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PAPER

Quantum solvability of a nonlinear δ-type mass profile system: coupling constant quantization

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Abstract

In this paper, we discuss the quantum dynamics of a nonlinear system that admits temporally localized solutions at the classical level. We consider a general ordered position-dependent mass Hamiltonian in which the ordering parameters of the mass term are treated as arbitrary. The mass function here is singular at the origin. We observe that the quantum system admits bounded solutions but importantly the coupling parameter of the system gets quantized which has also been confirmed by the semiclassical study as well.

1. Introduction

Several studies on physical systems with position-dependent effective mass have emerged in recent years due to their wide applications in the study of electronic properties of semiconductors [1], inhomogeneous crystals, quantum dots, quantum liquids [2–4] and so on. The time-independent Schrödinger equation gets generalized when the effective mass depends on the position and it is solved using both numerical and analytical techniques. Though difficult, it is of general interest to get exact solutions for such position-dependent mass Schrödinger equation (PDMSE) for specific potentials. Certain nonlinear systems, specifically quadratic Liénard type nonlinear oscillators, are found to possess position-dependent mass Hamiltonians. For example, Mathews-Lakshmanan oscillator and Higgs oscillator are considered to describe the dynamics of harmonic oscillators in curved space [5, 6]. Different studies have been carried out on these systems in the literature since their introduction in the literature[7–10]. While quantizing these position-dependent mass (PDM) quantum systems, one should consider (i) the possible choices of ordering between momentum and mass operators in their kinetic energy term and (ii) appropriate modification on the boundary conditions. The ordering may lead to Hermitian or non-Hermitian Hamiltonians. The most general ordering form had been introduced by Trabelsi *et al* [11]. In a recent study, it has been shown that the Mathews-Lakshmanan oscillator is exactly solvable for the general ordered form [12]. Motivated by the problem of ordering ambiguity of position-dependent mass Hamiltonian, two of the present authors studied the quantum dynamics of the Higgs oscillator and a k-dependent nonpolynomial oscillator by considering the general ordered form introduced by Trabelsi *et al.*, in [13].

Classically both the systems, Mathews-Lakshmanan oscillator and Higgs oscillator admit non-isochronous solutions. It is recently reported that certain quadratic Liénard type nonlinear oscillators can possess isochronous solutions as well [14]. We solved these nonlinear oscillators quantum mechanically and discussed their exact and quasi-exact solvable nature [15]. It is also worth mentioning that one can also derive a conservative description for the nonlinear oscillators of position dependent linearly damped Liénard type systems classically. Such studies have been carried out on generalized modified Emden equation in [16, 17]. The associated Hamiltonians obtained are non-standard. The Hamiltonian description for such a nonlinear oscillator, governed by a modified Emden equation with certain constraints on its parameters, paves a way to solve the system quantum mechanically. It is also shown that the Hamiltonian is invariant under combined

coordinate reflection and time reversal transformation and exhibits linear energy spectrum as that of the standard harmonic oscillator [18].

Based on all these studies, we are here interested to study the quantum dynamics of a quadratic Liénard type nonlinear oscillator which shows a special behavior at its classical level. In this work, we consider such a type of nonlinear system that exhibits temporally localized solutions [14]. It is observed that the associated Hamiltonian is of the form of position-dependent mass type. The mass profile has a resemblance to a δ-function form. A related model that has been used for describing electron systems in δ-doped semiconductors in the Thomas-Fermi field has been shown to be quantum mechanically exactly solvable [19]. In our work, we use a general ordering procedure to write down the appropriate quantum Hamiltonian in order to solve the underlying generalized Schrödinger equation. We also study the role of ordering parameters on obtaining well defined eigenfunctions as the mass function is not a continuous one here.

In this paper, we discuss the classical solvability of the system in section 2. In section 3, we implement a semiclassical quantization rule to analyze the quantum solvability of the system and find that the coupling parameter of the system gets quantized. The system is observed as a position-dependent mass one. We consider the generalized Schrödinger equation corresponding to a non-Hermitian ordered form to analyze the quantum solvability of the system which is discussed in section 4. Finally, we summarize our results.

2. A δ-type mass system and its classical dynamics

Consider a Hamiltonian of the form studied by Tiwari *et al* [14],

$$H = \frac{x^4\, p^2}{4} + \lambda x^2 \tag{1}$$

and the corresponding Lagrangian is

$$L = \frac{\dot{x}^2}{x^4} - \lambda x^2. \tag{2}$$

It is of the position-dependent mass form, $H = \frac{p^2}{2\, m(x)} + V(x)$, where the mass profile is of the form

$$m(x) = \frac{2}{x^4} \qquad \text{and} \qquad V(x) = \lambda x^2. \tag{3}$$

Here the mass is singular at x=0.

The equation of motion for the Hamiltonian H in (1) reads as

$$\ddot{x} - \frac{2}{x}\,\dot{x}^2 + \lambda x^5 = 0. \tag{4}$$

It can be integrated once on using the integrating factor, say $\frac{2\dot{x}}{x^4}$, as

$$\frac{\dot{x}^2}{x^4} + \lambda x^2 = C_1, \tag{5}$$

where C_1 is an integration constant. Integrating this equation (5) once more, we find that equation (4) admits the general solution,

$$x(t) = \frac{1}{\sqrt{\frac{\lambda}{C_1} + (C_2 + \sqrt{C_1}\, t)^2}}, \tag{6}$$

where C_2 is the second integration constant. For $\lambda > 0$, we have a temporally localized solution. And for $\lambda < 0$, we have a singular solution when $t = \frac{1}{\sqrt{C_1}}\left(\sqrt{\frac{|\lambda|}{C_1}} - C_2\right)$ in which case we consider that C_1 and C_2 are positive. The plot of $x(t)$ against t is depicted in figure 1 (*i*) for certain values of C_1, C_2 and λ. The figure 1 (*ii*) depicts the contour plot of $x(t)$ given in equation (6) for various values of λ with $C_1 = 1$, and $C_2 = -5$.

3. Semiclassical quantization

To understand the possibility of quantization of the above type of position-dependent mass system, we first apply the semiclassical quantization procedure to the system. The standard leading order WKB quantization condition for the potential having two turning points is [20],

$$\int_{x_1}^{x_2} p dx = \left(n + \frac{1}{2}\right)\hbar\, \pi, \qquad n = 0,\, 1,\, 2,..., \tag{7}$$

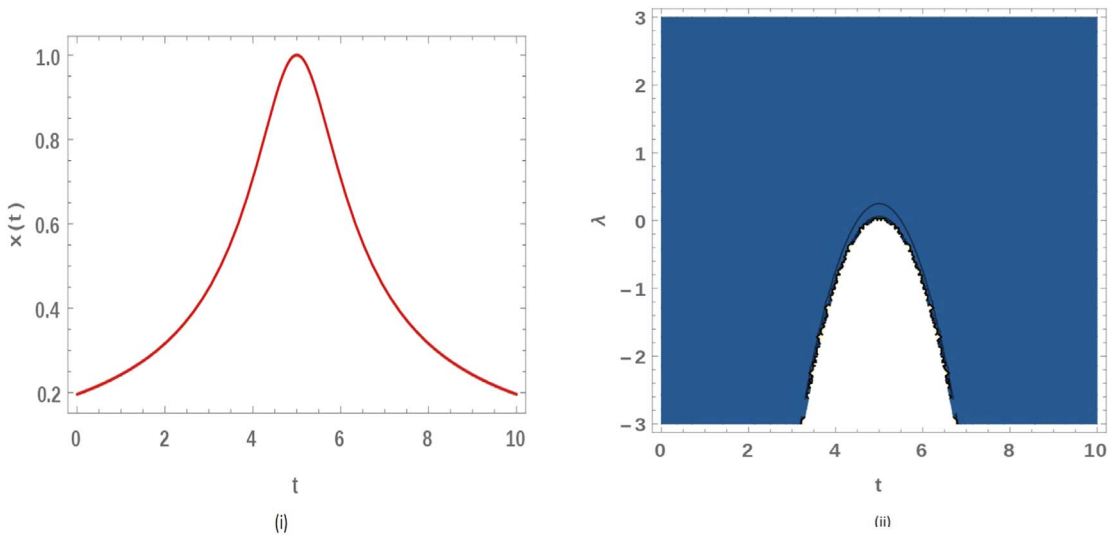

Figure 1. (i) The plot of $x(t)$ in equation (6) for $C_1 = 1,\ C_2 = -5$ and $\lambda = 1$ and (ii) the contour plot of $x(t)$ given in equation (6) for various values of λ with $C_1 = 1$, and $C_2 = -5$, where the blue shaded region denotes the possible values of λ for which the solutions are well defined and the white region denotes the values of λ for which the solutions are singular.

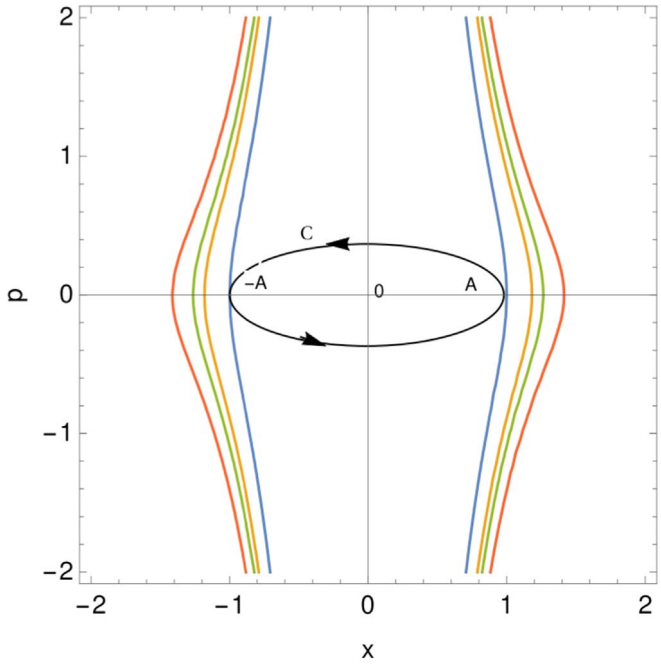

Figure 2. The phase portrait of Hamiltonian (1) for different values of energy $E = 0.5, 0.7, 0.8,\ 1$ with $\lambda = 0.5$.

where x_1 and x_2 are the classical turning points and the conjugate momentum, $p = \sqrt{2m(x)(E - V(x))}$. Here, $\hbar = \frac{h}{2\pi}$, where h is Planck's constant. From the Hamiltonian (1), with $H = E$, one can express the momentum as

$$p = \sqrt{\frac{4E}{x^4} - \frac{4\lambda}{x^2}}. \tag{8}$$

At the turning points, say $(x_1,\ x_2) = (-A, A)$, the momentum is zero, which is shown in the figure 2. Hence, from (1), the total energy, $H = E = \lambda A^2$ and the integral (7) becomes,

$$2\sqrt{\lambda}\int_{-A}^{A}\frac{\sqrt{A^2 - x^2}}{x^2}dx = \left(n + \frac{1}{2}\right)\hbar\ \pi, \quad n = 0,\ 1,\ 2,.... \tag{9}$$

To evaluate (9), consider the integral

$$I = \int_{-A}^{A} \frac{\sqrt{A^2 - x^2}}{x^2} dx. \tag{10}$$

One can also use the classical solution $x(t)$, (vide (6)) and evaluate the closed integral around contour C (given in figure 2) in the modified Bohr–Sommerfeld quantization rule [21],

$$\oint p dx = \left(n + \frac{1}{2}\right) h. \tag{11}$$

Here, the momentum, $p(t)$, takes the form as

$$p(t) = \frac{2\dot{x}(t)}{x(t)^4} = -2\sqrt{C_1}(C_2 + \sqrt{C_1}\, t)\sqrt{\frac{\lambda}{C_1} + (C_2 + \sqrt{C_1}\, t)^2}. \tag{12}$$

We integrate the integral (10) by considering $u = \sqrt{A^2 - x^2}$ and $dv = \frac{1}{x^2} dx$ and get

$$\begin{aligned} I &= \frac{\sqrt{A^2 - x^2}}{x} \Bigg|_{-A}^{A} - \int_{-A}^{A} \frac{dx}{\sqrt{A^2 - x^2}}, \\ &= 0 - \left[\arcsin\left(\frac{x}{A}\right)\right]_{-A}^{A}, \\ I &= -\pi. \end{aligned} \tag{13}$$

On substituting the integral (13) in (9), one obtains the following relation on the coupling parameter, λ, as

$$\lambda = \left(n + \frac{1}{2}\right)^2 \frac{\hbar^2}{4}, \qquad n = 0,\ 1,\ 2,\ 3,.... \tag{14}$$

Hence, the coupling parameter λ gets related with the quantum number n, as in (14).

While studying the quantum dynamics of the above type of position-dependent mass system (1) with a singular mass function, we meet with two difficulties: (i) how to define the configuration space and (ii) how to ensure the continuity of the eigenfunctions of the corresponding Schrödinger equation? We proceed to incorporate these two aspects in our further study as indicated below.

4. Quantization: general ordered form of Hamiltonian

We now consider the most general form of the associated Hamiltonian operator that provides a complete classification of Hermitian and non-Hermitian orderings [11],

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{N} w_i m^{\alpha_i} \hat{p} m^{\beta_i} \hat{p} m^{\gamma_i} + V(x), \tag{15}$$

where N is an arbitrary positive integer and $\hat{p}$ is the one dimensional momentum operator. The ordering parameters should satisfy the constraints, $\alpha_i + \beta_i + \gamma_i = -1,\ i = 1, 2, 3,...N$, and w_i's are real weights which are summed to be 1. The above form globally connects all the Hermitian orderings and also provides a complete classification of Hermitian and non-Hermitian orderings [11]. The operator $\hat{H}$ in (15) possesses $2N$ free ordering parameters, after taking into account the above constraints.

The corresponding Hamiltonian for the potential V can be written as

$$\hat{H} = \frac{1}{2}\hat{p}\frac{1}{m}\hat{p} + (\bar{\gamma} - \bar{\alpha})\frac{i\hbar}{2}\frac{d}{dx}\left(\frac{1}{m}\right)\hat{p} + \frac{\hbar^2}{2}\left[\bar{\gamma}\frac{d^2}{dx^2}\left(\frac{1}{m}\right) + \overline{\alpha\gamma}\left(\frac{m'^2}{m^3}\right)\right] + V, \tag{16}$$

where $\hat{p} = -i\hbar\frac{d}{dx}$. In (16), the over bar over the parameters represent their total value, $\bar{X} = \sum_i^N w_i X_i$.

The study on the effective-mass Hamiltonians for abrupt heterojunctions indicates that the single-term ordering forms of kinetic energy operator are viable candidates that ensure continuity of the associated matching conditions [22]. As the mass $m(x)$ is singular at $x = 0$, we use the single term of the general ordered form of the Hamiltonian as

$$\hat{H} = \frac{1}{2} m^{\alpha_1} \hat{p} m^{\beta_1} \hat{p} m^{\gamma_1} + V(x), \qquad \alpha_1 + \beta_1 + \gamma_1 = -1. \tag{17}$$

Here, we are considering non-Hermitian ordered form of the Hamiltonian (16) as the non-Hermitian ordered form can be related with the Hermitian ordered form through similarity transformation [23] as

$$\hat{H}_{her} = m^{\eta}\hat{H}\, m^{-\eta}, \qquad 2\,\eta = \gamma_1 - \alpha_1, . \tag{18}$$

Consequently, for (18) we have

$$\hat{H}_{her} = \frac{1}{2} m^{\frac{\gamma_1+\alpha_1}{2}} \hat{p} m^{\beta_1} \hat{p} m^{\frac{\gamma_1+\alpha_1}{2}} + V(x). \tag{19}$$

As the non-Hermitian ordered form (16) is being related with the Hermitian ordered form through similarity transformation (18), we use the non-Hermitian ordered form of the Hamiltonian in this present work and analyze the possibility of obtaining a complete set of solutions of the operator (16).

The time-independent Schrödinger equation for the non-Hermitian ordered Hamiltonian (17), $\hat{H}\psi = E\psi$, can be written as

$$\psi'' + (\gamma_1 - \alpha_1 - 1)\frac{m'}{m}\psi' + \left(\gamma_1 \frac{m''}{m} - (\alpha_1\gamma_1 + 2\gamma_1)\frac{m'^2}{m^2}\right)\psi + \frac{2m}{\hbar^2}(E - V(x))\psi = 0, \tag{20}$$

where $' = \frac{d}{dx}$.

As the above Hamiltonian depicts the dynamics of the one dimensional potential (1), we use the generalized position-dependent mass Schrödinger equation resulting from the non-Hermitian ordering (17), to study the solvability of the system (1). It results that

$$\psi'' + \frac{4(1 + \alpha_1 - \gamma_1)}{x}\psi' + \left[\frac{4\,E}{\hbar^2\, x^4} - \frac{16\alpha_1\gamma_1 + 12\gamma_1 + \frac{4\lambda}{\hbar^2}}{x^2}\right]\psi = 0. \tag{21}$$

By using the transformation, $\psi(x) = x^d\,\phi(x)$, where d is a parameter to be determined, we can reduce the equation (21) to the form

$$\phi'' + \frac{2d + 4(1 + \alpha_1 - \gamma_1)}{x}\phi' + \left[\frac{d(d + 3 + 4(\alpha_1 - \gamma_1)) - \left(16\alpha_1\gamma_1 + 12\gamma_1 + \frac{4\lambda}{\hbar^2}\right)}{x^2} + \frac{4\,E}{\hbar^2\, x^4}\right]\phi = 0. \tag{22}$$

We further use the transformation, $g(x) = \frac{1}{2\,x}$, so that equation (22) can be rewritten as

$$g^2\phi_{gg} + 2g[(2\gamma_1 - 2\alpha_1 - 1 - d)]\,\phi_g + [d(d + 3 + 4(\alpha_1 - \gamma_1)) - \left(16\alpha_1\gamma_1 + 12\gamma_1 + \frac{4\lambda}{\hbar^2}\right) + \frac{16\,E}{\hbar^2}\,g^2]\phi = 0, \tag{23}$$

where $\phi_g = \frac{d\,\phi}{dg}$.

In order to map equation (23) to the known form, we again use the transformation,

$$\tau = \frac{4\sqrt{E}}{\hbar}\,g, \tag{24}$$

with

$$d = 2\gamma_1 - 2\alpha_1 - \frac{3}{2}, \tag{25}$$

to transform equation (23) as

$$\tau^2\phi_{\tau\tau} + \tau\,\phi_\tau + (\tau^2 - \nu^2)\phi = 0, \tag{26}$$

where

$$\nu^2 = \left(2\alpha_1 + 2\gamma_1 + \frac{3}{2}\right)^2 + \frac{4\lambda}{\hbar^2}. \tag{27}$$

Equation (23) is of the form of Bessel's differential equation. Hence, the corresponding general solution is

$$\phi_\nu(\tau) = CJ_\nu(\tau) + DY_\nu(\tau), \tag{28}$$

where $J_\nu(\tau)$ and $Y_\nu(\tau)$ are the first and second kind of Bessel polynomials [24] and C and D are arbitrary constants. Now we can obtain the general solution for the equation (21) for the region $x \in (0, \infty)$ as

$$\psi_\nu(x) = \psi_\nu^{(+)}(x) = x^d\left[CJ_\nu\left(\frac{2\sqrt{E}}{\hbar\, x}\right) + DY_\nu\left(\frac{2\sqrt{E}}{\hbar\, x}\right)\right], \quad x \in (0, \infty). \tag{29}$$

And we can write down the general solution for the region $x < 0$, as

$$\psi_{\nu}^{(-)}(x) = (-|x|)^{d}\left[\widetilde{C}J_{\nu}\left(\frac{2\sqrt{E}}{\hbar\, x}\right) + \widetilde{D}Y_{\nu}\left(\frac{2\sqrt{E}}{\hbar\, x}\right)\right], \quad x \in (-\infty,\, 0), \tag{30}$$

where $\tilde{C}$ and $\tilde{D}$ are arbitrary constants and d (vide equation (25)).

Here we are interested to derive bounded solutions for the system (1) and so analyze the boundary conditions for the Bessel polynomials.

By choosing $d = 2\gamma_1 - 2\alpha_1 - 1$, equation (23) can now be reduced to the constant mass Schrödinger equation as

$$\phi_{gg} + \left[\frac{16\, E}{\hbar^2} - \frac{\frac{4\lambda}{\hbar^2} + (2\alpha_1 + 2\gamma_1 + 2)(2\alpha_1 + 2\gamma_1 + 1))}{g^2}\right]\phi = 0. \tag{31}$$

This equation can also be deduced by means of a point canonical transformation method, which relates the PDM Schrödinger equation with the canonical form of constant mass Schrödinger equation and it is a widely used method in solving position-dependent mass Schrödinger equations [25]. The potential of (31), $U(g) \propto \frac{1}{g^2}$, is similar to the effective potential that arose while studying the Efimov effect in the quantum three body system that describes the dynamics of two heavy particles interacting through a light particle [26].

4.1. Boundary conditions

In equation (29), when $x \to \infty$ the polynomials J_{ν} become zero for positive values of ν and become complex infinity for $\nu < 0$. And Y_{ν} becomes ∞ provided $\nu \neq 0$. Hence, we take $D = 0$ and $\nu > 0$ to get the solutions which are bounded as $x \to \infty$.

To proceed further, we now expand (29) around $x = \infty$,

$$\psi_{\nu}^{(+)}(x) = C\, x^{d}\, J_{\nu}\left(\frac{2\sqrt{E}}{\hbar\, x}\right) \approx_{x\to\infty} \frac{C}{\Gamma(\nu+1)}\left(\frac{E}{\hbar^2}\right)^{\nu/2} x^{d-\nu}. \tag{32}$$

The boundary condition on $\psi_{\nu}^{(+)}(x)$ at $x \to \infty$ fixes a constraint $d - \nu < 0$. As $\nu > 0$, the value of d fixes the lower bound of ν.

Secondly we analyze the bounded nature of $\psi_{\nu}^{(+)}(x)$ at x=0. When x approaches zero, $J_{\nu}\left(\frac{2\sqrt{E}}{\hbar\, x}\right)$ oscillates vastly as $\frac{2\sqrt{E}}{\hbar\, x}$ goes to ∞ . On expanding near zero, we obtain

$$\psi_{\nu}^{(+)}(x) = C\, x^{d}\, J_{\nu}\left(\frac{2\sqrt{E}}{\hbar\, x}\right) \approx_{x->0} C\sqrt{\frac{\hbar\, x}{\pi\,\sqrt{E}}}\, x^{d}\cos\left(\frac{2\sqrt{E}}{\hbar\, x} - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right). \tag{33}$$

Here we use the squeeze theorem which states that if a function $g(x)$ is squeezed between the functions $f(x)$ and $h(x)$ near a point a and if $f(x)$ and $h(x)$ have the same limit L at the point a, then $g(x)$ is trapped and will be forced to have also the same limit L at a [27]. Since near $x = 0$, the cosine function is not well defined as $-1 \leqslant \cos\left(\frac{2\sqrt{E}}{\hbar\, x} - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right) \leqslant 1$, in accordance with the squeeze theorem, if we consider the functions, $f(x) = \sqrt{x}$ and $h(x) = -\sqrt{x}$, then the $\lim_{x->0}\sqrt{x} = 0$ makes $\lim_{x->0}\sqrt{x}\cos\left(\frac{2\sqrt{E}}{\hbar\, x} - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right) = 0$.

- Hence, for the values of $d < 0$, the solutions $\psi_{\nu}^{(+)}(x)$ are not well defined near zero. It restricts that $d \geqslant 0$.
- But we have $d - \nu < 0$ which fixes the lower bound of ν. To consider the lower bound value of ν as the least of the value of ν, we consider $d = 0$.

Hence, the eigenfunction, equation (29) becomes

$$\psi_{\nu}^{(+)}(x) = CJ_{\nu}\left(\frac{2\sqrt{E}}{\hbar\, x}\right), \quad x \in (0,\, \infty). \tag{34}$$

Similarly, the eigenfunction, equation (30) takes the form,

$$\psi_{\nu}^{(-)}(x) = \tilde{C}J_{\nu}\left(\frac{2\sqrt{E}}{\hbar\, x}\right), \quad x \in (-\infty,\, 0). \tag{35}$$

We also consider that $\nu > 0$ from the fact that the Bessel functions $J_{\nu}(0)$ are not well defined at $\nu = 0$.

4.2. Parity

Now we use the parity condition on J_ν. The solution (35), defined in the region $x \in (-\infty, 0)$, may be symmetric or anti-symmetric with $\psi_\nu^{(+)}(x)$. Consider a point ϵ near $x = 0$, then we have

$$\tilde{C}\psi_\nu^{(-)}(x)|_{x=-\epsilon} = C\psi_\nu^{(+)}(x)|_{x=\epsilon}, \tag{36}$$

and so

$$(C - (-1)^\nu \tilde{C}) J_\nu\left(\frac{2\sqrt{E}}{\hbar\ \epsilon}\right) = 0. \tag{37}$$

The odd parity determines $\nu = 1, 3, 5,...$, odd integers, and so $\tilde{C} = -C$, whereas even parity leads to $\nu = 2, 4,...$, even integers, so that $\tilde{C} = C$.

Hence, the parity condition fixes

$$\nu = n, \quad n = 1, 2, 3,.... \tag{38}$$

As a result, we find that the coupling parameter (27) is now related with the quantum number 'n' as

$$\lambda = \left(n^2 - \left(2\alpha_1 + 2\gamma_1 + \frac{3}{2}\right)^2\right)\frac{\hbar^2}{4}, \quad n = 1, 2, 3,... \tag{39}$$

and so it is quantized which has also been confirmed by the semiclassical quantization method, vide equation (14).

Hence, the bound states from (34) and (35) become

$$\psi_n^{(+)}(x) = CJ_n\left(\frac{2\sqrt{E}}{\hbar\ x}\right), \quad x \in (0, \infty) \quad n = 1, 2, 3,.... \tag{40}$$

$$\psi_n^{(-)}(x) = C(-1)^n J_n\left(\frac{2\sqrt{E}}{\hbar\ |x|}\right), \quad x \in (-\infty, 0) \quad n = 1, 2, 3,.... \tag{41}$$

The parity nature of the eigenfunctions (40) and (41) restricts the coupling parameter to take discrete values, that is expressed in terms of quantum number 'n' in (39). Subsequently we analyze the energy eigenvalues in the following section.

4.2.1. Energy

As $m(x) = \frac{2}{x^4}$ is singular at $x = 0$, the eigenfunctions $\psi_n^{(\pm)}(x)$ (vide equations (40) and (41)) are restricted to be zero at that point $x = 0$, that is

$$\lim_{x->0} \psi_n^{(\pm)}(x) = 0. \tag{42}$$

Consequently, we have

$$\lim_{x->0} \sqrt{\frac{\hbar x}{\pi\sqrt{E}}}\ \cos\left(\frac{2\sqrt{E}}{\hbar\ x} - \frac{\pi}{2}\left(n + \frac{1}{2}\right)\right) = 0. \tag{43}$$

The above relation establishes that the energy eigenvalues are continuous, while the coupling parameter λ is quantized as in equation (39).

4.3. Normalizability condition of the states (40) and (41)

As the non-Hermitian ordered form of the Hamiltonian can be related with the Hermitian ordered form through similarity transformation, one can express the normalization condition for non-Hermitian ordered Hamiltonian as [23],

$$1 = \langle \psi_n^{(\pm)} m^{2\eta} | \psi_n^{(\pm)} \rangle, \tag{44}$$

where $\eta = \frac{\gamma_1 - \alpha_1}{2}$. On substituting (40) in (44), we can get

$$1 = C^2 2^{\gamma_1 - \alpha_1} \int_0^\infty \frac{1}{x^{4\gamma_1 - 4\alpha_1}} J_n\left(\frac{2\sqrt{E}}{\hbar\ x}\right) J_n\left(\frac{2\sqrt{E'}}{\hbar\ x}\right) dx. \tag{45}$$

As $d = 0$, we have $\gamma_1 - \alpha_1 = \frac{3}{4}$. By applying a simple transformation $\rho = \frac{1}{x}$ to (45), we can get

$$1 = C^2 2^{3/4} \int_0^\infty \rho\ J_n\left(\frac{2\sqrt{E}\ \rho}{\hbar}\right) J_n\left(\frac{2\sqrt{E'}\ \rho}{\hbar}\right) d\rho. \tag{46}$$

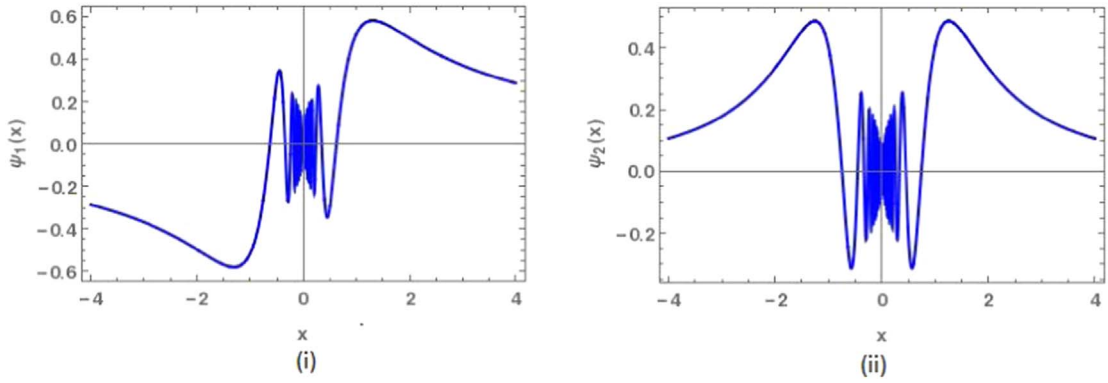

Figure 3. The plots of (i) $\psi_1(x)$ and (ii) $\psi_2(x)$ in equation (50).

On using the identity,

$$\int_0^\infty kJ_n(ka)J_n(kb)dk = \frac{1}{a}\delta(b-a), \qquad n = 0,\ 1,\ 2\ ,..., \tag{47}$$

we can obtain the condition

$$1 = \frac{C^2 2^{3/4}\hbar}{2\sqrt{E}}\delta\left(\frac{2\sqrt{E'}}{\hbar} - \frac{2\sqrt{E}}{\hbar}\right) \tag{48}$$

where $\delta(a-b)$ is the Dirac delta function which becomes infinity when $a = b$, otherwise it has zero value.

We now obtain,

$$C = \left(\frac{\sqrt{2\,E}}{\hbar\ \delta\left(\frac{2\sqrt{E}}{\hbar} - \frac{2\sqrt{E'}}{\hbar}\right)}\right)^{1/2}. \tag{49}$$

As the energy eigenvalue of the system is arbitrary and continuous, we have obtained the normalization constant in terms of Dirac delta function. This is analogous to the quantization of a free particle on a cone studied recently by Kowalski *et al* [28].

Hence, we obtained the bounded states (29) in both the regions, $x \in (0, \infty)$ and $x \in (-\infty, 0)$, as

$$\psi_n^{(\pm)}(x) = CJ_n\left(\frac{2\sqrt{E}}{\hbar\ x}\right), \quad n = 1,\ 2,\ 3\ ,.... \tag{50}$$

The first two states (unnormalized) are plotted in the figure 3.

One can reinterpret the normalization condition,

$$1 = \int_{-\infty}^{\infty} \psi_n^*(x)\psi_n(x)dx, \tag{51}$$

by omitting the singular region $(-\epsilon, \epsilon)$ and reconsidering the integral (44) by

$$1 = 2C^2\int_\epsilon^\infty \frac{1}{x^3}J_n\left(\frac{2\sqrt{E}}{\hbar\ x}\right)J_n\left(\frac{2\sqrt{E'}}{\hbar\ x}\right)dx, \tag{52}$$

in which we considered (50).

Let $\frac{1}{x} = \rho$. The integral (52) becomes

$$1 = 2C^2\int_0^{1/\epsilon} \rho\ J_n\left(\frac{2\sqrt{E}}{\hbar}\rho\right)J_n\left(\frac{2\sqrt{E'}}{\hbar}\rho\right)d\rho. \tag{53}$$

Now we use the identity [29]

$$\int_0^a \rho J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)d\rho = \frac{a^2}{2}[J_{\nu+1}(\alpha_{\nu m})]^2\delta_{nm}, \tag{54}$$

where δ_{nm} is Kronecker delta function that takes the value 1 when $n = m$ otherwise it takes zero. Here, $\alpha_{\nu m}$, $m = 1, 2, 3,... \infty$, is the mth zero of the Bessel function J_ν, that is $J_\nu(\alpha_{\nu m}) = 0$.

The integral (53) now becomes

$$1 = \frac{C^2}{\epsilon^2}\left[J_{n+1}\left(\frac{2\sqrt{E_n^N}}{\hbar\ \epsilon}\right)\right]^2 \tag{55}$$

which makes the energy eigenvalues to take the values,

$$E_n^N = \frac{\hbar^2}{4}\, j_n^{N^2}\, \epsilon^2, \qquad \epsilon \neq 0, \tag{56}$$

where j_n^N, $N = 1, 2, 3, ...\infty$, $n = 1, 2, 3, ...$ are zeroes of the Bessel function, J_n. The normalization constant reads as

$$C_n^N = \frac{\epsilon}{J_{n+1}\left(\frac{2\sqrt{E_n^N}}{\hbar\ \epsilon}\right)}. \tag{57}$$

The normalized eigenstates, vide (56) and (57), can be written as

$$\psi_n^N(x) = C_n^N J_n\left(\frac{2\sqrt{E_n^N}}{\hbar\ x}\right), \quad n = 1, 2, 3, ... \quad N = 1, 2, 3, ..., \quad \epsilon \neq 0. \tag{58}$$

We have observed that one can possibly obtain the normalized eigenfucntions with the corresponding eigenvalues by restricting the motion of the particle around a point near to the origin $\epsilon(\neq 0)$.

4.4. Hermitian ordering

In the previous section, we considered non-Hermitian ordered Hamiltonian (17) and solved the corresponding generalized Schrödinger equation that resulted in the general solution (50). In this sub-section, we discuss about the solution of the Hermitian ordered form of the Hamiltonian (19),

$$\hat{H}_{her} = \frac{1}{2} m^{\frac{\gamma_1+\alpha_1}{2}} \hat{p}\, m^{\beta_1} \hat{p}\, m^{\frac{\gamma_1+\alpha_1}{2}} + V(x). \tag{59}$$

Instead of solving the Schrödinger equation corresponding to the Hermitian ordered Hamiltonian (19), we can obtain the solution from the relation (18) that relates the non-Hermitian ordered form (16) with the Hermitian ordered form through similarity transformation.

$$\hat{H}\psi = m^{-\eta}\hat{H}_{her}\, m^{\eta}\psi, \qquad 2\eta = \gamma_1 - \alpha_1. \tag{60}$$

Let $m^{\eta}\psi = \phi$. As we have $2\eta = \gamma_1 - \alpha_1 = \frac{3}{2}$ from $d = 0$, we can write down the solution for (19) from (50),

$$\phi_n(x) = C m^{\eta} J_n\left(\frac{2\sqrt{E}}{\hbar\ x}\right) = C\, x^{-3/2} J_n\left(\frac{2\sqrt{E}}{\hbar\ x}\right), \quad n = 1, 2, 3, ..., \tag{61}$$

where the normalization constant C is the same as obtained in (49). The solution (61) is singular at $x = 0$. Hence, for the system (1), the non-Hermitian ordered form (17) only yields bounded solutions (50).

5. Conclusion

In this work, we considered a nonlinear system of the quadratic Liénard type which admits temporally localized solutions at the classical level. Depending upon the positive and negative values of the coupling parameter λ, the solution is well defined or has a singular value in its domain. To start with, we implemented the WKB quantization condition which ensures that the coupling parameter λ would be quantized. While studying the quantum dynamics of the system, we considered a single term of the general ordered position-dependent mass Hamiltonian as the mass function which is singular at the origin and solved the underlying Schrödinger equation. We observed that the quantum system admits bounded solutions. Specifically, we find that the coupling parameter of the system gets quantized. We believe that such an observation is quite new to the literature as far as the quantization is concerned. The position dependent mass with δ-type mass profile considered in this paper may find application in the field of semiconductor physics, as in the case of Thomas-Fermi potential with $\delta -$ doped semiconductor [19]. We believe that our study widens the scope of quantizing other solvable classical nonlinear oscillators exhibiting novel dynamical features in a broader sense.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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