'$.

Conversely, let D' be an inverse connected majority dominating set with respect to D and the conditions (i) and (ii) are true for every $v \in D'$. To prove that D' is minimal. Suppose D' is not minimal. Then either $|N[D' - v]| \geq \lceil \frac{p}{2} \rceil$, for any $v \in D'$ or the induced subgraph $\langle D' \rangle$ is not connected. We know that $|pn[v, D']| = |N[D']| - |N[D' - v]|$. If $|N[D' - v]| > \lceil \frac{p}{2} \rceil$. Then $|N[D' - v]| = |N[D']| - |pn[v, D']| > \lceil \frac{p}{2} \rceil$. Therefore $|pn[v, D']| < |N[D']| - \lceil \frac{p}{2} \rceil$ and $\langle D' \rangle$ is not connected which is a contradiction to the condition (ii).

Next, if $|N[D' - v]| = \lceil \frac{p}{2} \rceil$ then $|N[D' - v]| = |N[D']| - |pn[v, D']| = \lceil \frac{p}{2} \rceil$ and $|pn[v, D']| = |N[D']| - \lceil \frac{p}{2} \rceil$. which is absurd. Hence D' is a minimal inverse connected majority dominating set of G . ■

Theorem 5.4.2: The graph G has atleast two vertices v_i of degree $d(v_i) \geq \lceil \frac{p}{2} \rceil - 1$ if and only if $\gamma_{CM}^{-1}(G) = 1$.

Proof: Let $v_i \in V(G)$ such that $d(v_i) \geq \lceil \frac{p}{2} \rceil - 1, i \geq 2$. Then $|N[v_i]| \geq \lceil \frac{p}{2} \rceil, i \geq 2$. Hence $D = \{v_i\}$ is a γ_{CM} -set and $D' = \{v_j\} \subseteq V - D$ is a γ_{CM}^{-1} -set of G with respect to D , for any $v_j \in V(G)$. Hence $\gamma_{CM}^{-1}(G) = |D'| = 1$. Conversely, if $\gamma_{CM}^{-1}(G) = 1$ then by observation, $\gamma_{CM}(G) = 1$. It implies that $D = \{v_1\}$ and $D' = \{v_2\} \subseteq V - D$ are the γ_{CM} -set and γ_{CM}^{-1} -set of G . Also, $|N[v_1]| = |N[v_2]| \geq \lceil \frac{p}{2} \rceil$. Therefore, $d(v_i) \geq \lceil \frac{p}{2} \rceil - 1$, for $i \geq 2$. Hence the graph G has atleast two vertices $(v_i, v_j) \in V(G)$ with this condition. ■

Proposition 5.4.3: Let G be a connected graph with a γ_{CM} -set D . Then $\gamma_M^{-1}(G) = \gamma_{CM}^{-1}(G) = 1$ if and only if there exists atleast one majority dominating vertex in $(V - D)$.

Proof: Let D be a connected majority dominating set of G . Suppose there exists a majority dominating vertex $u \in (V - D)$, then

$d(u) \geq \lceil \frac{p}{2} \rceil - 1$. Let $D' = \{u\} \subseteq V - D$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $\langle D' \rangle$ connected in $\langle V - D \rangle$. It implies that D' is an inverse connected majority dominating set of G with respect to D and D' is also an inverse majority dominating set of G with respect to D . Hence $\gamma_M^{-1}(G) = \gamma_{CM}^{-1}(G) = 1$. Suppose there is more than one majority dominating vertex in G then also both inverse majority dominating sets and inverse connected majority dominating sets exist with respect to D and $\gamma_M^{-1}(G) = \gamma_{CM}^{-1}(G) = 1$. The converse is obvious. ■

Corollary 5.4.4: If a connected graph G with $p \geq 9$ vertices and there is no majority dominating vertices then $\gamma_M^{-1}(G) < \gamma_{CM}^{-1}(G)$.

Proof: Suppose $p = 7, 8$ and $d(u_i) \leq 2$, for all $u_i \in V(G)$. Then $\gamma_M^{-1}(G) = 2 = \gamma_{CM}^{-1}(G)$. If $p \geq 9$ and $d(u_i) \leq 2$, then the graph G is a path or a cycle structure. By the known results [2.3.1](2), $\gamma_M^{-1}(G) = \lceil \frac{p}{6} \rceil$ and $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 2$. Hence $\gamma_M^{-1}(G) < \gamma_{CM}^{-1}(G)$, if $p \geq 9$. Also, if $d(u_i) \geq 3$ then by the known results, $\gamma_M^{-1}(G) = \lceil \frac{p}{8} \rceil$ and $\gamma_{CM}^{-1}(G) > \lceil \frac{p}{8} \rceil$. Thus, $\gamma_M^{-1}(G) < \gamma_{CM}^{-1}(G)$. ■

Proposition 5.4.5: Let G be any connected graph with p vertices.

Then $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 2$ if and only if $G = P_p, C_p, p \geq 3$.

Proof: Let $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 2$. Then D' is an γ_{CM}^{-1} -set of G with respect to a connected majority dominating set D of G with $(\lceil \frac{p}{2} \rceil - 2)$ vertices. It implies that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $\langle D' \rangle$ is connected in G . Also D' is connected in $(V - D)$ then it gives that all vertices u_i such that $d(u_i) \leq 2$ in G . Hence $G = C_p$ or P_p , $p \geq 3$. For the converse part, by the Proposition (5.3.1) and results [5.3.4](1) we obtain the result. ■

5.5 Bounds of $\gamma_{CM}^{-1}(G)$

The following theorem gives lower and upper bounds for the inverse connected majority domination number of G , in terms of p and $\Delta'(G)$ which is a maximum degree in the set $(V - D)$.

Theorem 5.5.1: For any connected graph G , $\lceil \frac{p - \Delta'}{2} \rceil \leq \gamma_{CM}^{-1}(G) \leq \lceil \frac{p}{2} \rceil - \Delta'(G)$. The bounds are sharp.

Proof: Let D be a γ_{CM} -set of G this theorem is proved by induction on the maximum degree in $(V - D)$. Let $\Delta(G)$ and $\Delta'(G)$ be the maximum degree of G and the maximum degree in $\langle V - D \rangle$ respectively. If $\Delta'(G) = p - 1$ then $\Delta(G) = p - 1$. Let u_i be a maximum degree

vertex in $(V - D)$ and $D' = \{u_i\} \subseteq (V - D)$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$. It implies that $\gamma_{CM}^{-1}(G) = |D'| = 1 = \lceil \frac{p-\Delta'}{2} \rceil$. If $\Delta'(G) \geq \lceil \frac{p}{2} \rceil - 1$ then $D' = \{u_i\} \subseteq (V - D)$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$. Therefore, $\gamma_{CM}^{-1}(G) = \lceil \frac{p-\Delta'}{2} \rceil$. If $\Delta'(G) \leq \lceil \frac{p}{2} \rceil - 2$ then $D' = \{u_i, u_j\} \subseteq V - D$ with $d(u_i, u_j) = 1$ and $d(u_j) \leq \lceil \frac{p}{2} \rceil - 2$.

Now, $|N[D']| = d(u_i) + d(u_j) \geq \lceil \frac{p}{2} \rceil$ and the induced subgraph $\langle D' \rangle$ is connected. Hence $\gamma_{CM}^{-1}(G) = 2 = \lceil \frac{p}{4} \rceil + 1 \geq \lceil \frac{p-\Delta'}{2} \rceil$. This result is true for all $\Delta'(G) = p - 1, p - 2, \dots, 4, 3$. If $\Delta'(G) = 2$ then the graph becomes $G = P_p$ or C_p . By the Proposition (5.3.1), $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 2 = \lceil \frac{p}{2} \rceil - \Delta'(G)$. If $\Delta'(G) = 1$ then $\langle V - D \rangle$ has only isolates and $\gamma_{CM}^{-1}(G)$ does not exist. Hence $\lceil \frac{p-\Delta'}{2} \rceil \leq \gamma_{CM}^{-1}(G) \leq \lceil \frac{p}{2} \rceil - \Delta'(G)$. ■

Theorem 5.5.2: Let D be a γ_{CM} -set of a connected graph G . If $\Delta'(G) < \lceil \frac{p}{2} \rceil - 1$ and the induced subgraph $\langle V - D \rangle$ is connected then $\gamma_{CM}^{-1}(G)$ exists and $\gamma_{CM}^{-1}(G) \leq \lceil \frac{p}{2} \rceil - \Delta'(G)$ where $\Delta'(G)$ is a maximum degree in $(V - D)$.

Proof: Let D be a γ_{CM} -set of G with the vertex u_1 such that $d(u_1) = \Delta(G) = \text{maximum degree of } G$. Let $\Delta'(G) < \lceil \frac{p}{2} \rceil - 1$ and $d(u_2) = \Delta'(G)$ in $(V - D)$. Then the vertex u_2 dominates the vertices of $N[u_2]$

and atleast half of the vertices of $V(G)$ in $(\lceil \frac{V(G)}{2} \rceil - N(u_2))$ dominates themselves in the induced subgraph $\langle V - D \rangle$. Hence $D' = (\lceil \frac{V(G)}{2} \rceil - N(u_2))$ is a majority dominating set in $(V - D)$. Since $\Delta'(G) < \lceil \frac{p}{2} \rceil - 1$, choose the vertices such that $d(u_2, u_j) = 1$, where $u_j \in D', u_2 \in V - D$. Then the induced sub graph $\langle D' \rangle$ is connected in $\langle V - D \rangle$ and D' is an connected majority dominating set in $(V - D)$ with $(\lceil \frac{p}{2} \rceil - d(u_2))$. It implies that D' is an inverse connected majority dominating set of G . Hence in this case. Therefore $\gamma_{CM}^{-1}(G)$ exists and $\gamma_{CM}^{-1}(G) \leq |D'| = \lceil \frac{p}{2} \rceil - \Delta'(G)$. ■

Theorem 5.5.3: Let D be a γ_{CM} - set of a connected graph in G . If the induced subgraph $\langle V - D \rangle$ contains a component g_i with $|N[g_i]| \geq \lceil \frac{p}{2} \rceil$ then $\gamma_{CM}^{-1}(G)$ exists.

Proof: Let D be a γ_{CM} - set of G . Then the induced subgraph $\langle V - D \rangle$ is disconnected with 'n' components $g_i, i = 1, 2, \dots, n$. Let $|N[g_i]| \geq \lceil \frac{p}{2} \rceil$, for any g_i .

Case (i): Let $u \in V(g_i)$ and $d(u) = \Delta'(G)$. If $|N[u]| \geq \lceil \frac{p}{2} \rceil$ then $D' = \{u\}$ is a connected majority dominating set in $(V - D)$ and D' is an inverse connected majority dominating set of G . Therefore $\gamma_{CM}^{-1}(G)$ exists and $\gamma_{CM}^{-1}(G) = 1$.

Case (ii): If $|N[u]| < \lceil \frac{p}{2} \rceil$ then form a set $D' = \{u_1, u_2, \dots, u_i\}$ such that $d(u_i, u_j) = 1$ $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and the induced subgraph $\langle D' \rangle$ is connected in $(V - D)$. Hence D' is an inverse connected majority dominating set of G and $\gamma_{CM}^{-1}(G) = |D'|$. Hence $\gamma_{CM}^{-1}(G)$ exists. ■

Corollary 5.5.4: Let D be a γ_{CM} -set of G . If the induced subgraph $\langle V - D \rangle$ is disconnected with $|N[g_i]| < \lceil \frac{p}{2} \rceil$, for all components $g_i, i = 1, 2, \dots, n$ of $\langle V - D \rangle$ then $\gamma_{CM}^{-1}(G)$ does not exist.

Theorem 5.5.5: If a connected graph G contains a γ_{CM} -set D and $d(u_i) \leq 2$, for all $u_i \in (V - D)$ then $\gamma_{CM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 2$.

Proof: Let D be a γ_{CM} -set of G with all vertices $u_i \in V - D$ such that $d(u_i) \leq 2$. Since the induced subgraph $\langle D \rangle$ is connected, there may exist some isolates in $(V - D)$. It leads to all vertices u_i in $(V - D)$ are of degree $0 \leq d(u_i) \leq 2$ and the induced subgraph $\langle V - D \rangle$ is disconnected with some components g_i . Therefore, there exists atleast one component g_i with $|N[g_i]| \geq \lceil \frac{p}{2} \rceil$. Since $d(u_i) \leq 2$, for $u_i \in V - D$, there exists atleast one connected majority dominating set $D' \subseteq V - D$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $d(u_i, u_j) = 1$, for $u_i, u_j \in D'$ with $|D'| = \lceil \frac{p}{2} \rceil - 2$. Since $d(u_i, u_j) = 1$, the induced subgraph $\langle D' \rangle$

is connected. Hence D' is an inverse connected majority dominating set of G with respect to D and $\gamma_{CM}^{-1}(G) = |D'| = \lceil \frac{p}{2} \rceil - 2$. Suppose $d(u_i) \geq 3$ then $\gamma_{CM}^{-1}(G) < \lceil \frac{p}{2} \rceil - 2$. ■

Theorem 5.5.6: Let D be a γ_{CM} -set of a connected graph G . If the induced subgraph $\langle V - D \rangle$ contains only isolates then $\gamma_{CM}^{-1}(G)$ does not exist.

Proof: Let D be a γ_{CM} -set of G . In $\langle V - D \rangle$, if there are only isolates then $D = \{u\}$ where u is a full degree vertex of G . Since $\langle V - D \rangle$ is totally disconnected, there exists no connected majority dominating set $D' \subseteq (V - D)$ such that $D \cap D' = \emptyset$. Hence $\gamma_{CM}^{-1}(G)$ does not exist for this structure. ■

Remark 5.5.7: For a disconnected graph G , there is no connected dominating set and $\gamma_C(G)$ does not exist. But $\gamma_{CM}(G)$ and $\gamma_{CM}^{-1}(G)$ -sets exist for a disconnected graph.

Proposition 5.5.8: Let G be a disconnected graph with components $g_i, i \geq 2$ and D be a $\gamma_{CM}(G)$ -set of G . If $|N[g_i]| \geq \lceil \frac{p}{2} \rceil$, for any $g_i \in (V - D)$ then $\gamma_{CM}^{-1}(G)$ -set exists for this structure.

Proof: Let $g_1, g_2, \dots, g_i, i \geq 2$ be the componenets of G and $|V(G)| = |V(g_1) \cup \dots \cup V(g_i)| = p$ and D be a γ_{CM^-} set of G . Each component $g_i, i \geq 2$ is non-empty and each $g_i \in (V - D)$ may be an isolate or a connected subgraph of G . Let g_1 and g_2 be any two connected subgraphs g_i of G such that $|N[g_i]| \geq \lceil \frac{p}{2} \rceil$. Then connected majority dominating set D is formed by $V(g_1)$ and inverse connected majority dominating set $D' \subseteq V - D$ is formed by $V(g_2)$ such that $g_2 \subseteq (V - D)$ both are exist for G .

Suppose the induced subgraph $\langle V - D \rangle$ contains only isolates then inverse connected majority dominating set does not exist. Suppose $\langle V - D \rangle$ contains all the vertices u_i such that $d(u_i) \geq 1$ and $\Delta'(G) \leq \lceil \frac{p}{2} \rceil$ where $\Delta'(G)$ is the maximum degree in $(V - D)$. Then the inverse connected majority dominating set $D' \subseteq V - D$ with $|N[D']| \geq \lceil \frac{p}{2} \rceil$ the set D' contains the vertices of a component $g_i \in (V - D)$. Certainly, where $\gamma_{CM}^{-1}(G)$ exists and $\gamma_{CM}^{-1}(G) = |D'|$. ■

5.6 Algorithm and Application

The final section provides an algorithm to an ICMD set and ICMD number $\gamma_{CM}^{-1}(G)$ of a given graph G . A real life application is also discussed for this parameter.

5.6.1 Algorithm for an Inverse Connected Majority Dominating Set of G

To find an Inverse Connected Majority Dominating (ICMD) set for the given connected graph G with p vertices.

Let G be a given connected graph G with p vertices.

Step:1 Find the adjacency matrix $[A(G)]_{p \times p}$ for the given graph G .

Step:2 Change all the diagonal entries zero by the value one (because a vertex dominates itself)

$$\begin{array}{cccccc}
 u_1 & u_2 & \cdot & \cdot & \cdot & u_p & |N[u_i]| \\
 \\
 u_1 & \left(\begin{array}{cccccc} 1 & 1 & \cdot & \cdot & 1 & 0 \end{array} \right. \\
 u_2 & \left(\begin{array}{cccccc} 0 & 1 & \cdot & \cdot & 1 & 0 \end{array} \right. & \downarrow \\
 \cdot & \left(\begin{array}{cccccc} 1 & 0 & 1 & \cdot & \cdot & 1 \end{array} \right. \\
 \cdot & \left(\begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right. \\
 \cdot & \left(\begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right. \\
 u_p & \left(\begin{array}{cccccc} 0 & 1 & 1 & \cdot & 0 & 1 \end{array} \right) & p \times p
 \end{array}$$

In $[A(G)]_{p \times p}$, every row total is equal to the corresponding vertex degree $d(u_i) = |N(u_i)|$ and $|N[u_i]| = d(u_i) + 1$.

Step:3 Let D be a minimum connected majority dominating (CMD) set of G . Initially choose the set $D' \subseteq V - D$ and $D' = \emptyset$.

Step:4 Choose the maximum degree vertex $u_i \in (V - D)$ in the last column of $[A(G)]$ and if $|N[u_i]| > \lceil \frac{p}{2} \rceil$ then $D' = \{u_i\}$ is an ICMD set of G with respect to D . Otherwise go to step 5.

Step:5 Pickup all non-zero entries of the row of u_i and choose a maximum degree vertex $u_j \in V - D$ in that row itself among all non-zero entries vertices.

Step:6 Let $D' = \{u_i, u_j\} \subseteq V - D$ such that induced subgraph $\langle D' \rangle$ is connected. If $|N[D']| \geq \lceil \frac{p}{2} \rceil$ then D' is an ICMD set of G with respect to D . Otherwise go to step 7.

Step:7 Pick up all non-zero entries of the rows of u_i, u_j and choose the next maximum degree u_k in that rows u_i and u_j among all non-zero entries such that $d(u_i, u_k) = 1 = d(u_j, u_k)$.

Step:8 Let $D' = \{u_i, u_j, u_k\} \subseteq V - D$ such that $\langle D' \rangle$ is connected.

If $|N[D']| \geq \lceil \frac{p}{2} \rceil$ then D' is an ICMD set of G with respect to D . Otherwise go to step 7.

Apply the same procedure to get finally an ICMD set $D' \subseteq V - D$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $\langle D' \rangle$ is connected for the given graph G with respect to the CMD set D of G .

5.6.2 Algorithm for ICMD Number $\gamma_{CM}^{-1}(G)$

To find the Inverse Connected Majority Domination number $\gamma_{CM}^{-1}(G)$ for a given graph G . First of all, find a minimal inverse connected majority dominating set and then find its inverse connected majority domination number $\gamma_{CM}^{-1}(G)$ for the given connected graph G .

Let D be a minimum CMD set of G . By using algorithm [5.6.1](#), collect all the inverse connected majority dominating sets D' of the given connected graph G .

Step:1 Let $\mathcal{D}' = \{D'_1, D'_2, \dots, D'_t\}$ be the set of all Inverse Connected Majority Dominating (ICMD) sets of the graph G with respect to D .

Step:2 Test that the proper subset D_i'' of $D_i' \in \mathcal{D}'$, for each i , is an ICMD set of G with respect to D .

Step:3 If there exists an ICMD set D_i'' in any $D_i' \in \mathcal{D}'$ then D_i' is not a minimal ICMD set of G with respect to D and remove the set D_i' from the set $\mathcal{D}'$.

Step:4 If there is no proper subset D_i'' of the sets $D_i' \in \mathcal{D}'$ for any i , then the set D_i' is a minimal ICMD set of G with respect to D .

Step:5 Repeat the process to every $D_i' \in \mathcal{D}'$ and collect all minimal ICMD sets $D_i' \subseteq V - D$ for each i , from $\mathcal{D}'$ with respect to D .

Step:6 Let $S = \{D_1', D_2', \dots, D_n'\}$, $n \leq t$ be the set of all minimal ICMD sets of G with respect to D .

Step:7 Find the cardinality of each set $D_i' \in S$, $i = 1, 2, \dots, n$ and $n \leq t$.

Step:8 Pick up the minimum cardinality among all $D_i' \in S$, for $i = 1, 2, \dots, n$ and it gives the ICMD number of G , denoted by $\gamma_{CM}^{-1}(G) = |D_i'|$, for any $D_i' \in S$.

Step:9 Find the maximum cardinality among all $D'_i \in S$, for $i = 1, 2, \dots, n$ and it gives the upper ICMD number of G , denoted by $\Gamma_{CM}^{-1}(G) = |D'_i|$, for any $D'_i \in S$.

Hence the inverse connected majority domination number $\gamma_{CM}^{-1}(G)$ and the upper inverse connected majority domination number $\Gamma_{CM}^{-1}(G)$ for the given connected graph G with respect to D are determined.

5.6.3 Application

Suppose that V is a political party comprising P members. Of this, a few closely associated members (D) may try to influence and persuade other members covering atleast one half of the total strength and they form a first group / faction that will be taken as $N[D]$ such that $|N[D]| \geq \lceil \frac{p}{2} \rceil$. Similarly, in this same party another few homogenous persons (D') may try to discuss the party members and form another group covering similarly atleast one half of the total members, inclusive of first group members $N(D)$ and exclusive of the organizers or formers of first group (D). If first group could not reach a consensus

on any point, another group (D') will take advantage of it and exploit the situation.

In all these cases, the organisers of the group will not switch over / move from. All the organisers of the group D or D' are connected and likeminded with one another within the group only and then they could execute the programs effectively. This is the situation where the concept of connected majority dominating set D and an inverse connected majority dominating set $D' \subseteq (V - D)$ with respect to the connected majority dominating set D of a group structure G .

Chapter 6

Inverse Split Majority Dominating Set of a Graph

Abstract

This chapter introduces a new notion an inverse split majority domination of a graph G . For a connected graph G , the **inverse split majority dominating set** with respect to split majority dominating set of a graph G is defined and an **inverse split majority domination number**, denoted by $\gamma_{SM}^{-1}(G)$ is determined for various classes of graphs. Bounds of $\gamma_{SM}^{-1}(G)$ and relationship of $\gamma_{SM}^{-1}(G)$ with other domination parameters are also discussed in detail.

6.1 Introduction

In 1997, Kulli and Janagiram [31] introduced the concept split domination in graphs. In 2010, Ameen Bibi and Selvakumar [2] defined the notion inverse split and non-split domination in graphs. Now, the area of the research in an inverse domination with many branches has its long roots. In 2015, Joseline Manora and Veeramani [39, 40] studied the split majority domination and non-split majority domination of a graph. Particular values of the numbers $\gamma_{SM}(G)$ and $\gamma_{NSM}(G)$ for various classes of graphs, bounds of these numbers and some characterization theorems on these parameters are also established in their work.

In this thesis, the concept of an inverse majority dominating set in graphs was introduced and developed by the researcher by producing new results and new parameters. It has good applications in practical situations. Motivated by the concepts of the inverse split and non-split domination, the inverse split majority domination in graphs is defined and studied in detail by establishing many theorems.

The organization of this chapter is as follows, Section 6.1 is the introductory part of the chapter and it contains the motivation of

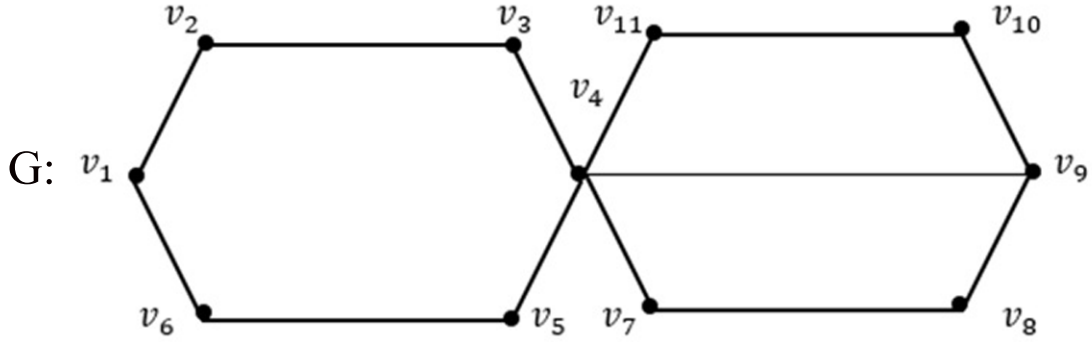
defining a new parameter, inverse split majority dominating set of a graph. Section 6.2 defines the concept of an inverse split majority dominating set with respect to the split majority dominating set and its number for a graph G illustrates the defined concept with examples. The inverse split majority domination number $\gamma_{SM}^{-1}(G)$ is determined for various classes of graphs in Section 6.3. Relationship of $\gamma_{SM}^{-1}(G)$ with other domination parameters are discussed in Section 6.4. In Section 6.5, bounds of $\gamma_{SM}^{-1}(G)$ are established with sharpness. Section 6.6 contains the main results and characterization theorems of an inverse split majority domination of a graph G .

6.2 Inverse Split Majority Dominating Sets in Graphs

Definition 6.2.1: Let G be any simple, finite, undirected and connected graph with p vertices and q edges. Let D be a minimum majority dominating set of a graph G . Let D' be an inverse majority dominating set of G with respect to D . Then $D'_1 \supseteq D'$ is called an inverse split majority dominating set of G if the induced subgraph $\langle V - D'_1 \rangle$ is disconnected. The inverse split majority domination

number, say $\gamma_{SM}^{-1}(G)$ of G is the minimum cardinality of a minimal inverse split majority dominating set of a graph G .

Example 6.2.2: Consider the following graph G with $p = 11$ vertices.



Let $D = \{v_4\}$ be a majority dominating set and a split majority dominating set of G . Then $\gamma_M(G) = \gamma_{SM}(G) = 1$. Then $D' = \{v_2, v_9\} \subseteq V - D$ is an inverse majority dominating set with respect to D and $\gamma_M^{-1}(G) = 2$. Since the induced subgraph $\langle V - D' \rangle$ is connected, choose $D'_1 = \{v_2, v_6, v_9\} \subseteq V - D$ and $\langle V - D'_1 \rangle$ is disconnected and $\gamma_{SM}^{-1}(G) = |D'_1| = 3$. Hence $\gamma_M^{-1}(G) < \gamma_{SM}^{-1}(G)$ and $\gamma_{SM}(G) < \gamma_{SM}^{-1}(G)$.

Next, the following inequalities are immediate using the property of this number $\gamma_{SM}^{-1}(G)$.

Proposition 6.2.3: For a connected graph G , $\gamma_M^{-1}(G) \leq \gamma_{SM}^{-1}(G)$ where $\gamma_M^{-1}(G)$ is an inverse majority domination number of G .

Proof: Since every inverse split majority dominating set of G is also an inverse majority dominating set of G , $\gamma_M^{-1}(G) \leq \gamma_{SM}^{-1}(G)$. ■

Proposition 6.2.4: For a connected graph G , $\gamma_{SM}(G) \leq \gamma_{SM}^{-1}(G)$ where $\gamma_{SM}(G)$ is the split majority domination number of G .

Proof: Since every inverse split majority dominating set of G is a split majority dominating set of G , the split majority domination number $\gamma_{SM}(G)$ is less than or equal to the inverse split majority domination number of G . Therefore, $\gamma_{SM}(G) \leq \gamma_{SM}^{-1}(G)$. ■

Proposition 6.2.5: Let $i_M^{-1}(G)$ be an inverse independent majority domination number of a connected graph G . Then $\gamma_M^{-1}(G) \leq i_M^{-1}(G) \leq \gamma_{SM}^{-1}(G)$.

Proof: By the proposition (4.2.7)(iii), we have $\gamma_M^{-1}(G) \leq i_M^{-1}(G)$ and by proposition (6.2.3), $\gamma_M^{-1}(G) \leq \gamma_{SM}^{-1}(G)$. Also, since every inverse split majority dominating set of G is an inverse independent

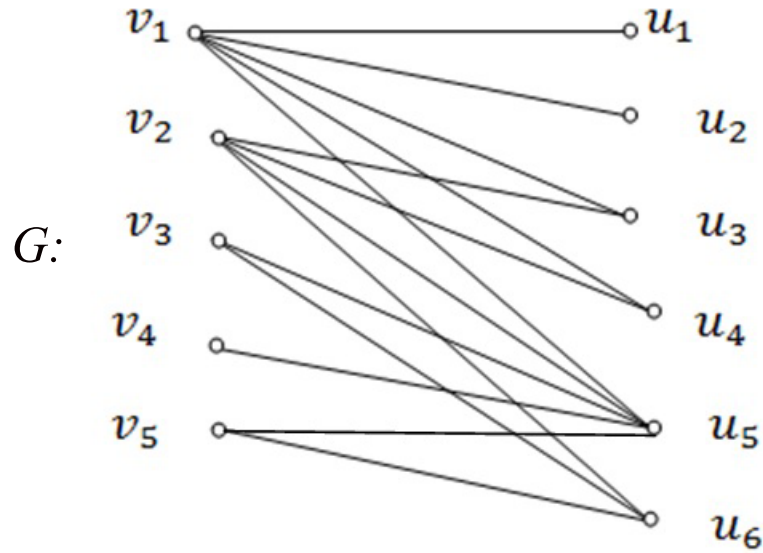
majority dominating set of G , $i_M^{-1}(G) \leq \gamma_{SM}^{-1}(G)$. Combining these inequalities, we obtain $\gamma_M^{-1}(G) \leq i_M^{-1}(G) \leq \gamma_{SM}^{-1}(G)$. ■

Proposition 6.2.6: For a connected graph G , $\gamma_M(G) \leq \gamma_M^{-1}(G) \leq \gamma_{SM}^{-1}(G)$.

Proof: By the proposition (2.2.6) and proposition (6.2.3), we obtain the result. ■

Example 6.2.7: For a connected graph G , $\gamma_M(G) < \gamma_M^{-1}(G) < \gamma_{SM}^{-1}(G)$.

Let $G = B_{5,6}$ be a bipartite graph with $p = 11$ vertices.



Let $V_1(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $V_2(G) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ with $d(v_1) = 5$. Let $D = \{v_1\}$ such that $|N[D]| = \lceil \frac{p}{2} \rceil$ and $\gamma_M(G) = 1$. Let $D' = \{v_2, v_3\} \subseteq (V - D)$ and $\gamma_M^{-1}(G) = 2 = i_M^{-1}(G)$. Let

$D'_1 = \{v_2, v_3, v_5\}$ and $D'_1 \supseteq D'$ such that $|N[D'_1]| > \lceil \frac{p}{2} \rceil$ and the induced subgraph $\langle V - D'_1 \rangle$ is disconnected. Hence D'_1 is an inverse split majority dominating set of G and $\gamma_{SM}^{-1}(G) = |D'_1| = 3$. Thus $\gamma_M(G) < \gamma_M^{-1}(G) < \gamma_{SM}^{-1}(G)$.

This section brings out the exact values of $\gamma_{SM}^{-1}(G)$ for some classes of graphs.

6.3 Inverse Split Majority Domination Number for Some Classes of Graphs

Results 6.3.1:

1. Let $G = K_p, p \geq 2$ be a complete graph. Then $\gamma_{SM}^{-1}(G) = 0$.
2. Let $G = K_{1,p-1}$ be a star with $p \geq 2$. Then $\gamma_{SM}^{-1}(G) = 0$.
3. For any cycle $C_p, p \geq 3$, $\gamma_{SM}^{-1}(G) = \begin{cases} 2, & \text{if } 3 \leq p \leq 6 \\ \lceil \frac{p}{6} \rceil, & \text{if } p \geq 7 \end{cases}$
4. For a corona graph $G = K_p \circ K_1$, $\gamma_{SM}^{-1}(G) = 1$, where K_p is a complete graph.
5. For the Petersen graph with $p = 10$ and $q = 15$, $\gamma_{SM}^{-1}(G) = 3$.

6. For the graph $G = K_p - \{e\}$, $\gamma_{SM}^{-1}(G) = p - 2$ where e is any edge in K_p .

7. For the Double star $G = D_{r,s}$, $r \leq s$ and $r, s \geq 2$,

$$\gamma_{SM}^{-1}(G) = \begin{cases} 1, & \text{if } s = r, r + 1, r + 2 \\ |e_i| + 1, & \text{if } s \geq r + 3. \end{cases}$$

Proposition 6.3.2: For a path P_p , $p \geq 2$, $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{6} \rceil$.

Proof: Let G be a path with $p \geq 2$ vertices. Let $D = \{u_2, u_5, u_8, \dots, u_t\}$ be a majority dominating set of G with $d(u_i, u_j) \geq 3$, for every $i \neq j$ and $|D| = t = \lceil \frac{p}{6} \rceil$. Choose $D' = \{u_3, u_6, u_9, \dots, u_t\} \subseteq V - D$ such that $d(u_i, u_j) \geq 3$, for every $i \neq j$ and $u_i, u_j \in D'$ with $|D'| = t = \lceil \frac{p}{6} \rceil$. Then $|N[D']| \geq \sum_{i=1}^t d(u_i) + t = 3t$. $|N[D']| \geq 3\lceil \frac{p}{6} \rceil \geq \lceil \frac{p}{2} \rceil$. Since $D' \subseteq V - D$ and $|N[D']| \geq \lceil \frac{p}{2} \rceil$, D' is an inverse majority dominating set of G with respect to D . Since the path P_p is minimally connected, the induced subgraph $\langle V - D' \rangle$ is disconnected and splitted into many components. It implies that D' is an inverse split majority dominating set of G with respect to D and $\gamma_{SM}^{-1}(G) \leq |D'| = \lceil \frac{p}{6} \rceil$.

Let D' be a γ_{SM}^{-1} -set of G with $|D'| = t = \gamma_{SM}^{-1}(G)$. Then by the definition of γ_{SM}^{-1} -set, $|N[D']| \geq \lceil \frac{p}{2} \rceil$. Since $D' \subseteq V - D$

and $\langle V - D' \rangle$ is disconnected, $|N[D']| \leq \sum_{i=1}^t d(u_i) + \gamma_{SM}^{-1}(G)$. Then $|N[D']| \leq \gamma_{SM}^{-1}(G)(d(u_i) + 1)$ and $\lceil \frac{p}{2} \rceil \leq 3\gamma_{SM}^{-1}(G)$. Since $\frac{1}{3}\lceil \frac{p}{2} \rceil = \lceil \frac{p}{6} \rceil$, if $p = 2r, 2r + 1$, $\gamma_{SM}^{-1}(G) \geq \lceil \frac{p}{6} \rceil$. Hence we obtain $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{6} \rceil$. ■

Proposition 6.3.3: For a Generalized Petersen $P(n, k)$, $n \geq 5$ and $0 < k < n$, $\gamma_{SM}^{-1}(G) = \lceil \frac{n}{4} \rceil + 1$.

Proof: Let $G = P(n, k)$ and $V(G) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ where $\{u_1, u_2, \dots, u_n\}$ lies in the outer cycle C_1 and $\{v_1, v_2, \dots, v_n\}$ lies in the inner cycle C_2 with $p = |V(G)| = 2n$. Let D be an γ_{SM} -set of G . Let $D' = \{v_1, u_n, u_2, u_5, \dots, u_t\} \subseteq V - D$ such that $d(u_i, u_j) \geq 3$, for $u_i, u_j \in C_1$ and v_1, u_n are adjacent to u_1 . Since G is a 3-regular graph and vertex connectivity number $\kappa(G) = 3$, the vertex u_1 is adjacent to $S = \{v_1, u_n, u_2\}$ such that $S \subseteq D'$ and $|N[S]| = 8$. It implies that the induced subgraph $\langle V - D' \rangle$ is disconnected. If $n \leq 8$ then the set $D' = \{v_1, u_2, u_n\} \subseteq V - D$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ is an inverse split majority dominating set of G and $\gamma_{SM}^{-1}(G) = |D'| = 3 = \lceil \frac{n}{4} \rceil + 1$, if $n \leq 8$.

Suppose $n \geq 9$ and $D' = \{v_1, u_n, u_2, u_5, \dots, u_t\} \subseteq V - D$ with $|D'| = t = \lceil \frac{n}{4} \rceil + 1$. Then $|N[D']| = \sum_{i=1}^{t-3} d(v_i) + |N[S]| + t - 3 = 3(t -$

3)+8+(t-3). Therefore $|N[D']| = 4(t-3)+8 = 4t-4 = 4(\lceil \frac{n}{4} \rceil + 1) - 4$.

$$|N[D']| = \begin{cases} 4\left(\frac{n}{4}\right) + 4 - 4 = n = \frac{p}{2}, & \text{if } n \text{ is even} \\ 4\left(\frac{n}{4}\right) + 5 - 4 = n + 1 > \frac{p}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Since $D' \subseteq (V - D)$, induced subgraph $\langle V - D \rangle$ is disconnected and

$|N[D']| \geq \lceil \frac{p}{2} \rceil$, D' is an inverse split majority dominating set of G .

Hence $\gamma_{SM}^{-1}(G) \leq |D'| = \lceil \frac{n}{4} \rceil + 1, n \geq 9$. Suppose $|D'| < t$ then

$|N[D']| < \lceil \frac{p}{2} \rceil$. It leads to D' would not be an inverse split majority

dominating set of G . Hence $\gamma_{SM}^{-1}(G) \geq |D'| = \lceil \frac{n}{4} \rceil + 1, n \geq 9$. Thus,

$\gamma_{SM}^{-1}(G) = \lceil \frac{n}{4} \rceil + 1, n \geq 5$. ■

Proposition 6.3.4: Let $G = W_p$ be a wheel graph of p vertices.

Then $\gamma_{SM}^{-1}(G) = 0$.

Proof: Let $V(G) = \{u_1, u_2, \dots, u_{p-1}, u_p\}$ where u_1 is a full degree

vertex and $d(u_i) = 3$, for all i . Let $D = \{u_1\}$ be a majority domi-

nating set of G . Choose a set $D' = \{u_2, u_5, \dots, u_{p-1}\} \subseteq (V - D)$ and

D' is an inverse majority dominating set of G such that $d(u_i, u_j) \geq 3$

for every $i \neq j$. By the Proposition (2.3.4), $\gamma_M^{-1}(W_p) = \lceil \frac{p-2}{6} \rceil$.

Since the graph $G = W_p$ has a full degree vertex u_1 and u_1 is adjacent to all the $(p - 1)$ vertices, the induced sub graph $\langle V - D' \rangle$

is not disconnected. In any way, one could not find a set D' such that $\langle V - D' \rangle$ is disconnected. By the result (1.4), $\gamma_{SM}(W_p) = 3$ and $D = \{u_1, u_2, u_5\}$ is a γ_{SM} -set of G . Now, choose a set $D' \subseteq V - D$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ but $\langle V - D' \rangle$ is not disconnected since D' does not contains full degree vertex u_1 . Hence $\gamma_{SM}^{-1}(G) = 0$. ■

Proposition 6.3.5: Let $G = K_{m,n}$ be a complete bipartite graph.

$$\text{Then } \gamma_{SM}^{-1}(G) = \begin{cases} n, & \text{if } m = n \\ n = \max(m, n), & \text{if } m < n \end{cases}$$

Proof: Case (i): When $m = n$. Let $V_1(G) = \{u_1, u_2, \dots, u_m\}$ and $V_2(G) = \{v_1, v_2, \dots, v_n\}$. Since all the vertices of G are majority dominating vertices, $D = \{u_1\}$ is a majority dominating set and $D' = \{v_1\} \subseteq V - D$ is an inverse majority dominating set of G but the induced subgraph $\langle V - D' \rangle$ remains connected since the graph G is a complete bipartite. Now, there exists a subset $D'_1 = \{v_1, v_2, \dots, v_n\} \subseteq V - D$ such that $|N[D'_1]| > \frac{p}{2}$ and the induced subgraph $\langle V - D'_1 \rangle$ is disconnected. This implies that D'_1 is a minimal inverse split majority dominating set of G . Thus $\gamma_{SM}^{-1}(G) = |D'_1| = n$.

Case (ii): When $m < n$. Let $D = \{u_1\}$ be a majority dominating set of G . Let $D' = \{u_2\} \subseteq V - D$, where $u_2 \in V_1(G)$. Then $|N[D']| =$

$n + 1 > \lceil \frac{p}{2} \rceil$ and $D' \subseteq V - D$. Therefore D' is a minimum inverse majority dominating set of G but the induced subgraph $\langle V - D' \rangle$ is connected. Now choose a subset $D'_1 = \{v_1, v_2, \dots, v_n\} \subseteq V - D$, where $v_i \in V_2(G)$ such that the induced subgraph $\langle V - D'_1 \rangle$ is disconnected with m components and $|N[D'_1]| > \lceil \frac{p}{2} \rceil$. Hence D'_1 is an inverse split majority dominating set of G and $\gamma_{SM}^{-1}(G) = |D'_1| = n$. Hence $\gamma_{SM}^{-1}(G) = \max\{m, n\} = n$, if $m < n$. ■

Example 6.3.6: Let $G = K_{3,10}$ and $V(G) = \{u_1, u_2, u_3, v_1, v_2, \dots, v_{10}\}$ with $d(u_i) = 10$ and $d(v_i) = 3$. Then $D = \{u_1\}$, $D' = \{u_2\}$ and $D'_1 = \{v_1, v_2, \dots, v_{10}\}$ are the majority dominating set an inverse majority dominating set and an inverse split majority dominating set of G respectively. Hence $\gamma_M(G) = 1 = \gamma_M^{-1}(G)$ and $\gamma_{SM}^{-1}(G) = |D'_1| = 10$, if $m < n$.

Proposition 6.3.7: Let $C_p, p \geq 3$ be any cycle. Then,

$$\gamma_{SM}^{-1}(G) = \begin{cases} 2, & \text{if } 3 \leq p \leq 6 \\ \lceil \frac{p}{6} \rceil, & \text{if } p \geq 7 \end{cases}$$

Proof: Case (i): When $3 \leq p \leq 6$. Let $G = C_p$, be a cycle with $d(v_i) = 2$ for all $v_i \in V(G)$. Let $D = \{v_2\}$ and $D' = \{v_3\} \subseteq V - D$ be

a majority dominating set and an inverse majority dominating set of G . Since $\langle V - D' \rangle$ is connected, choose $D'_1 = \{v_1, v_3\} \subseteq V - D$ such that $|N[D'_1]| > \frac{p}{2}$ and $\langle V - D'_1 \rangle$ is disconnected. Hence $\gamma_{SM}^{-1}(G) = |D'_1| = 2$.

Case (ii): When $p \geq 7$. By the similar arguments as in Proposition (6.3.2) of $G = P_p$, a path, we obtain $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{6} \rceil$. ■

Theorem 6.3.8: Let $G = S(K_{1,p-1})$ be a subdivision of a star with p vertices. Then $\gamma_{SM}^{-1}(G) = \lceil \frac{p-2}{4} \rceil$.

Proof: Let $V(G) = \{u, u_1, u_2, \dots, u_{\lfloor \frac{p}{2} \rfloor}, v_1, v_2, \dots, v_{\lfloor \frac{p}{2} \rfloor}\}$ where u is a central vertex, $v_1, v_2, \dots, v_{\lfloor \frac{p}{2} \rfloor}$ are pendants and $u_1, u_2, \dots, u_{\lfloor \frac{p}{2} \rfloor}$ are middle vertices of each edge of G . Since $d(u) = \lfloor \frac{p}{2} \rfloor - 1$, $D = \{u\}$ be a majority dominating set of G . Choose $D' = \{u_1, u_2, \dots, u_t\} \subseteq V - D$ with $|D'| = t = \lceil \frac{p-2}{4} \rceil$ such that $d(u_i, u_j) \geq 2$ for every $i \neq j$. Then $|N[D']| \geq 2t + 1$. Therefore $|N[D']| \geq 2\lceil \frac{p-2}{4} \rceil + 1 \geq \lfloor \frac{p}{2} \rfloor$ and $D' \subseteq V - D$. Hence D' is an inverse majority dominating set of G . Since the graph $S(K_{1,p-1})$ has cut vertices, the induced subgraph $\langle V - D' \rangle$ is disconnected. Hence D' is an inverse split majority dominating set of G and $\gamma_{SM}^{-1}(G) = |D'| = \lceil \frac{p-2}{4} \rceil$. ■

Theorem 6.3.9: Let G be a uniform caterpillar with p vertices and t pendants attached to each vertex of G . Then $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{2(t+3)} \rceil$.

Proof: Let P_n be a path with n vertices and t pendants attached to every vertex of G . Therefore $p = (t+1)n$ in which u_1 and u_n are end vertices of G and $(u_2, \dots, u_{n-1})$ are intermediate vertices of G with $d(u_1) = (t+1) = d(u_n)$ and $d(u_i) = (t+2)$, for $i = 2, \dots, (n-1)$. This theorem is proved by induction on the number of pendants ' t '. If $t = 1$ then $p = 2n$ and all intermediate vertices u_i such that $d(u_i) = 3$, for all $i = 2, \dots, (n-1)$. Let $D = \{u_2, u_5, \dots, u_r\}$ with $d(u_i, u_j) = 3$, for every $i \neq j$ and $u_i, u_j \in D$ such that $|D| = r = \lceil \frac{p}{8} \rceil$ with $|N[D]| \geq \lceil \frac{p}{2} \rceil$. Then D is a γ_M -set of G .

Let $D' = \{u_3, u_6, \dots, u_{r+1}\} \subseteq V - D$ with $d(u_i, u_j) = 3$ and $|D'| = r = \lceil \frac{p}{8} \rceil$. Then $|N[D']| = r|N[u_i]| = 4r = 4\lceil \frac{p}{8} \rceil \geq \lceil \frac{p}{2} \rceil$ and $D' \subseteq (V - D)$. Therefore D' is an inverse majority dominating set of G . Since $(V - D)$ has cut vertex, $\langle V - D' \rangle$ is disconnected. Therefore, D' is an inverse split majority dominating set of G and if $t = 1$, $\gamma_{SM}^{-1}(G) \leq |D'| = \lceil \frac{p}{8} \rceil = \lceil \frac{p}{2(t+3)} \rceil$. If $t = 2$, then $p = 3n$ and $d(u_i) = 4$, for $i = 2, 3, \dots, (n-1)$. Therefore choose $D' \subseteq (V - D)$ with $|D'| = \lceil \frac{p}{10} \rceil$. Since $(V - D)$ has cut vertex, $\langle V - D' \rangle$ is

disconnected. Therefore $\gamma_{SM}^{-1}(G) \leq |D'| = \lceil \frac{p}{10} \rceil = \lceil \frac{p}{2(t+3)} \rceil$, if $t = 2$.

This result is true for $t = 1, 2, 3, \dots, (t-1)$. If $t = (t-1)$ pendants then $p = tn$ vertices in G and $d(u_i) = (t-1+2) = t+1$, for $i = 2, \dots, (n-1)$. Hence $\gamma_{SM}^{-1}(G) \leq |D'| = \lceil \frac{p}{2(t+4)} \rceil = \lceil \frac{p}{2(t+3)} \rceil$, if $t = t-1$. Suppose G has t pendants at each of the ' n ' vertices. Then $p = (t+1)n$ and $d(u_i) = (t+2)$, for $i = 2, \dots, (n-1)$. Let $D' = \{u_3, u_5, \dots, u_r\} \subseteq (V - D)$ with $d(u_i, u_j) = 3$, for $i \neq j$ and $|D'| = r = \lceil \frac{p}{2(t+3)} \rceil$. Then $|N[D']| = r|N[u_i]| = r(t+1) = \lceil \frac{p}{2(t+3)} \rceil(t+3) \geq \lceil \frac{p}{2} \rceil$. Since $(V - D)$ has cut vertex, D' includes some cut vertices and $\langle V - D' \rangle$ is disconnected. Hence D' is an inverse split majority dominating set of G and $\gamma_{SM}^{-1}(G) \leq |D'| = \lceil \frac{p}{2(t+3)} \rceil$, if G has t - pendants and $p = n(t+1)$.

Suppose $|D_1| < |D'|$. Then $|N[D_1]| < \lceil \frac{p}{2} \rceil$ and $D_1 \subseteq V - D$. Also, $\langle V - D_1 \rangle$ is disconnected. But D_1 is not an inverse split majority dominating set of G . Hence $\gamma_{SM}^{-1}(G) > |D_1|$ and $\gamma_{SM}^{-1}(G) \geq |D'| = \lceil \frac{p}{2(t+3)} \rceil$ combining the above results, we obtain $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{2(t+3)} \rceil$. ■

Proposition 6.3.10: Let $G = F_n \circ K_1$ be a Corona graph with $n \geq 4$.

Then $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{10} \rceil$, where $F_n = P_{n-1} \vee K_1$.

Proof: Let $F_p = P_{n-1} \vee K_1$ and the Corona graph $G = F_n \circ K_1$

with $p = 2n = |V(G)|$. Let $V(G) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ where $u_i \in V(F_n)$ and v_i 's are pendants. Since $d(u_1) = \frac{p}{2}$, $D = \{u_1\}$ is a majority dominating set of G . Now, choose a set $D' = \{u_2, u_5, \dots, u_t\} \subseteq V - D$ with $|D'| = t = \lceil \frac{p}{10} \rceil$ such that $d(u_i, u_j) \geq 3$. Let $\Delta'(G)$ be the maximum degree of $(V - D)$ and $\Delta'(G) = d(u_i), u_i \neq u_1$. Since $d(u_i) = 4$, $|N[D']| = \sum_{i=1}^t d(u_i) + t - (t-1) = 4t + 1 = 4\lceil \frac{p}{10} \rceil + 1 \geq \lceil \frac{p}{2} \rceil$ and the induced subgraph $\langle V - D' \rangle$ is disconnected with ' t ' isolates. Hence D' is an inverse split majority dominating set and $\gamma_{SM}^{-1}(G) \leq |D'| = \lceil \frac{p}{10} \rceil$. Suppose choose the set D' with $|D'| = t - 1$. Then $|N[D']| < \lceil \frac{p}{2} \rceil$ and $D' \subseteq V - D$. It leads to the set D' is not an inverse split majority dominating set of G and $\gamma_{SM}^{-1}(G) \geq \lceil \frac{p}{10} \rceil$. Hence $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{10} \rceil$. ■

The following graph structure determines the two numbers $\gamma_{SM}^{-1}(G)$ and $\gamma_M^{-1}(G)$ are equal.

Theorem 6.3.11: For a binary tree T with k level and p vertices, $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{8} \rceil$ and $\gamma_{SM}^{-1}(G) = \gamma_M^{-1}(G)$.

Proof: This result is proved by induction on the level ' k ' of a tree T .

Let T be a binary tree with p vertices such that $d(u_1) = 2, d(u_i) = 3$,

where u_i 's are intermediate vertices and others are pendants. Let $D = \{u_1\}$ and $D' = \{u_2\}$ be a majority dominating set and an inverse majority dominating set of G . Since a tree T has a cutvertex, D' is also an inverse split majority dominating set of G .

At level 1, $p = 1 + 2^1 = 3$ and $\gamma_M^{-1}(G) = \gamma_{SM}^{-1}(G) = 1$. At level 2, $p = 3 + 2^2 = 7$. Since T has two intermediate vertices u_1 and u_2 , $\gamma_M(G) = |D| = \{u_1\} = 1$ and $\gamma_M^{-1}(G) = |D'| = \{u_2\} = 1$. Since u_2 is a cut vertex of T , D' is also an inverse split majority dominating set of G . Hence $\gamma_{SM}^{-1}(G) = 1 = \gamma_M^{-1}(G)$. At level 3, $p = 7 + 2^3 = 15$, there are 4 intermediate vertices $\{u_1, u_2, u_3, u_4\}$. Let $D = \{u_1, u_3\}$ and $D' = \{u_2, u_4\} \subseteq V - D$ such that $|N[D]| = |N[D']| = 8 = \lceil \frac{p}{2} \rceil$. Hence D is a majority dominating set of G and D' is an inverse majority dominating set of G . Since D' has cut vertex and $\langle V - D' \rangle$ is disconnected, D' is also an inverse split majority dominating set of G . It implies that $\gamma_M(G) = \gamma_M^{-1}(G) = \gamma_{SM}^{-1}(G) = 2$.

This result is true for $(k-1)$ level. Now, at level k , $p = [V(l_{k-1})] + 2^k$, where 2^k pendants and $(p-2^k-1)$ intermediate vertices. Let $D' = \{u_1, u_3, u_4\} \subseteq (V - D)$ such that $|D'| = t = \lceil \frac{p}{8} \rceil$ with $d(u_i, u_j) = 2$ and $|N[D']| = \sum_{i=1}^t d(u_i) + t = 4t = 4\lceil \frac{p}{8} \rceil \geq \lceil \frac{p}{2} \rceil$. Hence D' is an inverse majority dominating set of G and $\gamma_M^{-1}(G) = \lceil \frac{p}{8} \rceil$. Since $(V - D)$

contains cut vertices, induced subgraph $\langle V - D' \rangle$ is disconnected. It implies that D' is also an inverse split majority dominating set of G and $\gamma_{SM}^{-1}(G) = |D'| = \lceil \frac{p}{8} \rceil = \gamma_M^{-1}(G)$. ■

6.4 Relationship with Other Parameters

In this section, the relationship of $\gamma_{SM}^{-1}(G)$ with the inverse majority domination number $\gamma_M^{-1}(G)$ and connectivity number of $\kappa(G)$ of a graph G is discussed.

Theorem 6.4.1: Let D be a γ_M -set of G . If the induced subgraph $\langle V - D \rangle$ contains a cut vertex then $\gamma_{SM}^{-1}(G) = \gamma_M^{-1}(G)$.

Proof: Let D be a minimum majority dominating set of G . Let $u \in V - D$ be a cut vertex of G .

Case (i): When $d(u) \geq \lceil \frac{p}{2} \rceil - 1$. Then $D' = \{u\}$ is an inverse majority dominating set of G and $\gamma_{SM}^{-1}(G) = |D'| = \gamma_M^{-1}(G) = 1$.

Case (ii): When $d(u) \leq \lceil \frac{p}{2} \rceil - 2$. Then $\gamma_M^{-1}(G) \geq 2$. Since the γ_M^{-1} -set D' contains a cut vertex u , the induced subgraph $\langle V - D' \rangle$ is disconnected. It implies that $\gamma_{SM}^{-1}(G) \geq 2$. Since the cut vertex u of degree $d(u) \geq 2$, u is not a pendant. If $d(u) = 2$ and $u \in V - D$,

the subset $D' = \{u, u_1, \dots, u_t\} \subseteq V - D$ such that $d(u_i, u_j) \geq 3$, for $i \neq j$ and $|D'| = t + 1$ where $t = \lceil \frac{p}{6} \rceil - 1$. Then $|N[D']| = 3(t + 1) \geq \lceil \frac{p}{2} \rceil$. It implies that D' is an inverse majority dominating set of G and $\gamma_M^{-1}(G) = t + 1$. Since D' contains a cut vertex u and the induced subgraph $\langle V - D' \rangle$ is disconnected the set D' is an inverse split majority dominating set of G and $\gamma_{SM}^{-1}(G) = |D'| = t + 1$. Hence $\gamma_M^{-1}(G) = \gamma_{SM}^{-1}(G)$.

If $d(u) \geq 3$ and $u \in V - D$ then the set $D' = \{u, u_1, \dots, u_t\} \subseteq V - D$ with $d(u_i, u_j) \geq 3$, for $i \neq j$ and $u_i, u_j \in D'$ and $|D'| = t + 1$, where $t \geq \lceil \frac{p}{8} \rceil - 1$. Then $|N[D']| \geq 4(t + 1) \geq \lceil \frac{p}{2} \rceil$ implies that D' is an inverse majority dominating set of G and $\gamma_M^{-1}(G) = t + 1$. Since D' includes the cut vertex u , $\langle V - D' \rangle$ is disconnected and $\gamma_{SM}^{-1}(G) = |D'| = t + 1$. Hence, in all degrees of a cut vertex u , if the induced subgraph $\langle V - D \rangle$ contains a cut vertex ' u ' then $\gamma_{SM}^{-1}(G) = \gamma_M^{-1}(G)$. ■

Theorem 6.4.2: Let D be a minimum majority dominating set of a connected graph G . If the induced subgraph $\langle V - D \rangle$ does not contain a cut vertex then $\gamma_{SM}^{-1}(G) \geq \kappa(G)$ where $\kappa(G)$ is a vertex connectivity of G .

Proof: Let D be a γ_M -set of a connected graph G with p vertices.

Let $S \subseteq V - D$ and $S = \{u_1, u_2, \dots, u_t\}$ be a vertex cut of G . Then $\langle V - S \rangle$ is disconnected with atleast two components g_1 and g_2 and each vertex is the end vertex of every edge connecting the components g_1 and g_2 . Therefore the vertex connectivity number $\kappa(G) = |S| = t$.

Case (i): If $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $S \subseteq V - D$, then S is an inverse split majority dominating set of G and $\gamma_{SM}^{-1}(G) = |S| = \kappa(G)$.

Case (ii): If $|N[S]| < \lceil \frac{p}{2} \rceil$ then choose a subset $D_1 = \{S\} \cup \{S_1\}$, where $S = \{u_1, u_2, \dots, u_{t_1}\}$ with $|S| = t_1$ is a vertex cut and $S_1 = \{v_1, v_2, \dots, v_{t_2}\} \subseteq V - D$ with $|S_1| = t_2$ such that $|N[D_1]| \geq \lceil \frac{p}{2} \rceil$ and $|D_1| = t = t_1 + t_2$. If $\gamma_{SM}^{-1}(G) > \kappa(G) = |S|$. Since D_1 includes a vertex cut, the induced subgraph $\langle V - D_1 \rangle$ is disconnected. Since $S \subseteq V - D$ and $S_1 \subseteq V - D$, $D_1 \subseteq V - D$ and $|D_1| = t = t_1 + t_2$. It implies that D_1 is an inverse split majority dominating set of G and $|S| \subseteq |D_1|$. Therefore $|S| = \kappa(G) < \gamma_{SM}^{-1}(G) = |D_1|$. Thus $\gamma_{SM}^{-1}(G) > \kappa(G)$. Hence from case (i) and (ii), we obtain $\gamma_{SM}^{-1}(G) \geq \kappa(G)$, where $\kappa(G)$ is the vertex connectivity of G . This bound is sharp if $G = C_{19}$. By the results (6.3.1)(3), $\gamma_{SM}^{-1}(C_{19}) = \lceil \frac{p}{6} \rceil = 4$ and $\kappa(G) = 2$. Hence $\gamma_{SM}^{-1}(G) > \kappa(G)$. Also for Petersen graph, $\kappa(G) = 3$ and $\gamma_{SM}^{-1}(G) = 3$. ■

6.5 Bounds of $\gamma_{SM}^{-1}(G)$

The following theorem gives a lower and upper bound of a tree in terms of the number of vertices p , maximum degree $\Delta(G)$ and maximum degree $\Delta'(G)$ in $(V - D)$ where D is a γ_M - set of G .

Theorem 6.5.1: For any tree $T \neq K_{1,p-1}$, $\lceil \frac{p}{8} \rceil \leq \gamma_{SM}^{-1}(G) \leq \lceil \frac{p}{4} \rceil + 1$.

These bounds are sharp.

Proof: The theorem is proved by induction on the number of pendants ‘ e ’. Since every tree T has atleast two pendants, if $e = 2$ then $T = P_p$, a path. By the Proposition (6.3.2), $\gamma_{SM}^{-1}(P_p) = \lceil \frac{p}{6} \rceil > \lceil \frac{p}{8} \rceil$. If $e = 3$, then the tree T has the structure a caterpillar or a double star. By the result (6.3.9), we have $\gamma_{SM}^{-1}(T) = \lceil \frac{p}{8} \rceil$ and if T is a double star, by the result (6.3.1)(7), $\gamma_{SM}^{-1}(D_{r,s}) \geq \lceil \frac{p}{8} \rceil$. If $e = 4$, then T is a binary tree and by the Theorem (6.3.11), $\gamma_{SM}^{-1}(T) = \lceil \frac{p}{8} \rceil$.

This result is true for $e = 2, 3, 4, \dots, p - 3$. If $e = p - 2$ then $T = D_{r,s}$, a double star with $(r + s) = (p - 2)$. By the result (6.3.1)(7), $\gamma_{SM}^{-1}(G) = 1$, if $r = s$. The lower bound is sharp if T is a caterpillar with p vertices. The upper bound exists if $T = D_{1,10}$ is a double star with $p = 13$. Let $D = \{v\}$ be majority dominating set

$S = \{u, v_1, v_2, v_3, v_4\} \subseteq V - D$, where $d(u) = 2$ and v_i 's are pendants such that $|N[S]| \geq \lceil \frac{p}{2} \rceil$. It implies that S is an inverse split majority dominating set of G and $\gamma_{SM}^{-1}(G) = |S| = 5 = \lceil \frac{p}{4} \rceil + 1$. ■

Theorem 6.5.2. *If G is a connected graph and H is a connected spanning subgraph of G then $\gamma_{SM}^{-1}(G) \leq \gamma_{SM}^{-1}(H)$.*

Proof: Since G is connected, any connected spanning subgraph of G is a minimally connected subgraph H of G . It implies that $\gamma_{SM}(G) \leq \gamma_{SM}(H)$ and $\gamma_{SM}(G) \leq \gamma_{SM}^{-1}(G)$. Since H is a spanning subgraph of G , $\gamma_{SM}^{-1}(G) \leq \gamma_{SM}^{-1}(H)$. ■

Theorem 6.5.3. *Let $G = G_1 \circ G_2$ be a corona graph with p vertices where G_1 and G_2 are any two connected graphs. Then $\gamma_{SM}^{-1}(G) = \lceil \frac{p}{2(\Delta'(G)+1)} \rceil$ where $\Delta'(G)$ is a maximum degree of $(V - D)$.*

Proof: Let G_1 and G_2 be any two connected graphs with p_1 and p_2 such that $p_1 \geq p_2$. Let $V(G_1) = \{v_1, v_2, \dots, v_{p_1}\}$ and $V(G_2) = \{u_1, u_2, \dots, u_{p_2}\}$ with $|V(G_1)| = p_1$ and $|V(G_2)| = p_2$ and $|V(G)| = p = (p_1 \circ p_2) + p_1$. Let $\Delta(G)$ is the maximum degree of G and $\Delta'(G)$ be the maximum degree of $(V - D)$.

Case (i): When $\Delta(G) > \Delta'(G)$. Let $\Delta(G) = d(v_i) \geq \lceil \frac{p}{2} \rceil - 1$, for exactly one $v_1 \in V(G_1)$. Then $D = \{v_1\}$ be a majority dominating set of G . Since the corona graph G contains cutvertices, $\langle V - D \rangle$ is disconnected. Therefore D is a split majority dominating set of G and $\gamma_{SM}(G) = |D| = 1$. Now, take $\Delta'(G)$ is the maximum degree $\langle V - D \rangle$ with $\Delta'(G) < \lceil \frac{p}{2} \rceil - 1$ and $\Delta'(G) = d(v_i)$, for $v_i \neq v_1$.

Let $D' = \{v_2, v_5, \dots, v_t\} \subseteq V - D$ such that $d(x_i, x_j) \geq 3$, for $i \neq j$ and all $v_i \in V(G_1)$ with $|D'| = t = \lceil \frac{p}{2(\Delta'+1)} \rceil$. Then $|N[D']| = \sum_{i=1}^t d(v_i) + t = t\Delta'(G) + t = t(\Delta'(G) + 1) = \lceil \frac{p}{2} \rceil$. Since G contains cutvertices which are all $v_i \in V(G_1)$. Induced subgraph $\langle V - D' \rangle$ is disconnected. It implies that D' is an inverse split majority dominating set of G and $\gamma_{SM}^{-1}(G) \leq |D'| = \lceil \frac{p}{2(\Delta'+1)} \rceil$.

Case (ii): When $\Delta(G) = \Delta'(G)$. Let D be a γ_{SM} -set of G with $|D| = \lceil \frac{p}{2(\Delta+1)} \rceil$. Let $D' = \{v_2, v_5, \dots, v_t\} \subseteq V - D$ such that $d(v_i, x_j) \geq 3$, for $i \neq j$ with $|D'| = t = \lceil \frac{p}{2(\Delta'+1)} \rceil$. Then by the argument, $|N[D']| = \lceil \frac{p}{2} \rceil$ and D' is an inverse split majority dominating set of G . Hence $\gamma_{SM}^{-1}(G) \leq |D'| = \lceil \frac{p}{2(\Delta'+1)} \rceil$.

Suppose the set $D_1 \subseteq V - D$ such that $|D_1| < |D'|$. Then $|N[D_1]| < \lceil \frac{p}{2} \rceil$ and $\langle V - D_1 \rangle$ is disconnected. It implies that D_1 is not an inverse split majority dominating set of G . Hence $\gamma_{SM}^{-1}(G) \geq$

$|D_1| \geq |D'| = \left\lceil \frac{p}{2(\Delta'+1)} \right\rceil$. From the above inequalities, we obtained $\gamma_{SM}^{-1}(G) = \left\lceil \frac{p}{2(\Delta'(G)+1)} \right\rceil$. ■

The next result produces a bound for any connected graph in terms of p and $\Delta(G)$.

Theorem 6.5.4: For any connected graph $G \neq K_p$, a complete graph with p vertices, $\left\lceil \frac{p}{2(\Delta+1)} \right\rceil \leq \gamma_{SM}^{-1}(G) \leq p-2$. The bounds are sharp.

Proof: Since $\gamma_M^{-1}(G) \leq \gamma_{SM}^{-1}(G)$ and $\gamma_M^{-1}(G) \geq \left\lceil \frac{p}{2(\Delta+1)} \right\rceil$. Then $\gamma_{SM}^{-1}(G) \geq \left\lceil \frac{p}{2(\Delta+1)} \right\rceil$. Next inequality is proved by induction on $\Delta(G)$. If $\Delta = 2$, then $G = C_p$, a Cycle or P_p , a Path with p vertices. By the Proposition (6.3.2), $\gamma_{SM}^{-1}(G) = \left\lceil \frac{p}{6} \right\rceil = \left\lceil \frac{p}{2(\Delta+1)} \right\rceil$. If $\Delta = 3$, then G is a Caterpillar with one pendant at each vertex of the path. By the Theorem (6.3.9), $\gamma_{SM}^{-1}(G) = \left\lceil \frac{p}{8} \right\rceil = \left\lceil \frac{p}{2(\Delta+1)} \right\rceil$.

This result is true for all $\Delta = 2, 3, \dots, (p-2)$. If $\Delta = p-1$ and $G \neq K_p$, then $G = K_p - \{e\}$ and $\gamma_{SM}^{-1}(G) = \kappa(G) = p-2$. Thus $\gamma_{SM}^{-1}(G) = (p-2)$. Hence $\left\lceil \frac{p}{2(\Delta+1)} \right\rceil \leq \gamma_{SM}^{-1}(G) \leq (p-2)$. These bounds are sharp for $G = C_p$, cycle and $G = K_p - \{e\}$. ■

Theorem 6.5.5. *For any tree T , $\gamma_{SM}^{-1}(T) \leq \lceil \frac{p}{2} \rceil - k + 1$ where k is the degree of a cut vertex in $(V-D)$.*

Proof: Let D be a γ_M -set of T . Let v be a cut vertex in $(V-D)$ such that $d(v) = k$. If $k = \lceil \frac{p}{2} \rceil$, then the cut vertex v of degree $d(v) = \lceil \frac{p}{2} \rceil$. Let $D' = \{v\} \subseteq V - D$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $\langle V - D' \rangle$ is disconnected. It implies that D' is an inverse split majority dominating set of T and $\gamma_{SM}^{-1}(G) \leq |D'| = 1 = \lceil \frac{p}{2} \rceil - \lceil \frac{p}{2} \rceil + 1 = \lceil \frac{p}{2} \rceil - d(v) + 1$. It implies that $\gamma_{SM}^{-1}(G) \leq \lceil \frac{p}{2} \rceil - k + 1$. If $k = \lceil \frac{p}{2} \rceil - 1 = d(v)$ then $|N[v]| = \lceil \frac{p}{2} \rceil$. By the same argument as before, $\gamma_{SM}^{-1}(G) \leq \lceil \frac{p}{2} \rceil - k + 1$. If $k = \lceil \frac{p}{2} \rceil - 2 = d(v)$ then $|N[v]| = \lceil \frac{p}{2} \rceil - 1$.

Let $D' = \{v, v_1\} \subseteq (V - D)$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$. Since v is a cut vertex in $(V-D)$, $\langle V - D' \rangle$ is disconnected. It implies that D' is an inverse split majority dominating set of T and $\gamma_{SM}^{-1}(T) \leq |D'| = 2 = \lceil \frac{p}{2} \rceil - \lceil \frac{p}{2} \rceil + 2 = \lceil \frac{p}{2} \rceil - (\lceil \frac{p}{2} \rceil - 2)$.

Therefore $\gamma_{SM}^{-1}(T) \leq \lceil \frac{p}{2} \rceil - k$. Hence $\gamma_{SM}^{-1}(G) < \lceil \frac{p}{2} \rceil - k + 1$. This result is true for $k = \lceil \frac{p}{2} \rceil, \lceil \frac{p}{2} \rceil - 1, \lceil \frac{p}{2} \rceil - 2, \dots, 3$. When $k = 2 = d(v)$ then $|N[v]| = 3$. Let $D' = \{v, v_1, v_2, \dots, v_t\} \subseteq V - D$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $|D'| = \lceil \frac{p}{6} \rceil$. Since $v \in D'$ is a cut

vertex $\langle V - D' \rangle$ is disconnected. It implies that D' is an inverse split majority dominating set of T and $\gamma_{SM}^{-1}(G) \leq |D'| = \lceil \frac{p}{6} \rceil = \lceil \frac{p}{2} \rceil - \lceil \frac{p}{2} \rceil + \lceil \frac{p}{6} \rceil = \lceil \frac{p}{2} \rceil - \lceil \frac{p}{3} \rceil < \lceil \frac{p}{2} \rceil - k + 1$. Therefore $\gamma_{SM}^{-1}(T) < \lceil \frac{p}{2} \rceil - k + 1$. Hence $\gamma_{SM}^{-1}(T) \leq \lceil \frac{p}{2} \rceil - k + 1$.

The bound is sharp if $T = D_{r,s}, r \leq s$. If $T = D_{5,5}$ with $p = 12$ and $k = 6$. Then $\gamma_M(G) = 1$ and $\gamma_{SM}^{-1}(G) = 1 = \lceil \frac{p}{2} \rceil - k + 1$. If $T = D_{3,10}$ with $p = 15$ and $k = 4$ then $\gamma_M(G) = 1$ and $\gamma_{SM}^{-1}(G) = 3 < \lceil \frac{p}{2} \rceil - k + 1$. ■

The following section exhibits the necessary and sufficient conditions for the existence of a minimal inverse split majority dominating set with respect to a split majority dominating set D with two classifications such as $(V - D)$ has a cut vertex and $(V - D)$ contains no cut vertex.

6.6 Characterisation Theorem for Minimal Inverse Split Majority Dominating Set

Theorem 6.6.1: Let D be a γ_{SM} - set of a connected graph G and $(V - D)$ has a cut vertex. Then the inverse split majority dominating

set $D' \subseteq V - D$ is minimal if and only if for each $u \in D'$, either the following condition (a) or (b) holds.

(a) (i) If $|N[D']| > \lceil \frac{p}{2} \rceil$ then $|pn[u, D']| > |N[D']| - \lceil \frac{p}{2} \rceil$ and

(ii) $\langle (V - D') \cup \{u_i\} \rangle$ is connected, for all $u_i \in D'$.

(b) (i) If $|N[D']| = \lceil \frac{p}{2} \rceil$ then either u is an isolate of D' or $pn[u, D'] \cap$

$(V - D') \neq \emptyset$ and

(ii) $\langle (V - D') \cup \{u_i\} \rangle$ is connected, for all $u_i \in D'$.

Proof: Let D be a γ_{SM} -set of connected graph G and $(V - D)$ has a cut vertex. Assume that $D' \subseteq V - D$ is a minimal inverse split majority dominating set of G with respect to D .

Case(i) Let $u \in D'$. Since D' is minimal and $|N[D']| \geq \lceil \frac{p}{2} \rceil$, the set $(D' - \{u\})$ is not an inverse split majority dominating set of G . Then either $|N[D' - \{u\}]| < \lceil \frac{p}{2} \rceil$ or $\langle V - D' \rangle$ is connected. Let $|N[D']| > \lceil \frac{p}{2} \rceil$. Since, $|N[D' - \{u\}]| < \lceil \frac{p}{2} \rceil$, $|pn[u, D']| = |N[D']| - |N[D' - \{u\}]|$ and $|N[D']| - |pn[u, D']| < \lceil \frac{p}{2} \rceil$. It implies that $|pn[u, D']| > |N[D']| - \lceil \frac{p}{2} \rceil$, for some $u \in D'$. Hence the condition (a) (i) holds.

Since D' is a minimal inverse split majority dominating set of G , the induced subgraph $\langle V - D' \rangle$ is disconnected. If $D' = \{u_1\} \subseteq$

$(V-D)$ and $(V-D)$ has a cut vertex u_1 then $\langle V - D' \rangle$ is disconnected with atleast 2 components contains isolates. Now add this cut vertex u_1 to $\langle V - D' \rangle$, the induced subgraph $\langle (V - D') \cup \{u_1\} \rangle$ would be connected in G for some $u \in D'$. If $D' = \{u_1, \dots, u_i\}, i \geq 2$ and $D' \subseteq (V-D)$ contains a vertex cut ' S ' with $|S| \geq 2$ and $|N[D']| \geq \lceil \frac{p}{2} \rceil$ then the induced subgraph $\langle V - D' \rangle$ is disconnected with atleast two components. Now, if add all vertices $u_i \in D'$ to $(V - D')$ then $\langle (V - D') \cup \{u_i\} \rangle$ would be connected in G . Hence the condition (a)(ii) holds.

Case (ii): Let $|N[D']| = \lceil \frac{p}{2} \rceil$ and $D' \subseteq (V - D)$ and $u \in D'$. Suppose u is neither an isolate of D' nor u has a private neighbour in $\langle V - D' \rangle$. Then $pn[u, D'] = \emptyset$. Since $|N[D' - \{u\}]| = |N[D']| - |pn[u, D']|, |N[D' - \{u\}]| = |N[D']| = \lceil \frac{p}{2} \rceil$. It implies that $(D' - \{u\}) \subseteq (V-D)$ is an inverse split majority dominating set of G , which is a contradiction. Also, By the above arguments, $\langle (V - D') \cup \{u_i\} \rangle$ is connected in G , for all $u_i \in D'$.

Conversely, suppose one of the above conditions (a) or (b) is true. Let D' be an inverse split majority dominating set of G . Then prove that D' is a minimal inverse split majority dominating set of G .

Suppose D' is not minimal. Then $D_1 = (D' - \{u_1\}) \subseteq (V - D)$ is an inverse split majority dominating set of G , for some $u_1 \in D'$. It implies that $|N[D_1]| \geq \lceil \frac{p}{2} \rceil$ and $\langle V - D_1 \rangle$ is disconnected.

Case (i): Suppose the condition (a) (i) holds, for some $u \in D'$. Then $|N[D']| > \lceil \frac{p}{2} \rceil$ and $|pn[u, D']| > |N[D']| - \lceil \frac{p}{2} \rceil$. Since $|pn[u, D']| = |N[D']| - |N[D' - \{u\}]|$ and $D_1 = D' - \{u\}$, $|N[D']| - |N[D_1]| = |pn[u, D']| > |N[D']| - \lceil \frac{p}{2} \rceil$. It implies that $|N[D_1]| < \lceil \frac{p}{2} \rceil$, which is a contradiction to the assumption. Also, if the condition (a) (ii) holds then $\langle (V - D') \cup \{u_i\} \rangle$ is connected, for some $u_i \in D'$. If for any $u_1 \in D'$ and u_1 is a cut vertex in $(V - D)$, then $\langle (V - D') \cup \{u_1\} \rangle = \langle V - (D' \cup \{u_1\}) \rangle$ is connected. It implies that $\langle V - D_1 \rangle$ is connected for any $u_1 \in D'$, which is a contradiction. Hence D' is a minimal inverse split majority dominating set of G with respect to D .

Case (ii): Suppose the condition (b) (i) holds for some $u \in D'$. Then $|N[D']| = \lceil \frac{p}{2} \rceil$ and either u is an isolate of D' or $pn[u, D'] \cap (V - D') \neq \phi$. If u is an isolate of D' then $u \in pn[u, D']$ and $|pn[u, D']| \geq 1$. If $pn[u, D'] \cap (V - D') \neq \phi$ then $|pn[u, D']| \geq 2$. Since $|N[D']| - |N[D_1]| = |pn[u, D']| \geq 2$, $\lceil \frac{p}{2} \rceil - |N[D_1]| \geq 2$. Then

$|N[D_1]| \leq \lceil \frac{p}{2} \rceil - 2$, which is contradiction. Also by the above arguments the induced subgraph $\langle V - D_1 \rangle$ is connected, for any $u_1 \in D'$, which is a contradiction. Hence D' is a minimal inverse split majority dominating set of G with respect to D . ■

Theorem 6.6.2: Let D be a γ_{SM} - set of a connected graph G and $(V-D)$ has no cut vertex. Then the inverse split majority dominating set $D' \subseteq V - D$ is minimal if and only if for each $u \in D'$, one of the following conditions holds.

- (a) (i) If $|N[D']| = p$, $|pn[u, D']| \leq 1$ and
 - (ii) $\langle (V - D') \cup \{u_i\} \rangle$ is connected, for all $u_i \in D'$.
- (b) (i) If $|N[D']| \geq \lceil \frac{p}{2} \rceil$, $|pn[u, D']| \geq |N[D']| - \lceil \frac{p}{2} \rceil + 1$ and
 - (ii) $\langle (V - D') \cup \{u_i\} \rangle$ is connected, for all $u_i \in D'$.

Proof: Let D be a γ_{SM} - set of a connected graph G and $(V-D)$ has no cut vertex. Assume that $D' \subseteq (V - D)$ is a minimal inverse split majority dominating set of G with respect to D .

Claim (1): The condition (a) is true.

Case (i): Let $|N[D']| = p$ and let $u \in D'$. Since D' is minimal,

$|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $\langle V - D' \rangle$ is disconnected. Then $(D' - \{u\})$ is not an inverse split majority dominating set of G . It implies that either $|N[D' - \{u\}]| < \lceil \frac{p}{2} \rceil$ or $\langle V - (D' - \{u\}) \rangle$ is not disconnected. Since $(V - D)$ has no cut vertex, there exists a vertex cut $S \subseteq D'$ such that $|S| \geq 2$ and $|N[D']| = p$. Now $|pn[u, D']| = |N[D']| - |N[D' - \{u\}]|$. Since, $|N[D' - \{u\}]| \leq p - 1$, $|pn[u, D']| \leq p - (p - 1) \leq 1$. It implies that $|pn[u, D']| \leq 1$, for some $u \in D'$. Hence condition (a)(i) holds. By the above result suppose $\langle (V - D') \cup \{u_i\} \rangle$ is connected, for some $u_i \in D'$. Then $\langle (V - D') \cup \{u_i\} \rangle$ is connected for some $u_i \in D'$.

Since $(V - D)$ has no cut vertex, there exists a vertex cut $S \subseteq (V - D)$ and $S \subseteq D' \subseteq (V - D)$ such that $|S| \geq 2$. Let $S = \{u_1, u_2, \dots, u_t\} \subseteq D' \subseteq V - D$ and $D' = \{u_1, u_2, \dots, u_{t_1}, \dots, u_i\}$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$. By the above result, since $\langle V - D' \rangle$ is disconnected, it consists of atleast three components in G . If add all vertices $u_i \in D'$ to $\langle V - D' \rangle$ then only the induced subgraph $\langle (V - D') \cup \{u_i\} \rangle$ would be connected in G . Thus condition (a) (ii) holds.

Claim (2): The Condition (b) is true.

Case(ii): Let $|N[D']| \geq \lceil \frac{p}{2} \rceil$. Since D' is a minimal inverse split majority dominating set of G , $\langle V - D' \rangle$ is disconnected. Then $(D' - \{u_i\})$ is not an inverse split majority dominating set. It implies that either $|N[D' - \{u_i\}]| < \lceil \frac{p}{2} \rceil$ or $\langle V - (D' - \{u_i\}) \rangle$ is not disconnected, for some $u_i \in D'$.

Let $|N[D' - \{u_i\}]| < \lceil \frac{p}{2} \rceil$. Since $(V - D)$ has no cut vertex, there exists a vertex cut $S \subseteq (V - D)$ such that $|S| \geq 2$. Then $S \subseteq D' \subseteq (V - D)$ such that $|S| \leq |D'|$. Now, for some $u \in D'$, $|N[D' - \{u\}]| = |N[D']| - |pn[u, D']| < \lceil \frac{p}{2} \rceil - 1$ implies that $|pn[u, D']| \geq |N[D']| - \lceil \frac{p}{2} \rceil + 1$, for any $u \in D'$. Therefore condition (b) (i) holds, for any $u \in D'$. By the above result, if $\langle (V - D') \cup \{u_i\} \rangle$ is not disconnected then $\langle (V - D') \cup \{u_i\} \rangle$ is connected. By applying the argument as in case (i), we obtain the induced subgraph $\langle (V - D') \cup \{u_i\} \rangle$ is connected, for all $u_i \in D'$. Therefore condition (b) (ii) holds.

Conversely, assume that either the condition (a) or the condition (b) holds. Let D' be an inverse split majority dominating set.

To prove that D' is minimal inverse split majority dominating set of G . Suppose D' is not minimal. Then $(D' - \{u\}) \subseteq (V - D)$ is an inverse split majority dominating set of G , for some $u \in D'$. It implies

that $|N[D' - \{u\}]| \geq \lceil \frac{p}{2} \rceil$ and $\langle V - (D' - \{u\}) \rangle$ is disconnected.

Case (i): Suppose the condition (a) holds, for some $u \in D'$. Let $|N[D']| = p$ and $|pn[u, D']| \leq 1$. Since $|pn[u, D']| = |N[D']| - |N[D' - \{u\}]|$, $|N[D']| - |N[D' - \{u\}]| \leq 1$. It implies that $|N[D' - \{u\}]| \geq p - 1$, which is a contradiction. Also, if the condition (a) (ii) holds then $\langle (V - D') \cup \{u_i\} \rangle$ is connected, for all $u_i \in D'$. Since $(V - D)$ has no cut vertex and all $u_i \in D'$, $\langle (V - D') \cup \{u_i\} \rangle$ is connected. So, for any $u \in D'$, $\langle V - (D' \cup \{u\}) \rangle$ is connected, contradiction. Hence D' is a minimal inverse split majority dominating set of G with respect to D .

Case (ii): Suppose condition (b) holds, for each $u_i \in D'$. Let $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $|pn[u, D']| \geq |N[D']| - \lceil \frac{p}{2} \rceil + 1$. Since $|N[D']| - |N[D' - \{u\}]| = |pn[u, D']| \geq |N[D']| - \lceil \frac{p}{2} \rceil + 1$, implies that $|N[D' - \{u\}]| < \lceil \frac{p}{2} \rceil - 1$, which is a contradiction. By the similar argument as in case (i), $\langle V - (D' \cup \{u_i\}) \rangle$ is connected, for any $u_i \in D'$. Hence D' is a minimal inverse split majority dominating set of G with respect to D . ■

Theorem 6.6.3: Let D be a γ_{SM} -set of a graph G . Then $\gamma_{SM}^{-1}(G) = 0$ if and only if the graph G has atleast one full degree vertex.

Proof: Let D and D' be the γ_{SM} -set and γ_{SM}^{-1} -set of G respectively. Let $\gamma_{SM}^{-1}(G) = |D'| = 0$. Then there is no such set $D' \subseteq (V - D)$ such that the induced subgraph $\langle V - D' \rangle$ is disconnected. It implies that $\langle V - D' \rangle$ is connected. Since $\gamma_{SM}(G) = 1, D = \{u_1\}$ such that $|N[D]| \geq \lceil \frac{p}{2} \rceil$ and $\langle V - D \rangle$ is disconnected. Then this vertex $u_1 \in (V - D')$. Since $\langle V - D' \rangle$ is connected, u_1 is adjacent to all vertices of $\langle V - D' \rangle$. Also since $(V - D)$ is disconnected, all the remaining vertices are of degree $d(u_i) \leq 1, i \geq 2$ and $d(u_1) = p - 1$. It implies that u_1 is a full degree vertex of G .

Since $\langle V - D' \rangle$ is connected, the vertices of the set D are in $\langle V - D' \rangle$ and $\langle V - D \rangle$ is disconnected. Suppose there is no full degree vertex $u_1 \in V(G)$. Then u_1 is not adjacent to all the vertices of G . Now, there exists a set $D' \subseteq V - D$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $\langle V - D' \rangle$ is disconnected. It implies that D' is a γ_{SM}^{-1} -set of G and $\gamma_{SM}^{-1}(G) = |D'| \geq 2$, which is a contradiction to the assumption. Hence the graph G has atleast one full degree vertex.

Conversely, the theorem is proved by an induction on the number

of full degree vertices u_i . Suppose G has exactly one full degree vertex u_1 . Then $D = \{u_1\}$ is a majority dominating set of G and $\gamma_M(G) = 1$. Since $\delta(G) \geq 1$, $D' \subseteq V - D$ is an inverse majority dominating set of G with cardinality $|D'| \geq 2$ and the induced subgraph $\langle V - D' \rangle$ is not disconnected. Since u_1 is adjacent to all vertices of G and $D' \subseteq V - D$, $u_1 \in (V - D')$ and $D \subseteq V - D'$. Then $\langle V - D' \rangle$ is still connected to the full degree vertex u_1 . Hence one could not find an inverse split majority dominating set for G and $\gamma_{SM}^{-1}(G) = 0$.

If the graph G has two full degree vertices u_1 and u_2 then the subset $D = \{u_1\}$ and $D' = \{u_2\} \subseteq (V - D)$ are the majority dominating set and an inverse majority dominating set of G respectively. Since the induced subgraph $\langle V - D' \rangle$ contains the full degree vertex $u_1 \in D$, $\langle V - D' \rangle$ is again connected to the vertex u_1 , for any inverse majority dominating D' . Hence there is no existence for γ_{SM}^{-1} set of G and $\gamma_{SM}^{-1}(G) = 0$.

This result is true for $(p - 1)$ full degree vertices. Suppose all vertices of G are full degree then the graph G is complete. Then $\gamma_M(G) = |D| = 1$ and $\gamma_M^{-1}(G) = |D'| = 1$. Since every vertex is adjacent to all vertices of G , the induced subgraph $\langle V - D' \rangle$ would

never be disconnected, for any inverse majority dominating set $D' \subseteq V - D$. Hence there is no existence for γ_{SM}^{-1} - set of a graph G and $\gamma_{SM}^{-1}(G) = 0$. ■

Chapter 7

Inverse Non-split Majority Domination in Graphs

Abstract

This chapter introduces an **inverse non-split majority dominating set** of a graph G . **Inverse non-split majority domination number** $\gamma_{NSM}^{-1}(G)$ is determined for some families of graphs. Characterization theorem on minimal inverse non-split majority dominating sets is discussed. Bounds of an inverse non-split majority dominating set and some interesting results of $\gamma_{NSM}^{-1}(G)$ for connected and disconnected graphs are also established.

7.1 Introduction

Many domination parameters are obtained by combining domination with another graph theoretical property. In this chapter, the parameters are defined by imposing additional constraint on the complement of a dominating set. In 2000, Kulli and Janakiraman [32] introduced the new domination parameter namely the Non-Split dominating set of a graph G in domination theory. In 2008 [2], the inverse split dominating set and inverse non-split dominating set of a graph are defined and studied in detail by Amennal Bibi and Selvakumar. They produced many inequalities and theorems on these two inverse parameters. Then the concept “Non-Split Majority Dominating set of a graph” was studied by Joseline Manora and Veeramanikandan in 2015 [40].

The researcher is motivated by the above defined parameters and introduced new concept an “Inverse Non-split Majority Domination in Graphs”.

A study of the inverse non-split majority dominating set of any graph structure is made in this chapter to some extent. The organization of this chapter is as follows. Section 7.1 is the introduction

about the parameter. In Section 7.2, the concept of an inverse non-split majority dominating set for a graph G is defined and illustrates the concept with an example and some inequalities are also established. The inverse non-split majority domination number $\gamma_{NSM}^{-1}(G)$ is determined for various families of graphs in Section 7.3. Main results of $\gamma_{NSM}^{-1}(G)$ and characterization theorem for a minimal inverse non-split majority dominating set of a graph G are determined in Section 7.4. For a connected and disconnected graphs G , bounds of $\gamma_{NSM}^{-1}(G)$ is discussed in the final Section 7.5.

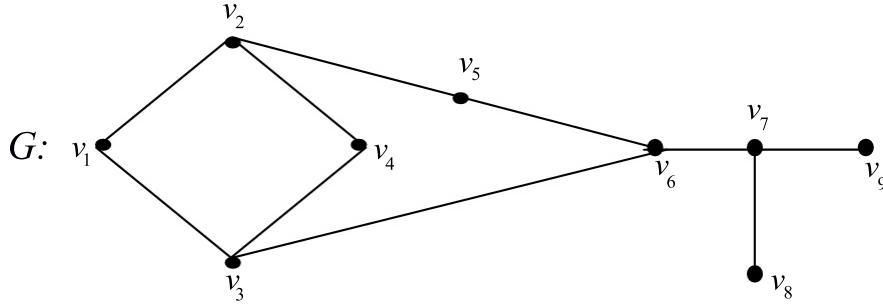
7.2 Inverse Non-Split Majority Dominating Sets

Definition 7.2.1: Let G be a graph with p vertices and q edges and let D be a minimum non-split majority dominating set of G . Then the set $D' \subseteq V - D$ is called an inverse non-split majority dominating set of G with respect to D if $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and the induced subgraph $\langle V - D' \rangle$ is connected.

Definition 7.2.2: The minimum and maximum cardinality of a minimal inverse non-split majority dominating set D' of G with

respect to D is called the inverse non-split majority domination number, denoted by $\gamma_{NSM}^{-1}(G)$ and upper inverse non-split majority domination number by $\Gamma_{NSM}^{-1}(G)$ respectively.

Example 7.2.3: Consider the graph G with $p = 9$ vertices.



Let D and D' be a majority and inverse majority dominating set of G . In the graph G , $D = \{v_1, v_2\}$ and $D' = \{v_6, v_7\} \subseteq V - D$. Hence, $\gamma_M(G) = |D| = 2$ and $\gamma_M^{-1}(G) = |D'| = 2$.

Let $D_1 = \{v_1, v_2\}$ be a non-split majority dominating set of G . Let $D'_1 = \{v_6, v_7, v_8, v_9\} \subseteq V - D_1$ and $D'_2 = \{v_3, v_4, v_9\} \subseteq V - D_1$ be any two minimal inverse non-split majority dominating set of G with respect to D_1 . Hence $\gamma_{NSM}(G) = |D_1| = 2$, $\gamma_{NSM}^{-1}(G) = |D'_2| = 3$ and $\Gamma_{NSM}^{-1}(G) = |D'_1| = 4$.

Observation 7.2.4:

- 1) For any connected graph G , $\gamma_M^{-1}(G) \leq \gamma_{NSM}^{-1}(G)$.

- 2) For any connected graph G , $\gamma_{NSM}(G) \leq \gamma_{NSM}^{-1}(G)$.
- 3) For any connected graph G , $\gamma_{SM}^{-1}(G) \leq \gamma_{NSM}^{-1}(G)$.
- 4) For any connected graph G , $\gamma_M^{-1}(G) \leq \gamma_{SM}^{-1}(G) \leq \gamma_{NSM}^{-1}(G)$.

7.3 $\gamma_{NSM}^{-1}(G)$ for Some Families of Graphs

The inverse non-split majority domination number $\gamma_{NSM}^{-1}(G)$ for some classes of graph is determined in the following results.

Proposition 7.3.1: For a path $P_p, p \geq 2$, then $\gamma_{NSM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 1$.

Proof: Let G be a path with $p \geq 2$. Let $D = \{u_1, u_2, \dots, u_{\lceil \frac{p}{2} \rceil - 1}\}$ be a non-split majority dominating set of G with $|D| = \lceil \frac{p}{2} \rceil - 1$. Choose $D' = \{u_p, u_{p-1}, \dots, u_{\lceil \frac{p}{2} \rceil + 1}\} \subseteq V - D$ such that $d(u_i, u_j) = 1$, for $u_i, u_j \in D'$ and $i \neq j$ and $|D'| = p - (\lceil \frac{p}{2} \rceil + 1) + 1 = \lceil \frac{p}{2} \rceil - 1$. Then $|N[D']| = |D'| + 1 = \lceil \frac{p}{2} \rceil$. Since $d(u_i, u_j) = 3$, for $u_i, u_j \in D'$ and u_i 's are at the end of a path, $\langle V - D' \rangle$ is connected. Hence D' is an inverse non-split majority dominating set and $\gamma_{NSM}^{-1}(G) \leq |D'| = \lceil \frac{p}{2} \rceil - 1$.

Suppose a subset $D_1 \subseteq (V - D)$ with $|D_1| < |D'|$ such that $|D_1| = \lceil \frac{p}{2} \rceil - 2$. Then $|N[D_1]| = |D_1| + 1 < \lceil \frac{p}{2} \rceil$. Since all vertices in D_1 are of distance one, the induced subgraph $\langle V - D' \rangle$ is connected but

$|N[D_1]| < \lceil \frac{p}{2} \rceil$. It implies that D_1 is not an inverse non-split majority dominating set of G . Hence $\gamma_{NSM}^{-1}(G) > |D_1|$ and $\gamma_{NSM}^{-1}(G) \geq |D'| = \lceil \frac{p}{2} \rceil - 1$. Thus, $\gamma_{NSM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 1$. ■

Proposition 7.3.2: Let $G = S(K_{1,n})$ be a subdivision of a star with $p = (2n + 1)$ vertices. Then $\gamma_{NSM}^{-1}(G)$ does not exist.

Proof: Let $V(G) = \{u, u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ where u is a central vertex $v_1, v_2, \dots, v_n$ are pendants and $u_1, u_2, \dots, u_n$ are middle vertices of degree $d(u_i) = 2$ and $|V(G)| = 2n + 1 = p = \text{odd}$. Let $D = \{v_1, v_2, \dots, v_t\}$ be a γ_{NSM} -set of G with $|D| = t = \lceil \frac{p}{4} \rceil$ such that $\langle V - D \rangle$ is connected. Choose the set $D' \subseteq V - D$ and $D' = \{v_{t+1}, \dots, v_n, u_n\}$ such that $|D'| = n - t + 1$. Then $|N[D']| = 2(n - t) + 1$. When $t = \lceil \frac{p}{4} \rceil$ and $p = 2n + 1$, $|N[D']| = n - 1 < \lceil \frac{p}{2} \rceil = n + 1$. Then $|N[D']| < \lceil \frac{p}{2} \rceil$ and $\langle V - D' \rangle$ is connected. Therefore, one could not form a set $D' \subseteq (V - D)$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $\langle V - D' \rangle$ is connected in G . Otherwise the induced subgraph $\langle V - D' \rangle$ is not connected. Hence the inverse non-split majority dominating set does not exist to this structure G . ■

Proposition 7.3.3: Let $G = W_p$ be a wheel graph of p vertices.

Then $\gamma_{NSM}^{-1}(G) = \lceil \frac{p-2}{6} \rceil$.

Proof: Let $G = W_p = C_{p-1} \vee K_1$ and $V(G) = \{u_1, u_2, \dots, u_{p-1}, u_p\}$ where u_1 is a full degree vertex and $d(u_i) = 3$, for all i . Let $D = \{u_1\}$ be a majority dominating set of G . By the proposition (2.3.4), $\gamma_M^{-1}(W_p) = \lceil \frac{p-2}{6} \rceil$. Choose a set $D' = \{u_2, u_5, \dots, u_{p-1}\} \subseteq (V - D)$ and D' is an inverse majority dominating set of G such that $d(u_i, u_j) \geq 3$ for every $i \neq j$ with $|D'| = \lceil \frac{p-2}{6} \rceil$. Since the graph $G = W_p$ has a full degree vertex u_1 , u_1 is adjacent to all the $(p-1)$ vertices and the induced subgraph $\langle V - D' \rangle$ is connected with a vertex u_1 . Hence D' is also an inverse non-split majority dominating set of G and $\gamma_{NSM}^{-1}(G) = |D'| = \lceil \frac{p-2}{6} \rceil$. ■

Proposition 7.3.4: Let $G = H \circ K_1$ be a corona graph with $p = 2n$ vertices. Then $\gamma_{NSM}^{-1}(G) = \lceil \frac{n}{2} \rceil$.

Proof: Let $G = H \circ K_1$, where H is any connected graph with n vertices. Let $V(G) = \{v_1, \dots, v_n, u_1, \dots, u_n\}$ for $v_i \in V(H)$ and u_i 's are pendants and $|V(G)| = p = 2n$. By the result (1.4), $\gamma_{NSM}(G) = \lceil \frac{p}{2} \rceil$. Let $D = \{u_1, \dots, u_{\frac{n}{2}}\} \subseteq V(G)$ with $|D| = \lceil \frac{p}{4} \rceil$ be a non-split majority dominating set of G . If $\left(\frac{p}{2}\right) = n$ is even, choose the set $D' = \{u_{\frac{n}{2}+1}, \dots, u_n\} \subseteq V - D$ such that $|D'| = n - \frac{n}{2} = \frac{n}{2}$ and $|N[D']| = 2|D'| = n = \lceil \frac{p}{2} \rceil$. Since $\{u_{\frac{n}{2}+1}, \dots, u_n\}$ are all pendants,

the induced subgraph $\langle V - D' \rangle$ is connected. If $\left(\frac{p}{2}\right) = n$ is odd, $D' = \{u_{\frac{n}{2}+1}, \dots, u_n, v_n\} \subseteq V - D$, where u'_i s are pendants and $v_n \in V(H)$ with $|D'| = \lceil \frac{n}{2} \rceil$ and $|N[D']| = 2(|D'| - 1) + d(v_n) = 2\left(\lceil \frac{n}{2} \rceil - 1\right) + 1 = 2\left(\frac{n}{2}\right) + 1 = n + 1$. Then $|N[D']| = n + 1 > \frac{p}{2}$. Since u_n is adjacent to $v_n \in V(H)$ and others are only pendants in D' , the induced subgraph $\langle V - D' \rangle$ is connected. Hence in both cases, D' is an inverse non-split majority dominating set of G with respect to D and $\gamma_{NSM}^{-1}(G) \leq |D'| = \lceil \frac{n}{2} \rceil$.

Suppose $D'_1 = \{v_1, \dots, v_{\lceil \frac{n}{2} \rceil - 1}\} \subseteq V - D$ such that $|D'_1| = \lceil \frac{n}{2} \rceil - 1$ and $|D'_1| < |D'|$. Since $v_i \in V(H)$, the induced graph $\langle V - D'_1 \rangle$ is not connected and $\langle V - D'_1 \rangle$ contains isolates. Therefore $|N[D'_1]| < \lceil \frac{p}{2} \rceil$ and D'_1 is not an inverse non-split majority dominating set of G . Hence $\gamma_{NSM}^{-1}(G) > \lceil \frac{n}{2} \rceil - 1$ and $\gamma_{NSM}^{-1}(G) \geq |D'| = \lceil \frac{n}{2} \rceil$. Thus, $\gamma_{NSM}^{-1}(G) = \lceil \frac{n}{2} \rceil$. ■

Proposition 7.3.5: Let G be a caterpillar graph with p vertices.

$$\text{Then } \gamma_{NSM}^{-1}(G) = \begin{cases} \lceil \frac{p}{4} \rceil, & \text{if } \left(\frac{p}{2}\right) \text{ is even} \\ \left(\frac{p}{2}\right) - 1, & \text{if } \left(\frac{p}{2}\right) \text{ is odd.} \end{cases}$$

Proof: Let G be a caterpillar with one pendant at each vertex and

$V(G) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ where $v_1, v_2, \dots, v_n$ are pendants and $u_1, u_2, \dots, u_n$ are the vertices of a path in G with $p = 2n$.

By the result (1.4), $\gamma_{NSM}(G) = \lceil \frac{p}{4} \rceil$ and let $D = \{v_1, v_2, \dots, v_t\}$ be a non – split majority dominating set of G with $|D| = t = \lceil \frac{p}{4} \rceil$ such that $\langle V - D \rangle$ is connected and $|N[D]| \geq \binom{p}{2}$.

Case (i): when $\lceil \frac{p}{2} \rceil = n$ is even. Choose the set $D' = \{v_{t+1}, \dots, v_n\} \subseteq V - D$ with $|D'| = n - t = \lceil \frac{p}{4} \rceil$ and $v_{t+1}, \dots, v_n$ are pendants. Then $|N[D']| = 2|D'| = 2\lceil \frac{p}{4} \rceil \geq \frac{p}{2}$. Therefore $|N[D']| = \binom{p}{2}$ and the induced subgraph $\langle V - D' \rangle$ is connected because $v_i \in D'$ are all pendants. Hence D' is an inverse non-split majority dominating set of G with respect to D and $\gamma_{NSM}^{-1}(G) \leq |D'| = \lceil \frac{p}{4} \rceil$.

Suppose the set $|D_1| < |D'|$ and $|D_1| = |D'| - 1$. Then $|N[D_1]| < \binom{p}{2}$ and $\langle V - D_1 \rangle$ is connected but D_1 is not an inverse majority dominating set with respect to D . Hence $\gamma_{NSM}^{-1}(G) > |D_1|$ and $\gamma_{NSM}^{-1}(G) \geq |D'| = \lceil \frac{p}{4} \rceil$. Thus $\gamma_{NSM}^{-1}(G) = \lceil \frac{p}{4} \rceil$, if $\lceil \frac{p}{2} \rceil$ is even.

Case (ii): when $\lceil \frac{p}{2} \rceil = n$ is odd. Choose the set $D' = \{v_{t+1}, \dots, v_n\} \subseteq V - D$ with $|D'| = n - t$ where $t = |D| = \lceil \frac{p}{4} \rceil$. Then $|N[D']| = \lceil \frac{p}{2} \rceil - 1 < \frac{p}{2}$. Now choose another set $D' = \{v_{t+1}, \dots, v_n, u_{t+1}, \dots, u_n\} \subseteq V - D$ with $|D'| = 2(n - t)$. Then $|N[D']| = |D'| + 1 = 2(n - t) + 1 = p - 2\lceil \frac{p}{4} \rceil + 1 = \lceil \frac{p}{2} \rceil$. Since all the vertices v_i and u_i of D' are

adjacent, $\langle V - D' \rangle$ is connected. Hence D' is an inverse non-split majority dominating set of G with respect to D and $\gamma_{NSM}^{-1}(G) \leq |D'| = 2(n - t) = \lceil \frac{p}{2} \rceil - 1$. Hence $\gamma_{NSM}^{-1}(G) \leq \lceil \frac{p}{2} \rceil - 1$. Then applying the same argument as in case (i), D_1 is not an inverse non-split majority dominating set of G with $|D_1| < |D'|$. Therefore $\gamma_{NSM}^{-1}(G) \geq |D'| = \lceil \frac{p}{2} \rceil - 1$. Thus $\gamma_{NSM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 1$, if $\frac{p}{2}$ is odd. ■

Results 7.3.6:

- 1) For a Complete graph $G = K_p$, $\gamma_{NSM}^{-1}(G) = 1$.
- 2) For a Cycle C_p , $\gamma_{NSM}^{-1}(C_p) = \lceil \frac{p}{2} \rceil - 2$, $p \geq 3$.
- 3) Let $G = K_{m,n}$, $m, n \geq 2$ and $m \leq n$ be a Complete bipartite graph. Then $\gamma_{NSM}^{-1}(G) = 1$.
- 4) For a Star $G = K_{1,p-1}$, $\gamma_{NSM}^{-1}(G) = \lfloor \frac{p-1}{2} \rfloor$, $p \geq 3$.
- 5) If $G = \overline{mK_2}$, $m \geq 1$ then $\gamma_{NSM}^{-1}(G) = 1$.
- 6) For a Double star $G = D_{r,s}$, $r \leq s$, $\gamma_{NSM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 1$, if $s = r, r + 1, r + 2$.
- 7) For $G = K_p - \{e\}$ and $K_p - \{2e\}$, $\gamma_{NSM}^{-1}(G) = 1$.

7.4 Results on $\gamma_{NSM}^{-1}(G)$

This following theorem characterizes a minimal inverse non-split majority dominating set of a graph G .

Theorem 7.4.1: Let D be a non-split majority dominating set of graph G with p vertices. Then an inverse non-split majority dominating set $D' \subseteq V - D$ of G is minimal with respect to D if and only if for each vertex $v \in D'$, one of the following conditions holds.

- (i) If $|N[D']| > \lceil \frac{p}{2} \rceil$, then $|pn[v, D']| > |N[D']| - \lceil \frac{p}{2} \rceil$ and $N(v) \cap (V - D') \neq \phi$.
- (ii) If $|N[D']| = \lceil \frac{p}{2} \rceil$ then either v is an isolate of D' or $N(v) \cap (V - D') = \phi$ or $|pn[v, D']| \cap (V - D') \neq \phi$.

Proof: Let D be a γ_{NSM} -set of a graph G . Suppose D' is minimal non-split majority dominating set of G with respect to D . Then $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and the induced subgraph $\langle V - D' \rangle$ is connected. Let $v \in D'$.

Case (i): Suppose $|N[D']| > \lceil \frac{p}{2} \rceil$. Since D' is minimal, the set $D'_1 = D' - \{v\}$ is not an inverse non-split majority dominating set of G with respect to D . It implies that either $|N[D'_1]| < \lceil \frac{p}{2} \rceil$ or

$\langle V - D'_1 \rangle$ is not connected.

Subcase (i): When $|N[D'_1]| < \lceil \frac{p}{2} \rceil$. Since $|pn[v, D']| = |N[D']| - |N[D'_1]|$, $|pn[v, D']| > |N[D']| - \lceil \frac{p}{2} \rceil$, for every $v \in D'$. Hence the condition (i) holds.

Subcase (ii): When $\langle V - D'_1 \rangle$ is not connected. Then for any $v \in D'$, $\langle V - (D' - \{v\}) \rangle = (\langle V - D' \rangle \cup \{v\})$ is disconnected. This implies that $N(v) \subseteq V - D'$ and $N(v) \cap (V - D') \neq \phi$, for any $v \in D'$. Thus the condition (i) holds.

Case (ii): Assume that $|N[D']| = \lceil \frac{p}{2} \rceil$. Suppose that v is neither an isolate of D' nor v has a private neighbour in the induced subgraph $\langle V - D' \rangle$ that is, $pn[v, D'] \cap (V - D') = \phi$. Then $|N[D' - v]| = |N[D']| = \lceil \frac{p}{2} \rceil$ which implies that $|D' - \{v\}|$ is an inverse non-split majority dominating set, which is a contradiction to the assumption. Hence the condition (ii) holds.

Conversely assume that the above condition (i) hold. Let D' be an inverse non-split majority dominating set of G . Suppose D' is not minimal. Then $D'_1 = D' - \{v\}$ is an inverse non-split majority dominating set of G for some $v \in D'$ with respect to D . Then

$$|N[D'_1]| \geq \left\lceil \frac{p}{2} \right\rceil \text{ and } \langle V - D'_1 \rangle \text{ is connected.} \quad (7.1)$$

Suppose the condition (i) holds for $v \in D'$. Then $|N[D']| > \lceil \frac{p}{2} \rceil$ and $|pn[v, D']| > |N[D']| - \lceil \frac{p}{2} \rceil$. Since $|N[D']| - |N[D' - v]| = |pn[v, D']|$ and $|N[D']| - |N[D' - v]| > |N[D']| - \lceil \frac{p}{2} \rceil$. It implies that $|N[D' - v]| < \lceil \frac{p}{2} \rceil$ and $|N[D'_1]| < \lceil \frac{p}{2} \rceil$, which is a contradiction. If $|N[D']| > \lceil \frac{p}{2} \rceil$ then $N(v) \cap (V - D') \neq \phi$. It implies that $N(v) \subseteq (V - D')$ and $\langle (V - D') \cup \{v\} \rangle = \langle V - D'_1 \rangle$ is disconnected, which is a contradiction. Hence D' is minimal.

Suppose the condition (ii) holds for $v \in D'$. Then $|N[D']| = \lceil \frac{p}{2} \rceil$ and v is an isolate of D' or $N(v) \cap (V - D') = \phi$ or $|pn[v, D']| \cap (V - D') \neq \phi$. Since $|N[D']| - |pn[v, D']| = |N[D' - v]|$ and by the result (7.1), we obtain,

$$|N[D']| \geq |pn[v, D']| + \left\lceil \frac{p}{2} \right\rceil. \quad (7.2)$$

If v is an isolate of D' , then $|pn[v, D']| \geq 1$. Hence, the result (7.2) becomes $|N[D']| \geq \lceil \frac{p}{2} \rceil + 1$, which is a contradiction to $|N[D']| = \lceil \frac{p}{2} \rceil$. If the condition $N(v) \cap (V - D') = \phi$ holds, for any $v \in D'$ then $\langle V - D' \rangle \cup \{u\}$ is disconnected. It implies that $\langle V - D'_1 \rangle$ is disconnected, which is a contradiction to the result (7.1). Suppose $|pn[v, D']| \cap (V - D') \neq \phi$, then $N(v) \subseteq (V - D')$. By the result (7.2),

$|N[D']| \geq |pn[v, D']| + \lceil \frac{p}{2} \rceil$. Since $|N[D']| = \lceil \frac{p}{2} \rceil$, $|pn[v, D']| \leq 0$, which is contradiction to $|pn[v, D']| \geq 1$. Hence D' is a minimal inverse non-split majority dominating set with respect to D for G . ■

Proposition 7.4.2: If any tree T contains exactly $(p - 1)$ pendants then $\gamma_{NSM}^{-1}(T) = \lceil \frac{p}{2} \rceil - 1$.

Proof: Since G contains exactly $(p - 1)$ pendants, $V(T) = \{v, v_1, v_2, \dots, v_{p-1}\}$ where v is a central vertex and $(v_1, v_2, \dots, v_{p-1})$ are pendants. Let $D = \{v_1, v_2, \dots, v_t\}$ be a non-split majority dominating set of G with $|D| = t = \lceil \frac{p}{2} \rceil - 1$. Choose $D' = \{v_p, v_{p-1}, \dots, v_t\} \subseteq V - D$ with $|D'| = \lceil \frac{p}{2} \rceil - 1$. Then $|N[D']| = |D'| + 1 = \lceil \frac{p}{2} \rceil$. Since the graph G has a full degree vertex v and $v \in (V - D')$, the induced subgraph $\langle V - D' \rangle$ is connected. Hence D' is an inverse non-split majority dominating set of G with respect to D and $\gamma_{NSM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 1$. ■

Theorem 7.4.3: For any tree T , every γ_{NSM}^{-1} -set contains atleast one pendant.

Proof: Let T be any tree with $p \geq 2$ vertices and e pendants. This theorem is proved by induction on the number of pendants ' e '. Let D

be a γ_{NSM} -set of G . Since every tree T has atleast two pendants, $e = 2$ and $T = P_p$, a path. By Proposition (7.3.1), $\gamma_{NSM}^{-1}(P_p) = \lceil \frac{p}{2} \rceil - 1$ and the γ_{NSM}^{-1} -set $D' = \{u_p, u_{p-1}, \dots, u_{\lceil \frac{p}{2} \rceil + 1}\} \subseteq V - D$, where u_p is a pendant of a path. If $e = 3$, the tree structure is like a caterpillar T and T has $e \geq 3$ pendants. By Proposition (7.3.5), the γ_{NSM}^{-1} -set $D' = \{v_{t+1}, \dots, v_n\} \subseteq V - D$, if $\frac{p}{2}$ is even and $v_{t+1}, \dots, v_n$ are all pendants so that $\langle V - D' \rangle$ is connected. If $\frac{p}{2}$ is odd, the γ_{NSM}^{-1} -set $D' = \{v_{t+1}, \dots, v_n, u_{t+1}, \dots, u_n\} \subseteq V - D$ where $v_{t+1}, \dots, v_n$ are pendants and $d(u_i) \geq 2$, $t + 1 \leq i \leq n$. Hence every γ_{NSM}^{-1} -set D' contains atmost $\lceil \frac{p}{4} \rceil$ pendants.

The result is true for $e = 2, 3, \dots, (p - 2)$ pendants. Suppose $e = (p - 1)$ pendants. Then the structure becomes $T = K_{1,p-1}$, a star with $(p - 1)$ pendants. By results (7.3.6)(4), the γ_{NSM}^{-1} -set $D' = \{v_p, v_{p-1}, \dots, v_{\lceil \frac{p}{2} \rceil - 1}\} \subseteq V - D$ and all vertices of D' are pendants so that the remaining vertices in the $\langle V - D' \rangle$ is connected with a central vertex. Thus, in all cases of $e \geq 2$, every γ_{NSM}^{-1} -set D contains atleast one pendant vertex in a tree T . ■

Corollary 7.4.4: If G be a regular graph with p vertices then $\gamma_{NSM}(G) = \gamma_{NSM}^{-1}(G)$.

7.5 Bounds on $\gamma_{NSM}^{-1}(G)$

This section provides some bounds and exact values of an inverse non-split majority domination number of a connected and disconnected graph G .

Theorem 7.5.1: Let D and D' be the γ_{NSM} -set and γ_{NSM}^{-1} -set of G respectively. Then $\gamma_{NSM}^{-1}(G) = 1$ if and only if the graph G contains atleast two vertices of degree $\geq \lceil \frac{p}{2} \rceil$ and the induced graph $\langle V - D' \rangle$ is connected.

Proof: Let $\gamma_{NSM}^{-1}(G) = 1$. Then $\gamma_{NSM}(G) = 1, D = \{u_1\}$ and $D' = \{u_2\} \subseteq V - D$ are the γ_{NSM} -set and γ_{NSM}^{-1} -set of G with respect to D . Hence the induced subgraphs $\langle V - D \rangle$ and $\langle V - D' \rangle$ are connected and $|N[D]| = |N[D']| \geq \lceil \frac{p}{2} \rceil$. It implies that the graph G contains atleast two vertices u_1 and u_2 such that $d(u_i) \geq \lceil \frac{p}{2} \rceil$ for $i \geq 2$ and the induced subgraph $\langle V - D' \rangle$ is connected. The converse is obvious. ■

Theorem 7.5.2: If a connected graph G contains atmost $(p - 1)$ pedants then $\gamma_{NSM}^{-1}(G) \leq \lceil \frac{p}{2} \rceil - 1$.

Proof: Let ‘ e ’ denotes the number of pendants of G . when $e = 1$, then the graph G becomes any connected graph or unicyclic graph with only one pedant vertex u . Let D be a non-split majority dominating set of G containing that pedant vertex u such that $\langle V - D \rangle$ is connected. Suppose G has at least two majority dominating vertices v_1 and v_2 with one pedant, then $d(v_i) \geq \lceil \frac{p}{2} \rceil - 1$, for $i = 1, 2$. An inverse non-split majority dominating set D' with respect to D contains either v_1 or v_2 such that the induced subgraph $\langle V - D' \rangle$ is connected. Hence $\gamma_{NSM}^{-1}(G) \geq 1$ and $\gamma_{NSM}^{-1}(G) < \lceil \frac{p}{2} \rceil - 1$.

When $e = 2$, the graph $G = P_p$, a path, by Proposition (7.3.1), $\gamma_{NSM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 1$. Therefore, the result is true for $e = 1, 2, \dots, (\frac{p}{2} - 1)$ pedants. When $e = \frac{p}{2}$, then the graph structure becomes like a caterpillar or a corona graph $G = H \circ K_1$, for any connected graph H . By the proposition (7.3.5), $\gamma_{NSM}^{-1}(G) = \lceil \frac{p}{4} \rceil < \lceil \frac{p}{2} \rceil - 1$, if $(\frac{p}{2})$ is even and $\gamma_{NSM}^{-1}(G) = (\frac{p}{2}) - 1$, if $(\frac{p}{2})$ is odd. Hence in every case, $\gamma_{NSM}^{-1}(G) \leq \lceil \frac{p}{2} \rceil - 1$. Suppose $G = H \circ K_1$ with $p = 2n$ and $(\frac{p}{2})$ pendants then by the proposition (7.3.4), $\gamma_{NSM}(G) = \gamma_{NSM}^{-1}(G) = \lceil \frac{n}{2} \rceil$. Hence $\gamma_{NSM}^{-1}(G) = \lceil \frac{p}{4} \rceil < \lceil \frac{p}{2} \rceil - 1$.

Where $e = p - 2$, the graph becomes a double star $D_{r,s}$ with $r \leq s$ and $s = r, r+1, r+2$. In this graph u_1 and u_2 are majority dominating

vertices with r and s pendants. Then $|D_1| = r$ and $|D_2| = s$ are the $\gamma_{NSM}(G)$ and $\gamma_{NSM}^{-1}(G)$ respectively and by the result (7.3.6)(6), $\gamma_{NSM}^{-1}(G) = s = \lceil \frac{p}{2} \rceil - 1$. If $s \geq r + 3$ with $(p - 2)$ pendants then $\gamma_{NSM}^{-1}(G)$ does not exist. When $e = p - 1$, then $G = K_{1,p-1}$. By the result (7.3.6)(4), $\gamma_{NSM}^{-1}(G) = \lceil \frac{p}{2} \rceil - 1$. Thus, in all cases, if any graph G has atmost $(p - 1)$ pendants then $\gamma_{NSM}^{-1}(G) \leq \lceil \frac{p}{2} \rceil - 1$. ■

Theorem 7.5.3: Let G be a disconnected graph with $p = 2n$ vertices. If G has exactly two components g_1 and g_2 with $|N[g_i]| = \frac{p}{2}$ then $\gamma_{NSM}^{-1}(G) = \frac{p}{2}$.

Proof: Let $G = g_1 \cup g_2$ and $|V(G)| = p = 2n$. Since $|N[g_i]| = \frac{p}{2}$, for $i = 1, 2$. Let D be a γ_{NSM} -set of G and $D = \{v_1, v_2, \dots, v_n\}$ such that the induced subgraph $\langle V - D \rangle = \langle g_2 \rangle$ is connected. Let $D' = \{v_{n+1}, v_{n+2}, \dots, v_{2n}\} \subseteq (V - D)$ with $|N[D']| = n = \frac{p}{2}$ and the induced subgraph $\langle V - D' \rangle = \langle g_1 \rangle$ is connected. It implies that D' is an inverse non-split majority dominating set of G and $\gamma_{NSM}^{-1}(G) = \frac{p}{2}$. ■

Corollary 7.5.4: If a disconnected graph G contains more than two components then γ_{NSM}^{-1} -set does not exist for G .

Corollary 7.5.5: If a disconnected graph G contains exactly two components g_1 and g_2 such that $|N[g_1]| < \lceil \frac{p}{2} \rceil$ and $|N[g_1]| > \lceil \frac{p}{2} \rceil$ then γ_{NSM}^{-1} -set does not exist for G .

Conclusion

The research work primarily concentrates on inverse majority dominating sets of a graph G . The researcher has related his newly defined parameters with other graph theoretical parameters and extensive works on this defined parameters for a graph G are studied. Exact values of an inverse majority domination number $\gamma_M^{-1}(G)$ are determined for various classes of graphs. Also, inverse split majority domination number $\gamma_{SM}^{-1}(G)$ and inverse non-split majority domination number $\gamma_{NSM}^{-1}(G)$, inverse connected majority domination number $\gamma_{CM}^{-1}(G)$, inverse independent majority domination number $i_M^{-1}(G)$ are introduced and studied to some extent. Algorithms and Applications to inverse majority dominating sets and inverse connected majority dominating sets are also discussed. Further it is also opened the gateway for doing more work on inverse majority domination in graphs by imposing more conditions on them. It is very fruitful area in the field of domination theory.

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REPRINTS

RESULTS ON THE INVERSE MAJORITY DOMINATION AND MAJORITY INDEPENDENCE NUMBER OF A GRAPH

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ABSTRACT. In this article, the relationship between Inverse Majority Domination number $\gamma_M^{-1}(G)$ and Majority Independence number $\beta_M(G)$ of a graph G is discussed for some classes of graphs. In particular, $\gamma_M^{-1}(G)$ and $\beta_M(G)$ for cubic and cubic bipartite graph are studied with examples. Also characterization theorem for this relation and some results are determined.

Keywords: Majority Dominating Set, Inverse Majority Domination Number, Majority Independence Number, Cubic Bipartite Graphs.

AMS Subject Classification: 05C69

1. INTRODUCTION

Domination in graph provide numerous applications both in the position or location and protection strategies. This concepts was introduced by Claude Berge in 1958 [1]. In 1962, Ore used the name “Dominating set” and “Domination number”. In 1977, Cockayne and Hedetniemi made an interesting and extensive survey of the results known at that time about dominating sets in graphs. The survey paper of Cockayne and Hedetniemi has generated a lot of interest in the study of domination in graphs. Domination has a wide range of application in radio stations, modeling social networks, coding theory, nuclear power plants problems. One of the fastest growing areas in graph theory is the study of domination and related subset problems such as independence, covering, matching and inverse domination.

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ of finite order and edge set $E(G)$. Let $v \in V(G)$. The neighborhood of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. If $S \subseteq V(G)$, then the open neighborhood of S is the set $N_G(S) = N(S) = \bigcup_{v \in S} N_G(v)$. The closed neighbourhood of S is $N_G(S) = N[S] = S \cup N(S)$.

A set $S \subseteq V(G)$ of vertices in a graph $G = (V, E)$ is a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S . A dominating set S is called a minimal dominating set if no proper subset of S is a dominating set. The minimum cardinality of a minimal dominating set is called the domination number and

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the maximum cardinality of a minimal dominating set is called the upper domination number in a graph G . It is denoted by $\gamma(G)$ and $\Gamma(G)$ respectively.

The concept of inverse domination was introduced by V. R. Kulli. If a non-empty subset $D \subseteq V(G)$ is called the minimum dominating set, then if $v - D$ contains a dominating set D' , then D' is called the inverse dominating of G with respect to D and $\gamma'(G)$ is the inverse domination number of G .

A set $D \subseteq V(G)$ of vertices in a graph $G = (V, E)$ is called a Majority Dominating set of G [4] if at least half of the vertices of $V(G)$ are either in S or adjacent to the elements of S . A majority dominating set [4] D is minimal if no proper subset of D is a majority dominating set of a graph G . The minimum cardinality of a minimal majority dominating set of G is called majority domination number of G , is denoted by $\gamma_M(G)$ and the minimum majority dominating set of G is denoted by $\gamma_M - set$. If a vertex u of degree satisfies the condition $d(u) \geq \lceil \frac{p}{2} \rceil - 1$, then the vertex u is called majority dominating vertex of G . All full degree vertices are majority dominating vertices but all majority dominating vertices are not full degree vertices.

A set $S \subseteq V(G)$ of vertices in a graph G is said to be a Majority Independent set [5] if it induces a totally disconnected subgraph with $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $|p_n[v, S]| > |N[S]| - \lceil \frac{p}{2} \rceil$, for every $v \in S$. If any vertex set S is properly containing S is not majority independent then S is called a maximal majority independent set. The maximum cardinality of a maximal majority independent set of G is called majority independence number of G and it is denoted $\beta_M(G)$.

2. BASIC RESULTS ON $\gamma_M(G), \beta_M(G)$ AND $\gamma_M^{-1}(G)$

Definition 2.1. [7] Let G be simple and finite graph with p vertices and q edges and D be a minimum majority dominating set of G . If the set $(V - D)$ contains a majority dominating set say D' then the set D' is called Inverse Majority Dominating set with respect to D . The Inverse Majority Domination number [7] $\gamma_M^{-1}(G)$ of a graph G is the minimum cardinality of a minimal inverse majority dominating set of G .

Proposition 2.1. [4] For any graph G , $\gamma_M(G) = 1$ if and only if G has at least one vertex u with degree $d(u) \geq \lceil \frac{p}{2} \rceil - 1$.

Proposition 2.2. [6]

1. For any graph G , $\beta_M(G) = 1$ if and only if G has all vertices u with degree $d(u) \geq \lceil \frac{p}{2} \rceil - 1$, for all $u \in V(G)$.
2. For a cubic bipartite graph G , the majority independence number $\beta_M(G) = \lceil \frac{p}{4} \rceil - 1$.
3. For any cubic graph G , $\beta_M(G) = \lceil \frac{p}{8} \rceil$.
4. For a Fan graph $G = F_p$, $\beta_M(G) = \lceil \frac{p}{6} \rceil$, $p \geq 3$.
5. For a star graph $G = K_{(1, p-1)}$, $\beta_M(G) = \lfloor \frac{(p-2)}{2} \rfloor$, $p \geq 2$.

Proposition 2.3. [7]

1. For any graph G , $\gamma_M(G) \leq \gamma_M^{-1}(G)$.
2. For a path P_p , $p \geq 2$ and cycle C_p , $p \geq 3$ with p vertices, $\gamma_M^{-1}(G) = \lceil \frac{p}{6} \rceil$.
3. For a wheel W_p , $p \geq 5$, $\gamma_M^{-1}(G) = \lceil \frac{(p-2)}{6} \rceil$.
4. For a complete graph K_p , $\gamma_M^{-1}(G) = 1$.
5. For a fan graph F_p , $\gamma_M^{-1}(G) = 1$.
6. For a star graph $G = K_{(1, p-1)}$, $\gamma_M^{-1}(G) = \lfloor \frac{(p-2)}{2} \rfloor$, $p \geq 2$.

Proposition 2.4. [7] *For any graph G , $\gamma_M^{-1}(G) = 1$ if and only if G has atleast one majority dominating vertex u in $(V - D)$, where D is a minimum majority dominating set of G .*

3. INVERSE MAJORITY DOMINATING SET AND MAJORITY INDEPENDENT SET

Example 3.1. *Consider the following graph $G = T_{5k}$, $k = 5$ with $p = 25$ vertices. The graph G contains five P_5 paths which is connected in the middle vertex of each path P_5 . The vertex set is labeled as $\{y_1, \dots, y_{10}\}$ are pendants, $\{x_1, \dots, x_{10}\}$ are two degree vertices and $\{a, b, c, d, e\}$ are middle vertices of each P_5 .*

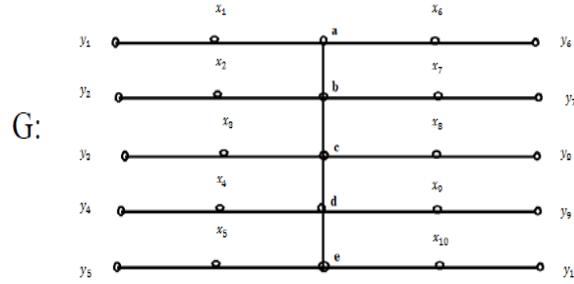


FIGURE 1

In G , $D_1 = \{a, b, c, d\}$ is a majority dominating set $\Rightarrow \gamma_M(G) = |D_1| = 4$. $D_2 = \{x_1, x_2, x_3, e\}$ is a inverse majority dominating set of G with respect to $D_1 \Rightarrow \gamma_M^{-1}(G) = |D_2| = 4$. $D_3 = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$ is a majority independent set of $G \Rightarrow \beta_M(G) = |D_3| = 7$. Hence $\gamma_M^{-1}(G) < \beta_M(G)$.

Proposition 3.1. *For any graph G , it satisfies the following inequalities.*

- i) $\gamma_M^{-1}(G) \leq \beta_M(G)$ and
- ii) $\gamma_M(G) \leq \gamma_M^{-1}(G) \leq \beta_M(G)$.

Proof: i) Let D be a minimum majority dominating set and D' be an inverse majority dominating set with respect to D of a graph G . Since any maximal majority independent set S of G , $S \subseteq (V - D)$ is also a inverse majority dominating set of G . Hence $\gamma_M^{-1}(G) \leq \beta_M(G)$.

ii) By Proposition (2.4), (i) $\gamma_M(G) \leq \gamma_M^{-1}(G)$ and by proposition(3.2)(i) $\gamma_M^{-1}(G) \leq \beta_M(G)$. We obtain the inequality (ii). $\square$

Proposition 3.2. *If a graph G has a full degree vertex and others are pendants then $\gamma_M^{-1}(G) = \beta_M(G)$.*

Proof: Let G be a graph with p vertices in which u is a full degree vertex and $(p - 1)$ pendants. Therefore, $D = \{u\}$ is γ_M set of G . Then $D' = \{u_1, u_2, \dots, u_{\lceil \frac{p}{2} \rceil - 1}\} \subseteq (V - D)$ is a inverse majority dominating set with $|D'| = \lceil \frac{p}{2} \rceil - 1$. Since all vertices in D' are pendants, $|N[D']| = \lceil \frac{p}{2} \rceil$. It implies that $\gamma_M^{-1}(G) = |D'| = \lceil \frac{p}{2} \rceil - 1$. By proposition(2.3) (5), $\beta_M(G) = \lfloor \frac{(p-1)}{2} \rfloor$, $p \geq 2$. When p is odd and even $\lceil \frac{p}{2} \rceil - 1 = \lfloor \frac{(p-1)}{2} \rfloor$. D' is also a majority independent set of G and $\beta_M(G) = \lceil \frac{p}{2} \rceil - 1$.

Hence $\gamma_M^{-1}(G) = \beta_M(G)$. $\square$

Corollary 3.1. *Let G be a totally disconnected graph with even number of vertices. Then $\gamma_M^{-1}(G) = \beta_M(G) = \frac{p}{2}$.*

Corollary 3.2. *Let G be a disconnected graph without isolates. Then $\gamma_M^{-1}(G) = \beta_M(G)$.*

4. $\gamma_M^{-1}(G)$ AND $\beta_M(G)$ FOR CUBIC BIPARTITE GRAPHS

Definition 4.1. *Let G be a cubic bipartite graph with a partition of the vertex set $V_1(G)$ and $V_2(G)$ such that $|V_1| + |V_2| = p$. A cubic bipartite graph G with minimum number of vertices is $K_{3,3}$. Also graphs that have an odd number of vertices cannot be a cubic bipartite graph. We now concentrate the cubic bipartite graphs when $p = 6, 8, 10, 12, 14, 16, \dots$*

Theorem 4.1. *For all cubic and cubic bipartite graph G , $\gamma_M^{-1}(G) = \lceil \frac{p}{8} \rceil$.*

Proof: Let $V(G) = \{v_1, v_2, \dots, v_p\}$ be the vertex set of the given graph G . Let D be a majority dominating set of G and $D' = \{v_1, v_2, \dots, v_t\}$ be the inverse majority dominating set of G with respect to D with $|D'| = t = \gamma_M^{-1}(G)$. Then

$$|N[D']| \geq \lceil \frac{p}{2} \rceil \text{ and } D' \subseteq (V - D) \quad (1)$$

Then, $|N[D']| \leq \sum_{i=1}^t d(v_i) + t = 4t = 4\gamma_M^{-1}(G)$

By (1), $\lceil \frac{p}{2} \rceil \leq 4\gamma_M^{-1}(G)$. Therefore,

$$\gamma_M^{-1}(G) \geq \lceil \frac{p}{8} \rceil \quad (2)$$

Suppose $D' = \{u_1, u_2, u_{\lceil \frac{p}{8} \rceil}\}$ is a subset of vertices in $(V - D)$ such that $N[u_i] \cap N[u_j] = \emptyset, i \neq j$ and $|D'| = \lceil \frac{p}{8} \rceil$. Then $|N[D']| = 4\lceil \frac{p}{8} \rceil$.

In all cases of $p = r(\text{mod } 8)$, when $0 \leq r \leq 7$, $|N[D']| = 4\lceil \frac{p}{8} \rceil \geq \lceil \frac{p}{2} \rceil$.

And $D' \subseteq V - D$. Therefore D' is a Inverse Majority Dominating set of G .

Hence,

$$\gamma_M^{-1}(G) \leq |D'| = \lceil \frac{p}{8} \rceil \quad (3)$$

from (1) and (2) we obtain $\gamma_M^{-1}(G) = \lceil \frac{p}{8} \rceil$.

The following results are the characterization theorem of an inverse majority dominating sets and majority independent set. $\square$

Proposition 4.1. $\gamma_M(G) = \gamma_M^{-1}(G) = \beta_M(G) = 1$ if and only if the cubic bipartite graph G has all vertices $u \in V(G)$ of degree $d(u) \geq \lceil \frac{p}{2} \rceil - 1$.

Proof: Let G be a cubic bipartite graph with p vertices. If the given graph G has vertices of degree $d(u) \geq \lceil \frac{p}{2} \rceil - 1$, for all $u \in V(G)$, then every vertex is a majority dominating vertex of G .

Therefore, $D = \{u\}$ is a minimal majority dominating set of G and $D' = \{v\} \subseteq V - D$ is a minimal inverse majority dominating set of G , also any one vertex of G forms a majority Independent set of G . Hence the result

For the converse, by the Propositions (2.2), (2.3) and (2.5), we get the condition. $\square$

Observation 4.1. 1. For a cubic bipartite graph with $p = 6$ or 8 ,

$$\gamma_M(G) = \gamma_M^{-1}(G) = \beta_M(G) = 1.$$

2. For a cubic bipartite graph with $p = 10$ or 12 ,

$$\gamma_M(G) = \gamma_M^{-1}(G) = \beta_M(G) = 2.$$

3. The following graph G_1 is a cubic bipartite with $p = 14$,

In G_1 ; $D_1 = \{v_1, v_4\}$ is a Majority Dominating set of G_1 and $D_2 = \{v_2, v_6\}$ is a Inverse Majority Dominating set of G_1 with respect to D_1 . $\gamma_M(G_1) = \gamma_M^{-1}(G_1) = 2$. Also,

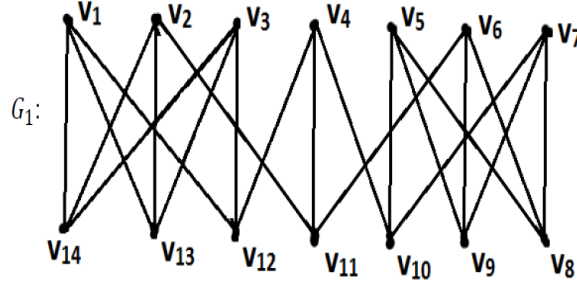


FIGURE 2

$S = \{v_1, v_2, v_3\}$ is a majority independent set of G_1 . Hence, $\beta_M(G) = 3$.

4. For $p = 16$, we have

$$\gamma_M(G) = \gamma_M^{-1}(G) = 2 \quad \text{and} \quad \beta_M(G) = 3.$$

5. For $p = 18$, We have

$$\gamma_M(G) = \gamma_M^{-1}(G) = 3 \quad \text{and} \quad \beta_M(G) = 4.$$

6. For $p = 20$, we have

$$\gamma_M(G) = \gamma_M^{-1}(G) = 3 \quad \text{and} \quad \beta_M(G) = 4.$$

7. For $p = 22$, we have

$$\gamma_M(G) = \gamma_M^{-1}(G) = 3 \quad \text{and} \quad \beta_M(G) = 4.$$

8. For $p = 24$, we have

$$\gamma_M(G) = \gamma_M^{-1}(G) = 3 \quad \text{and} \quad \beta_M(G) = 5.$$

and so on...

Theorem 4.2. Let G be an cubic bipartite graph with p vertices. The subsets D and D' and S are the majority domination, inverse majority domination and majority independent sets of G respectively. Then $\gamma_M^{-1}(G) < \beta_M(G)$, when $p \geq 14$ If and only if

- (i) $|p_n[v, D']| \geq 3$, for every $v \in D'$ and
- (ii) $|p_n[v, S]| \leq 2$, for every $v \in S$.

Proof: Let D be the majority dominating set of G . Let $\gamma_M^{-1}(G) < \beta_M(G)$. Then the inverse majority domination number $\gamma_M^{-1}(G) = |D'|$ and the majority independence number $\beta_M(G) = |S|$. Also, D' is the minimum inverse majority dominating set with respect to D of G and S is the maximum majority independent set of a cubic bipartite graph G . Let $V_1(G)$ and $V_2(G)$ be the bipartition of $V(G)$. $V_1(G) = \{v_1, v_2, \dots, v_{p_1}\}$ $V_2(G) = \{u_1, u_2, \dots, u_{p_2}\}$ with $p = (p_1 + p_2)$. By the theorem (4.2), $\gamma_M^{-1}(G) \leq \lceil \frac{p}{8} \rceil$. Let $D' = \{v_1, v_2, \dots, v_{\lceil \frac{p}{8} \rceil}\} \subseteq V_1(G)$ such that $d(v_i, v_j) \geq 2$ for any v_i and $v_j \in D'$ and $i \neq j$. Since each vertex has degree $d(v_i) = 3$, $N(v_i) = (u_i, u_j, u_k) \subseteq V_2(G)$,

For each vertex $v_i \in D'$, $|N(v_i) \cup N(v_j)| \neq \emptyset$, for any v_i, v_j and $i \neq j$. Then there exists atmost one vertex u such that $N(v_i) \cup N(v_j) = \{u\}$.

$|N(v_i) \cup N(v_j)| \geq 1$ and the private neighbour of each vertex $v_i \in D'$ is $\{v_i, u_i, u_j\}$ or $\{v_i, u_i, u_j, u_k\} \Rightarrow |p_n[v_i, D']| \geq 3$, for each $v_i \in D'$. Hence the condition (i) is true, Similarly By proposition (2.3) (2), For any cubic bipartite graph G , $\beta_M(G) \leq \lceil \frac{p}{4} \rceil - 1$.

Let $S = \{v_1, v_2, \dots, v_t\} \subseteq V_1(G)$ be a majority independent set of G where $t = \lceil \frac{p}{4} \rceil - 1$ such that $d(v_i, v_j) = 2$ for any i, j and $i \neq j$. For every vertex $v_i \in S$,

$N(v_i) \cup N(v_j) = \{(u_i, u_j) \text{ or } (u_i, u_j, u_k)\} \Rightarrow |N(v_i) \cup N(v_j)| = 2 \text{ or } 3$, for any $v_i, v_j \in S$. Since $d(v_i) = 3$, $|p_n[v_i, S]| = 2 \text{ or } 1$ respectively.

$\Rightarrow |p_n[v, S]| \leq 2$, for all $v_i \in S$. Hence the condition (ii) holds.

Let D' be a Inverse Majority Dominating set of a cubic bipartite graph G . Since $|p_n[v, D']| \geq 3$, for all $v \in D'$, $|N(v_i) \cup N(v_j)| = 1 \text{ or } \emptyset$, for $i \neq j$ and for $v_{(i)}, v_{(j)} \in D'$. Then $|N(v)| = 3 \text{ or } 4$ and $|p_n[v, D']| = 3 \text{ or } 4$, for every $v \in D'$. It implies that each vertex $v \in D'$ dominates atleast 3 vertices.

Hence, we get an inverse majority dominating set D' with minimum cardinality for G . The inverse majority dominating number is $\gamma_M^{-1}(G) = |D'|$. Let S be a majority independent set of a cubic bipartite graph of G . Since $|p_n[v_i, S]| \leq 2$, $|N(v_i) \cup N(v_j)| = 3 \text{ or } 2$, for any $v_i, v_j \in S$ and $i \neq j$. Then we obtain a majority independent set with maximum cardinality for G . Majority independence number of $G = |S| = \beta_M(G)$. Since $\gamma_M^{-1}(G) \leq \lceil \frac{p}{8} \rceil$ and $\beta_M(G) \leq \lceil \frac{p}{4} \rceil - 1$, we get $\gamma_M^{-1}(G) < \beta_M(G)$. $\square$

Corollary 4.1. *Let G be a cubic bipartite graph with $p \leq 13$ vertices and D be majority dominating set of G . If $|p_n[v, D']| \geq 3$, for all $v \in D'$ and $|p_n[v, S]| = 3$, for atleast one vertex $v \in S$ then $\gamma_M^{-1}(G) = \beta_M(G)$, where D' and S are the inverse majority dominating set and majority independent set of G .*

Proof. Let D be a minimum majority dominating set and S be a maximal majority independent set of G . Since for atleast one vertex $v \in S$, $|p_n[v, S]| = 3$, $|N(v_i) \cup N(v_j)| = 1$, for any v_i and $v_j \in S$. Then $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $|p_n[v, S]| > |N[S]| - \lceil \frac{p}{2} \rceil$, for all $v \in S$. $\Rightarrow S$ is also a minimal majority dominating set which is in $(V - D)$. Hence $\gamma_M^{-1}(G) \geq \beta_M(G)$. By proposition (3.2) (i), for any graph G , $\gamma_M^{-1}(G) \leq \beta_M(G)$. Thus, $\gamma_M^{-1}(G) = \beta_M(G)$. $\square$

5. $\gamma_M^{-1}(G)$ AND $\beta_M(G)$ FOR SOME FAMILIES OF GRAPHS

Proposition 5.1. *If $G = K_p$ is a complete graph with p vertices, $\gamma_M(G) = \gamma_M^{-1}(G) = \beta_M(G) = 1$.*

Proof: Since the graph G is complete, it is a regular graph of degree $(p - 1)$. Each vertex of G is a full degree vertex. The majority dominating set, the Inverse majority dominating set and also majority Independent set are all equal to any one vertex $\{v\}$ of G .

Hence $\gamma_M(G) = \gamma_M^{-1}(G) = \beta_M(G) = 1$ $\square$

Proposition 5.2. *For a wheel graph $W_p, p \geq 5, \beta_M(G) = \lceil \frac{(p-2)}{6} \rceil$, if $5 \leq p \leq 18$ and*

$$\beta_M(G) = \left\lceil \frac{(p-3)}{4} \right\rceil, \quad \text{if } p \geq 19$$

Theorem 5.1. *Let $G = W_p$ be a wheel of $p \geq 5$ vertices. Then*

(i). $\gamma_M^{-1}(G) = \beta_M(G) = \lceil \frac{(p-2)}{6} \rceil$, if $5 \leq p \leq 18$ and

(ii). $\gamma_M^{-1}(G) < \beta_M(G)$, if $p \geq 19$.

Proof: By the proposition (2.4), and the proposition (5.2),

$$\gamma_M^{-1}(G) = \beta_M(G) = \lceil \frac{(p-2)}{6} \rceil, 5 \leq p \leq 18.$$

Thus the result (i) is true. Also when $p \geq 19$, using the above results, We get $\gamma_M^{-1}(G) < \beta_M(G)$. $\square$

Proposition 5.3. *For a cycle $C_p, p \geq 3$,*

(i). $\beta_M(G) = \lceil \frac{p}{6} \rceil$, if $3 \leq p \leq 16$ and (ii). $\beta_M(G) = \lceil \frac{(p-4)}{4} \rceil$, if $p \geq 17$.

Theorem 5.2. *Let $G = C_p$, be a cycle $p \geq 3$ vertices Then*

(i). $\gamma_M^{-1}(G) = \beta_M(G) = \lceil \frac{p}{6} \rceil, 3 \leq p \leq 16$

(ii). $\gamma_M^{-1}(G) < \beta_M(G) = \lceil \frac{(p-4)}{4} \rceil, p \geq 17$

Proof: By the proposition (2.4) (2) and the proposition (5.4), we obtain

$\gamma_M^{-1}(G) = \beta_M(G) = \lceil \frac{p}{6} \rceil, 3 \leq p \leq 16$. Hence result (i) is true.

Also, when $p \geq 17$,

Using the above results, We get $\gamma_M^{-1}(G) = \lceil \frac{p}{6} \rceil$ and $\beta_M(G) = \lceil \frac{(p-4)}{4} \rceil$.

Hence $\gamma_M^{-1}(G) < \beta_M(G)$. □

Proposition 5.4. *For a path $(P_p)p \geq 2, \beta_M(G) = \lceil \frac{p}{4} \rceil$, if $2 \leq p \leq 10$,*

$\beta_M(G) = \lceil \frac{(p-2)}{6} \rceil$, if $p > 11$.

Theorem 5.3. *Let G be a path of $p \geq 2$, vertices then*

$\gamma_M^{-1}(G) = \beta_M(G) = \lceil \frac{p}{6} \rceil$, if $2 \leq p \leq 10$, and

$\gamma_M^{-1}(G) < \beta_M(G) = \lceil \frac{(p-2)}{6} \rceil$, if $p \geq 11$.

Proof: By the proposition (2.4) (2) and the proposition (5.6), we get,

$\gamma_M^{-1}(G) = \beta_M(G) = \lceil \frac{p}{6} \rceil$, if $2 \leq p \leq 10$,

Also, when $p \geq 11, \gamma_M^{-1}(G) < \beta_M(G) = \lceil \frac{(p-2)}{6} \rceil$. □

Proposition 5.5. *Let F_p , be a fan with $p \geq 4$ vertices. Then*

(i). *When $p \equiv 1(mod \ 6), \gamma_M^{-1}(G) < \beta_M(G)$ and*

(ii). *When $p \not\equiv 1(mod \ 6), \gamma_M^{-1}(G) = \beta_M(G)$.*

Proof: By the proposition (2.3) and proposition (2.4)

$$\gamma_M^{-1}(G) = \lceil \frac{(p-1)}{6} \rceil \quad \text{and} \quad \beta_M(G) = \lceil \frac{p}{6} \rceil. \quad (4)$$

(i). When $p \equiv 1(mod \ 6)$. Since $\lceil \frac{(p-1)}{6} \rceil < \lceil \frac{p}{6} \rceil$, $\gamma_M^{-1}(G) < \beta_M(G)$.

(ii). When $p \not\equiv 1(mod \ 6)$, Then $p \equiv 0, 2, 3, 4, 5(mod 6)$

$\lceil \frac{(p-1)}{6} \rceil = \lceil \frac{p}{6} \rceil$, if $p = 6r, 6r + 2, 6r + 3, 6r + 4, 6r + 5$

By using (4), We obtain, $\gamma_M^{-1}(G) = \beta_M(G)$. □

Proposition 5.6. *If the cubic graph G is a Generalised Petersen $P(n, k)$ graph. Then*

$\gamma_M^{-1}(G) = \beta_M(G)$.

Proof: Since G is a 3-regular graph and by theorem (2.3) (3), We have $\beta_M(G) = \lceil \frac{p}{8} \rceil$.

Also, by theorem (4.2), For a Generalised Petersen graph $P(n, k)$, $\gamma_M^{-1}(G) = \lceil \frac{p}{8} \rceil = \beta_M(G)$.

□

Proposition 5.7. *Let $G = K_{(1,p-1)}$ be a star with $p \geq 2$ vertices. Then $\gamma_M^{-1}(G) = \beta_M(G)$.*

Proof: By the Proposition (2.4) (6) and Proposition (2.3) (5), We have $\gamma_M^{-1}(G) =$

$\lfloor \frac{(p-1)}{2} \rfloor = \beta_M(G)$. □

Result 5.1. *There exists a graph G for which $\gamma_M(G) = \gamma_M^{-1}(G) = 2$ and $\beta_M(G) = 2t = \lceil \frac{p}{2} \rceil - \gamma_M^{-1}(G)$, where $t \geq 3$.*

Proof: The graph G is obtained by adding one pendant at each vertex of a complete graph and then add a pendant each time at each vertex of K_4 . Finally we obtain a new structure with $p = 4 + 4t$, where t is the number of pendants at each time at one vertex of K_4 .

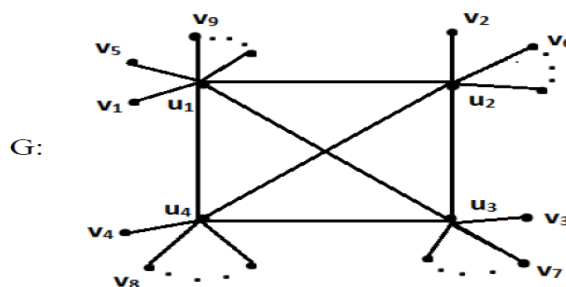


FIGURE 3

Let $|V(G)| = p = 4 + 4t$, when $t = 1$ then the vertex set $V(G) = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ where $(u_1, u_2, u_3, u_4) \subseteq V(K_4)$ and other vertices are pendants. Then $p = 4 + 4 = 8$, and $\gamma_M(G) = |\{u_1\}|$ and $\gamma_M^{-1}(G) = |\{u_2\}|$. $\gamma_M(G) = \gamma_M^{-1}(G) = 1$ and $\beta_M(G) = |v_1, v_2| = 2$, where v_1 and v_2 are adjacent to u_1 and u_2 in G .

when $t = 2, p = 4 + 8 = 12, \gamma_M(G) = \gamma_M^{-1}(G) = 1$ and $\beta_M(G) = |\{v_1, v_2, v_5, v_6\}| = 4 = 2t$, if $t = 2$.

when $t = 3, p = 4 + (4 \times 3) = 16$. In G_1 , there are 3 pendants at each vertex of K_4 .

Let $D = \{u_1, u_2\}$ be a majority dominating set of G and $D' = \{u_3, u_4\} \subseteq V - D$ is a inverse majority dominating set of G Therefore $\gamma_M(G) = \gamma_M^{-1}(G)$. Now $S = \{v_1, v_5, v_9, v_2, v_6, v_{10}\}$ such that $|N[S]| = 8$ and $|p_n[v, S]| = 1 > |N[S]| - \lceil \frac{p}{2} \rceil = 0$, for all $u \in S$. Hence $\beta_M(G) = |S| = 6 = 2t$, if $t = 3$ and so on.

Also, when $t = 3, \lceil \frac{p}{2} \rceil - \gamma_M^{-1}(G) = 8 - 2 = 6 = \beta_M(G)$,

Thus, $\beta_M(G) = 2t = \lceil \frac{p}{2} \rceil - \gamma_M^{-1}(G)$.

In general In this structure, the difference between $\gamma_M^{-1}(G)$ and $\beta_M(G)$ is very large. Hence $\gamma_M^{-1}(G) < \beta_M(G)$ and the difference between these two numbers is very large when p is large and $t \geq 3$. $\square$

6. CONCLUSION

In this article, we have discussed the relation between inverse majority domination number and majority independence number of a graph is discussed. Also some classes of graphs, characterisation theorem for this relation are studied

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INVERSE MAJORITY DOMINATION NUMBER ON SUBDIVISION GRAPHS

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Abstract

In this article, Majority domination number $\gamma_M(G)$ and Inverse Majority domination number $\gamma_M^{-1}(G)$ are found for some special graphs and its subdivision graphs. Then $\gamma_M^{-1}(G)$ for some families of the subdivision graphs $S(G)$ is determined. Some results on $\gamma_M(S(G))$ and $\gamma_M^{-1}(S(G))$ are also studied.

1. Introduction

The Domination theory in graphs was defined by Ore and Berge, in 1977, Cockayne et al., developed the domination concept and it has been discussed extensively in their seminal paper. Then many eminent graph theorists defined various domination parameters and produced many interesting results in this area. Also the new parameter inverse domination in graphs was initiated by Kulli et al., in 1991.

Let G be a simple, *un* directed and finite with p vertices and q edges. $N(V) = \{u \in V(G)/uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ be the open

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neighbourhood and the closed neighbourhood of v respectively.

A set $S \subseteq V(G)$ of vertices in a graph $G = (V, E)$ is called a majority dominating (MD) set of G if at least half of the vertices of $V(G)$ are either in S or adjacent to the elements of S the Majority dominating set S is minimal if no proper subset of S is a majority dominating set of a graph G .

A subdivision of an edge $e = uv$ of a graph G is the replacement of an edge e by a path (u, v, w) . The graph obtained from a graph G by subdividing every edge e of G exactly once and is called the subdivision graph of G denoted by $S(G)$.

Let G be simple and finite graph with p vertices and q edges and D be a minimum majority dominating set of G . If the set $(V - D)$ contains a majority dominating set say D' then D' is called Inverse majority dominating (IMD) set with respect to D .

1.2. Results on $\gamma_M(G)$ and $\gamma_M^{-1}(G)$ [3] and [5]

The following are the results on $\gamma_M(G)$ and $\gamma_M^{-1}(G)$

1. For $G = P_p$, $p \geq 2$ and cycle C_p , $p \geq 3$, $\gamma_M(G) = \left\lceil \frac{p}{6} \right\rceil$.
2. Let $G = K_{m, n}$, $m, n \geq 2$. Then $\gamma_M(G) = 1$.
3. For a Path P_p , $p \geq 2$ and cycle C_p , $p \geq 3$, $\gamma_M^{-1}(G) = \left\lceil \frac{p}{6} \right\rceil$
4. Let $G = K_{m, n}$, $m, n \geq 2$. Then $\gamma_M(G) = 1$.
5. Let $G = K_{1, p-1}$, $\gamma_M^{-1}(G) = \left\lfloor \frac{p-1}{2} \right\rfloor$, $p \geq 2$.
6. For a $G = K_p$, $\gamma_M^{-1}(K_p) = 1$.
7. Let $G = mk_2$. Then $\gamma_M^{-1}(G) = \left\lceil \frac{p}{4} \right\rceil$, where $p = 2m$.

8. For any regular graph with p vertices then $\gamma_M(G) = \gamma_M^{-1}(G)$.

2. Inverse Majority Domination number for some Special Graphs

Proposition 1. *Let G be the Dodecahedron graph and $G' = S(G)$ be the subdivision graph. Then*

(i) $\gamma_M(G) = 3 = \gamma_M^{-1}(G)$ and

(ii) $\gamma_M(G') = 7 = \gamma_M^{-1}(G')$.

Proof. Let G be the platonic solid dodecahedron, Then G is a 3-regular graph $p = 20$ and $q = 30$. Let $D = \{v_1, v_4, v_6\}$ such that $d(u_i, v_j) = 3$. Then $|N[D]| = 11 > \left\lceil \frac{p}{2} \right\rceil$. Hence D is a MD set of G and $\gamma_M(G) = 3$. Let $D' = \{v_2, v_5, v_8\} \subseteq V - D$, Such that $d(u_i, v_j) = 3$. By similar argument, D' is a IMD-set of G and $\gamma_M^{-1}(G) = 3$.

Let G' be a subdivision graph of dodecahedron with $p' = 50$ and $V(G') = \{v_1, \dots, v_{20}, u_1, u_2, \dots, u_q\}$ with $d(v_i) = 3, i = 1, 2, \dots, 20$ and $d(u_i) = 2, i = 1, 2, \dots, 30$. Let $S = \{v_1, v_3, v_6, v_7, v_{11}, v_{13}, v_{17}\}$, such that $d(u_i, v_j) = 4$, for $\forall i, j, i \neq j$. Then $|N[S]| = 28 > \left\lceil \frac{p}{2} \right\rceil$. It implies that S is a majority dominating set of G' and $\gamma_M(G') = |S| = 7$. Next choose $S' = \{v_2, v_4, v_5, v_7, v_{18}, v_8, v_{15}\}$ such that $d(u_i, v_j) = 4$, and $S' \subseteq V - S$. By the similar argument, S' is a Inverse majority dominating set of G and $\gamma_M^{-1}(G') = 7$. Hence, $\gamma_M(G') = 7 = \gamma_M^{-1}(G')$.

Proposition 2. *Let G be the Tetrahedron graph and Octahedron graph. Then*

(i) $\gamma_M(G) = 1 = \gamma_M^{-1}(G)$ and

(ii) $\gamma_M(G') = 2 = \gamma_M^{-1}(G')$.

Proof. The proof is obvious.

Proposition 3. Let G be an icosahedral graph and $S(G)$ be the subdivision graph of G . Then

- (i) $\gamma_M(G) = 1 = \gamma_M^{-1}(G)$
- (ii) $\gamma_M(G') = 2 = \gamma_M^{-1}(G')$, if $G' = S(G)$.

Proof. (i) Let G be an icosahedral graph with $p = 12$ vertices and it is a 5-regular graph. Since each vertex dominates six vertices $\gamma_M(G) = 1 = \gamma_M^{-1}(G)$.

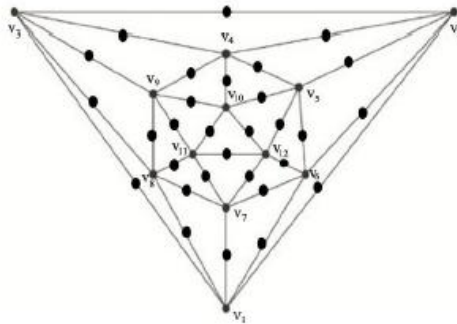


Figure 1. Subdivision of $G - S(G)$.

Let G' be the subdivision graph of icosahedral with $p' = 42$ and $V(G') = \{v_1, v_2, \dots, v_{12}, u_1, u_2, \dots, u_{30}\}$. Now G' is not a regular graph with $d(v_i) = 5$, for all $i = 1, 2, \dots, 12$ and $d(v_j) = 2$, for all $i = 1, 2, \dots, 30$. Let $S = \{v_1, v_4, v_{11}, v_{12}\}$ such that $d(u_i, v_j) = 4$. Then $|N[S]| = \sum d(v_i) + |S|$
 $(3 \times 5) + 4 + 4 > \left\lceil \frac{p}{2} \right\rceil$ And $|N[S]| = 23 > \left\lceil \frac{p}{2} \right\rceil$. Then S is a majority dominating set of G' and $\gamma_M(G') = 4$.

Choose $S' = \{v_2, v_9, v_7, v_{10}\} \subseteq V - S$. By the similar argument, $\gamma_M^{-1}(G') = 4$.

Proposition 4. Let G and G' be the Frucht graph and its subdivision graph respectively.

(i) $\gamma_M(G) = 2 = \gamma_M^{-1}(G)$ and

(ii) $\gamma_M(G') = 4 = \gamma_M^{-1}(G')$

Proof. The proof is obvious.

Proposition 5. (i) Let G be a Doyle graph. Then $\gamma_M(G) = 3 = \gamma_M^{-1}(G)$ and

(ii) If $G' = S(G)$ is the subdivision of G then $\gamma_M(G') = 8 = \gamma_M^{-1}(G')$.

Proof. Let G be a Doyle graph with $p = 27$ vertices and G is a 4-regular graph. The vertex set $V(G)$ can be partitioned into three vertex sets V_1 , V_2 and V_3 , each comprising of 9 vertices with degree 4 and V_1 , V_2 and V_3 are vertices of a outer cycle C_1 , inner circle C_2 , innermost cycle C_3 respectively.

Let $D = \{v_2, v_5, v_8\}$ and $D' = \{v_2, v_5, v_8\} \subseteq V - D$. Since G is a 4-regular graphs, $|N[D]| = |N[D']| = \sum d(v_i) + 3 = 15 > \left\lceil \frac{p}{2} \right\rceil$.

Hence, the sets D and D' are the MD-set and the IMD-set of G respectively and $\gamma_M(G) = \gamma_M^{-1}(G) = 3$.

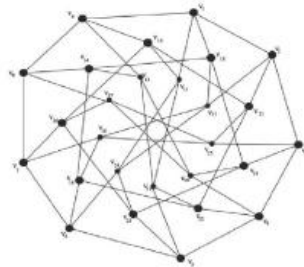


Figure 2. G : Doyle graph.

Let G' be the subdivision graph of a Doyle graph with $p' = 72$ vertices. Let $V(G') = \{v_1, v_2, \dots, v_{27}, u_1, u_2, \dots, u_{45}\}$ where $d(u_j) = 2, j = 1$ to 45 and $d(v_i) = 2, i = 1$ to 27 and $V_1(G') = \{v_1, \dots, v_9\}$, $V_2(G') = \{v_{10}, \dots, v_{18}\}$, $V_3(G') = \{v_{19}, \dots, v_{27}\}$

Let $S = \{v_1, v_4, v_7, v_{13}, v_{17}, v_{18}, u_{20}, u_{25}\}$ such that $d(u_i, v_j) = 4$ for

$i \neq j$ and $v_i, v_j \in S$.

Then $|N[S]| = \sum_{v_i \in S} d(v_i) + |S| = (8 \times 4) + 8 = 40 > \left\lceil \frac{p}{2} \right\rceil$. Hence S is a MD-set of G' and $\gamma_M(G') = |S| = 8$. In $V - S$, choose $S' = \{v_2, v_5, v_8, v_{10}, v_{13}, v_{16}, v_{18}, v_{22}\}$ and $|N[S']| = 40 > \left\lceil \frac{p}{2} \right\rceil$. Hence, S' is a IMD-set of G' and $\gamma_M(G') = |S| = 8$. Hence, $\gamma_M(G') = \gamma_M^{-1}(G') = 8$.

Proposition 6. (i) Let G be a Folkman graph. Then $\gamma_M(G) = 2 = \gamma_M^{-1}(G)$ and (ii) $\gamma_M(G') = 6$ and $\gamma_M^{-1}(G') = 8$.

Proof. Let G be a Folkman graph with $p = 20$ vertices. It is a bipartite, 4-regular, Hamiltonian graph and it is a four edge connected perfect graph. Let $D = \{u_1, v_1\}$ and $D' = \{u_3, v_3\} \subseteq V - D$ such that $d(u_i, v_j) \geq 4$. Since each vertex dominates 5 vertices, $|N[D]| = |N[D']| = 10 = \left\lceil \frac{p}{2} \right\rceil$. Hence $\gamma_M(G) = \gamma_M^{-1}(G) = 2$.

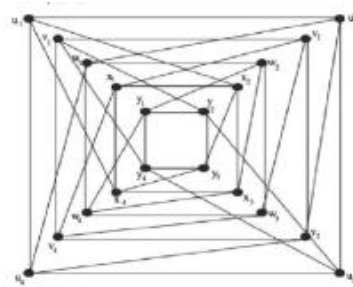


Figure 3. G: Folk man graph.

Let G' be a subdivision graph of a Folkman graph with the vertex set $V(G') = \{u_1, \dots, u_4, v_1, \dots, v_4, w_1, \dots, w_4, x_1, \dots, x_4, y_1, \dots, y_4, z_1, z_2, \dots, z_{40}\}$ and $|V(G')| = 60$, where the vertices u_i, v_i, w_i, x_i and y_i are in the outer square to inner square of totally 5 squares in G'_R and $z_i, i = 1, \dots, 40$ denotes the newly added vertices in G . Let $S = \{u_1, v_1, w_1, x_1, y_1, y_3\}$ in which all are non-adjacent vertices in G' and $|N[S]| = \sum d(v_i) + 6 = 30$

$= \left\lceil \frac{p}{2} \right\rceil$. Hence S is a majority dominating set of G' and $\gamma_M(G') = |S| = 6$. In $V - S$, Choose $S' = \{u_3, v_3, w_3, x_3, u_4, v_4, w_4, x_4\} \subseteq V - S$ and $|S'| = 8$. It implies that set S' is an inverse majority dominating set of G' and $\gamma_M^{-1}(G') = |S'| = 8$.

Proposition 7. (i) Let G be a Levi graph, Then $\gamma_M(G) = 4 = \gamma_M^{-1}(G)$ and (ii) Let $G' = S(G)$ be the subdivision of G . Then $\gamma_M(G') = 9 = \gamma_M^{-1}(G')$.

Proof. Let G be a Levi graph with $p = 30$ and $q = 45$, and it is not a regular graph.

Let $V(G) = \{w_1, w_2, \dots, w_{10}, v_1, \dots, v_{10}, u_1, \dots, u_{10}\}$ in which the vertices w_i, v_i and u_i , for $i = 1, \dots, 10$ from a outer circle to inner circle and $d(w_i) = 3, d(v_i) = 2, d(u_i) = 4, i = 1, \dots, 10$.

Let $D = \{u_1, u_3, w_2, w_5\}$ and $|N[D]| = \sum d(u_i) + \sum d(w_i) + 4 = 18 > \left\lceil \frac{p}{2} \right\rceil$. Hence D is a MD-set of G and $\gamma_M(G) = |D| = 4$.

Next, choose $D' = \{u_5, u_6, w_1, w_8\} \subseteq V - D$. Then $|N[D']| = 18 > \left\lceil \frac{p}{2} \right\rceil$.

It implies that D' is aIMD-set of G and $\gamma_M^{-1}(G) = 4$.

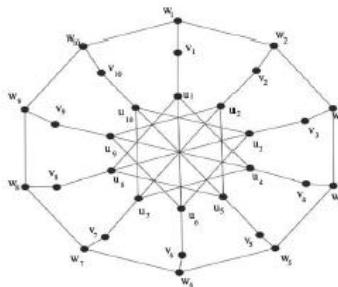


Figure 4. G : Levi graph.

Let G' be the subdivision graph of a Levi graph G with $p' = 75$.

$V(G') = \{w_1, \dots, w_{10}, v_1, \dots, v_{10}, u_1, \dots, u_{10}, x_1, x_2, \dots, x_{45}\}$ where $d(w_i) = 3$, $d(v_i) = 2$, $d(u_i) = 4$. $i = 1$ to 10 and $d(x_j) = 2$, $j = 1$ to 45.

Let $S = \{u_1, u_2, u_3, w_1, w_3, w_5, w_9, u_9\} \subseteq V(G')$ and $|N[D']| = \sum_1^3 d(u_i) + \sum_1^5 d(w_i) + d(u_9) + |S| = 39 > \left\lceil \frac{p}{2} \right\rceil$. It implies that $\gamma_M(G') = |S| = 9$.

Next, choose, $S' = \{u_4, u_5, u_6, w_2, w_4, w_6, w_8, w_{10}, u_{10}\} \subseteq V - S$.

By the above calculations, $|N[S']| = 39 > \left\lceil \frac{p}{2} \right\rceil$.

Hence S' is a IMD-set of G' and $\gamma_M^{-1}(G') = |S| = 9$.

Thus, $\gamma_M(G') = 9 = \gamma_M^{-1}(G')$.

Proposition 8. *Let G be Platonic Solid Cube and $S(G)$ be the subdivision of G . Then*

- (i) $\gamma_M(G) = 1 = \gamma_M^{-1}(G)$ and
- (ii) $\gamma_M(G) = 3 = \gamma_M^{-1}(G)$, if $G' = S(G)$.

Proof. The proof is obvious.

3. $\gamma_M^{-1}(G)$ for Some Classes of Subdivision Graphs

Proposition 1. *Let $G = S(C_p)$ be a subdivision graph of a cycle with p vertices then $\gamma_M(G) = \gamma_M^{-1}(G) = \left\lceil \frac{p}{3} \right\rceil$.*

Proof. Let $G = S(C_p)$ and $V(G) = \{u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_p\}$ and $|V(G)| = 2p$. Then $S(C_p)$ is also a cycle with $2p$ vertices. By the result (1.3) 1 and (3) $\gamma_M(C_p)$ and $\gamma_M^{-1}(C_p) = \left\lceil \frac{p}{6} \right\rceil$, where $|V(C_p)| = p$. Hence $\gamma_M^{-1}(G) = \left\lceil \frac{2p}{6} \right\rceil = \left\lceil \frac{p}{3} \right\rceil = \gamma_M(C_p)$. Thus $\gamma_M(G) = \gamma_M^{-1}(G) = \left\lceil \frac{p}{3} \right\rceil$.

Proposition 2. *If G is a subdivision of a complete graph then $\gamma_M(G) = 2$*

and $\gamma_M^{-1}(G) = 2$.

Proof. Since K_p is complete graph, there are p vertices and $\frac{p(p-1)}{2} = q$ edges. Let $G = S(K_p)$ with $V(G) = \{u_1, \dots, u_p, v_1, v_2, \dots, v_q\}$ and $|V(G)| = p' = (p + q)$, where $d(u_i) = p - 1$, for $\forall u_i, \dots, u_p$ and $d(v_i) = 2, \forall i$ or $v_i, \dots, v_q, p' = \frac{p(p+1)}{2}$ and $\left\lceil \frac{p}{2} \right\rceil = \left\lceil \frac{p(p+1)}{4} \right\rceil$. Let $D = \{u_1, u_3\}$ such that $d(u_i, v_j) = 4$. Then $|N[D]| = (p - 1) + (p - 2) + 2 = 2p - 1$ $|N[D]| = 2p - 1 > \left\lceil \frac{p}{2} \right\rceil$. It implies that D is a MD-set of G and $\gamma_M(G) = |D| = 2$.

Let $D' = \{u_2, u_4\} \subseteq V - D$ such that $d(v_2, v_4) = 4$. Then $|N[D']| = (p - 1) + (p - 2) + 2 = 2p - 1 > \left\lceil \frac{p}{2} \right\rceil \Rightarrow D'$ is a IMD-set of G and $\gamma_M^{-1}(G) = |D'| = 2$. Hence $\gamma_M(G) = 2 = \gamma_M^{-1}(G)$.

Proposition 3. Let $G = S(K_{1, p-1})$. Then $\gamma_M(G) = 1$ and $\gamma_M^{-1}(G) = \left\lceil \frac{p-1}{2} \right\rceil$.

Proof. Let $V(G) = \{u, u_1, \dots, u_{p-1}, v_1, \dots, v_{p-1}\}$ where $d(u_i) = 2$, for $\forall u_i$ and v_i 's are all pendants, $d(u) = p - 1$ and $|V(G)| = 2p - 1 = p'$. Since u is a MD vertex of G , $\gamma_M(G) = 1$.

Let $D' = \{v_1, \dots, v_{\left\lceil \frac{p-1}{2} \right\rceil}\} \subseteq V - D$. Then $|N[D']| = 2 \left\lceil \frac{p-1}{2} \right\rceil + 1 = 2 \left(\frac{p-1}{2} \right) + 1p$ or $p + 1$ $|N[D']| = p$ or $p + 1$, if p is odd or even. $|N[D']| \geq \left\lceil \frac{p'}{2} \right\rceil$. It implies that D' is a IMD-set of G . Hence $\gamma_M^{-1}(G) = \left\lceil \frac{p-1}{2} \right\rceil$.

Proposition 4. Let $G = S(P_p)$ be a subdivision graph of a path with

$p \geq 2$ vertices. Then $\gamma_M^{-1}(G) = \left\lceil \frac{p}{3} \right\rceil$.

Proof. Since the subdivision graph of path P_p is also a path, the result is obvious.

Proposition 5. For a graph $G = S(W_p)$ be a subdivision graph of a wheel with $p \geq 5$ vertices. Then $\gamma_M^{-1}(G) = \gamma_M(G) = \left\lceil \frac{p'}{8} \right\rceil$, where $p' = (p + q)$.

Proof. Let $G = S(W_p)$ and $V(G) = \{u, u_1, u_2, \dots, u_{p-1}, v_1, v_2, \dots, v_q\}$, and $|V(G)| = p + q = p'$, where u is a central vertex, $d(v_i) = 2$, for $\forall v_i, i = 1, \dots, q$ and $d(u_i) = 3$, for $\forall u_i, i = 1, 2, \dots, p - 1$. Let $D = \{u_1, u_3, \dots, u_t\}$ such that $t = \left\lceil \frac{p'}{8} \right\rceil$ and $d(u_i, u_j) \geq 4$, for $i \neq j$. Then $|N[D]| = \sum_{i=1}^t d(v_i) + t = 4t = 4 \left\lceil \frac{p'}{8} \right\rceil \geq \left\lceil \frac{p'}{2} \right\rceil$. Then D is a MD-set of G and $\gamma_M(G) = |D| = \left\lceil \frac{p'}{8} \right\rceil$. Now, choose the set $D' = \{u_2, u_5, \dots, u_t\} \subseteq V - D$ Such that $t = \left\lceil \frac{p'}{8} \right\rceil$ and $d(u_i, u_j) \geq 4$, for $i \neq j$. By the above argument, the set D' is a IMD- set of G .

$$\gamma_M^{-1}(G) \leq |D'| = \left\lceil \frac{p'}{8} \right\rceil \quad (1)$$

Suppose $D_1 \subseteq V - D$ be a set with $|D_1| < |D'| = \left\lceil \frac{p'}{8} \right\rceil$. Then $|N[D_1]| < \left\lceil \frac{p'}{2} \right\rceil$. It implies that D_1 is not a IMD-set of G and

$$\gamma_M^{-1}(G) > |D_1| \text{ and } \gamma_M^{-1}(G) \geq \left\lceil \frac{p'}{8} \right\rceil \quad (2)$$

Hence It implies that $\gamma_M^{-1}(G) \geq \left\lceil \frac{p'}{8} \right\rceil$, and $\gamma_M(G) \geq \left\lceil \frac{p'}{8} \right\rceil$, where $p' = p + q$.

4. Results on $\gamma_M(G')$ and $\gamma_M^{-1}(G')$

Observation 1.

(1) A full degree vertex of a graph becomes a majority dominating vertex in the subdivision graph $S(G)$ of G .

(2) The degree of a vertex of G will never change in the subdivision graph $S(G)$ and the degree of a newly added vertex is always two in $S(G)$.

(3) The regular graph G is not a regular in $S(G)$ except $G = C_p$.

(4) Any pendant edge becomes a path K_2 .

Proposition 2. *Let G and G' be the disconnected graph and its Subdivision graph with p and p' vertices respectively. Then $\gamma_M(G') = \gamma_M^{-1}(G')$.*

For Example

(i) Let $G = 5K_2$ By the result, $\gamma_M(G) = \left\lceil \frac{p}{4} \right\rceil$, $\gamma_M(G) = 3 = \gamma_M^{-1}(G) = 3$.

Let the subdivision graph $S(G) = G' = 5P_3$ with $p' = 15$. Now, the set $D = \{u_1, u_2, u_3\}$, where $d_G(u_i) = 2$, for $\forall, u_i$ and D is a MD-set of G' and $\gamma_M(G') = 3$. And Now the set $D' = \{u_4, u_5, u_1\} \subseteq V - D$ is a IMD- set of G' and $\gamma_M(G') = \gamma_M^{-1}(G')$.

(ii) Let $G = 5K_3$ with $p = 15$ vertices. Then $D = \{v_1, v'_1, v''_1\}$ is a MD-set of G and $\gamma_M^{-1}(G) = \gamma_M(G) = 3$. Let $G' = 5K_6$ be the subdivision graph of G with $p = 30$. Now, the set $D = \{u_1, u'_1, u''_1, u'_3, v_3\}$ is a MD-set of G' and $\gamma_M(G') = |D| = 5$. Now the set $D' = \{u_2, u'_2, u''_2, v'_1, v_1\} \subseteq V - D$ is a IMD-set of G' and $\gamma_M(G') = \gamma_M^{-1}(G') = |D'| = 5$. Hence $\gamma_M(G') = \gamma_M^{-1}(G')$.

Proposition 3. *If the graph G is regular then $\gamma_M(G') = \gamma_M^{-1}(G')$.*

Proof. Since G is a regular graph, and G by the result [1.3] (10), $\gamma_M(G) = \gamma_M^{-1}(G)$. Let $G' = S(G)$ be the subdivision graph of G . Then G' is

not a regular graph except $G = C_p$, a cycle. Since the degree of the vertices of G is equal to the degree of the vertices of G' except the newly added vertex u_i with $d(u_i) = 2$, for $\forall, u_i, i = 1, 2, \dots, q$. To get the minimality select the vertices v_i of G with the distance $d(v_i, v_j) \geq 4$ in G' then it will form a minimum MD-set and minimum IMD-set in G' with the same cardinality. Hence, $\gamma_M(G') = \gamma_M^{-1}(G')$.

Proposition 4. *If the graph G contains a full degree vertex and others are pendants then $\gamma_M(G') < \gamma_M^{-1}(G')$.*

Proof. By the observation (4.1) (1), any full degree vertex v of G becomes a majority dominating vertex v of G' . Then $D = \{v\}$ is a majority dominating set of G' and $\gamma_M(G') = 1$. Since G has pendant vertices, $d(u_i) = 2$, for every $u_i, i = 1, 2, \dots, q$, and the vertices u_i of $G' \subseteq V - D$ will form an inverse majority dominating set of G' with respect to D . It implies that $\gamma_M^{-1}(G') > 1$. Hence $\gamma_M(G') < \gamma_M^{-1}(G')$.

Proposition 4.5. *If the graph G has exactly one majority dominating vertex and other are pendants then $\gamma_M(G') > 1$ and $\gamma_M(G') < \gamma_M^{-1}(G')$.*

Proof. Let $|V(G)| = p$ and $|V(G')| = p + q = p'$. Since the majority dominating vertex v of degree $d_G(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1$, $d_G(v) < \left\lceil \frac{p}{2} \right\rceil - 1$. Then the majority dominating set D will contain at least two vertices of G' . Hence $\gamma_M(G') = |D| > 1$. Since other vertices u_i are pendants, the newly added vertices $d(u_i) = 2$. Such that for $i = 1, 2, \dots, q$. Now the IMD-set (D') contains the vertices of $u_i \subseteq V - D$ of G' . It implies that $\gamma_M^{-1}(G') > 2$. Hence $\gamma_M(G') < \gamma_M^{-1}(G')$.

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Inverse independent majority dominating set of a graph

J. Joseline Manora¹ and S. Vignesh²

Abstract

In this research article, an Inverse Independent Majority dominating set of a graph G is introduced. Inverse independent majority domination number $i_M^{-1}(G)$ is determined for some families of graphs. Some characterisation theorem on $i_M^{-1}(G)$ are discussed. Bounds of the inverse independent majority domination number $i_M^{-1}(G)$ and its relationship with other domination parameters are also established.

Keywords

Independent majority dominating set, Inverse independent majority domination number.

AMS Subject Classification

05C69.

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1. Introduction

Domination as a graph theoretic concept was introduced by C. Berge in 1958 and O. Ore in 1962. E.J.Cockayne and S.T.Hedetniemi presented a paper on domination and it has been studied extensively in this article in 1977. T.W.Haynes et al published a book entitled "Fundamentals of Domination in Graphs" [2] contains various domination parameters. In 1991, Kulli and Sigarkanti [7] initiated a new parameter inverse domination in graphs.

By a graph, we mean a finite, simple graph which is undirected and nontrivial. Let $G = (V, E)$ be a graph of order p and size q . For every vertex $v \in V(G)$, the open neighbourhood $N(v) = \{u \in V(G) / uv \in E(G)\}$ and the closed neighbourhood $N[v] = N(v) \cup \{v\}$. The open neighbourhood of a set $S \subseteq V(G)$

is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$.

A set $D \subseteq V(G)$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D . The minimum cardinality of a minimal dominating set is known as the domination number is denoted by $\gamma(G)$. Let D be a minimum dominating set in a graph G . If the set $V - D$ contains a dominating set D' then D' is called an inverse dominating set of G [7] with respect to D . The cardinality of a minimum inverse dominating set is defined as the inverse domination number of a graph G and it is denoted by $\gamma^{-1}(G)$.

A subset S of $V(G)$ is called an independent set if no two vertices in S are adjacent. The minimum cardinality of a maximal independent set is called the independent domination number of G and is denoted by $i(G)$. Let D be a minimum independent dominating set of G . If $(V - D)$ contains an independent dominating set D' then the set D' is called an Inverse independent dominating set with respect to D . The inverse independent domination number $i^{-1}(G)$ [7] is the minimum cardinality of a minimal inverse independent dominating set of G .

A set $S \subseteq V(G)$ of vertices in a graph $G = (V, E)$ is called a majority dominating set of G if atleast half of the vertices of $V(G)$ are either in S or adjacent to the elements of S . The majority dominating set S is minimal if no proper subset of S is a majority dominating set of a graph G . The minimum cardinality of a minimal majority dominating set of G is called majority domination number of G , denoted by $\gamma_M(G)$ [4]. Let

G be simple and finite graph with p vertices and q edges and D be a minimum majority dominating set of G . If the set $(V - D)$ contains a majority dominating set say D' then D' is called an Inverse majority dominating set [6] with respect to D . The inverse majority domination number $\gamma_M^{-1}(G)$ of a graph G is the minimum cardinality of all Inverse majority dominating sets of a graph G .

A majority dominating set D of a graph G is called an independent majority dominating (IMD) set if the induced subgraph $\langle D \rangle$ has no edges. The independent majority domination number $i_M(G)$ [3] of a graph G is the minimum cardinality of a minimal independent majority dominating set of G .

Result 1.1 ([3]). *The following are the results of some classes of graphs regarding $\gamma_M(G)$, $i_M(G)$ and $\gamma_M^{-1}(G)$*

- (i) For a Path P_p , $p \geq 2$ and cycle C_p , $p \geq 3$, $\gamma_M(G) = \lceil \frac{p}{6} \rceil$.
- (ii) $i_M(G) = \lceil \frac{p}{4} \rceil$, if $G = mK_2$
- (iii) If G is without isolated vertices then $i_M(G) \leq \lfloor \frac{p}{4} \rfloor$.
- (iv) For a $G = K_{1,p-1}$, $\gamma_M^{-1}(G) = \lfloor \frac{p-1}{2} \rfloor$, $p \geq 2$
- (v) For a $G = K_p$, $\gamma_M^{-1}(K_p) = 1$.
- (vi) Let $G = mK_2$. Then $\gamma_M^{-1}(G) = \lfloor p \rfloor$ where $p = 2m$
- (vii) For a Path P_p , $p \geq 2$ and Cycle C_p , $p \geq 3$, $\gamma_M^{-1}(G) = \lceil \frac{p}{6} \rceil$.

2. Inverse Independent Majority Dominating Set

Definition 2.1. Let G be a graph with p vertices and q edges. Let D be a minimum Independent Majority Dominating set of a graph G . If the set $(V - D)$ contains an Independent Majority Dominating set D' of G , then the set D' is called an Inverse Independent Majority Dominating set of G with respect to D . The Inverse Independent Majority Dominating number $i_M^{-1}(G)$ of a graph G is the minimum cardinality of a minimal Inverse Independent Majority Dominating set of G .

Example 2.2. Consider the following graph G with $p = 20$ vertices.

Let $D = \{a, b\}$ be a majority dominating set and $D' = \{c, d\}$ is an Inverse majority dominating set with respect to D such that $D' \subseteq V - D$. Then $\gamma_M(G) = 2$ and $\gamma_M^{-1}(G) = 2$. Let $D_M = \{a, v_1, v_2\}$ be an independent majority dominating set and $D'_M = \{b, v_5, v_6\} \subseteq V - D_M$ is an inverse independent majority dominating set with respect to D_M . Then $i_M(G) = 3 = i_M^{-1}(G)$.

Hence, $\gamma_M(G) < i_M(G)$ and $\gamma_M^{-1}(G) < i_M^{-1}(G)$. Thus $\gamma_M(G) < i_M(G)$ and $\gamma_M^{-1}(G) < i_M^{-1}(G)$.

Example 2.3. Consider the following graph G with $p = 20$ vertices.

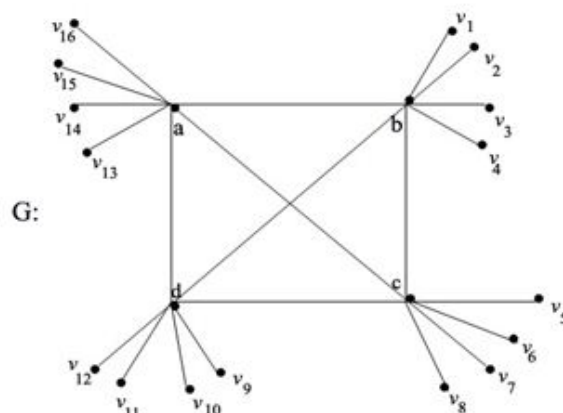


Figure 1

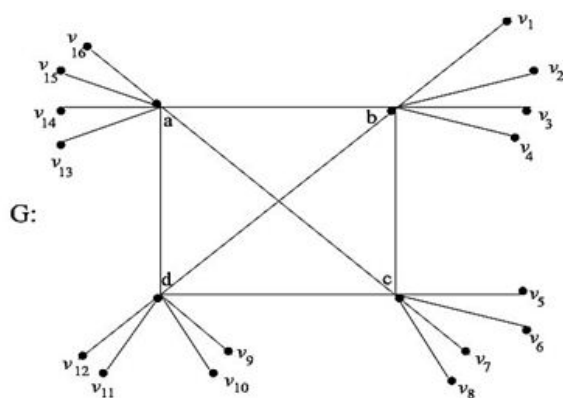


Figure 2

Let $D = \{a, b\}$ be a majority dominating set and $D' = \{c, d\}$ is an Inverse majority dominating set with respect to D such that $D' \subseteq V - D$. Then $\gamma_M(G) = 2$ and $\gamma_M^{-1}(G) = 2$. Let $D_M = \{b, v_6\}$ be an independent majority dominating set and $D'_M = \{d, v_1, v_2\} \subseteq V - D_M$ is an inverse independent majority dominating set with respect to D_M . Then $i_M(G) = 2$ and $i_M^{-1}(G) = 3$. Hence $i_M(G) < i_M^{-1}(G)$.

Proposition 2.4. (i) For any graph G , $i_M(G) \leq i_M^{-1}(G) \leq i^{-1}(G)$.

(ii) For any graph G , $\gamma_M^{-1}(G) \leq i_M^{-1}(G) \leq i^{-1}(G)$.

Proof. (i) Since every inverse independent majority dominating set of G is an independent majority dominating set of G the independent majority domination number $i_M(G)$ and inverse $i_M(G) \leq i_M^{-1}(G)$. Since every inverse independent dominating set of G is an inverse independent majority dominating set of G , their numbers satisfies $i_M^{-1}(G) \leq i^{-1}(G)$, we obtain $i_M(G) \leq i_M^{-1}(G) \leq i^{-1}(G)$.

(ii) Since every inverse independent majority dominating set of G is an inverse majority dominating set of G , their



number gives an in equality, $\gamma_M^{-1}(G) \leq i_M^{-1}(G)$, by the inequality (i), we obtain $\gamma_M^{-1}(G) \leq i_M^{-1}(G) \leq i^{-1}(G)$ $\square$

3. Inverse Independent Majority Domination Number for Some Classes of Graphs

Result 3.1. 1. Let $G = K_p, p \geq 2$ be a complete graph. Then $i_M^{-1}(G) = 1$.

2. Let $G = K_{1,p-1}$ be a star with $p \geq 2$. Then $i_M^{-1}(G) = \left\lfloor \frac{p-1}{2} \right\rfloor$.

3. If $G = P_p$ a path, $p \geq 2$ and C_p a cycle, $p \geq 3$ then $i_M^{-1}(G) = \left\lfloor \frac{p}{6} \right\rfloor$.

4. If $G = W_p$ a wheel with $p \geq 5$ then $i_M^{-1}(G) = \left\lfloor \frac{p-2}{6} \right\rfloor$.

5. If $G = F_p$ be a fan graph with $p \geq 4$, then $i_M^{-1}(G) = \left\lfloor \frac{p-2}{6} \right\rfloor$.

6. Let $G = K_{m,n}, m, n \geq 2$ be a complete bipartite graph. Then $i_M^{-1}(G) = 1$

7. Let $G = mK_2, m \geq 1$. Then $i_M^{-1}(G) = \left\lfloor \frac{p}{4} \right\rfloor$

Proposition 3.2. Let $G = D_3(C_t)$ be a windmill graph with only three cycles of different size $t \geq 4$. Then $i_M(G) = \left\lfloor \frac{p-2}{6} \right\rfloor$ and $i_M^{-1}(G) = \left\lfloor \frac{p}{6} \right\rfloor$. Also $i_M(G) < i_M^{-1}(G)$

Proof. Let $G = D_3(C_t)$ be a windmill graph with only three cycles and each cycle is of size $t \geq 4$. Then the vertex set

$$V(G) = \{u, c_{11}, c_{12}, \dots, c_{21}, c_{22}, \dots, c_{31}, c_{32}, \dots\}.$$

All these three cycles C_1, C_2, C_3 meet at a vertex u . When $t = 4$ and $G = D_3(C_4)$. Then

$$V(G) = \{u, c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}, c_{31}, c_{32}, c_{33}\}$$

and $p = 10$. Here $i_M(G) = |\{u\}| = 1$ and

$$i_M^{-1}(G) = |\{c_{12}, c_{22}\}| = 2.$$

When $t = 5$ and $G = D_3(C_5)$. Then $|V(G)| = p = 13$. Here $i_M(G) = |\{u\}| = 1$ and $i_M^{-1}(G) = |\{c_{12}, c_{22}, c_{32}\}| = 3$. If $t = 6$ then $G = D_3(C_6)$ and $p = 16$. Then i_M -set $D = \{u, c_{13}\}$ and i_M^{-1} -set $D' = \{c_{12}, c_{22}, c_{32}\}$. It implies that $i_M(G) = 2$ and $i_M^{-1}(G) = 3$. In general, Let $V(G) = p = 3t - 2$ and $\left\lfloor \frac{p}{2} \right\rfloor = \left\lfloor \frac{3t}{2} \right\rfloor - 1$. Let, $D = \{u, c_{13}, c_{23}, \dots, c_{ij}\}$ such that $d(x_i, x_j) \geq 3$, where the vertices $x_i, x_j \in D$ and $|D| = \left\lfloor \frac{p-2}{6} \right\rfloor$. Then $|N[D]| = |N[u]| + 3(|D| - 1) = 7 + 3 \left\lfloor \frac{p-2}{6} \right\rfloor - 3$. Therefore $|N[D]| = 3 \left\lfloor \frac{p-2}{6} \right\rfloor + 4 \geq \left\lfloor \frac{p}{2} \right\rfloor$. Since $d(x_i, x_j) \geq 3$,

the induced subgraph $\langle D \rangle$ has only isolates. Hence D is an Independent Majority Dominating set of and

$$i_M(G) \leq |D| = \left\lfloor \frac{p-2}{6} \right\rfloor \quad (3.1)$$

$\square$

Suppose $|D_1| < |D|$ and $|D_1| = |D| - 1$. Then $|N[D_1]| < \left\lfloor \frac{p}{2} \right\rfloor$ and $\langle D_1 \rangle$ has only isolates. But D_1 is not an Independent Majority Dominating set of G and

$$i_M(G) > |D_1| \geq |D| = \left\lfloor \frac{p-2}{6} \right\rfloor \quad (3.2)$$

Combining (3.1) and (3.2), we obtain $i_M(G) = \left\lfloor \frac{p-2}{6} \right\rfloor$. Let

$$D' = \{C_{13}, C_{16}, \dots, C_{1i}, C_{23}, C_{26}, \dots, C_{2j}, C_{33}, C_{36}, \dots, C_{3k}\}$$

such that $d(x_i, x_j) \geq 3$ with $|D'| = \left\lfloor \frac{p}{6} \right\rfloor$. Then

$$|N[D']| = \sum_{x=1}^3 d(C_{xy}) = 3|D'| = 3 \left\lfloor \frac{p}{6} \right\rfloor \geq \left\lfloor \frac{p}{2} \right\rfloor.$$

Also, $D' \subseteq V - D$ and $\langle D' \rangle$ has only isolates. Then

$$i_M^{-1}(G) \leq |D'| = \left\lceil \left\lfloor \frac{p}{6} \right\rfloor \right\rceil \quad (3.3)$$

Suppose $|D'_1| < |D'|$ and $|D'_1| = |D'| - 1$. Apply the above argument, we obtain $|N[D'_1]| < \left\lfloor \frac{p}{2} \right\rfloor$ and $D'_1 \subseteq V - D_1$. But D'_1 is not an inverse independent majority dominating set of G . Hence $i_M^{-1}(G) > |D'_1|$ and

$$i_M^{-1}(G) \geq |D'| = \left\lfloor \frac{p}{6} \right\rfloor \quad (3.4)$$

From the results (3.3) and (3.4), $i_M^{-1}(G) = \left\lfloor \frac{p}{6} \right\rfloor$. Since $i_M(G) = \left\lfloor \frac{p-2}{6} \right\rfloor$ and $i_M^{-1}(G) = \left\lfloor \frac{p}{6} \right\rfloor$, $i_M(G) < i_M^{-1}(G)$.

Theorem 3.3. Let $G = D_t(7)$ be a windmill graph with $t \geq 2$ the number of 7-cycles. Then

(i)

$$i_M(G) = \left\lfloor \frac{p}{6} \right\rfloor - \left\lfloor \frac{\Delta+1}{3} \right\rfloor, \text{ if } p \equiv 1, 2 \left(\text{mod } \left(\frac{\Delta+1}{3} \right) \right) \\ = \left\lfloor \frac{p}{6} \right\rfloor - \left\lfloor \frac{\Delta+1}{3} \right\rfloor + 1, \text{ if } p \equiv 0 \left(\text{mod } \left(\frac{\Delta+1}{3} \right) \right)$$

(ii) $i_M^{-1}(G) = \left\lfloor \frac{p}{6} \right\rfloor$.

(iii) $i_M(G) < i_M^{-1}(G)$

Proof. By the same argument of Proposition (3.3), the results (i) and (ii) are obtained. Then from (ii) and (iii) it implies that $i_M(G) < i_M^{-1}(G)$. $\square$



4. Characterisation Theorems on Inverse Independent Majority Dominating Set.

Theorem 4.1. Let D be a i_M -set of a connected graph G . Then $i_M^{-1}(G) = 1$ if and only if the set $(V - D)$ contains at least one majority dominating vertex of G .

Proof. Let D be a i_M -set of a connected graph G . Let $i_M^{-1}(G) = 1$. Then $D' = \{u\}$ is an inverse independent majority dominating set of G and $D' \subseteq (V - D)$. Since D' is an inverse independent majority dominating set, $|N[u]| \geq \lceil \frac{p}{2} \rceil$. It implies that the degree a vertex "u" such that $d(u) \geq \lceil \frac{p}{2} \rceil - 1$ and u is a majority dominating vertex of G such that $u \in V - D$. Hence $(V - D)$ contains the majority dominating vertex 'u' of G . Suppose $(V - D)$ contains more than one majority Dominating vertex in $V(G)$, then one majority Dominating vertex is enough to find an inverse independent majority dominating set for G . Thus the set $(V - D)$ contains at least one majority dominating vertex of G . The converse is obvious. $\square$

Theorem 4.2. Let D be a i_M -set of a connected graph G . Then $i_M^{-1}(G) \geq 2$ if and only if all vertices u_i are of degree such that $d(u_i) < \lceil \frac{p}{2} \rceil - 2$, for every i and $u_i \in (V - D)$.

Proof. Let $i_M^{-1}(G) \geq 2$. Let D be a i_M -set of G . Then the inverse independent majority dominating set D' which contains more than one vertex. Suppose $D' = \{u_1, u_2\}$ such that $|N[D']| \geq \lceil \frac{p}{2} \rceil$ and $D' \subseteq (V - D)$. Then $\lceil \frac{p}{2} \rceil \leq |N[D']| = \Sigma[d(u_1) + d(u_2)] + |D'|$. It implies that either $d(u_1) < \lceil \frac{p}{2} \rceil - 2$ and $d(u_2) \geq 1$ or vice versa. Since $D' \subseteq (V - D)$, all vertices u_i with degree $d(u_i) < \lceil \frac{p}{2} \rceil - 2$. Conversely, if the set $(V - D)$ consists of the vertices with the degree $d(u_i) < \lceil \frac{p}{2} \rceil - 1$, then any single vertex u_i could not form an inverse independent majority dominating set for G . Hence $i_M^{-1}(G) \geq 2$. $\square$

Theorem 4.3. $i_M^{-1}(G) = i^{-1}(G)$ if and only if the graph G has at least two full degree vertices.

Proof. Assume that

$$i_M^{-1}(G) = i^{-1}(G) \quad (4.1)$$

Let D and D' be an independent dominating set and an inverse independent dominating set of G . Suppose $\Delta(G) \neq p - 1$ then $i^{-1}(G) \neq 1$ and $i^{-1}(G) = 9 = |D'| \geq 2$ is an inverse independent domination number of G . Let $D' = D_1 \cup D_2$ where $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$ such that $D_1 \cap D_2 = \emptyset$. Since $|N[D']| = \phi$ and $D_1 \cap D_2 = \emptyset$, $|N[D_1]| \geq \lceil \frac{p}{2} \rceil$ or $|N[D_2]| \geq \lceil \frac{p}{2} \rceil$. Hence D_1 or D_2 is an inverse independent majority dominating set of G and $i_M^{-1}(G) \leq |D_1|$ or $i_M^{-1}(G) \leq |D_2|$. Since $D_1 \neq \emptyset$, and $D_2 \neq \emptyset$, $i_M^{-1}(G) < |D'| = i^{-1}(G)$, which is a contradiction to the assumption (1). Hence $\Delta(G) = p - 1$. It implies that the graph G contains at least one full degree vertex "u" with $d(u) = p - 1$. Suppose G has exactly one vertex u with $d(u) = p - 1$, then $i(G) = |\{u\}| = 1$ and $i_M(G) = |\{u\}| = 1$ but $i^{-1}(G) \geq 2$ and $i_M^{-1}(G) \geq 2$. Suppose the graph G contains any vertex v of degree $\lceil \frac{p}{2} \rceil - 1 \leq d(v) < p - 1$, then

$D_1 = \{u\} \subseteq V - D$ such that $|N[D_1]| \geq \lceil \frac{p}{2} \rceil$ and D_1 is an inverse majority dominating set of G . It implies that $i_M^{-1}(G) = |D_1| = 1$ $i^{-1}(G) > |D_1| = 1$. Hence $i_M^{-1}(G) < i^{-1}(G)$ contradiction to the assumption(1). Hence the graph G has at least two full degree vertices. The converse is obvious. $\square$

Result 4.4. Given any positive integer $k > 1$, there exists a graph G such that $i_M^{-1}(G) = 1$.

Proof. Let $k \geq 2$ be any positive integer. Then there is a complete graph $G = K_k$, with k -vertices. By the result (3.1)(1), $i_M^{-1}(G) = 1$, for any $k \geq 2$. $\square$

Result 4.5. Given any positive integer $k \geq 1$. There exists a graph G with $p = 2k + 1$ for which $i^{-1}(G) - i_M^{-1}(G) = k$.

Proof. Let $k \geq 1$ be any given integer. Let $G = K_{1,p-1}$, p is odd. By the result (3.1)(2) $i_M^{-1}(G) = \frac{p-1}{2}$ and $i^{-1}(G) = p - 1$. If $p = 2k + 1$, $k \geq 1$, Then $i_M^{-1}(G) = \frac{2k}{2} = k$ and $i^{-1}(G) = 2k$. Hence $i^{-1}(G) - i_M^{-1}(G) = 2k - k = k$. $\square$

Theorem 4.6. Let D and D' be an Independent majority dominating set and an inverse independent majority dominating set of G . Then its numbers $i_M(G) = i_M^{-1}(G) = 1$ if and only if the graph G contains at least two majority dominating vertices.

Proof. Let $i_M(G) = i_M^{-1}(G) = 1$. Since $i_M(G) = 1$, $D = \{u\}$ such that $|N[u]| \geq \lceil \frac{p}{2} \rceil$ and the graph G contains a vertex u' with $d(u) \geq \lceil \frac{p}{2} \rceil - 1$. Since $i_M^{-1}(G) = 1$, $D' = \{v\} \subseteq V - D$ and $(V - D)$ contains a vertex v with $d(v) \geq \lceil \frac{p}{2} \rceil - 1$. It implies that the vertices u and v are the majority dominating vertices of G . Suppose the graph G contains more than two vertices then $|D| = |D'| = 1$. Hence G contains at least two majority dominating vertices. The converse is obvious. $\square$

5. Bounds of the Inverse Independent Majority Domination Number

Proposition 5.1. For any graph G , $1 \leq i_M^{-1}(G) \leq \frac{p}{2}$. The bound is sharp when $G = \overline{K_p}$, p is even.

Proof. If a graph G has all vertices are of full degree then each single vertex forms an independent majority dominating set and an inverse independent majority dominating set for G . Hence $i_M(G) = i_M^{-1}(G) = 1$. Suppose the graph G contains all vertices are of degree $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$, then each single vertex forms an independent majority dominating set and an inverse independent majority dominating set for G . Hence $i_M(G) = i_M^{-1}(G) = 1$.

If the graph G is minimally connected graph then $\delta(G) \geq 1$ and $\Delta(G) \leq p - 1$. Then $i_M(G) = 1$ and $i_M^{-1}(G) \geq 2$. Suppose the graph G is disconnected without isolates, then $i_M(G) \geq 2$ and $i_M^{-1}(G) \geq 2$. If the graph G has components with isolates then i_M -set D $i_M(G) < (\frac{p}{2})$ and $i_M^{-1}(G) < (\frac{p}{2})$. The bound is sharp. Let $G = \overline{K_p}$, p is even then $i_M(G) = \frac{p}{2}$ and $i_M^{-1}(G) = \frac{p}{2}$. Hence $1 \leq i_M^{-1}(G) \leq (\frac{p}{2})$. $\square$



Proposition 5.2. Let $G = (V, E)$ be any graph with p vertices. Then $i_M^{-1}(G) \leq \left\lceil \frac{i^{-1}(G)}{2} \right\rceil$.

Proof. Let D_M and D be an independent majority dominating set and independent dominating set of G respectively. Suppose $D' \subseteq V - D$ is an inverse independent dominating set of G . Then D' is an i^{-1} -set of G with $|N[D']| = |V(G)|$. Let $D' = D_1 \cup D_2$ where $|D_1| = \left\lfloor \frac{i^{-1}(G)}{2} \right\rfloor$ and $|D_2| = \left\lceil \frac{i^{-1}(G)}{2} \right\rceil$. Now $N[D] = (N[D_1] - N[D_2]) \cup (N[D_2])$ and $|N[D]| = |N[D_1] - N[D_2]| \cup |N[D_2]|$. It implies that either $|N[D_1] - N[D_2]| \geq \left\lceil \frac{p}{2} \right\rceil$ or $|N[D_2]| \geq \left\lceil \frac{p}{2} \right\rceil$. If $|N[D_2]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $D_2 \subseteq V - D_M$ then D_2 is an inverse independent majority dominating set of G . If $|N[D_1] - N[D_2]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $D_1 \subseteq V - D_M$ then D_1 is an inverse independent majority dominating set of G when p is even. Hence $i_M^{-1}(G) \leq |D_2| = \left\lceil \frac{i^{-1}(G)}{2} \right\rceil$. Thus $i_M^{-1}(G) \leq \left\lceil \frac{i^{-1}(G)}{2} \right\rceil$. $\square$

Example 5.3. Let G be a subdivision of star $S(K_{1,10})$ by dividing each edge exactly once. Then $p(G) = 21$ and $q(G) = 20$. The inverse independent domination number $i^{-1}(G) = 11$ and Inverse independent majority domination number $i_M^{-1}(G) = 5$. Hence $i_M^{-1}(G) < \left\lceil \frac{11}{2} \right\rceil = \left\lceil \frac{i^{-1}(G)}{2} \right\rceil$.

Theorem 5.4. Let G be any connected graph with $p \geq 2$ vertices. Then $i_M^{-1}(G) = p - \kappa(G)$ if and only if G is a complete graph of order p where $\kappa(G)$ is a vertex connectivity of G .

Proof. Let G be any connected graph and

$$i_M^{-1}(G) = p - \kappa(G) \quad (5.1)$$

Let D and D' be an independent majority dominating set and an inverse independent majority dominating set of G . Let u be a vertex of G such that $d(u) = \Delta(G) = |N(u)|$. Since $N[V(G) - N(u)] = V(G) - N(u) = D'$, D' is an inverse independent majority dominating set of G . Hence $i_M^{-1}(G) \leq |D'| = |V(G)| - |N(u)| = p - \Delta(G)$. by $p - \kappa(G) = i_M^{-1}(G) \leq p - \Delta(G)$ It implies that $\kappa(G) \geq \Delta(G)$. By known inequality “for any graph G , $\kappa(G) \leq \delta(G) \leq \Delta(G)$,” combining these results, we obtain $\kappa(G) = \delta(G) = \Delta(G)$. Hence G is a regular graph say κ -regular graph.

Let $D = \{u_1\}$ be a γ_M -set of G . Let u_2 be any vertex of κ -regular graph G . Then the vertex u_2 dominates $(\kappa + 1)$ vertices of G . If $(\kappa + 1) < \left\lceil \frac{p}{2} \right\rceil$ then $D_1 = [N[u] \cup S] \subseteq V - D$ is an inverse independent majority dominating set of G where S is a set of $\left[\left\lceil \frac{p}{2} \right\rceil - (\kappa + 1)\right]$ vertices disjoint from the vertices of $N[u]$. Therefore, $i_M^{-1}(G) \leq 1 + \left\lceil \frac{p}{2} \right\rceil - (\kappa + 1) = \left\lceil \frac{p}{2} \right\rceil - \kappa$. Then by (5.1), $p - \kappa(G) \leq \left\lceil \frac{p}{2} \right\rceil - \kappa(G)$, which is impossible. Hence $(\kappa + 1) \geq \left\lceil \frac{p}{2} \right\rceil$ and $(\kappa + 1)$ vertices are dominated by one and only one vertex u'_2 and $D_1 = \{u_2\}$. It implies that $i_M^{-1}(G) = 1$. By (5.1) $p - \kappa(G) = 1 \Rightarrow \kappa(G) = (p - 1)$ is a vertex connectivity of G . Thus, the corresponding graph G is a complete graph of order p . Conversely, let $G = K_p$ then the vertex connectivity of

$$G = \kappa(G) = p - 1 \quad (5.2)$$

The independent majority dominating set D and inverse independent majority dominating set D' set of a complete graph G is $D = \{u_1\}$ and $D' = \{u_2\}$ respectively, for $u_1, u_2 \in V(G)$ and $u_2 \subseteq V - D$. It implies that $i_M^{-1}(G) = |D'| = 1$ and $i_M^{-1}(G) = 1 = p - (p - 1)$. Therefore, By (2), $i_M^{-1}(G) = p - \kappa(G)$. Hence $i_M^{-1}(G) = p - \kappa(G)$ if and only if $G = K_p$. $\square$

Proposition 5.5. For any graph G with p vertices, $i_M^{-1}(G) = \gamma_M^{-1}(G) = 1$ if and only if the graph G contains atleast two vertices of degree $\geq \left\lceil \frac{p}{2} \right\rceil - 1$.

Proof. Let G be any graph with p vertices Let

$$i_M^{-1}(G) = \gamma_M^{-1}(G) = 1 \quad (5.3)$$

Then $i_M(G) = \gamma_M(G) = 1$. It implies that $D = \{u_1\}$ is an Independent majority dominating set and majority dominating set of G and $|N[D]| \geq \left\lceil \frac{p}{2} \right\rceil$. Hence the vertex u_1 has degree $d(u_1) \geq \left\lceil \frac{p}{2} \right\rceil - 1$. By the hypothesis (1), $D' = \{u_2\}$ is an Inverse independent majority dominating set and an inverse majority dominating set of G with $|N[D']| \geq \left\lceil \frac{p}{2} \right\rceil$. It implies that the vertex u_2 is of degree $d(u_2) \geq \left\lceil \frac{p}{2} \right\rceil - 1$. Hence there are two vertices u_1 and u_2 in G with $d(u_i) \geq \left\lceil \frac{p}{2} \right\rceil - 1$, $i = 1, 2$. These two vertices are enough to obtain the value $i_M^{-1}(G) = 1 = \gamma_M^{-1}(G)$. Thus, the graph G contains atleast two vertices of degree $\geq \left\lceil \frac{p}{2} \right\rceil - 1$. The converse is obvious. $\square$

Proposition 5.6. If a graph G is a Path or a Cycle $p \geq 7$ with $i_M^{-1}(G) \geq 2$ and $\gamma_M^{-1}(G) \geq 2$, then $i_M^{-1}(G) = \gamma_M^{-1}(G)$.

Proof. Let $G = P_p$ or $G = C_p$, $p \geq 7$. Then the inverse independent majority domination number $i_M^{-1}(G) \geq 2$ and the Inverse majority domination number $\gamma_M^{-1}(G) \geq 2$. Let D_1 and D_2 be the Independent majority dominating set and majority dominating set of G . By the result (3.1)(3), $i_M(G) = \left\lceil \frac{p}{6} \right\rceil$. Since $|D_1| \geq 2$, $D_1 = \{u_i, u_j\}$ such that $d(u_i, u_j) \geq 3$ and $D'_1 = \{(u_{i+1}, u_{j+1})\} \subseteq V - D_1$ such that $d(u_{i+1}, u_{j+1}) \geq 3$ and Then $|N[D'_1]| \geq \left\lceil \frac{p}{2} \right\rceil$ is an inverse independent majority dominating set of G . By the proposition (3.1), [1], $i_M^{-1}(G) = \left\lceil \frac{p}{6} \right\rceil = |D'_1|$ $p \geq 7$. Similarly since $|D_2| \geq 2$, D_2 and D'_2 are taken by the above argument in G . Then By the result (1.2)(vii), $\gamma_M^{-1}(G) = \left\lceil \frac{p}{6} \right\rceil$, $p \geq 7$. Hence $i_M^{-1}(G) = \gamma_M^{-1}(G) = \left\lceil \frac{p}{6} \right\rceil$, $p \geq 7$. $\square$

Proposition 5.7. For a tree T with p vertices, $\left\lfloor \frac{p}{6} \right\rfloor \leq i_M^{-1}(T) \leq \left\lceil \frac{p}{2} \right\rceil - 1$. These bounds are sharp.

Proof. This result is proved by induction on the number of pendants “ e ”. Since every tree has $e \geq 2$ pendants, when $e = 2$ then $T = P_p$, a path of p vertices. By the result (3.1)[3], $i_M^{-1}(P_p) = \left\lceil \frac{p}{6} \right\rceil$. If $e = 3$ then G is any connected graph with three pendants. The graph G is either $T = K_{1,3}$, a star or $T = D_{1,2}$, a double star or a Caterpillar with $e=3$ pendants. Then If $T = K_{1,3}$ then $i_M^{-1}(G) = 1 = \left(\frac{p}{2}\right) - 1$, $T = D_{1,2}$ then, $i_M^{-1}(G) = 1 < \left\lceil \frac{p}{2} \right\rceil - 1$ and $i_M^{-1}(G) = 2 < \left\lceil \frac{p}{2} \right\rceil - 1$, if T is Caterpillar. Suppose the tree T takes the Caterpillar structure with $(p - e)$ intermediate vertices and $e = 3$ pendants. Then



$i_M(G) = i_M^{-1}(G) = \left\lfloor \frac{p}{6} \right\rfloor$. This is true for $e = 2, 3, 4, \dots, (p-2)$ pendants. Then $i_M^{-1}(T) \geq \left\lfloor \frac{p}{1} \right\rfloor$ and $i^{-1}(T) \leq \left\lfloor \frac{p}{2} \right\rfloor - 1$. If $e = p-1$ pendants then the graph T becomes $T = K_{1,p-1}$, a star. By the result (3.1)(2), $i_M^{-1}(K_{1,p-1}) = \left\lfloor \frac{p-1}{2} \right\rfloor$. If p is odd then $i_M^{-1}(K_{1,p-1}) = \left\lfloor \frac{p}{2} \right\rfloor - 1$. Hence for any tree T with p vertices, $\left\lfloor \frac{p}{6} \right\rfloor \leq i_M^{-1}(T) \leq \left\lfloor \frac{p}{2} \right\rfloor - 1$ $\square$

Theorem 5.8. For any graph with p vertices and

$$\Delta(G), i_M^{-1}(G) \geq \left\lceil \frac{p}{2(\Delta+1)} \right\rceil$$

The bound is sharp.

Proof. Let $D = \{v_2, v_4, \dots, v_{i_M}\}$ be a i_M -set of a graph G . Then $|N[D]| \geq \left\lfloor \frac{p}{2} \right\rfloor$ and $\langle D \rangle$ has only isolates such that $d(v_i, v_j) \geq 2$, for $v_i, v_j \in D, i \neq j$. Let $D' = \{v_3, v_5, \dots, v_t\} \subseteq V - D$ be a i_M^{-1} -set of G with $|D'| = t$. Then $|N[D']| \geq \left\lfloor \frac{p}{2} \right\rfloor$ and D' contains independent vertices. Also each vertex of D' is adjacent to the vertex of D and D' contains the maximum degree vertex $\Delta(G)$. Then $|N[D']| = \sum_{i=1}^t d(v_i) + t$ i.e. $\left\lfloor \frac{p}{2} \right\rfloor \leq |N[D']| = t(\Delta(G)) + 1$. Therefore $\left\lfloor \frac{p}{2} \right\rfloor \leq |D'|(\Delta + 1)$. It implies that $\left\lfloor \frac{p}{2} \right\rfloor \leq i_M^{-1}(G)(\Delta + 1)$. Hence $i_M^{-1}(G) \geq \frac{\left\lfloor \frac{p}{2} \right\rfloor}{(\Delta + 1)} = \left\lceil \frac{p}{2(\Delta + 1)} \right\rceil$. The bound is sharp for $G = K_p$ a complete graph, and a path $P_p, p \geq 2$. By the result (3.1)[1], $i_M^{-1}(K_p) = 1 = \left\lceil \frac{p}{2(\Delta + 1)} \right\rceil$ where $\Delta = p-1$ and $i_M^{-1}(P_p) = \left\lfloor \frac{p}{6} \right\rfloor = \left\lceil \frac{p}{2(\Delta + 1)} \right\rceil$ where $\Delta = 2$. $\square$

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