A STUDY ON DOMINATION PARAMETERS OF GRAPHS

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By

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List of Publications

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2 Joseline Manora J and Mekala R, Results on Majority Dom-Chromatic Set of a Graph, **Turkish World Mathematical Society Journal of Applied and Engineering Mathematics**, V.11, Special Issue (2021), 30-41, ISSN: 2146-1147.

3 Joseline Manora J and Mekala R, Majority Dominating Chromatic Partition Number of a Graph, **Malaya Journal of Matematik**, Vol. S, No. 1, 437 - 440, 2021, ISSN(O):2321-5666.

4 Joseline Manora J, and Mekala R, Changing and Unchanging of Majority Dominating Chromatic Number When Removal of a Single Vertex, **Advances and Applications in Mathematical Sciences**, Volume 21, Issue 2, December 2021, 951-963 ©‘ 2021 Milli Publications, India, ISSN 0974-6803.

5 Joseline Manora J and Mekala R, Changing and Unchanging of Majority Dominating Chromatic Number by Edge Deletion,

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9 Joseline Manora J and Mekala R, Majority dom-chromatic Set of Special Graphs, communicated to **Journal of Graph Theory**.

List of Notations

Notation	Meaning
$G = (V, E)$	Graph with vertex set V and edge set E
$\bar{G}$	Complement graph of G
S	Majority Dom-chromatic Set
$d(v)$	Degree of the vertex v
$\delta(G)$	Minimum degree of the graph G
$\Delta(G)$	Maximum degree of the graph G
$\lceil x \rceil$	Smallest integer greater than or equal to x
$\lfloor x \rfloor$	Largest integer less than or equal to x
$N(u)$	Open Neighborhood of a vertex u
$N[u]$	Closed Neighborhood of a vertex u
$\|N[u]\|$	Cardinality of Closed Neighborhood of a vertex u

Notation	Meaning
$pn[u, S]$	Private Neighbour of the vertex u with respect to the set S
C_p	Cycle with p vertices
P_p	Path with p vertices
F_p	Fan with p vertices
K_p	Complete graph with p vertices
W_p	Wheel graph with p vertices
$K_{m,n}$	Complete Bipartite graph
$G_1 \circ G_2$	Corona graph of G_1 and G_2
$P_i \times P_j$	Grid graph
$P_i \times C_j$	Cylinder graph
$C_i \times C_j$	Torus graph
$G_1 \circ_v G_2$	Rooted Product graph of G_1 and G_2
$D_n^{(m)}$	Dutch Windmill graph
$S(G)$	Subdivision of a graph G
$\gamma(G)$	Domination Number of G
$\chi(G)$	Chromatic Number of G
$d(G)$	Domatic Number of G

Notation	Meaning
$\gamma_c(G)$	Connected Domination Number of G
$\gamma_M(G)$	Majority Domination Number of G
$d_M(G)$	Majority Domatic Number of G
$\gamma_{ch}(G)$	Dom-Chromatic number of G
$d_{ch}(G)$	Dom-Chromatic Partition Number of G
$\gamma_{CM}(G)$	Connected Majority Domination Number of G
$\gamma_{M\chi}(G)$	Majority Dom-Chromatic Number of G
$d_{M\chi}(G)$	Majority Dom-Chromatic Partition Number of G
$\gamma_{CM\chi}(G)$	Connected Majority Dom-Chromatic Number of G
$CVR_{M\chi}$	Changing Vertex Removal
$UVR_{M\chi}$	Unchanging Vertex Removal
$CER_{M\chi}$	Changing Edge Removal
$UER_{M\chi}$	Unchanging Edge Removal
$CEA_{M\chi}$	Changing Edge Addition
$UEA_{M\chi}$	Unchanging Edge Addition

Contents

Abstract

A STUDY ON DOMINATION PARAMETERS OF GRAPHS

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Graph theory had witnessed an unprecedented growth in the twentieth century. A major impetus for this growth has certainly been the wide applicability of graph theory especially in computer science and in many areas. Graphs serve as Mathematical models to analyze successfully many concrete real-world problems. The Swiss Mathematician Leonard Euler learned of the society's frustrating phenomenon of seven bridges of Konigsberg River and in 1736, he wrote an article about the "Konigsberg Bridge Problem". Later, his work is considered by many to be the beginning of the field of Graph Theory. This field began to blossom in the twentieth century as more and more, modelling possibilities were recognized and the growth continues. It is interesting to note that specific applications have increased in number and in scope, the theory itself has developed beautifully as well. Perhaps Domination Theory is one of the fastest-growing area of graph theory with variety of domination parameters and its applications.

In 1958, Berge [5] introduced the concept 'domination' and this inception made drastic change in the field of Graph Theory and Ore [49] who gave the formal mathematical definition to the topic of domination in 1962. Cockayne and Hedetniemi [14] published their article "Towards a theory of domination in graphs" in 1977. This paper became the point of interest for many researchers to step into domination. Then many eminent mathematicians have expansively developed this theory and numbers of domination parameters are formed by the combination of common property and some specific graph theoretic property. Domination has a wide range of applications in radio stations, modelling social networks, coding theory and nuclear power plants problems.

Graph colouring and domination are major areas in graph theory. These concepts also give rise to a number of practical applications in real life. In recent years, several graph-theoretic parameters that

combine the concepts of domination and colouring have been investigated by several graph theorists effectively. One such parameter is the concept of dom-colouring which was introduced by Janakiraman and Poobalaranjani [31] in 2012. To find a dominating set having the same chromatic number as that of the graph, the chromatic preserving set (cp - set) is introduced to serve this purpose. Thus, a dom-chromatic set is a dominating cp-set. Then Swaminathan and Joseline Manora [40] introduced the concept "Majority Domination" with respect to sets with the idea of dominating atleast half of the vertices of a graph in 2006. They elucidated the parameter in various levels by establishing many results [41].

The researcher has motivated by these concepts to introduce new graph theoretical parameter "Majority Dominating Chromatic (MDC) Set of a graph" and "Majority Dominating Chromatic Number" on graphs. In this type of domination, the elements of the dominating set must be a majority dominating set S and the chromatic number of a graph must equal to the chromatic number of an induced subgraph $\langle S \rangle$. This new parameter is also called majority dom-chromatic sef of G. Thus, majority dom-chromatic sets play a vital role in domination theory. The relationship among majority domination, cpn – set

and chromatic number with dom-chromatic sets and the newly defined parameter majority dom-chromatic sets have yet to be closely studied in the context of domination theory.

This thesis entitled "**A Study on Domination Parameters of Graphs**" consists of six chapters. The organization of the thesis follows the pattern given below.

In the first chapter, the researcher presents the basic definitions and results on domination theory which are necessary to write this thesis. Survey of the literature, the motivation and the scope of the thesis are also mentioned.

In chapter 2, the new parameter Majority dom-chromaticset of a graph has been defined with an example. Then Majority dom-chromatic number $\gamma_{M\chi}(G)$ is determined for some families of graphs. The necessary and sufficient condition for a minimal Majority dom-chromatic number is produced. Also the lower and upper bounds on $\gamma_{M\chi}(G)$ is given. In the next section, some results on Majority dom-chromatic set of a graph are determined and some beautiful inequalities on $\gamma_{M\chi}(G))$ are also investigated. Then $\gamma_{M\chi}(G)$ for bipartite graph is studied and bounds on $\gamma_{M\chi}(G)$ for bipartite graph is established. Finally algorithm of majority dom-chromatic set, its

number $\gamma_{M\chi}$ and application of majority dom-chromatic set are discussed.

Chapter 3 includes the exact values of majority dom-chromatic number for product graphs such as Grid, Cylinder and Torus are investigated. Also the parameter values of $\gamma_{M\chi}(G)$ for Corona, Generalized Petersen graph $P(n,k)$, rooted product graphs and disconnected graphs with p vertices are determined.

In chapter 4, majority dom-chromatic partition set of G and its number $d_{M\chi}$ of G is defined. The particular value of $d_{M\chi}(G)$ for some classes of graphs is found and bounds on majority dom-chromatic partition number are also discussed. The majority dom-chromatic partition number $d_{M\chi}$ for Generalized Peterson graph, friendship graph and Dutch windmill graphs has been determined. The relationship among $d_c(G), d_M(G), d_{ch}(G)$ and $d_{M\chi}(G)$ has been investigated in terms of maximum degree of a vertex.

Chapter 5 deals with the effects of the majority dom-chromatic number when the graph G is modified by removing a vertex. The classification of $V^0_{M\chi}(G), V^-_{M\chi}(G)$ and $V^+_{M\chi}(G)$ are defined and characterization theorems on $CVR_{M\chi}$ and $UVR_{M\chi}$ are studied. In next section, the changing and unchanging of $\gamma_{M\chi}(G)$ due to the deletion of an edge is investigated. The edge critical classifications of $E^0_{M\chi}(G), E^-_{M\chi}(G)$ and $E^+_{M\chi}(G)$ are discussed. The characterization theorems on connected and disconnected graphs are determined for $CER_{M\chi}$ and $UER_{M\chi}$. In the last section, the effects of the majority dom-chromatic number when the graph G is modified by adding an edge e from the complement of G between any two vertices of a graph are discussed. Then the classifications namely $\xi^\circ_{M\chi}(G), \xi^+_{M\chi}(G)$ and $\xi^-_{M\chi}(G)$, for any edge $e \in E(G^c)$ are investigated for connected as well as disconnected graphs.

Chapter 6 discusses the definition of the connected majority dom-chromatic set of a connected and disconnected graphs. The majority dom-chromatic number $\gamma_{CM\chi}(G)$ is determined for product graphs such as Grid and Cylinder. The comparison of $\gamma_{CM}(G), \gamma_{cch}(G)$ and $\gamma_{CM\chi}(G)$ is studied. Also some inequalities of $\gamma_{CM\chi}(G)$ is established in terms of diameter of a graph.

Finally, the total summary of the research work in the thesis highlighting all new findings developed using the newly coined concept of majority dom-chromatic set of a graph. Also some references which are needed to the Thesis are given at the end.

Chapter 1

Prolegomenon

Abstract

This chapter is introductory in nature which unlocks the fundamental theoretical background of the thesis. This chapter comprises the details of chronological survey of all literature, basic concepts of graph theory, domination theory and objectives of the study. The motivation, scope and organization of the thesis are also given at the end.

In this chapter, the basic definitions and results are given which are needed in the subsequent chapters.

1.1 Introduction

Graph theory had witnessed an unprecedented growth in the twentieth century. A major impetus for this growth has certainly been the wide applicability of graph theory especially in computer science and in many areas. Graphs serve as Mathematical models to analyse successfully many concrete real-world problems.

It is interesting to note that specific applications have increased in number and in scope, the theory itself has developed beautifully as well. Perhaps Domination Theory is one of the fastest-growing area of graph theory with variety of domination parameters and its applications.

1.2 Survey of Literature

In 1892, W. Rouse Ball [52] studied some basic types of problems on N-Queen problem. In 1958, Claude Berge [5] wrote a book on graph theory, in which he defined for the first time the concept of the domination number. In the year 1962, Oystein Ore [49] published his book *Theory of Graphs on Graph Theory*. In this he used for the first time, the name "Domination Number".

In 1976, More contributions on the theory of domination was given by Walikar and Acharya [60] and these results were published in National Academic Science. This concept survived almost in hibernation until 1975 when Cockayne and Hedetniemi [14] published their paper *Towards a Theory of Domination in Graphs* which appeared in Networks in 1977. This survey paper brought to light new ideas and potentially of being applied in variety of areas. Some thirty years later more than 2000 research papers have been published on this topic, and the number of papers is steadily growing. The researcher is inspired by the explosive growth of this field of study. He is also motivated by a desire to put some order into this huge collection of research papers, to organize the study of dominating sets in graphs into meaningful subareas, and to attempt the place of the study of dominating sets in even broader mathematical and algorithmic contexts.

Walikar, Acharya and Sampathkumar are some of the Indian mathematicians who have made substantial contribution to the study of domination theory in graphs. More than fifty types of domination parameters have been studied by different authors. In 1979, Walikar et. al. [60] published a technical report as lecturer notes on -MRI. In 1990, Hedetniemi and Laskar [26] published their *Bibliography on domination in graphs and some basic definitions of domination parameters*. This book contained about 400 references at that time. In 1991, the concept was then developed by Carrington, Harary and Haynes [11] published an article on "*Changing and Unchanging the domination number of a graph G*". Further in 1991, ElZahar and Pareek [17], determined domination number of Cartesian Products of graphs. In 1995, [9] Broere, Hattingh, Henning and Mcrae introduced the concept of majority dominating function in graphs and gave a detailed account of results in the book *Domination in Graphs: Advanced Topics* (chapter 4, 91-104). Towards the end of 1998, Haynes, Hedetniemi and Slater [24] brought out a comprehensive two volumes of text book - *Fundamentals of Domination in Graphs and Domination in Graphs: Advance Topics* which contain more than 1200 bibliographical entries. Within last 25 years many researchers worked in this domination field at different aspects and produced so many results with new types of domination parameters.

The idea of dominating half of the vertex set is a crucial one and it gives the inspiration for defining majority dominating sets instead of functions. In 2006, Swaminathan and Joseline Manora [40] introduced the new parameter "Majority Dominating Sets of a Graph" in domination theory. Further in 2011, this concept was further developed into many area of domination. In 2011, [41] many results on majority dominating sets are introduced in research paper and in 2011 [39], they studied various parameters in this area such as majority domatic number [37], vertex and edge critical graphs [38, 39] on majority domination number.

Graph colouring and domination are major areas in graph theory. These concepts also give rise to a number of practical applications in real life. In recent years, several graph-theoretic parameters that combine the concepts of domination and colouring have been investigated by several graph theorists effectively. One such parameter is the concept of dom-colouring which was introduced by [31] Janakiraman and Poobalaranjani in 2010. To find a dominating set having the same chromatic number as that of the graph, the chromatic preserving set (cp - set) is introduced to serve this purpose. Thus, a dom-chromatic set is a dominating cp-set.

1.3 Preliminaries on Graph Theory

Definition 1.3.1. *A **graph** is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G, called edges. The vertex set and the edge set of G are respectively denoted by $V(G)$ and $E(G)$.*

If $e = \{u, v\}$ is an edge, we write $e = uv$ and we say e joins the vertices u and v; u and v are adjacent vertices; u and v are incident with e. If two vertices are not joined by an edge, then we say that they are non-adjacent.

Definition 1.3.2. *The number of elements in the vertex set of a graph is called the **order** of G and is denoted by n. The number of elements in the edge set of a graph is called the **size** of G and is denoted by m. A graph with n vertices and m edges is called as (**n, m**)-**graph**. The (1, 0)-graph is called as **trivial graph**.*

Definition 1.3.3. *A graph H is called a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A **spanning subgraph** of G is a subgraph H with $V(H) = V(G)$. For any set S of vertices of G, the **induced subgraph** $\langle S \rangle$ is the maximal subgraph of G with vertex set S.*

Definition 1.3.4. *If G is a graph with the vertex v then $G - v$ is the* ***induced subgraph*** *$\langle V(G) - v \rangle$ of G and obtained from G by removing v and the edges incident with v. If $e \in E(G), G - e$ is the spanning subgraph with edge set $E(G) - \{e\}$ and it is obtained from G by removing the edge e from G.*

Definition 1.3.5. *The degree of a vertex v in a graph G is the number of edges of G incident with v and is denoted by deg(v) or d(v). The maximum and the minimum degrees of the vertices of G are respectively denoted by* $\boldsymbol{\Delta(G)}$ *and* $\boldsymbol{\delta(G)}$. *A vertex of degree 0 in G is called an isolated vertex, and a vertex of degree 1 is called a* ***pendant vertex*** *or an* ***end vertex*** *of G. Any vertex adjacent to a pendent vertex is called a* ***support****.*

Definition 1.3.6. *A graph G is said to be* ***regular*** *graph of degree r if every vertex of G has degree r Such graphs are called* ***r-regular*** *graphs. A 3-regular graph is called a* ***cubic graph****.*

Definition 1.3.7. *A graph G is* ***Complete*** *if every pair of its vertices are adjacent. A complete graph on p vertices is denoted by K_p. A clique of a graph is a maximal complete subgraph.*

Definition 1.3.8. *A **bipartite graph** is a graph G whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge in G has one end vertex in V_1 and the other end vertex in V_2. The vertex set (V_1, V_2) is called a bipartition of G. Further, if every vertex of V_1 is adjacent to every vertex of V_2 then G is called a complete bipartite graph. The **complete bipartite** graph with bipartition (V_1, V_2) such that $|V_1| = r$ and $|V_2| = s$ is denoted by $K_{r,s}$.The graph $K_{1,p-1}$ is called a star. When $r \geq 2$ the vertices of degree 1 of a star are called **claws** of the star and the vertex of degree $(p-1)$ is called the **centre** of the star.*

Definition 1.3.9. *A **double star** is a graph obtained by taking two stars and joining the vertices of maximum degrees with an edge. It is denoted by $D_{r,s}$.*

Definition 1.3.10. *A graph G is said to be **connected** if any two distinct vertices of G are joined by a path. A maximal connected subgraph of G is called a component of G. Thus, a disconnected graph has at least two components.*

Definition 1.3.11. *A **subdivision** of an edge uv of a graph G is obtained by introducing a new vertex w and replacing the edge uv with edges uw and wv. The graph obtained from G by subdividing each edge of G exactly once is called the subdivision graph (or subdivision) of G and is denoted by S(G).*

Definition 1.3.12. *The distance between two vertices u and v in a graph G is the length of a shortest $u - v$ path in G. It is denoted by $d(u, v)$. The **diameter** of a connected graph G is the length of any longest geodesic. The diameter of G is denoted by $diam(G)$.*

Definition 1.3.13. *For any real number x, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x and $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.*

Definition 1.3.14. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. Then their **union** $G_1 \cup G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$.*

Definition 1.3.15. *An **open neighbourhood $N(v)$** of a vertex v in a graph G is the set of all vertices adjacent to v in G. The **closed neighbourhood $N[v]$** of v is the set $N(v) \cup \{v\}$.*

Definition 1.3.16. *The* ***open neighbourhood*** $\boldsymbol{N(S)}$ *of a set* S *of vertices is the set of all vertices adjacent to the vertices in* S*. The* ***closed neighbourhood*** $\boldsymbol{N[S]}$ *of* S *is the set* $\boldsymbol{N(S) \cup S}$*. If* $x \in S$*, a* ***private neighbour of*** x *with respect to* S *is a vertex* v *such that* $v \in N[x] - N[S - \{x\}]$*.*

Definition 1.3.17. *For* $S \subseteq V$*, a vertex* $v \in S$ *is called an* ***enclave*** *of* S *if* $N[v] \subseteq S$*.*

Definition 1.3.18. *A subset* S *of* $V(G)$ *is said to be a* ***dominating set*** *of* G *if every vertex in* $V - S$ *is adjacent to at least one vertex in* S*. A dominating set is called minimal dominating set if no proper subset of* S *is a dominating set. The minimum cardinality of the minimal dominating set of* G *is called the* ***domination number*** *of* G*, denoted by* $\gamma(G)$ *and* $\Gamma(G)$ *denotes the maximum cardinality of a minimal dominating set of* G *and* $\Gamma(G)$ *is called the* ***upper domination number*** *of* G*.*

Theorem 1.3.19. *A dominating set* S *of a graph* G *is* ***minimal*** *if and only if for every* $u \in S$ *one of the following conditions holds. (i)* $N(u) \cap S = \phi$ *(ii) There is a vertex* $v \in V - S$ *such that* $N(v) \cap S = \{u\}$*.*

Theorem 1.3.20. *Every connected graph G of order $n \geq 2$ has a dominating set S whose complement $V - S$ is also a dominating set.*

Theorem 1.3.21. *If G is a graph with no isolated vertices then the complement $V - S$ of every minimal dominating set S is a dominating set.*

Definition 1.3.22. *A **domatic partition** (d-partition) of a graph G is a partition of $V(G)$ into dominating sets. The maximum cardinality of a partition $V(G)$ into dominating sets is called the domatic number and is denoted by $d(G)$.*

Definition 1.3.23. *A dominating set S is said to be a **connected dominating set** if the subgraph $\langle S \rangle$ induced by S is connected in G. A connected dominating set S is minimal if no proper subset of S is a connected dominating set. The minimum cardinality of the minimal connected dominating set of G is called the **connected domination number**, denoted by $\gamma_C(G)$.*

Definition 1.3.24. *A graph G is said to be a **CVR- graph** if $\gamma(G - v) \neq \gamma(G)$, for every $v \in V(G)$ and graph G is said to be a **UVR-graph** if $\gamma(G - v) = \gamma(G)$, for every $v \in V(G)$*

Definition 1.3.25. *A graph G is said to be a* ***CER- graph*** *if* $\gamma(G - e) \neq \gamma(G)$, *for every* $e \in E(G)$ *and the graph G is said to be a* ***UER-graph*** *if* $\gamma(G - e) = \gamma(G)$, *for every* $e \in E(G)$.

Definition 1.3.26. *A graph G is said to be a* ***CEA- graph*** *if* $\gamma(G + e) \neq \gamma(G)$, *for every* $e \in E(G)$ *and the graph G is said to be a* ***UEA-graph*** *if* $\gamma(G + e) = \gamma(G)$, *for every* $e \in E(G)$.

Definition 1.3.27. *A subset S of $V(G)$ is said to be a* ***majority dominating set*** *if at least half of the vertices of $V(G)$ are either in S or adjacent to elements of S i.e.* $|N[S]| \geq \lceil \frac{V(G)}{2} \rceil$. *A majority dominating set S is minimal if no proper subset of S is a majority dominating set. The minimum cardinality of a minimal majority dominating set is called* ***majority domination number*** *and denoted by* $\gamma_M(G)$.

Definition 1.3.28. *A* ***majority domatic partition*** *of a graph G is a partition of the vertex set $V(G)$ into majority dominating sets of G. The maximum number of sets of majority domatic partition of G is called the* ***majority domatic number of*** *G, denoted by* $d_M(G)$.

Definition 1.3.29. *A subset $S \subseteq V(G)$ is a **connected majority dominating(CMD)** set if S is a majority dominating set and the induced subgraph $\langle S \rangle$ is connected in G. The minimum cardinality of the minimal connected majority dominating set S of G is called the **connected majority domination number** and denoted by $\gamma_{CM}(G)$.*

Definition 1.3.30. *Let G be any graph with p vertices and let $u \in V(G)$. Then u is said to be **Majority Dominating (MD) vertex** if $d(u) \geq \lceil \frac{p}{2} \rceil - 1$.*

Definition 1.3.31. *A vertex v of a graph G is said to be a **full degree vertex** or a **dominating vertex** if it is adjacent to all other vertices in G. i.e., $d(v) = p - 1$.*

Definition 1.3.32. *A graph G is said to be a $\boldsymbol{CVR_M}$- graph if $\gamma_M(G - v) \neq \gamma_M(G)$, for every $v \in V(G)$ and the graph G is said to be a $\boldsymbol{UVR_M}$- graph if $\gamma_M(G - v) \neq \gamma_M(G)$, for every $v \in V(G)$.*

Definition 1.3.33. *A graph G is said to be a $\boldsymbol{CER_M}$- graph if $\gamma(G - e) \neq \gamma(G)$, for every $e \in E(G)$ and the graph G is said to be a $\boldsymbol{UER_M}$- graph if $\gamma(G - e) = \gamma(G)$, for every $e \in E(G)$.*

Definition 1.3.34. *The* ***chromatic number*** $\chi(G)$ *is the minimum k such that G is k-colourable. If $\chi(G) = k$ then G is said to be* ***k-chromatic***. *If $\chi(G) = k$, but $\chi(G) < k$ for every proper subgraph H of G then G is said to be a k- critical graph.*

Definition 1.3.35. *A graph G is said to be* ***vertex-color-critical*** *graph or* $\boldsymbol{\chi -}$ ***critical*** *if $\chi(G - v) < \chi(G)$, for every $v \in V(G)$ and called* **edge-critical** *if $\chi(G - e) < \chi(G)$, for every $e \in E(G)$. A graph is called* **color-critical** *graph if which each vertex and edge are critical.*

Definition 1.3.36. *A set $S \in V(G)$ is said to be a* ***chromatic preserving set or a cp-set*** *if $\chi(\langle S \rangle) = \chi(G)$ and the minimum cardinality of a cp-set in G is called the* ***chromatic preserving number*** *or cp-number of G and is denoted by cpn(G). A cp-set of cardinality cpn(G) called cpn-set.*

Definition 1.3.37. *A subset S of $V(G)$ is said to be a* ***dom-chromatic set*** *or dc-set if S is a dominating set and $\chi(\langle S \rangle) = \chi(G)$. The minimum cardinality of a dom-chromatic set in a graph G is called the* ***dom-chromatic number*** *or dc-number of G and is denoted by $\gamma_c h(G)$ or $\gamma_\chi(G)$.*

Definition 1.3.38. *A **dom-chromatic partition** of a graph G is a partition of $V(G)$ into dom-chromatic sets. The maximum cardinality of a partition of $V(G)$ into dom-chromatic sets is the dom-chromatic partition number and denoted by $d_{ch}(G)$.*

Definition 1.3.39. *A dom-chromatic set S is said to be **connected dom-chromatic set** if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected dom-chromatic set S is called connected **dom-chromatic number** and is denoted by $\gamma_{cch}(G)$ or $\gamma_{c\chi}(G)$.*

Definition 1.3.40. *A graph G is said to be a **CVR_{ch} graph** if $\gamma_{ch}(G - v) \neq \gamma_{ch}(G)$, for every $v \in V(G)$ and the graph G is said to be a **UVR_{ch}- graph** if $\gamma_{ch}(G - v) = \gamma_{ch}(G)$, for every $v \in V(G)$.*

Definition 1.3.41. *A graph G is said to be a **CER_{ch} graph** if $\gamma_{ch}(G - e) \neq \gamma_{ch}(G)$, for every $e \in E(G)$ and the graph G is said to be a **UER_{ch} graph** if $\gamma_{ch}(G - e) = \gamma_{ch}(G)$, for every $e \in E(G)$.*

Definition 1.3.42. *A graph G is said to be a **CEA_{ch} graph** if $\gamma_{ch}(G + e) \neq \gamma_{ch}(G)$, for every $e \in E(G)$ and the graph G is said to be a **UEA_{ch} graph** if $\gamma_{ch}(G + e) = \gamma_{ch}(G)$, for every $e \in E(G)$.*

Results 1.3.43: (i) For $G = K_{1,p-1}, D_{r,s}, W_p, F_p, \gamma_M(G) = 1$.

(ii) For any path $G = P_p$ and any cycle $C_p, \gamma_M(G) = \lceil \frac{p}{6} \rceil$.

(iii) Let G be a cycle with p vertices. a) Then $\gamma_{ch}(G) = p$,

b) $$\gamma_{ch}(G) = \begin{cases} \frac{(p+3)}{3}, & \text{if } p \equiv 0(mod3) \\ \frac{(p+2)}{3}, & \text{if } p \equiv 1(mod3) \\ \frac{(p+4)}{3}, & \text{if } p \equiv 2(mod3) \end{cases}$$

(iv) For a path $G = P_p, \gamma_c(G) = p - 2$.

(v) For a Cycle $G = C_p$,

$$\chi(G) = \begin{cases} 2, & \text{if } p \text{ is even} \\ 3, & \text{if } p \text{ is odd} \end{cases}$$

(vi) For a G be a Wheel graph. Then

$$\chi(G) = \begin{cases} 3, & \text{if } p \text{ is odd} \\ 4, & \text{if } p \text{ is even} \end{cases}$$

(vii) Let G be a tree of diameter 3. Then $\gamma_{ch}(G) = p - \Delta(G)$.

1.4 Motivation and Scope of the Thesis

✠ In 2006, Swaminathan and Joseline Manora [40] introduced the concept "Majority Domination" with respect to sets with the idea of dominating atleast half of the vertices of a graph. Further, the concept of dom-colouring which was introduced by Janakiraman and Poobalaranjani in 2010 [31] finding a dominating set having the same chromatic number as that of the graph, the chromatic preserving set (cp - set).The researcher has motivated by these concepts to introduce new graph theoretical parameter "Majority Dom-Chromatic Sets in Graphs".

✠ In 1997, Cockayne and Hedetneimi [14] introduced "Domatic Number of a Graph" and the same concept was extended to majority domatic number of a graph by Swaminathan and Joseline Manora [37] in 2010. Then dom-chromatic partition number is studied by Janakiraman and Poobalaranjani in [31] 2012. The researcher has discussed this idea to Majority dom-chromatic partition number of a graph to some extent.

✠ In 1982, Harary [21] introduced and suggested the changing and unchanging dominating invariants for graphs. This concept was extended to majority dominating sets by Swaminathan and Joseline Manora in [38, 39] 2011 and 2013. They studied and produced many results in critical vertex and critical edge with respect to majority domination number of a graph G. Changing and unchanging properties of Dom-chromatic properties due to vertex deletion, edge deletion from G and edge addition was introduced by Janakiraman, Poobalaranjani [31] in 2012. With the help of these articles, the researcher has extended to find Changing and unchanging properties of Majority Dom-chromatic number of a graph G when removal of a single vertex, an edge deletion from a graph and edge addition to $E(G)$ from the complement $E(G^c)$.

✠ In 1979, the concept "Connected Domination Number in Graphs" was introduced by Sampathkumar and Walikar [53] and they produced many interesting results in their article. In 2012, the parameter "connected dom-chromatic number" was studied by Janakiraman and Poobalaranjani. In 2017, Joseline Manora and Muthukani Vairavel [34] introduced "Connected majority dominating set of a graph". Further the researcher has defined Connected majority dom-chromatic set and Connected majority dom-chromatic number $\gamma_{CM_\chi}(G)$. Using this parameter, many theorems and bounds on $\gamma_{CM_\chi}(G)$ are established in this research work.

- ✠ The relationship among majority domination, cpn-set and chromatic number with dom-chromatic sets and the newly defined parameter majority dom-chromatic sets have yet to be closely studied in the context of domination theory.

1.5 Objectives of The Thesis

- ✠ To introduce a new parameter majority dom-chromatic set (MDC-set) in a graph and majority dom-chromatic number $\gamma_{M\chi}$ of a graph.
- ✠ To obtain the lower and upper bounds of majority dom-chromatic number of a graph in terms of order and size of a graph G.
- ✠ To determine some inequalities on $\gamma_{M\chi}(G)$ and the $\gamma_{M\chi}$ for complement of G.
- ✠ To find the existence of a MDC set in the case of disconnected graphs.
- ✠ To find the exact values of majority dom-chromatic number for some families of graphs, product graphs, rooted product graphs and some special graph structures.

- ✠ To study the necessary and sufficient for a minimal MDC set of a graph G.
- ✠ To define another parameter majority dom-chromatic partition set and its number $d_{M\chi}(G)$ of a graph G.
- ✠ To establish the bounds of $d_{M\chi}(G)$ and the exact values of $d_{M\chi}(G)$ for various classes of graphs.
- ✠ To investigate the changing and unchanging properties of the removal of a single vertex from the graph G with respect to majority dom-chromatic number of G.
- ✠ To study the effects of a single edge deletion in G with respect to majority dom-chromatic number $\gamma_{M\chi}$ of a graph G.
- ✠ To investigate the changes in the value of majority dom-chromatic number when adding an edge from the complement of G.
- ✠ To find another parameter connected majority dom-chromatic set and its number $\gamma_{CM\chi}(G)$ of a graph G.
- ✠ To establish an Algorithm for a majority dom-chromatic set and majority dom-chromatic number $\gamma_{M\chi}$ of a given graph G.

1.6 Organisation of the Thesis

This thesis entitled "**A Study on Domination Parameters of Graphs**" consists of six chapters. The organisation of the thesis follows the pattern given below.

1. **Prolegomenon**.
2. **Majority Dom-Chromatic Set of a Graph**.
3. **Majority Dom-Chromatic Set of Special Graph Structures**.
4. **Majority Dom-Chromatic Partition Number of Graphs**.
5. **Changing and Unchanging Properties of Majority Dom-Chromatic Number**.
6. **Connected Majority Dom-Chromatic Set of a Graph**.

   Conclusion.

In the first chapter, the researcher presents the basic definitions and results on domination theory which are necessary to write this thesis. Survey of the literature, the motivation and the scope of the thesis are also mentioned.

In chapter 2, the new parameter Majority dom-chromatic set of a graph has been defined with an example. Then Majority dom-chromatic number $\gamma_{M\chi}(G)$ is determined for some families of graphs. The necessary and sufficient condition for a minimal Majority dom-chromatic number is produced. Also the lower and upper bounds on $\gamma_{M\chi}(G)$ are given. The content of this section is published in "**Bulletin of Pure and Applied Sciences**".

In the next section, some results on Majority dom-chromatic set of a graph are determined and some beautiful inequalities on $\gamma_{M\chi}(G)$ are also investigated. This work is published in "**Turkish World Mathematical Society Journal of Applied and Engineering Mathematics**" **(Indexed in SCOPUS)**. Then $\gamma_{M\chi}(G)$ for bipartite graph are studied and bounds on $\gamma_{M\chi}(G)$ for bipartite graph is established. The content of this section is communicated in "**International Journal on Soft Computing**".

Chapter 3 includes the exact values of majority dom-chromatic number for product graphs such as Grid, Cylinder and Torus. Also the particular values of $\gamma_{M\chi}(G)$ for Corona, Generalized Petersen graph $P(n, k)$, rooted product graphs and disconnected graphs with p vertices are determined. This work is communicated to "**Journal of Graph Theory**".

In chapter 4, majority dom-chromatic partition set of G and its number $d_{M\chi}$ of G is defined. The particular value of $d_{M\chi}(G)$ for some classes of graphs is found and bounds on majority dom-chromatic partition number are also discussed. The majority dom-chromatic partition number $d_{M\chi}$ for Generalized Petersen graph, friendship graph and Dutch windmill graphs has been determined. The relationship among $d_C(G), d_M(G), d_{ch}(G)$ and $d_{M\chi}(G)$ has been investigated in terms of maximum degree of a vertex. This work is published in "**Malaya Journal of Matematik**".

Chapter 5 deals with the effects of the majority dom-chromatic number when the graph G is modified by removing a vertex. The classification of $V^0_{M\chi}(G), V^-_{M\chi}(G)$ and $V^+_{M\chi}(G)$ are defined and characterization theorems on $CVR_{M\chi}$ and $UVR_{M\chi}$ are studied. This content is published in "**Advances and Applications in Mathematical Sciences**" **(Indexed in WEB of Science)**. In next section, the changing and unchanging of $\gamma_{M\chi}(G)$ due to the deletion of an edge is determined. The edge critical classifications of $E^0_{M\chi}(G), E^-_{M\chi}(G)$ and $E^+_{M\chi}(G)$ are discussed. The characterization theorems on connected and disconnected graphs are determined for $CER_{M\chi}$ and $UER_{M\chi}$. This work is published in "**International**

Journal of Recent Scientific Research". In the last section, the effects of the majority dom-chromatic number when the graph G is modified by adding an edge e from the complement of G between any two non-adjacent vertices of a graph are discussed. Then the classifications namely $\xi^{\circ}_{M\chi}(G), \xi^{+}_{M\chi}(G)$ and $\xi^{-}_{M\chi}(G)$, for any edge $e \in E(G^c)$ are investigated for connected as well as disconnected graphs. This concept is accepted in "**Communications in Mathematics and Applications**".

Chapter 6 discusses the definition of the connected majority dom-chromatic set of a connected and disconnected graphs. The majority dom-chromatic number $\gamma_{CM\chi}(G)$ is determined for product graphs such as Grid, Cylinder and Torus. The comparison of the parameters $\gamma_{CM}(G), \gamma_{cch}(G)$ and $\gamma_{CM\chi}(G)$ are studied. Also some inequalities of $\gamma_{CM\chi}(G)$ are established in terms of diameter of a graph. This concept is published in "**Advances and Applications in Mathematical Sciences**" **(Indexed in WEB of Science)**.

Finally, the total summary of the research work in the thesis highlighting all new findings developed using the newly coined concept of majority dom-chromatic set of a graph.

Chapter 2

Majority Dom-Chromatic Set of a Graph

Abstract

This chapter introduces a new notion majority dom-chromatic set (MDC-set) of a graph G. For a graph G, the majority dom-chromatic number $\gamma_{M\chi}(G)$ is investigated for some families of graphs. Bounds on $\gamma_{M\chi}(G)$ and its relationship with other graph theoretic parameters are studied. Some inequalities on majority dom-chromatic sets of a connected and disconnected graph G are determined. Also characterization theorems on $\gamma_{M\chi}(G)$ and majority dom-chromatic number for the complement of a bipartite graphs are investigated.

The contents of this chapter are published in

1. **Bulletin of Pure and Applied Sciences, Vol. 38E (Math & Stat.), No.1, 289-296 (2019), ISSN 0970 6577.**
2. **Turkish World Mathematical Journal of Applied and Engineering Mathematics, Vol. 11, Special Issue (2021), 30-41, ISSN 2146-1147.**

2.1 Introduction

In 2006, Swaminathan and Joseline Manora [40] introduced the concept "Majority Domination" with respect to sets with the idea of dominating atleast half of the vertices of a graph. They elucidated the parameter in various levels by establishing many results. They produced the exact values of $\gamma_M(G)$ for some classes of graphs. Also they developed some inequalities for $\gamma_M(G)$ and interesting results on it.

Graph coloring and domination are major areas in graph theory. These concepts also give rise to a number of practical applications in real life. In recent years, several graph-theoretic parameters that combine the concepts of domination and coloring have been investigated by several graph theorists effectively. One such parameter is the concept of dom-coloring which was introduced by [31] Janakiraman and Poobalaranjani. To find a dominating set having the same chromatic number as that of the graph, the chromatic preserving set (cp - set) is introduced to serve this purpose. Thus, a dom-chromatic set is a dominating cp-set. Its number $\gamma_{ch}(G)$ was defined and the exact values for various classes of graphs are determined.

They established bounds of $\gamma_{ch}(G)$ and more results on $\gamma_{ch}(G)$ for connected and disconnected graphs.

These two parameters $\gamma_M(G)$ and $\gamma_{ch}(G)$ gave the motivation to introduce new graph theoretical parameter "Majority Dominating Chromatic (MDC) Set of a graph" and "Majority Dominating Chromatic Number" on graphs. In this type of domination, the elements of the dominating set must be a majority dominating set S and the chromatic number of a graph must equal to the chromatic number of an induced subgraph S of G. This parameter is also called majority dom-chromatic set of G. Thus, majority dom-chromatic sets play a vital role in domination theory.

Organization of this chapter is as follows. The introduction is given in section 2.1 and it contains the motivation of defining the parameter majority dom-chromatic number in graphs. In section 2.2, the concept of majority dom-chromatic set of a graph G and its number $\gamma_{M\chi}(G)$ are defined with examples. The exact values of $\gamma_{M\chi}(G)$ for various families of graphs are determined in section 2.3. In the subsequent section 2.4 and section 2.5, characterization theorems, bounds on $\gamma_{M\chi}(G)$ and some inequalities on majority dom-chromatic set of connected and disconnected graphs are obtained. The relationship of $\gamma_{M\chi}(G)$ with other domination parameters $\gamma_M(G)$, $\gamma_{ch}(G)$ and $\gamma(G)$ is studied in section 2.6 with an example. In section 2.7 and section 2.8, results on $\gamma_{M\chi}(G)$ for the complement of graph G, connected bipartite and disconnected bipartite graphs. In section 2.9, bounds on $\gamma_{M\chi}(G)$ for both connected and disconnected bipartite graphs are investigated with regard to $diam(G)$ and $\Delta(G)$. Finally in section 2.10, algorithm for a MDC – set and its number $\gamma_{M\chi}(G)$ are given and real life application for this parameter is also given.

2.2 Majority Dom-Chromatic Set of a Graph

In this section, majority dom-chromatic set of a graph G, minimal majority dom-chromatic set and its number are defined. An example illustrating these definitions are also given.

Definition 2.2.1: A subset S of $V(G)$ is said to be Majority Dominating Chromatic Set (MDC- set) if (i) S is a majority dominating set of G and (ii) the induced subgraph of $\langle S\rangle$ satisfies $\chi(\langle S\rangle) = \chi(G)$. It is also called majority dom-chromatic set of G. The majority dom-chromatic set S is minimal if no proper subset S' of S such that S' is majority dom-chromatic set of G.

Definition 2.2.2: The minimum cardinality of a minimal majority dom-chromatic set of G is called a majority dom-chromatic number and is denoted by $\gamma_{M\chi(G)}$.

Example 2.2.3: Consider the following graph with $p = 11$ vertices.

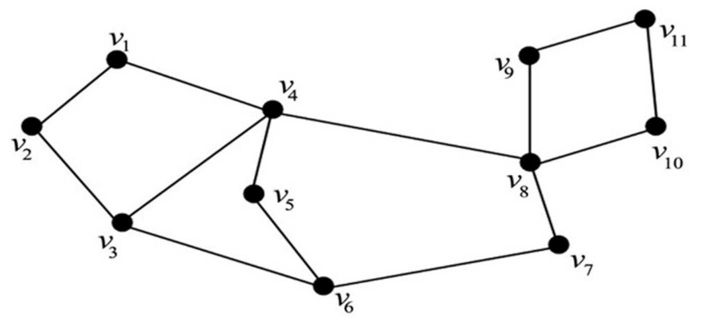

Figure 2.1: G

The chromatic number of G in Fig. (2.1) is $\chi(G) = 3$ and $\gamma_{ch}(G) = 7$.

(i) The sets $S_1 = \{v_1, v_4, v_5, v_6, v_7, v_8\}$, $S_2 = \{v_4, v_5, v_6, v_7, v_8\}$ and $S_3 = \{v_4, v_5, v_6, v_7, v_8, v_{11}\}$ are majority dom-chromatic sets where as $D = \{v_8, v_{11}\}$ is a majority dominating set of G. Therefore $\gamma_M(G) = 2$.

(ii) The set $S_2 = \{v_4, v_5, v_6, v_7, v_8\}$ is the minimal majority dom-chromatic set of G. Hence $\gamma_{M\chi}(G) = |S_2| = 5$.

Observation 2.2.4: (i) Since $V(G)$ is the majority dominating set and $\chi(\langle V(G)\rangle) = \chi(G)$, majority dom-chromatic set exists for all graphs.

(ii) For a vertex χ-critical (vertex color critical) graph, the vertex set $V(G)$ itself is the only majority dom-chromatic set for G.

For example, C_p, p is odd and K_p are vertex color critical graphs.

Proposition 2.2.5: For any graph $G, \gamma_{M\chi}(G) \leq \gamma_{ch}(G)$.

Proof: Since all the dom-chromatic sets of a graph G are majority dom-chromatic sets of $G, \gamma_{M\chi}(G) \leq \gamma_{ch}(G)$. ∎

Proposition 2.2.6: For any graph $G, \gamma_M(G) \leq \gamma_{M\chi}(G)$.

Proof: Since every MDC set of G is a majority dominating set of $G, \gamma_M(G) \leq \gamma_{M\chi}(G)$. ∎

Corollary 2.2.7: For any graph $G, \gamma_M(G) \leq \gamma_) M\chi(G) \leq \gamma_{ch}(G)$.

Example 2.2.8: (i) For the graph G in Fig (2.1), $\gamma_{M\chi}(G) = 5$, $\gamma_{ch}(G) = 7$ and $\gamma_M(G) = 2$. Hence, $\gamma_M(G) < \gamma_{(M\chi)}(G) < \gamma_{ch}(G)$.

(ii) For a star $G = K_{1,p-1}, \gamma_{M\chi}(G) = \gamma_{ch}(G) = 2$ and $\gamma_M(G) = 1$. Hence, $\gamma_M(G) < \gamma_{M\chi}(G) \leq \gamma_{ch}(G)$.

Theorem 2.2.9: For any graph G with an isolate, there exists a $\gamma_{M\chi}$- set of G not containing that isolate.

Proof: Let v be an isolate of G. If S is a $\gamma_{M\chi}$- set of G containing v then $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $\chi(\langle S \rangle) = \chi(G)$.

Case (i): If $|N[S]| > \lceil \frac{p}{2} \rceil$ then $|N[S - \{v\}]| \geq \lceil \frac{p}{2} \rceil$ and $\chi(\langle S - v \rangle) = \chi(G)$. It implies that $S - \{v\}$ is a $\gamma_{M\chi}$- set. Hence $S - \{v\} = S'$ is a $\gamma_{M\chi}$- set of G without an isolate v.

Case (ii): If $|N[S]| = \lceil \frac{p}{2} \rceil$ then $|N[S - \{v\}]| \leq \lceil \frac{p}{2} \rceil - 1$ and $v \notin N[S]$. Now, if $|N[S - \{v\}] \cup \{v_1\}| \geq \lceil \frac{p}{2} \rceil$, for any $v_1 \in V(G)$ then $S' = S - \{v\} \cup \{v_1\}$. Also, $\chi(\langle S' \rangle) = \chi(\langle S \rangle$ and $|S'| = |S| = \gamma_{M\chi}(G)$. Hence S' is a $\gamma_{M\chi}$- set of G without an isolate v. ∎

2.3 Majority Dom-Chromatic Number of Some Standard Graphs

In this section, the exact value of the majority dom-chromatic number $\gamma_{M\chi}(G)$ is determined for some classes of graphs.

2.3.1 Results on $\gamma_M\chi(G)$

(i) Let $G = mK_2, m \geq 1$ with $p = 2m$. Then $\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil + 1, p \geq 2$.

(ii) Let $G = \bar{K}_p$ be a totally disconnected graph of p vertices. Then $\gamma_{M\chi}(\bar{K}_p) = \lceil \frac{p}{2} \rceil$.

(iii) For the Petersen graph $P(10, 15), \gamma_{M\chi}(P) = 5$.

(iv) For a double star graph, $D_{r,s}, \gamma_{M\chi}(G) = 2$, if $r \leq s$.

(v) Let G be a caterpillar in which exactly one pendant at each vertex

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{8} \rceil + 1, \text{if } p \equiv 0, 5, 6, 7 (mod\ 8) \\ \lceil \frac{p}{8} \rceil, \text{if } p \equiv 1, 2, 3, 4 (mod\ 8). \end{cases}$$

Proposition 2.3.2: (i) Let $G = K_p, p \geq 1$ be a complete graph. Then $\gamma_{M\chi}(G) = p$.

(ii) Let $G = K_{1,p-1}$ be a star. Then $\gamma_{M\chi}(G) = 2$.

Proof: (i) Since $G = K_p$ is color critical, By observation (2.2.4)(ii), the vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_p\}$ is the MDC set of G. Hence $\gamma_{M\chi}(G) = |V(G)| = p$.

(ii) For a star $G = K_{1,p-1}, \chi(G) = 2$. The set $S = \{v_1, v_2\}$, where $d(v_1) = p-1$ and $d(v_1, v_2) = 1$ is the subset of G. Since $|N[S]| \geq \lceil \frac{P}{2} \rceil$ and $\chi(\langle S \rangle) = \chi(G), S$ is the MDC set of G and $\gamma_{M\chi}(G) = |S| = 2$. ∎

Proposition 2.3.3: Let $G = C_p$ be a cycle of p vertices, $p \geq 3$. Then

$$\gamma_{M\chi}(G) = \begin{cases} \left\lceil \frac{p}{6} \right\rceil, & \text{if } p \equiv 2(mod\ 6) \\ \left\lceil \frac{p}{6} \right\rceil + 1, & \text{if } p \equiv 0, 4(mod\ 6) \\ p, & \text{if } p \text{ is odd.} \end{cases}$$

Proof: Let $\{v_1, v_2, v_3, \cdots, v_p\}$ be a set of vertices of C_p and $d(v_i) = 2$, for all $v_i \in V(G)$. By the result (1.3.43) (v)

$$\chi(C_p) = \begin{cases} 2, & \text{if } p \text{ is even} \\ 3, & \text{if } p \text{ is odd.} \end{cases} \tag{2.1}$$

Case: **(i)** When $p \equiv 2(mod\ 6)$. Since C_p is even, $\chi(G) = 2$. Let $S = \{v_1, v_2, v_5, \cdots, v_{\gamma_{M\chi}(G)}\}$ be a majority dom-chromatic set of G such that $d(v_1, v_2) = 1$ and $d(v_i, v_j) = 3, i \notin j, i, j = 2, 5, \cdots, \gamma_{M\chi}(G)$ and $v_i, v_j \in S$. So that the induced sub graph $\langle S \rangle$ contains K_2 or $K_2 \cup tK_1, t > 0$. Then $|N[S]| \geq \lceil \frac{p}{2} \rceil$, where $|S| = \gamma_{M\chi}(G)$. By (2.1) and since $\chi(K_2) = 2, \chi(\langle S \rangle) = \chi(G)$.

Then $|N[S]| \leq \sum_{i=1}^{|S|} d(v_i) + \gamma_{M\chi}(G) - 1 \leq 3\gamma_{M\chi}(G) - 1$, and $\left\lceil \frac{p}{2} \right\rceil \leq |N[S]| \leq 3\gamma_{M\chi}(G) - 1$. Implies that $\gamma_{M\chi}(G) \geq \frac{1}{3}\left(\left\lceil \frac{p}{2} \right\rceil + 1\right)$. If $p = 6r + 2$ then $\frac{1}{3}\left(\left\lceil \frac{p}{2} \right\rceil + 1\right) = \frac{1}{3}\left(\left\lceil \frac{6r+2}{2} \right\rceil + 1\right) = \left\lceil \frac{p}{6} \right\rceil$. Therefore, $\gamma_{M\chi}(G) \geq \left\lceil \frac{p}{6} \right\rceil$.

Suppose the set $S = \{v_1, v_2, \cdots, v_t\} \subseteq V(G)$ with $d(v_i, v_j) = 3, i \notin j$ and exactly one pair $d(v_1, v_2) = 1$ and $|S| = |t| = \lceil \frac{p}{6} \rceil$. Then $|N[S]| = 3\left\lceil \frac{p}{6} \right\rceil - 2 = 3\left\lceil \frac{6r+2}{6} \right\rceil - 2 = 3\left(\frac{p-2}{6}\right) + 1 \geq \left\lceil \frac{p}{2} \right\rceil$. Since $d(v_1, v_2) = 1$, the induced subgraph $\langle S \rangle$ contains K_2. It implies that $\chi(\langle S \rangle) = 2 = \chi(G)$. Hence, the set S is a majority dom-chromatic set of G. Thus, $\gamma_{M\chi}(G) \leq |S| = \left\lceil \frac{p}{6} \right\rceil$. Combine the results, we obtain the result.

Case: (ii) Let $p \equiv 0, 4(mod\ 6)$.

Let $S = \{v_1, v_2, v_5, \ldots, v_{\gamma_{M\chi}(G)}\}$ be a majority dom-chromatic set of G with the same properties as in case (i). Now, since $(\frac{p}{2}) \equiv 2(mod\ 3), |N[S]| \leq \sum\limits_{i=1}^{\gamma_{M\chi}(G)} d(v_i) + \gamma_{M\chi}(G) - 4 \leq 3\gamma_{M\chi}(G) - 4$ and $\lceil \frac{p}{2} \rceil \leq |N[S]| \leq 3\gamma_{M\chi}(G) - 4$. It implies that $\gamma_{M\chi}(G) \geq \frac{1}{3}\left(\lceil \frac{p}{2} \rceil + 4\right)$. If $p = 6r$ then $\frac{1}{3}\left(\lceil \frac{6r}{2} \rceil + 4\right) = \left(\frac{p}{6} + 2\right) = \lceil \frac{p}{2} \rceil + 1$ and if $p = 6r + 4$ then $\frac{1}{3}\left(\lceil \frac{6r+4}{2} \rceil + 4\right)$. Therefore, $\gamma_{M\chi}(G) \geq \lceil \frac{p}{6} \rceil + 1$. Applying the same argument as in case (i), we obtain, $\gamma_{M\chi}(G) \leq \lceil \frac{p}{6} \rceil + 1$. Combining these results, $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil + 1$, if $p \equiv 0, 4(mod\ 6)$.

Case: (iii) When p is odd. Then by the result (1.3.43)(v), $\chi(C_p) = 3$. By observation (2.2.4)(ii), C_p is vertex χ - critical graph, and the vertex set $V(G)$ is the majority dom-chromatic set of G. Hence $\gamma_{M\chi}(G) = |V(G)| = p$, if p is odd. ∎

Corollary 2.3.4: Let G be a path on p vertices. Then its majority dom-chromatic number is

$$\gamma_{M\chi}(P_p) = \begin{cases} \lceil \frac{p}{6} \rceil, & \text{if } p \equiv 1, 2(mod\ 6) \\ \lceil \frac{p}{6} \rceil + 1, & \text{if } p \equiv 0, 3, 4, 5(mod\ 6). \end{cases}$$

Proof: Applying the same arguments as in proposition (2.3.3), we obtain the result. ∎

Proposition 2.3.5: For a complete bipartite graph $G = K_{m,n}$, $\gamma_{M\chi}(G) = 2$.

Proof: Let $G = K_{m,n}$. Then $\gamma_M(G) = 1$. Since $\chi(G) = 2, S = \{u_1, v_1\}$ is a majority dominating chromatic set of G such that $u_1 \in V_1$ and $v_1 \in V_2$ and $\chi(\langle S \rangle) = \chi(G)$. Therefore, $\gamma_{M\chi}(G) = 2$. ∎

Proposition 2.3.6: Let $G = W_p = C_{p-1} \vee K_1$ be a wheel graph with p vertices, $p \geq 5$. Then

$$\gamma_{M\chi}(W_p) = \begin{cases} 3, \text{if } p \text{ is odd} \\ p, \text{if } p \text{ is even.} \end{cases}$$

Proof: Let $G = W_p = C_{p-1} \vee K_1$. From the results in (1.3.43)(i),(vi),

$$\gamma_M(W_p) = 1 \text{ and } \chi(W_p) = \begin{cases} 3, \text{if } p \text{ is odd} \\ 4, \text{if } p \text{ is even.} \end{cases}$$

When $(p-1)$ is odd, C_{p-1} is vertex color critical graph, the vertex set $V(C_{p-1})$ is majority dom-chromatic set for the graph. When $(p-1)$ is even, two colors are enough to C_{p-1}. Therefore,

$$\gamma_{M\chi(G)} = \begin{cases} 3, & \text{if } (p-1) \text{ is even} \\ p, & \text{if } (p-1) \text{ is odd.} \end{cases}$$

Then for a graph $G = W_p$, we obtain the required result. ∎

Proposition 2.3.7: For a Fan graph with p vertices, $\gamma_{M\chi}(F_p) = 3,\ p \geq 3$.

Proof: Let F_p=$P_{p-1} \vee K_1$. Since $G = F_p$ has a full degree vertex and G contains triangles, $\gamma_M(G)$=1 and $\chi(G)$=3. Hence, $\gamma_{M\chi}(G) = 3$. ∎

2.4 Characterization Theorems and Bounds on $\gamma_{M\chi}(G)$

In this section, the characterization of a minimal majority dom-chromatic set of a graph G and bounds on $\gamma_{M\chi}(G)$ are discussed.

Theorem 2.4.1: Let $G(p, q)$ be any graph. A majority dom-chromatic set S of G is minimal if and only if for each $u \in S$, one of the following conditions holds.

(i) $\chi(\langle S - \{u\}\rangle) < \chi(G)$

(ii) $S - \{u\}$ is not a majority dominating set of G.

Proof: Let S be a minimal majority dom-chromatic set for G. Then G is a majority dominating set and $\chi(\langle S\rangle) = \chi(G)$. To prove that for each $u \in S$, either (i) or (ii) holds. Suppose $\chi(\langle S - \{u\}\rangle) = \chi(G)$. Then for any $u \in S$, $\langle S - \{u\}\rangle$ is a majority dom-chromatic set of G, which is a contradiction to S is minimal. Therefore condition (i) holds. Suppose for any vertex $u \in S, \langle S - \{u\}\rangle$ is a majority dominating set of G. Then the induced subgraph $\langle S - \{u\}\rangle$ such that $\chi(\langle S - \{u\}\rangle) = \chi(G)$, it is a contradiction to the assumption. Hence condition (ii) holds.

Conversely, suppose that S is not a minimal majority dom-chromatic set, then there exists a vertex $u \in S$ such that $\langle S - \{u\}\rangle$ is a majority dom-chromatic set of G. It implies that, $\langle S - \{u\}\rangle$ is a majority dominating set and $\chi(\langle S - \{u\}\rangle) = \chi(G)$, for any vertex $u \in S$, which is a contradiction to the conditions (i) and (ii). Hence the result. ∎

Proposition 2.4.2: A graph G is vertex color critical with order p if and only if $\gamma_{M\chi}(G) = p$.

Proof: Since every $\gamma_{M\chi}$- set is a γ_M- set of $G, \gamma_{M\chi}(G) \geq \gamma_M(G)$ and since any majority dom-chromatic set of G contains atleast one vertex from each color class $\gamma_{M\chi}(G) \geq \chi(G)$. Thus, the lower bound follows. For a color critical graph, $V(G)$ is the only MDC set of G and hence $\gamma_{M\chi}(G) \leq p$. The lower bound is sharp for $G = K_p$ or $G = \bar{K}_p$ and the upper bound attains for $G = C_p$, when p is odd. ∎

Proposition 2.4.3: Let G be any graph with p vertices. Then $\gamma_{M\chi}(G) = 1$ if and only if $G = K_1$ or $\bar{K}_2$.

Proof: Assume that $\gamma_{M\chi}(G) = 1$. Then by proposition (2.4.2), $\max\{\chi(G), \gamma_M(G)\} \leq \gamma_{M\chi}(G) = 1$. It implies that $\gamma_M(G) = 1$ and $\chi(G) = 1$. Then there is no edge in G. Hence $G = \bar{K}_p$, which is totally disconnected graph. But by the result in (**??**)(ii), $\gamma_{M\chi}(\bar{K}_p) = \lceil \frac{p}{2} \rceil$. So, when $p = 2, \gamma_{M\chi}(\bar{K}_2) = 1$. It implies that $G = \bar{K}_2$ or K_1. The converse is obvious. ∎

Proposition 2.4.4: Let G be any graph of order p. Then $\gamma_{M\chi}(G) = p$ if and only if G is vertex color critical.

Proof: Assume that G is a vertex colorcritical graph. Then $\chi(G-v)$ $< \chi(G)$, for any $v \in V(G)$. It implies that $\chi(G) = p$ and $\chi(G -$

$v) = p - 1$. Then $\gamma_M(G) \geq 1$. Since $\gamma_M(G) \geq 1$ and $\chi(G) = p$, the set $S = \{v_1, v_2, \ldots, v_p\}$ is the majority dom-chromatic set of G with $|S| = p$. Thus $\gamma_M\chi(G) \leq |S| = p$. By the proposition (2.4.2), $\gamma_{M\chi}(G) \geq \max\{\gamma_M(G), \chi(G)\}$. Then $\gamma_M\chi(G) \geq p$. Hence $\gamma_M\chi(G) = p$.

Conversely, the graph G on p vertices with $\gamma_M\chi(G) = p$. It means that $\chi(G) = p$ and $\gamma_M(G) \geq 1$. Hence $S = \{v_1, v_2, \ldots, v_p\}$ is a majority dom-chromatic set for G and $|S| = p$. Hence, $\chi(\langle S\langle) = p = \chi(G)$. It is clear that the graph G is either K_p or an odd cycle.

Claim: $\chi(G - v) < \chi(G)$, for any $v \in V(G)$.

Suppose that $G_1 = K_p$ or $G_2 = C_p, p$ is odd. Then by the proposition (2.3.2)(i), we would have $\gamma_M\chi(G_1) = p$ and by proposition (2.3.3), $\gamma_{M\chi}(G_2) = p, p$ is odd. It follows easily that $\chi(G_1) = p$ and $\chi(G_2) = 3, p$ is odd. For a subgraph $H = (G_1 - v), \chi(\langle H\rangle) = p - 1 < \chi(G_1)$. It shows that $G_1 = K_p$ is vertex color critical graph.

Consider now $H = (G_2 - v)$, the induced subgraph $\langle H\rangle$ is a path and its chromatic number $\chi(\langle H) = 2 < \chi(G_2)$. It implies that $G_2 = C_p, p$ is odd, is vertex color critical. As a result we obtain G is a vertex color critical graph. ∎

The following theorem gives the characterization of $\gamma_{M\chi}(G) = p - q$, where G is any graph with p vertices and q edges.

Theorem 2.4.5: Let G be a graph with p vertices and q edges. Then $\gamma_{M\chi}(G) = p - q$ if and only if $G = K_p, p = 1$.

Proof: The sufficiency follows by the fact that $\gamma_{M\chi}(G) = p-q, \gamma_{M\chi}(G) \geq 1, (p-q) \geq 1$.

Case: (i) Let the graph G be connected. Then $q \geq p-1 \Rightarrow (p-q) \leq 1$. Hence we get $p - q = 1$. Obviously G is a tree. In view of this property $\chi(G) = 2$ and and by the result (1.3.43)(ii) $1 \leq \gamma_M(G) \leq \lceil \frac{p}{6} \rceil$. Also by proposition (2.4.2), $\gamma_{M\chi}(G) \geq \max\{\chi(G), \gamma_M(G)\}$.

since $p - q = 1 = \gamma_{M\chi}(G)$, the two numbers $\chi(G)$ and $\gamma_M(G)$ must be one. When G is a tree and it has $\chi(G) = 2$ and $\gamma_M(G) = 1$, then the graph becomes $G = K_2$ and then $\gamma_{M\chi}(G) = 2$, but it is contradiction to $\gamma_{M\chi}(G) = p - q = 1$. Hence $G \notin K_2$.

Case: (ii) Suppose G is disconnected. Then the results (2.3.1)(ii), $\lceil \frac{p}{4} \rceil \leq \gamma_{M\chi}(\bar{K}_p) \leq \lceil \frac{p}{2} \rceil$. The lower bound is attained for $G = mK_2$. If $m = 1, \gamma_{M\chi}(K_2) = 2 \neq p-q = 1$ and the upper bound is attained for $G = \bar{K}_p$ when $p = 2$ then $\gamma_{M\chi}(\bar{K}_2) = 1 \neq p - q = 2$. Hence $G \neq \bar{K}_2$ or K_2. It follows that the graph must be $G = K_1$.

The converse is obvious. ∎

Next, result is the characterization of $|V - S| = 0$, where S is a MDC set of vertex color critical graph G.

Theorem 2.4.6: A majority dom-chromatic set S belongs to a vertex color critical graph if and only if $|V - S| = 0$.

Proof: Suppose $|V - S| = 0$. Then the majority dom-chromatic set $S = \{u_1, u_2, \cdots, u_p\} \subseteq V(G)$. It implies that $|S| = |V(G)|$. Suppose we remove one vertex from S then S may not be a majority dom-chromatic set of G. Hence G is vertex color critical graph.

Conversely by the proposition (2.4.4), if G is vertex color critical graph with p vertices then $\gamma_{M\chi}(G) = p$. Hence $|V - S| = 0$. ∎

Proposition 2.4.7: A majority dom-chromatic set S belongs to a vertex color critical graph if and only if $|V - S| = 0$.

Proof: If G is vertex color critical graph with p vertices, it follows that $\gamma_{M\chi}(G) = p$. Hence $|V - S| = 0$.

Conversely, $|V - S| = 0$. Then $|S| = |V(G)| = p$. Suppose if $\chi(G - v) < \chi(G)$, immediately it follows that G is vertex color critical graph. ∎

Theorem 2.4.8: Let G be a graph of order p with $\chi(G) \geq 3$ and it has no triangles. Then $\gamma_{M\chi}(G) \geq 5$.

Proof: Let $\chi(G) \geq 3$ and G has no triangles. Then $G \neq K_p$, complete graph and G is not a tree. Therefore, G contains a cycle. If $\chi(G) \geq 3$, then G contains only odd cycles with atleast $p \geq 5$. By the proposition (2.3.3), $\gamma_{M\chi}(C_p) = p,\ p$ is odd, $p \geq 5,$ and $\gamma_M(G) \geq 1$. Since $\chi(G) \geq 3, p \geq 5$, we obtain $\gamma_{M\chi}(G) \geq 5$. ∎

2.5 Inequalities on MDC Set

In this section, inequality between the sum of the degrees of all vertices of a majority dom-chromatic set S of G and the complement of S i.e. $(V - S)$ in a graph G is discussed. We determine some inequalities such as $|V - S| \leq \sum\limits_{v_i \in S} d(v_i)$ and $|V - S| \geq \sum_{v_i \in S} d(v_i)$ with respect to the majority dom-chromatic set S of a connected graph G.

Theorem 2.5.1: If S is a majority dom-chromatic set with two majority dominating vertices of a connected graph G then

$$|V - S| \leq \sum_{v_i \in S} d(v_i).$$

Proof: Let $v \in V(G)$ be a majority dominating vertex such that $d(v) \geq \lceil \frac{p}{2} \rceil - 1$ and $S = \{v_1, v_2\}$ be a majority dom-chromaticset with only two majority dominating vertices of G.

Case (i): The graph G is a tree. Then $d(v_i) \geq \lceil \frac{p}{2} \rceil - 1, i = 1, 2$ for all $v_i \in S$. It implies that $\chi(G) = 2, \gamma_M(G) = 1$ then

$$\sum_{v_i \in S} d(v_i) = d(v_1) + d(v_2) \geq \left\lceil \frac{p}{2} \right\rceil - 1 + \left\lceil \frac{p}{2} \right\rceil - 1$$

$$= \begin{cases} p-2, & \text{if } p \text{ is even} \\ p, & \text{if } p \text{ is odd} \end{cases}$$

$\sum\limits_{v_i \in S} d(v_i) = p - 2$ or p. Therefore, $|V - S| = p - 2 \leq \sum\limits_{v_i \in S} d(v_i)$.

Case (ii): The graph G is not a tree and G contains two majority dominating vertices. Then G is not complete but G consist of triangles. It implies that $\chi(G) = 3, \gamma_M(G) = 1$. Then $S = \{v_1, v_2, v_3\}$ be a majority dom-chromatic set of G where v_2 and v_3 are joined with a majority dominating vertex v_1 such that $d(v_1) = \Delta(G)$. Therefore, $\sum\limits_{v_i \in S} d(v_i) = d(v_1) + d(v_2) + d(v_3) \geq \lceil \frac{p}{2} \rceil + 4$. Hence, $|V - S| = p - 3 \leq \sum\limits_{v_i \in S} d(v_i)$. It implies that $|V - S| \leq \sum\limits_{v_i \in S} d(v_i)$. ∎

Example 2.5.2: Consider the following Hajas graph G with $p = 10$.

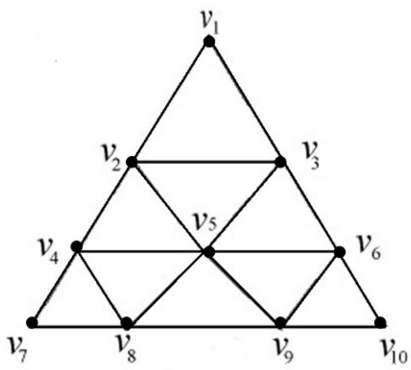

Figure 2.2: G

For $G, \chi(G) = 3, \gamma_M(H) = 1$. Then $S = \{v_2, v_3, v_5\}$ is the majority dom-chromatic set of G and $\sum\limits_{v \in S} d(v) = 4+4+6 = 14$ and $|V-S| = 7$.

Theorem 2.5.3: Let G be a non-trivial connected graph with atleast one full degree vertex. If S is a majority dom-chromatic set of G then $|V - S| < \sum\limits_{u_i \in S} d(u_i)$.

Proof: The graph G contains atleast one full degree vertex $u \in V(G)$. Then $d(u) = p - 1$.

Case (i): The graph G is a tree. Consider $S = \{u_1, u_2\}$ be the majority dom-chromatic set of G and $\chi(G) = 2$. Hence $|V - S| \leq p - 2$. Also $\sum\limits_{u_i \in S} d(u_i) = d(u_1) + d(u_2) \geq p - 1 + 1 = p$.
Hence, $|V - S| < \sum\limits_{u_i \in S} d(u_i)$.

Case (ii): The graph G is complete. Then the graph G contains vertices are of full degree vertices. Since $\chi(G) = p,\ S = \{u_1, u_2, \cdots, u_p\}$

is a majority dom-chromatic set of G. Therefore, $|V - S| = 0$ and $\sum_{u_i \in S} d(u_i) = p(p-1)$. It implies that, $|V - S| < \sum_{u_i \in S} d(u_i)$.

Case (iii) The graph G is not complete.

Subcase (i): If G has only one full degree vertex u and it is not a tree then G contains a triangle. Since $\chi(G) = 3, S = \{u_1, u_2, u_3\}$ is a majority dom-chromatic set of G. It implies that $|V - S| = p - 3$ and $\sum_{u_i \in S} d(u_i) = (p-1) + 3 + 3 = p + 5$. Hence, $|V - S| < \sum_{u_i \in S} d(u_i)$.

Subcase (ii): Suppose the graph G has two full degree vertices u_1 and u_2, then G contains a triangle. Hence, $\chi(G) = 3$. Let $S = \{u_1, u_2, u_3\}$ be a majority dom-chromatic set of G. Then $|V - S| = p - 3$ and $\sum_{u_i \in S} d(u_i) = (p-1) + (p-1) + 2 = 2p$. It implies that, $|V - S| < \sum_{u_i \in S} d(u_i)$. In all cases, the vertices of S majority dominates the graph G and also addition with its coloring number. Thus,

$|V - S| < \sum_{u_i \in S} d(u_i)$. ∎

Theorem 2.5.4: If a connected graph G contains only majority dominating vertices then $|V - S| \leq \sum_{u_i \in S} d(u_i)$, where S is the MDC-set of G.

Proof: Let G be a connected graph which contains only majority dominating vertices. Then $\gamma_M(G) = 1$ and $\chi(G) \geq 2$. Consider the set $S = \{u_1, u_2, \cdots, u_t\}$ with $|S| = t$ be a majority dom-chromatic set of G and $\gamma_{M\chi} = |S| \geq 2$. Then $|V - S| \leq p - 2$. Since G contains only majority dominating vertices, $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$, for each $u_i \in S$.

Case (i): The graph G has no triangles. Let $S = \{u_1, u_2\}$ be a majority dom-chromatic set of G. Then $\sum\limits_{u_i \in S} d(u_i) = d(u_1) + d(u_2) \geq \lceil \frac{p}{2} \rceil - 1 + \lceil \frac{p}{2} \rceil - 1 \geq p - 2$ and $|V - S| \leq \sum\limits_{u_i \in S} d(u_i)$.

Hence $|V - S| \leq \sum\limits_{u_i \in S} d(u_i)$.

Case (ii): The graph G has triangles. Then $\gamma_M(G) = 1$ and $\chi(G) \geq 3$. It implies that $S = \{u_1, u_2, u_3\}$ is a majority dom-chromatic set of G. Hence $|V - S| = p - 3$. Then $\sum\limits_{u_i \in S} d(u_i) \geq 3\left(\lceil \frac{p}{2} \rceil - 1\right) \geq \frac{3p}{2}$ or $\frac{3p}{2}$.

Hence, $|V - S| \leq \sum\limits_{u_i \in S} d(u_i)$. ∎

Theorem 2.5.5: If a connected graph G has no majority dominating vertices then $|V - S| \geq \sum\limits_{u_i \in S} d(u_i)$, where S is the MDC set of G.

Proof: Let S be the majority dom-chromatic set of a connected graph G of p vertices and q edges. Since the graph G has no majority dominating vertices, it contains all vertices with $d(u_i) < \lceil \frac{p}{2} \rceil - 1$. Assume that $S = \{u_1, u_2, \cdots\}$ be the majority dom-chromatic set of G. Then $|V - S| \leq p - 2, p > 6$.

Also, $\sum_{u_i \in S} d(u_i) = d(u_1) + d(u_2) + \cdots \leq \lceil \frac{p}{2} \rceil - 2 + \lceil \frac{p}{2} \rceil - 2 + \cdots \leq 2\lceil \frac{p}{2} \rceil - 4 \leq (p-2)$ or $(p-4)$.

Hence we obtain, $|V - S| \geq \sum_{u_i \in S} d(u_i)$. ∎

Theorem 2.5.6. *If a MDC set S contains a majority dominating vertex v and other vertices u_i such that $d(u_i) \leq \lceil \frac{p}{2} \rceil - 3$ then*

$$|V - S| > \sum_{u_i \in S} deg(u_i).$$

Proof: Let u be the majority dominating vertex such that $d(u) = \lceil \frac{p}{2} \rceil - 1$ and other vertices u_i with degree $d(u_i) \leq \lceil \frac{p}{2} \rceil - 3$ in G. Then $\gamma_M(G) = |\{u\}| = 1$ and $\chi(G) = 2$. Therefore $S = \{u, u_1\}$ is a MDC set of G and $|V - S| \leq p - 2$.

$$\text{Then } \sum_{u_i \in S} deg(u_i) = d(u) + d(u_1) \leq \left\lceil \frac{p}{2} \right\rceil - 1 + \left\lceil \frac{p}{2} \right\rceil - 3$$
$$\leq \begin{cases} \frac{p}{2} - 1 + \frac{p}{2} - 3 = p - 4, & \text{if } p \text{ is even} \\ \frac{p}{2} + \frac{p}{2} + 1 - 4 = p - 3, & \text{if } p \text{ is odd} \end{cases}$$

Therefore $\sum_{u_i \in S} deg(u_i) \leq (p-4)$ or $(p-3)$. Hence $|V - S| > \sum_{u_i \in S} deg(u_i)$. ∎

Corollary 2.5.7: If the majority dom-chromatic set S contains a majority dominating vertex and pendants only then $|V - S| > \sum_{u_i \in S} d(u_i)$.

Theorem 2.5.8: Let G be a connected graph with atleast one vertex v such that $\lceil \frac{p}{2} \rceil - 1 \leq d(v) \leq \lceil \frac{p}{2} \rceil + 2$. Then $|V - S| > \sum\limits_{v_i \in S} d(v_i)$, where S is majority dom-chromatic set which contains a vertex v.

Proof: Let $\lceil \frac{p}{2} \rceil - 1 \leq d(v) \leq \lceil \frac{p}{2} \rceil + 2,$ for any vertex, $v \in V(G)$

Case (i): The graph G is a tree. Let $S = \{v, u_1\}$ be a majority dom-chromatic set in which v is a pendent. Suppose $d(v) = \lceil \frac{p}{2} \rceil - 1$. Then $|V - S| = p - 2$. Now, $\sum\limits_{v_i \in S} d(v_i) = d(v) + d(u_1) = \lceil \frac{p}{2} \rceil - 1 + 1 = \lceil \frac{p}{2} \rceil$. It implies that, $|V - S| = p - 2 > \sum\limits_{v_i \in S} d(v_i)$.
Suppose $d(v) = \lceil \frac{p}{2} \rceil + 2$. Then $\sum\limits_{v_i \in S} d(v_i) = d(v) + d(u_1) = \lceil \frac{p}{2} \rceil + 2 + 1 = \lceil \frac{p}{2} \rceil + 3 < |V - S|$. But $\sum\limits_{v_i \in S} d(v_i)$ takes the value from $\lceil \frac{p}{2} \rceil$ to $\lceil \frac{p}{2} \rceil + 3$. Hence $|V - S| > \sum\limits_{v_i \in S} d(v_i)$.

Case (ii): The graph G is not a tree. Let S be a majority dom-chromatic set of G and let $S = \{v, v_1\}$ where v is a majority dominating vertex and v_1 is not a pendent of G. Then

$$\sum_{v_i \in S} d(v_i) = d(v) + d(v_1) \geq \left\lceil \frac{p}{2} \right\rceil - 1 + 2 = \left\lceil \frac{p}{2} \right\rceil + 1, \text{ if } d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1.$$

$$\sum_{v_i \in S} d(v_i) = \left\lceil \frac{p}{2} \right\rceil + 2 + 2 \geq \left\lceil \frac{p}{2} \right\rceil + 4, \text{ if } d(v) \leq \left\lceil \frac{p}{2} \right\rceil + 2.$$

Hence $|V - S| = p - 2 > \sum\limits_{v_i \in S} d(v_i)$. ∎

2.6 Relationship of $\gamma_{M\chi}(G)$ with $\gamma_M(G)$ and $\gamma_{ch}(G)$

Proposition 2.6.1: Let G be a complete bipartite graph with a majority dominating vertex. Then $\gamma_{M\chi}(G) = 2$ and $\gamma_M(G) < \gamma_{M\chi}(G)$.

Proof: Let $G = K_{m,n}, m \leq n$, be a complete bipartite graph.

Case (i): Since G has a majority dominating vertex $u_1, \gamma_M(G) = 1$ and $\chi(G) = 2$. Then $S = \{u_1, v_1\}$ is a majority dom-chromatic set of G, where $u_1 \in V_1(G)$ and $v_1 \in V_2(G)$. It implies that $\gamma_{M\chi}(G) = 2$ and $\gamma_M(G) < \gamma_{M\chi}(G)$.

Case (ii): If G is not a complete bipartite graph then G may contains pendants. Since G has a majority dominating vertex $u_1 \in V(G)$ and $\chi(G) = 2, S = \{u_1, v_1\}$ is a majority dom-chromatic set of G where $u_1 \in V_1(G)$ and $v_1 \in V_2(G)$. It implies that $\gamma_{M\chi} = 2$ and $\gamma_M(G) = 1$. Hence $\gamma_M(G) < \gamma_{M\chi}(G)$. ∎

Theorem 2.6.2: Let G be any graph on p vertices with dom-chromatic number $\gamma_{ch}(G)$. Then $\gamma_{M\chi}(G) \leq \lceil \frac{\gamma_{ch}(G)}{2} \rceil$, if $\gamma_{ch}(G)$ is odd and $\gamma_{M\chi}(G) = \left(\frac{\gamma_{ch}(G)}{2}\right) + 1$ if $\gamma_{ch}(G)$ is even.

Proof: Let S be the minimum dom-chromatic set of G. Then $\gamma_{ch}(G) = |S|$ and $|N[S]| = |V(G)|$. Let $S = S_1 \cup S_2$, where $|S_1| = \lceil \frac{\gamma_{ch}(G)}{2} \rceil$ and $|S_2| = \lfloor \frac{\gamma_{ch}(G)}{2} \rfloor$. Therefore $|N[S_1] \cup N[S_2] = ||N[S]| = |V(G)|$.

Case (i): Since every γ_{ch}- set of G is also a $\gamma_{M\chi}-$ set of G and $\gamma_{ch}(G) = |S|$ is odd, $|S| = |S_1| \cup |S_2|$ when $|N[S_1]| \geq \lceil \frac{p}{2} \rceil$ and $|N[S_2]| < \lceil \frac{p}{2} \rceil$. Since $\chi(\langle S \rangle) = \chi(G), \chi(\langle S_1 \rangle) = \chi(G)$ and $\chi(\langle S_2 \rangle) \neq \chi(G)$. It implies that S_1 is a $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G) \leq |S_1| = \lceil \frac{\gamma_{ch}(G)}{2} \rceil$, if $\gamma_{ch}(G)$ is odd.

Case (ii): Let $\gamma_{ch}(G) = |S|$ be even. Then $S = S_1 \cup S_2$ with $|S_1| = |S_2|$. But $|N[S]| = p$ and $|N[S_1]| < \lceil \frac{p}{2} \rceil$ and $|N[S_2]| > \lceil \frac{p}{2} \rceil$. If S_1 contains the vertices u_i, u_j such that $d(u_i, u_j) = 1$ then $\chi(\langle S_1 \rangle) = \chi(G)$. If S_2 contains the vertices u_i, u_j such that $d(u_i, u_j) \geq 3$ then $\chi(\langle S_1 \rangle) \neq \chi(G)$. Hence both S_1 and S_2 are not $\gamma_{M\chi}$- set of G. Let $S' = S_1 \cup \{u_k\}$, for any $u_k \in V - S_1$. Then $|N[S']| \geq \lceil \frac{p}{2} \rceil$ and $\chi(\langle S' \rangle) = \chi(G)$. It implies that S' is a $\gamma_{M\chi}$- set of G and $|S'| = |S|+1$. Therefore $\gamma_{M\chi}(G) = \left(\frac{\gamma_{ch}(G)}{2}\right) + 1$ if $\gamma_{ch}(G)$ is even. ∎

Example 2.6.3: (i) Let P_p be a path with $p \equiv 0(mod\ 6)$. Consider $G = P_{18}$ then $\gamma_{M\chi}(G) = 4$, $\gamma_{ch}(G) = 7$ and $\lceil \frac{\gamma_{ch}(G)}{2} \rceil = \lceil \frac{7}{2} \rceil = 4$. Hence $\gamma_{M\chi}(G) = \lceil \frac{\gamma_{ch}(G)}{2} \rceil$.

(ii) Let $G = S(K_1, t)$, then $S_1 = \{u_1, u_2, \cdots, u_t\}$ is a dom-chromatic set of G. It implies that $\gamma_{ch}(G) = |S_1| = t+1$ and $S_2 = \{u, u_1\}$ is a MDC of G. Hence $\gamma_{M\chi}(G) = 2$ and $\gamma_{M\chi}(G) < \lceil \frac{\gamma_{ch}(G)}{2} \rceil$.

Construction 2.6.4: For every integer $k \geq 0$, there exists a graph G such that $\lceil \frac{\gamma_{ch}(G)}{2} \rceil - \gamma_{M\chi}(G) = k$.

Proof: Let G be the subdivision of a Star $K_{1,2k+2}$ by dividing each edge exactly ones. Then $|V(G)| = 2(2k+2)+1, \gamma_{ch}(G) = 2k+2+1$ and $\gamma_{M\chi}(G) = 2$. Then $\lceil \frac{\gamma_{ch}(G)}{2} \rceil - \gamma_{M\chi}(G) = k+2-2 = k$. ∎

Observation 2.6.5: Let G be any Connected graph with p vertices. Let $\chi(G), \gamma_M(G)$ and $\gamma_{M\chi}(G)$ be the chromatic number of majority domination number and MDC number respectively. Then $\chi(G)$ and $\gamma_M(G)$ are not comparable. (i) $\gamma_M(G) < \chi(G) < \gamma_{M\chi}(G)$ and $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

2.7 $\gamma_{M\chi}$ for Complement of Graph

Proposition 2.7.1: Let the complete bipartite graph G with $diam(G) = 3$. Then $\gamma_{M\chi}(G) = \gamma_{M\chi}(\bar{G})$ if and only if $G = K_{2,2}$, where $\bar{G}$ is the complement of G.

Proof: Let the equality holds and let uv be the dominating edge of G. Let $|N[u]| = m, |N[v]| = n$ and $p = m + n$. In the graph $\bar{G}$, both $N(u)$ and $N(v)$ are of cardinality 2. The set $\{N(u) \cup N(v)\}$ is a K_{m+n-2} graph, $\chi(\bar{G}) = m+n-2$ and $\{N(u) \cup N(v)\}$ be the majority dom-chromatic set for $\bar{G}$. It implies that $\gamma_{M\chi}(\bar{G}) = m+n-2$. Since $\gamma_{M\chi}(G) = \gamma_{M\chi}(\bar{G}), \frac{m+n}{2} = m+n-2$. It implies that $m+n = 4$. Hence the graph must be $K_{2,2}$. The converse is obvious. ∎

Proposition 2.7.2: If the graph $G = K_p$ is the vertex color critical then $1 \leq \gamma_{M\chi}(\bar{G}) \leq \lceil \frac{p}{2} \rceil$.

Proof: Since the complete graph $G = K_p$ is the vertex color critical graph, $1 \leq \gamma_{M\chi}(G) \leq p$. The complement of K_p is $\bar{G} = \overline{K_p}$. By the proposition (2.3.1)(ii), the majority dom-chromatic number is $\gamma_{M\chi}(\bar{G}) = \lceil \frac{p}{2} \rceil$ and the lower bound attains for $\bar{G} = \overline{K_2}$. ∎

Proposition 2.7.3: Let $G = K_{m,n}, m \leq n$ and $m, n \geq 3$ be a complete bipartite graph. Then majority dom-chromatic number of a complement $\bar{G}$ is $\gamma_{M\chi}(\bar{G}) \geq \lceil \frac{p}{2} \rceil$ and $\gamma_{M\chi}(G) < \gamma_{M\chi}(\bar{G})$.

Proof: Let $\bar{G} = K_m \cup K_n$ be the complement of G where K_m and K_n both are complete graphs with m and n vertices.

Case (i): Suppose $m = n, n+1, n+2$. Since K_m and K_n are vertex color critical and $p = m+n, \gamma_{M\chi}(\bar{G}) = n, n+1, n+2$. Hence $\gamma_{M\chi}(\bar{G}) = \max\{m,n\}$.

Case (ii): Let $m < n$ and $n \geq m+3$. Since K_m and K_n are vertex color critical and $p = m+n, m < \lceil \frac{p}{2} \rceil$ and $n > \lceil \frac{p}{2} \rceil$. Hence $\gamma_{M\chi}(\bar{G}) = \max\{m,n\}$. If $G = K_{m,n}, m \leq n$, then by the proposition (2.3.5), $\gamma_{M\chi}(G) = 2$. By case (i), $\gamma_{M\chi}(\bar{G}) = n$ or $n+1 = \lceil \frac{p}{2} \rceil$ and $\gamma_{M\chi}(\bar{G}) = n+2 > \lceil \frac{p}{2} \rceil$. Then, $\gamma_{M\chi}(\bar{G}) = n$, if $m < n$. It implies that $\gamma_{M\chi}(\bar{G}) > \lceil \frac{p}{2} \rceil$. Hence, $\gamma_{M\chi}(G) < \gamma_{M\chi}(\bar{G})$, if $m, n \geq 3$. ∎

Proposition 2.7.4: Let G be a bipartite graph with $diam(G) \geq 6$. Then $\gamma_{M\chi}(\bar{G}) > \gamma_M(\bar{G}) + 1$, if $\bar{G}$ is the complement of G and $\gamma_M(\bar{G})$ is the majority domination number of $\bar{G}$.

Proof: If $diam(G) \geq 6$, then $G = P_p, p \geq 7$. The complement $\bar{G}$ contains two vertices with degree $\bar{d}(u_i) = p-2, i = 1, p$ and $\bar{d}(v_i) = p-3, i = 2, \cdots, p-1$. It gives that there are atleast two vertices with degree $\bar{d}(u_i) \geq \lceil \frac{p}{2} \rceil - 1$ and the majority domination number of $\bar{G}$ is $\gamma_M(\bar{G}) = 1$. Since $\bar{G}$ contains a triangle, $\chi(\bar{G}) = 3$ and $\gamma_{M\chi}(\bar{G}) \geq 3$. Hence, $\gamma_{M\chi}(\bar{G}) > \gamma_M(\bar{G}) + 1$. ∎

2.8 MDC Number for Bipartite Graph

In this section, the characterization theorems of $\gamma_{M\chi}(G)$, where the graph G is a bipartite are investigated.

Theorem 2.8.1: Let G be a connected bipartite graph with p vertices. Then $\gamma_{M\chi}(G) = 2$ if and only if $G_1 = K_{m,n}, m \leq n$, a Path $G_2 = P_i, i \leq 8$ and $G_3 = B_{X,Y}$ such that $|N[u_1] \cup N[v_1]| \geq \frac{p}{2}$ and $d(u_1, v_1) = 1$, where $u_1 \in V_1(G)$ and $v_1 \in V_2(G)$.

Proof: Let $\gamma_{M\chi}(G) = 2$. Then $\chi(G) = 2 = \chi(\langle S \rangle)$, where S is a majority dom-chromatic set of G with $|S| = 2$.

Case (i): Suppose $diam(G) = 1$ then the graph $G = K_p$. Since K_p is vertex color critical, $\gamma_{M\chi}(G) = p$. By assumption, the only graph $G = K_2 = K_{1,1} = G_1$ is a complete bipartite.

Case (ii): Suppose $diam(G) = 2$ then the graph G becomes $K_{m,n}, m \leq n, P_3$ and $K_{1,p-1}$, a star. Since $\gamma_{M\chi}(G) = 2$, we obtain the graph structures such as $G_1 = C_4 = K_{2,2}$ and $G_1 = K_{1,p-1}, G_2 = P_3$ and also $G_3 = B_{X,Y}$ includes the following structure with $diam(G) = 2$.

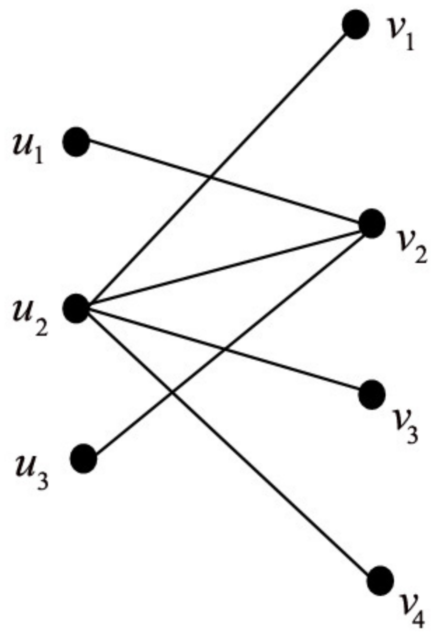

Figure 2.3: G_3

For the graph structure G_3, $S = \{u_2, v_2\} \subseteq V(G)$ such that $d(u_2, v_2) = 1, |N[S]| = |N[u_2] \cup N[v_2]| \geq \lceil \frac{p}{2} \rceil$ and $\chi(\langle S \rangle) = 2 = \chi(G)$. It implies that S is a majority dom-chromatic set of G_3. Hence $G_3 = B_{X,Y}$ with these properties.

Case (iii): Suppose $diam(G) = 3$. The bipartite graph G becomes P_4 and $D_{r,s}$, a double star. Since $\gamma_{M\chi}(G) = 2$, by the corollary (2.3.4), $\gamma_{M\chi}(P_4) = 2$. Hence $G_2 = P_4$. In $D_{r,s}, r \leq s$, by assumption, $S = \{u_1, v_1\}$ is the subset of G such that $d(u_1) \leq \lceil \frac{p}{2} \rceil - 1$, and $d(v_1) \geq \lceil \frac{p}{2} \rceil - 1$ with $d(u_1, v_1) = 1$, where $u_1 \in V_1(G)$ and $v_1 \in V_2(G)$ and $|N[S]| = |N[u_2] \cup N[v_2]| \geq \lceil \frac{p}{2} \rceil$. Also $\chi(\langle S \rangle) = 2 = \chi(G)$. Hence

S is a majority dom-chromatic set of G. It implies that $G_2 = B_{X,Y} = D_{r,s}, r \leq s$.

Case (iv) Suppose $diam(G) \geq 4$. Then the bipartite graphs are $P_p, p \geq 5$ and any bipartite graph $B_{X,Y}$. By the corollary (2.3.4), $\gamma_{M\chi}(P_p) = \lceil \frac{p}{6} \rceil = 2, p = 5, 6, 7, 8$ and $\gamma_{M\chi}(P_p) > 2$, if $p \geq 9$. Since $\gamma_{M\chi}(G) = 2$, the only bipartite graph $G_2 = P_5$ to P_8. For a bipartite graph $B_{X,Y}$, if $S = \{u_1, v_1\} \subseteq V(G)$ such that $|N[u_1] \cup N[v_1]| \geq \lceil \frac{p}{2} \rceil$ and $d(u_1, v_1) = 1$, where $u_1 \in V_1(G)$ and $v_1 \in V_2(G)$ with $diam(G) = 4$, then S is a majority dom-chromatic set of $B_{X,Y}$. Also, clearly $\chi(\langle S \rangle) = 2 = \chi(G)$ and satisfies the assumption. Hence the bipartite graph $G_3 = B_{X,Y}$ with the above said properties and also the only bipartite graphs are $G_2 = P_5$ to P_8.

Conversely, let $G_1 = K_{m,n}, m \leq n$ which is complete bipartite with $p = m + n$. Then by the proposition (2.3.5), $\gamma_{M\chi}(G) = 2$ and for a path $G_2 = P_i, i = 2, \cdots, 8$, by corollary (2.3.4), $\gamma_{M\chi}(G) = 2$. Let $G_3 = B_{X,Y}$ be a graph with bipartition $V_1(G)$ and $V_2(G)$. Let $u_1 \in V_1(G)$ and $v_1 \in V_2(G)$ such that $d(u_1, v_1) = 1$. Since $|N[u_1] \cup N[v_1]| \geq \frac{p}{2}$ and $\chi(\langle S \rangle) = 2 = \chi(G)$, $S = \{u_1, v_1\}$ is a majority dom-chromatic set of G and $\gamma_{M\chi}(G_3) = 2$. ∎

Proposition 2.8.2: Let G be any bipartite graph $B_{X,Y}$ with p vertices and without isolates. Then $\gamma_{M\chi}(G) \leq \lceil \frac{p}{4} \rceil + 1$ and $\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil + 1$ if and only if $G = K_{1,j}$ if $j = 1, 2, 3, K_{2,2}, P_4$ and $mK_2, m \geq 1$.

Proof: Let $G = B_{X,Y}$ be a bipartite graph with $\{u_1, u_2, \cdots, u_m\}$ and $\{v_1, v_2, \cdots, v_n\}$ and $|V(G)| = p = m + n$.

Case(i): Suppose $G = K_{m,n}$ is a complete bipartite with $m \leq n$. Let $S = \{u_1, v_1\}$, where $u_1 \in V(X)$ and $v_1 \in V(Y)$. Then $|N[S]| = |N[u_1]| + |N[v_1]| = (n + 1) + (m + 1) \geq \lceil \frac{p}{2} \rceil$. Therefore S is a majority dominating set of G. Since G is complete bipartite, $\chi(G) = 2 = \chi(\langle S \rangle)$. It implies that S is a majority dom-chromatic set of G. Hence $\gamma_{M\chi}(G) \leq |S| = 2 = \lceil \frac{p}{4} \rceil + 1$, where $p = 2, 3, 4$. Thus the graph becomes $G = K_{1,1}, K_{1,2}, K_{1,3}$ and $K_{2,2}$. When $p \geq 5$, for $G = K_{m,n}, m \leq n$, by the proposition (2.3.5), $\gamma_{M\chi}(G) = 2 < \lceil \frac{p}{4} \rceil + 1$. Hence, $\gamma_{M\chi}(G) \leq \lceil \frac{p}{4} \rceil + 1$, for $G = K_{m,n}, m \leq n$.

Case (ii): The graph G is not complete and connected bipartite. Then the minimally connected bipartite graph is a path $P_p, p \geq 2$. By the corollary (2.3.4), $\gamma_{M\chi}(P_p) = \lceil \frac{p}{6} \rceil$ or $\lceil \frac{p}{6} \rceil + 1$. Hence in this structure, when $p = 2, 3, 4, \gamma_{M\chi}(G) = 2 = \lceil \frac{p}{6} \rceil + 1 = \lceil \frac{p}{4} \rceil + 1$. When $p \geq 5, \gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil$ or $\lceil \frac{p}{6} \rceil + 1 < \lceil \frac{p}{4} \rceil + 1$. Hence, $\gamma_{M\chi}(G) \leq \lceil \frac{p}{4} \rceil + 1$, if $p \geq 2$.

Case (iii): The graph G is not complete and disconnected bipartite. Then the graph structure becomes mK_2, mP_4, mC_4 and mP_6. In such cases, by the result (2.3.1)(i), $\gamma_{M\chi}(mK_2) = \lceil \frac{p}{4} \rceil + 1$ and all other graphs the majority dom-chromatic number is $\gamma_{M\chi}(G) < \lceil \frac{p}{4} \rceil + 1$. Hence $\gamma_{M\chi}(G) \leq \lceil \frac{p}{4} \rceil + 1$. From the above cases, we obtain $\gamma_{M\chi}(G) \leq \lceil \frac{p}{4} \rceil + 1$.

Conversely, let $\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil + 1$. By case (i), if G is a complete bipartite graph, we obtain the graphs $G = K_{1,j}, j = 1, 2, 3$ and $K_{2,2}$. By case (ii), if G is not complete bipartite then the graphs are $G = P_2 = K_{1,1}, P_3 = K_{1,2}$ and P_4. Also by case (ii), if G is not complete and disconnected bipartite, the graph $G = mK_2, m \geq 1$. Hence $\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil + 1$ if and only if $G = K_{1,j}, j = 1, 2, 3, K_{2,2}, P_4$ and $mK_2, m \geq 1$. ∎

Proposition 2.8.3: Let G be any connected bipartite graph with p vertices. Then $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$ if and only if $G = P_3, P_4, C_4$ and $K_{1,3}$.

Proof: Assume that $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$. Since G is connected bipartite graph, $\chi(G) \geq 2$.

Case (i): If $diam(G) = 1$, then $G = K_2$ and $\gamma_{M\chi}(G) = 2 = p$, which is a contradiction to the assumption. Hence $G \neq K_2$.

Case (ii): If $diam(G) = 2$, then $G = P_3, C_4, K_{1,n}$. By the corollary (2.3.4), $\gamma_{M\chi}(P_3) = 2 = \lceil \frac{p}{2} \rceil$. By proposition (2.3.3), $\gamma_{M\chi}(C_4) = \lceil \frac{p}{2} \rceil$. Suppose $G = K_{1,3}$, by the proposition (2.3.5), $\gamma_{M\chi}(G) = 2 = \lceil \frac{p}{2} \rceil$.

Case (iii): If $diam(G) = 3$, then $G = P_4$ and $D_{r,s}$. By the corollary (2.3.4), $\gamma_{M\chi}(G) = 2 = \lceil \frac{p}{2} \rceil$. For $D_{r,s}$, by the proposition(**??**)(iv), $\gamma_{M\chi}(G) = 2 = \lceil \frac{p}{2} \rceil$, when $r = s = 1$.

Case (iv): If $diam(G) \geq 4$, then $G = P_p, C_p, p \geq 5$ and any other graphs. By the corollary (2.3.4), $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil + 1 = 2 < \lceil \frac{p}{2} \rceil$, which is a contradiction to the assumption.

Thus, from the above 4 cases, G must be P_3, P_4, C_4 and $K_{1,3}$. The converse is obvious. ∎

Observation 2.8.4: Suppose G is a disconnected bipartite graph. If the graph structures are $G_1 = K_{1,3} \cup mK_2, m$ is even and $m \geq 2, G_2 = mP_p, m = 4, p = 3$ and $G_3 = mK_{1,3}, m = 3$ then $\gamma_{M\chi}(G) = \frac{p}{4}$.

Corollary 2.8.5: Let G be a disconnected bipartite graph. If the graph structure is $K_{1,3} \cup mK_2, m$ is odd then $\gamma_{M\chi}(G) = \frac{p}{4} + 1$.

Proposition 2.8.6: Let G be a disconnected bipartite graph without isolates. Then $\gamma_{M\chi}(G) = \frac{p}{2}$ if and only if $G = mK_2, 1 < m \leq 3$.

Proof: Let $\gamma_{M\chi}(G) = \frac{p}{2}$. Since G is a disconnected bipartite graph, let $G_1, G_2, \cdots, G_k$ are the components of G and $V(G) = V(G_1) \cup \cdots \cup V(G_k)$.

Case (i): All components are of $diam(G) = 1$. Then the graph $G = mK_2$. By the assumption, when $G = mK_2$ if $m = 2$ and 3 then $G = 2K_2$ and $3K_2$. It implies that $\gamma_{M\chi}(G) = 2 = \frac{p}{2}$, if $m = 2$ and $\gamma_{M\chi}(G) = 3 = \frac{p}{2}$, if $m = 3$. Suppose $m \geq 4$, then by the proposition (**??**)(i), $\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil + 1 < \frac{p}{2}$. It is a contradiction to the assumption.

Case (ii): Suppose G contains the components which are of $diam(G) = 1$ and 2. Then $G = K_{1,t} \cup mK_2$, where $G_1 = K_{1,t}, G_2 = mK_2$ and $V(G) = \{u, u_1, \cdots, u_t, v_1, \cdots, v_{2m}\}$with $p = 1 + t + 2m$.

Subcase(i): If $|t| \geq \lceil \frac{p}{2} \rceil - 1$ and $2m = p - \left(\lceil \frac{p}{2} \rceil - 1\right)$ then the majority dom-chromatic set $S = \{u, u_1\}$ where $u, u_1 \in V(G_1)$ such that $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $\chi(G_1) = 2 = \chi(\langle S \rangle)$. It implies that S is a majority dom-chromatic set of G and $\gamma_{M\chi}(G) = 2 < \frac{p}{2}$, if $|t| \geq \lceil \frac{p}{2} \rceil - 1$, which is a contradiction to the assumption. Therefore $G \neq K_{1,t} \cup mK_2$.

Subcase (ii): If $|t| \leq \lceil \frac{p}{2} \rceil - 2$ then the MDC-set $S = \{u, u_1, v_1, v_2, \cdots, v_k\}$, where $|k| = \lceil \frac{p}{2} \rceil - (1 + t)$ such that $|N[S]| = 1 + t + 2k \geq \lceil \frac{p}{2} \rceil$.

Also $\chi(G) = 2 = \chi(\langle S \rangle)$. Hence $\gamma_{M\chi}(G) = |S| = (2+k) < \frac{p}{2}$, it is a contradiction. Hence the graph $G \neq K_{1,t} \cup mK_2$.

Case (iii): If the components G_i of G with $diam(G_i) \geq 2, i = 1, 2, \cdots, k$ then $\gamma_{M\chi}(G) < \frac{p}{2}$. From the above cases, we get the graph structures become $G = mK_2, 1 < m \leq 3$. Conversely, let $G = mK_2, m \leq 3$. Then by the proposition (**??**)(i), $\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil + 1 = \frac{p}{2}$. ■

Corollary 2.8.7: Let G be a disconnected graph which is not bipartite with isolates. Then $\gamma_{M\chi}(G) \leq \lceil \frac{p}{2} \rceil$ and $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$ if and only if $G = pK_1$.

2.9 Bounds of $\gamma_{M\chi}(G)$ for Bipartite Graphs

In this section, the bounds of $\gamma_{M\chi}(G)$ with respect to $\gamma_M(G)$ for a bipartite graph are established.

Proposition 2.9.1: Let G be a connected bipartite graph with p vertices. Then $\gamma_{M\chi}(G) = p$ if and only if $G = K_p, p = 2$.

Proof: Let G be a connected bipartite graph with p vertices. Since $\gamma_{M\chi}(G) = p$, then the graph must be a vertex color critical. The only connected bipartite vertex color critical graph is K_2. It implies that $G = K_2$. The converse is obvious. ∎

Proposition 2.9.2: If the graph G is bipartite with $diam(G) \leq 2$ then $\gamma_{M\chi}(G) \leq p - \Delta(G) + 1$ and $\gamma_{M\chi}(G) = p - \Delta(G) + 1$ if and only if $G = K_2, P_3$ and $K_{1,p-1}, p \geq 2$.

Proof: Let G be a bipartite graph with $diam(G) \leq 2$. The theorem is proved by induction on $\Delta(G)$. If $\Delta(G) = 1$, the graph G becomes K_2. By the proposition (2.3.2)(i), $\gamma_{M\chi}(G) = 2 = p - \Delta(G) + 1$, if $G = K_2$. If $\Delta(G) = 2$, the graph structures becomes P_p, a path and $K_{2,2}$. Since $diam(G) \leq 2$, and by corollary (2.3.4), $\gamma_{M\chi}(G) = 2 = p - \Delta(G) + 1$, if $G = p_3$ and $\gamma_{M\chi}(K_{2,2}) = 2 < p - \Delta(G) + 1$. Suppose $\Delta(G) = 3$. Then $G = K_{3,3}$. By the proposition(2.3.5), $\gamma_{M\chi}(K_{3,3}) = 2 < p - \Delta(G) + 1$. If $\Delta(G) \geq 4$ then the graph G becomes $K_{m,n}, m = n \geq 4$. By the proposition (2.3.5), $\gamma_{M\chi}(G) = 2 < p - \Delta(G) + 1$.

This is true for $\Delta(G) = 1, 2, 3, \cdots, (p-2)$. Suppose $\Delta(G) = p - 1$. Then the only bipartite graph $G = K_{1,p-1}$. By the proposition (**??**)(iv), $\gamma_{M\chi}(G) = 2 = p - \Delta(G) + 1$. Hence from the above cases, $\gamma_{M\chi}(G) \leq p - \Delta(G) + 1$. Also, $\gamma_{M\chi}(G) = p - \Delta(G) + 1$ is true if and only if $G = K_2, P_3$ and $K_{1,p-1}, p \geq 2$. ∎

Proposition 2.9.3: Let G be a bipartite graph with $diam(G) = 3$. Then $\gamma_{M\chi}(G) \leq p - \Delta(G)$. Also $\gamma_{M\chi}(G) = p - \Delta(G)$ if and only if $G = P_4$ and $D_{r,s}, r = 1$ and $s = p - 3$.

Proof: Let G be a bipartite graph with $diam(G) = 3$. By the result (1.3.43)(vii), $\gamma_{ch}(G) \leq p - \Delta(G)$. Since $\gamma_{M\chi}(G) \leq \gamma_{ch}(G), \gamma_{M\chi}(G) \leq \gamma_{ch}(G) \leq p - \Delta(G)$ and $\gamma_{M\chi}(G) \leq p - \Delta(G)$. Let $\gamma_{M\chi}(G) = p - \Delta(G)$.

Case (i): Since $diam(G) = 3$, the graph G has a dominating edge uv with some pendants at u and v. Let $V(G) = \{u, v, u_1, \cdots, u_r, v_1, v_2, \cdots, v_s\}$ where $u_i, i = 1, \cdots, r$ and $v_j, j = 1, \cdots, s$ are pendants with $r \leq p - 3$ and $s \geq 1$. Clearly, since G is bipartite, $\chi(G) = 2$. By the assumption, $S = \{u, v, v_1, \cdots, v_t\}$ is a majority dom-chromatic set with $|S| = p - \Delta(G)$.

Subcase (i): Let $d(u) = p - 2$ and $d(v) = 2$. Since G has a dominating edge $e = uv, \gamma_{M\chi}(G) = |S| = 2$. By the assumption, $\gamma_{M\chi}(G) = p - \Delta(G)$. It implies that $2 = p - d(u) \Rightarrow 2 = p - (p - 2)$. It gives the structure of the graph G with $d(u) = p - 2, d(v) = 2$ and the graph is $G = D_{r,s}, r < s$ with $r = 1$ and $s = p - 3$.

Subcase (ii): Let $d(u) \leq p - 3$ and $d(v) \geq 3$. The majority dom-chromatic set for the graph G is $S = \{u, v\}$. It implies that $\gamma_{M\chi}(G) =$

$|S| = 2$. By the assumption, $\gamma_{M\chi}(G) = p - \Delta(G) = p - d(u) = p - (p-3) = 3$. Hence, $\gamma_{M\chi}(G) < p - \Delta(G)$.

Subcase (iii): If $d(u) = p-2$ and $d(v) = p-2 = \Delta(G)$ then the majority dom-chromatic set becomes $S = \{u, v\}$. It implies that $\gamma_{M\chi}(G) = |S| = 2$. By the assumption, $\gamma_{M\chi}(G) = p - \Delta(G) = p - d(u) \Rightarrow 2 = p - (p-2)$. Since $d(u) = p-2$ and $d(v) = p-2, r = s = 1 \Rightarrow p = r+s+2 = 4$. Hence the graph G with $p = 4$ vertices and $diam(G) = 3$ is P_4.

Case (ii): Suppose G has no dominating edge $e = uv$. Then the graph G is a wounded spider with $diam(G) = 3$ and the graph contains a vertex u with $d(u) = \frac{p}{2} = \Delta(G)$ and $d(u_i) \leq 2, u_i \in (V(G) - \{u\})$. Hence $S = \{u, u_1\}$ be the majority dom-chromatic set of G with $d(u_1) = 2$, where $d(u, u_1) = 1$ and $\gamma_{M\chi}(G) = |S| = 2$. By the assumption, $\gamma_{M\chi}(G) = p - \Delta(G) = p - \frac{p}{2} = \frac{p}{2}$. Therefore $\gamma_{M\chi}(G) < p - \Delta(G)$.

Thus, $\gamma_{M\chi}(G) = p - \Delta(G)$ if and only if $G = P_4$ and $D_{r,s}, r = 1$ and $s = p-3$. Hence the result. ∎

Proposition 2.9.4: If G be a bipartite graph of $diam(G) = 3$ then $\gamma_{M\chi}(G) = \gamma_M(G) + 1$.

Proof: Let G be a connected bipartite graph with $diam(G) = 3$. Then the graph G has the structure with two central vertices u and v which are adjacent with some pendants. Then $G = P_4$ and $G = D_{r,s}, r \leq s$ where r and s number of pendants at u and v respectively. Since u and v are MD vertices of G, $\gamma_M(G) = |\{v\}| = 1$.

Case (i): If $s = r, r+1, r+2$ then both u and v are adjacent to some number of pendant vertices. Since $\chi(G) = 2, S = \{u, v\}$ be the majority dom-chromatic set of G and $\gamma_{M\chi}(G) = |S| = 2$. Hence $\gamma_{M\chi}(G) = \gamma_M(G) + 1$.

Case(ii): If $r < s$ and $s \geq r + 3$. Choose $S = \{u, v\}$, where u and v are central vertices of G. Then $|N[S]| = d(u) + d(v) = r + s + 2 = p > \lceil \frac{p}{2} \rceil$. Therefore, S is majority dominating set of G. Also, $\chi(G) = 2 = \chi(\langle S \rangle)$. Hence S will be the majority dom-chromatic set of G and $\gamma_{M\chi}(G) = |S| = 2$. Since $\gamma_M(G) = 1, \gamma_{M\chi}(G) = \gamma_M(G) + 1$. This result is true for $G = P_4$. ∎

Proposition 2.9.5: Let G be a bipartite graph of $diam(G) \leq 5$. Then $\gamma_{M\chi}(G) = \gamma_M(G) + 1$.

Proof: Since the graph G is bipartite, the graph structures are $P_p, p \leq 6, K_{1,n}, C_4$ and K_2.

Case (i): Suppose $diam(G) = 1$, then the bipartite graph G becomes only K_2. Then $\gamma_M(G) = 1$ and $\chi(G) = 2$ and by proposition (2.3.2), $\gamma_{M\chi}(G) = 2 = \gamma_M(G) + 1$.

Case (ii): If $diam(G) = 2$, then the graph structures becomes $G = P_3$ or $K_{1,n}$. By the result (1.3.43)(ii), $\gamma_M(G) = 1$. Also, by corollary (2.3.4), $\gamma_{M\chi}(G) = 2$. In both graphs, $\gamma_{M\chi}(G) = \gamma_M(G) + 1$.

Case (iii): Let $diam(G) = 3$. Then the graph becomes $G = P_4$ or and $D_{r,s}$. By proposition (2.9.4), the result is true.

Case (iv): When $diam(G) = 4$ and 5, the bipartite graph is $P_p, p \leq 6$. By the result (1.3.43)(ii), $\gamma_M(G) = 1$. Since $\chi(G) = 2$, the set $S = \{v_2, v_3\}$ be the majority dom-chromatic set of G, where $v_2, v_3 \in V(P_5)$. Hence $\gamma_{M\chi}(G) = 2 = \gamma_M(G) + 1$. Hence for all cases, $\gamma_{M\chi}(G) = \gamma_M(G) + 1$. ∎

Proposition 2.9.6: Let G be a bipartite graph with $diam(G) \geq 6$. Then (i) $\gamma_{M\chi}(G) = \gamma_M(G)$, if $p = 1, 2(mod\ 6)$

(ii) $\gamma_{M\chi}(G) = \gamma_M(G) + 1$, if $p = 0, 3, 4, 5(mod\ 6)$.

Proof: If the bipartite graph G with $diam(G) \geq 6$, then $G = P_p$, a Path with $p > 6$. By the result (1.3.43)(ii), $\gamma_M(G) = \lceil \frac{p}{6} \rceil$, for all $p \geq 7$ and by corollary (2.3.4),

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{6} \rceil = \gamma_M(G), \text{ if } p \equiv 1, 2 (mod\ 6) \\ \lceil \frac{p}{6} \rceil + 1 = \gamma_M(G) + 1, \text{ if } p \equiv 0, 3, 4, 5 (mod\ 6). \end{cases}$$

Hence the result. ∎

Proposition 2.9.7: Let G be a 3-regular bipartite graph with p vertices. Then

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{8} \rceil, \text{ if } p \equiv 2, 4 (mod\ 8) \\ \lceil \frac{p}{8} \rceil + 1, \text{ if } p \equiv 0, 6 (mod\ 8). \end{cases}$$

Proof: Let $V_1(G) = \{v_1, v_2, \cdots, v_{\frac{p}{2}}\}$ and $V_2(G) = \{u_1, u_2, \cdots, u_{\frac{p}{2}}\}$ with $p = 2m$.

Case (i): Let $p \equiv 2, 4 (mod\ 8)$. Let $S = \{v_1, u_1, v_j, v_{j+1}, \cdots, v_{j+r}\}$ be the subset of G with $|S| = t = \gamma_{M\chi}(G)$ such that $d(v_1, u_1) = 1$ and $d(v_i, u_1) \geq 4$. Then $|N[S]| = |N[v_1] + N[u_1]| + \sum_{j=1}^{t-2} d(v_j) - (t-2) = 6 + 4(t-2) = 4t - 2 \geq \lceil \frac{p}{2} \rceil$.

Let $p = 8r+2$. Then $|N[S]| = 4t-2 = 4\lceil \frac{p}{8} \rceil - 2 = \frac{p}{2} - 2 + 2 = \lceil \frac{p}{2} \rceil$. Let $p = 8r + 4$. Then $|N[S]| = 4t - 2 = 4\lceil \frac{p}{8} \rceil - 2 = \frac{p}{2} - 2 + 2 = \lceil \frac{p}{2} \rceil$. Since $d(v_1, u_1) = 1$, the induced subgraph $\langle S \rangle$ contains K_2 and $\chi(\langle S \rangle) = 2 = \chi(G)$. Thus S is a majority dom-chromatic set of G and $\gamma_{M\chi}(G) \leq |S| = \lceil \frac{p}{8} \rceil$.

Suppose that $S = \{v_1, u_1, v_j, \cdots, v_{j+r}\}$ with $|S| = t = \gamma_{M\chi}(G)$ such that $d(v_1, u_1) = 1, d(v_i, v_j) \geq 4$ and $|N[S]| \geq \lceil \frac{p}{2} \rceil$. Since S contains the induced subgraph K_2 and $\chi(\langle S \rangle) = 2 = \chi(G)$. Therefore $|N[S]| \leq 4t = 4\gamma_{M\chi}(G)$. Since $|N[S]| \geq \lceil \frac{p}{2} \rceil, \lceil \frac{p}{2} \rceil \leq 4\gamma_{M\chi}(G)$. It implies that $\gamma_{M\chi}(G) \geq \frac{1}{4}\lceil \frac{p}{2} \rceil$. Hence $\gamma_{M\chi}(G) \geq \lceil \frac{p}{8} \rceil$. Combining these two results, $\gamma_{M\chi}(G) = \lceil \frac{p}{8} \rceil$, if $p \equiv 2, 4(mod\ 8)$.

Case (ii): Let $p \equiv 0, 6(mod\, 8)$. Let $S_1 = \{v_1, u_1, v_j, \cdots, v_{j+r}\}$ be the subset of $V(G)$ with $|S_1| = t_1 = \lceil \frac{p}{8} \rceil + 1 = \gamma_{M\chi}(G)$ and $\chi(\langle S_1 \rangle) = 2$. Let $p = 8r$. Then $|N[S_1]| = 4t - 2 = 4\left(\lceil \frac{p}{8} \rceil + 1\right) - 2 = 4\lceil \frac{p}{8} \rceil + 2 > \lceil \frac{p}{2} \rceil$. Let $p = 8r + 6$. Then $|N[S_1]| = 4t_1 - 2 = 4\left(\lceil \frac{p}{8} \rceil + 1\right) - 2 = 4\lceil \frac{p}{8} \rceil + 2 > \lceil \frac{p}{2} \rceil$. Hence $|N[S_1]| \geq \lceil \frac{p}{2} \rceil$ and $\chi(\langle S_1 \rangle) = \chi(G)$. Therefore S_1 is a majority dom-chromatic set of G and $\gamma_{M\chi}(G) \leq |S_1| = t_1 = \lceil \frac{p}{8} \rceil + 1$. Applying the same arguments as in case (i), $\gamma_{M\chi}(G) \geq \lceil \frac{p}{8} \rceil + 1$. Hence $\gamma_{M\chi}(G) = \lceil \frac{p}{8} \rceil + 1$, if $p \equiv 0, 6(mod\ 8)$. ∎

2.10 Algorithm and Applications for a MDC Set of Graph G

2.10.1 Algorithm for MDC Set of a Graph

To find a MDC set for the given graph G with pvertices andq edges.

Step 1: Find the chromatic number χ for the given graph G.

Step 2: Choose a vertex v such that $d(v) = \Delta(G)$ and the set $S = \phi$.

Step 3: Select the vertex set $S \subseteq V(G)$ which contains the vertex v and obtain its induced subgraph $\langle S \rangle$.

Step 4: Find the chromatic number for the induced subgraph $\langle S \rangle$ and verify $\chi(\langle S \rangle) = \chi(G)$. If it is not true then go to step (3). If it is true then go to step (5).

Step 5: Find the neighborhoods of S and check whether the set S satisfies $|N[S]| \geq \lceil \frac{p}{2} \rceil$ or not. If $|N[S]| \geq \lceil \frac{p}{2} \rceil$ then the set S is the MDC set of G.

Step 6: If $|N[S]| < \lceil \frac{p}{2} \rceil$, choose a vertex u from $V - S$ with the next maximum degree and form a new set $S_1 = \{v, v_i, u\}$ such that $d(v_i, u) \geq 3, d(v, u) \geq 3$ where $v_i \in S$. Then go to step (5).

Hence from the above steps we could find all MDC sets of a given graph S.

2.10.2 Algorithm for $\gamma_M\chi$ of Graph G

Using Algorithm (2.10.1), find out all the Majority Dom-Chromatic sets for the given graph G.

Step 1: Let $S' = \{S_1, S_2, \cdots, S_t\}$ be the set of all majority dom-chromatic sets of G.

Step 2: Verify that the proper subset S_i' of each $S_i \in S', i = 1, 2, \cdots, t$ is a majority dom-chromatic sets of G.

Step 3: If the proper subset S_i' of S_i is a majority dom-chromatic sets of G then the set S_i is not a minimal majority dom-chromatic set of G.

Step 4: Suppose there exists no such majority dom-chromatic subset S_i' in the set S_i then S_i is a minimal majority dom-chromatic sets of G.

Step 5: Repeat the process to every $S_i \in S'$ and collect all the minimal majority dom-chromatic set of G.

Step 6: Let $S' = \{S_1, S_2, \cdots, S_r\}, r \leq t$ be the set of all minimal majority dom-chromatic set of G.

Step 7: Find the cardinality of each set $S_i \in S', i = 1, 2, \cdots, r$. Pick up the minimum cardinality of S_i among all S_i's in S'.

Step 8: The minimum cardinality of $S_i \in S'$ is the majority dom-chromatic set of G. It is denoted by $\gamma_M \chi(G)$.

2.10.3 Applications of MDC Number

When frequencies are assigned to towers, frequencies assigned to all towers at the same location must be different. How to assign frequencies with this constraints ? What is the minimum number of frequencies needed? Due to the minimum financial constraints this has to be done at as minimum cost. It is possible that, it does not bother about facilities reduction or increasing the number of locations. It is possible to identify the best thing that, MDC set can be done to the villages if the geographical structure is known.

If the villages are marked as vertices and roads are marked as edges and finding the majority dom-chromatic number of the graph representing the communal structure. Majority dom-chromatic concept can be used in security system also.

Suppose we product a building at all entries by attaching various security devices with the least number at the entrance, the building may be represented by a graph with the entries as vertices and adjacency can be done if two the entries can be viewed form one another. Hence finding majority dom-chromatic number gives the best solution.

Chapter 3

Majority Dom-Chromatic Number for Special Graphs

Abstract

In this chapter, majority dom-chromatic sets are discussed for various graph structures. The majority dom-chromatic (MDC) number $\gamma_{M\chi}$ is determined for Corona graphs, Cartesian Product graphs, Generalized Petersen graphs and Rooted product graphs. The characterization on MDC number is established for disconnected graphs with isolates and without isolates. Also some inequalities on the complement of a MDC set S namely, $|V - S|$ and $\sum_{u_i \in S} d(u_i)$ are investigated for disconnected graphs.

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3.1 Introduction

In 1970, Faudre and Schelp [18] studied a product graphs in "The Domination Number for the Product of Graphs" and in 1997, Gravier and Mollard [19] studied Cartesian Products of Paths in "On Domination Numbers of Cartesian Products of Paths". Then in 2012, Jankiraman and Poobalaranjani [31] studied Cartesian Product graphs with respect to $\gamma_{CM}(G)$. In 2017, Joseline Manora and Muthukani Vairavel [35, 36] determined many results on product graphs with respect to $\gamma_{CM}(G)$. They produced the exact values of $\gamma_{CM}(G)$ for some standard graphs. These concepts gave the motivation to investigate $\gamma_{CM}(G)$ on product graphs and corona graphs.

Organization of this chapter is as follows. The introduction of this chapter is given in section 3.1. In section 3.2, the exact values of $\gamma_{M\chi}$ for Corona graphs are determined. In section 3.3, the particular values of MDC number are investigated for Cartesian Product graphs. The MDC number for generalized Petersen graphs and Rooted Product graphs are studied in section 3.4 and 3.5. In section 3.6, the characterisation on $\gamma_{M\chi}$ and some inequalities on $|V - S|$ and $\sum\limits_{u_i \in S} d(u_i)$ are investigated for disconnected graphs.

3.2 $\gamma_{M\chi}$ for Corona Graph

In this section, the majority dom-chromatic number $\gamma_{M\chi}$ for corona graphs with respect to cycles, complete graph and complete bipartite graph are determined.

Proposition 3.2.1: Let $G = C_n \circ K_2$ be a corona graph with p vertices. Then $\gamma_{M\chi}(G) = \lceil \frac{p}{10} \rceil + 2$.

Proof: Let $G = C_n \circ K_2$ be a corona graph. Let $V(G) = \{u_1, u_{11}, u_{12}, u_2, u_{21}, u_{22}, \ldots, u_n, u_{n1}, u_{n2}\}$ and $|V(G)| = p = 3n$, where $u_i \in V(C_n)$ and $u_{ij} \in V(K_2), i \neq j, 1 \leq i \leq n, j = 1, 2$. Since this graph structure contains n triangles, $\chi(G) = 3$. Let $S = \{u_1, u_{11}, u_{12}, u_4, \ldots, u_t\}$ be the subset of $V(G)$ with $|S| = \lceil \frac{p}{10} \rceil + 2$ such that $d(u_i, u_j) \geq 3$ and $\{u_1, u_{11}, u_{12}\}$ is the triangle in G. Since every vertex in S have degree 4, $|N(S)| = 5t - 10 = 5\left(\lceil \frac{p}{10} \rceil + 2\right) - 10 \geq \lceil \frac{p}{2} \rceil$. Since S contains a triangle, $\chi(\langle S \rangle) = 3 = \chi(G)$. Hence S is a majority dom-chromatic set of G and $\gamma_{M\chi}(G) \leq \lceil \frac{p}{10} \rceil + 2$.

Suppose $S' = S - \{u_i\}$ with $|S'| = |S| - 1 = \lceil \frac{p}{10} \rceil + 1$. Then $|N(S')| = 5t - 10 = 5\left(\lceil \frac{p}{10} \rceil + 1\right) - 10 < \lceil \frac{p}{2} \rceil$. Hence S' would not be a majority dominating set for G and $\gamma_{M\chi}(G) > |S'| \geq \lceil \frac{p}{10} \rceil + 2$. Thus, $\gamma_{M\chi}(G) = \lceil \frac{p}{10} \rceil + 2$. ∎

Proposition 3.2.2: Let $G = K_t \circ K_{m,n}$ be a Corona graph with p vertices and $t \geq 3, m, n \geq 3$. Then $\gamma_{M\chi}(G) = t$.

Proof: Let $G = K_t \circ K_{m,n}$ be a Corona graph with $p = t(m+n+1)$ vertices. Since this graph structure contains a vertex color critical graph K_t as a subgraph, $\chi(G) = t, t \geq 2$. Let $\{v_1, v_2, \ldots, v_t\} \subseteq V(K_t)$ and $\{u_1, u_2, \ldots, u_m, w_1, w_2, \ldots, w_n\} \subseteq V(K_{m,n})$ be the vertex sets of G. Since K_t is vertex color critical as a subgraph, any $\gamma_{M\chi}$- set S of G must contain the full vertex set of K_t. Since each vertex of K_t is adjacent to all vertices of $K_{m,n}, \chi(\langle S\rangle) = t = \chi(G)$ and $|N(S)| > \lceil \frac{p}{2} \rceil$. Hence S is a majority dom-chromatic set of G and $\gamma_{M\chi}(G) = t$. ■

Example 3.2.3: Consider the graph $G = K_3 \circ K_{3,2}$ with $p = 18$ vertices.

Since G contains a vertex color critical $K_3, \chi(G) = 3$. Hence $S = \{v_1, v_2, v_3\} \subseteq V(K_3)$ is a $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G) = 3$.

Proposition 3.2.4: Let $G = C_m \circ C_n$ be a Corona graph with p vertices and $m = 3$, $n > 3$. Then $\gamma_{M\chi}(G) = m$.

Proof: Let $G = C_m \circ C_n$ be a corona graph with p vertices and $m = 3, n > 3$. Let $V(G) = \{u_1, u_{11}, u_{12}, \ldots, u_{1n}, u_2, u_{21}, u_{22}, \ldots, u_{2n}, u_3, u_{31}, \ldots, u_{3n}\}$ and $|V(G)| = p = m(n+1)$, where $u_i \in V(C_m)$ and

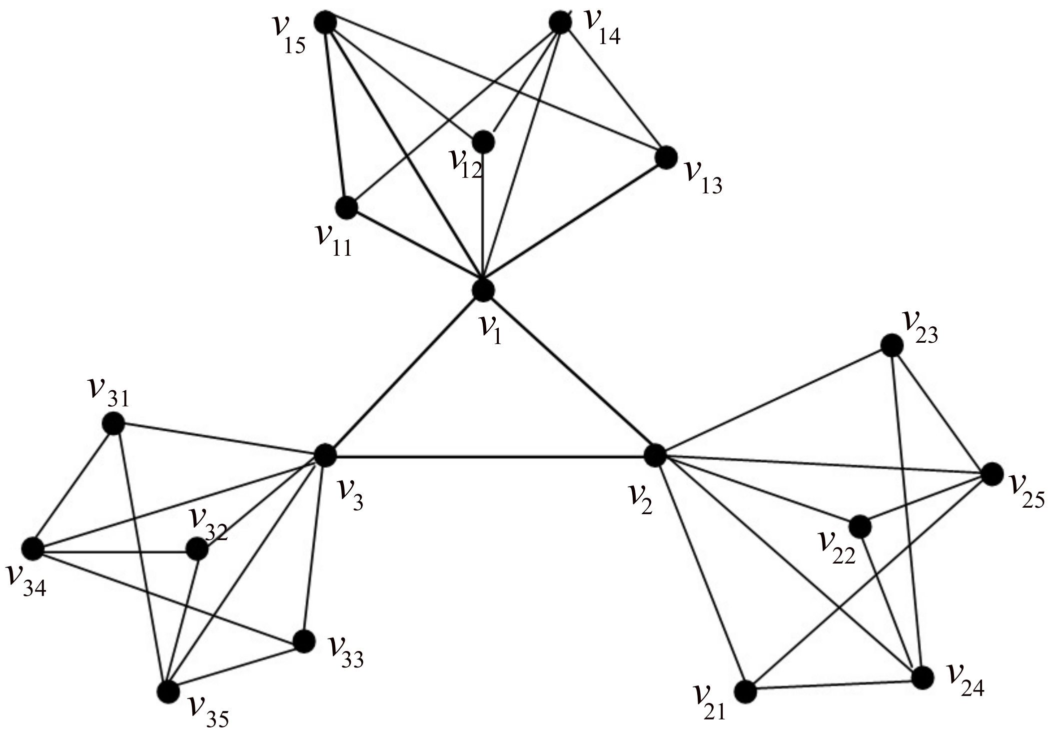

Figure 3.1: $G : K_3 \circ K_{3,2}$

$u_{ij} \in V(C_n)$. Since G contains triangles, $\chi(G) = 3$ and $\gamma_{M\chi}(G) \geq 3$. Let $S = \{u_1, u_2, u_3\}$ be the subset of $V(G)$, where $\{u_1, u_2, u_3\} \in V(C_m)$. Since the degree of each vertex in C_m is $(n+2), |N[S]| = \sum_{i=1}^{3} d(u_i) = (n+1) + (n+1) + (n+1) = 3(n+1) = 3\left(\frac{p}{m}\right) > \lceil \frac{p}{2} \rceil$.

Then the set S is majority dominating set of G and since $\{u_1, u_2, u_3\}$ forms a triangle, $\chi(\langle S \rangle) = 3 = \chi(G)$. Hence the set S is majority dom-chromatic set of G and $\gamma_{M\chi}(G) = m = 3$. ∎

Corollary 3.2.5: Let $G = C_3 \circ C_3$ be a Corona graph with two cycles. Then $\gamma_{M\chi}(G) = 4$.

Proof: Since the graph $G = C_3 \circ C_3$ contains a clique K_4 as a subgraph, $\chi(G) = 4$. So that any $\gamma_{M\chi}$- set of G must contain the full vertex set of K_4 and $\gamma_{M\chi}(G) = 4$. ■

3.3 $\gamma_{M\chi}$ for Product Graphs

In this section, MDC number is determined for grid $G = P_i \times P_j$, cylinder $G = P_i \times C_j$ and torus $G = C_i \times C_j$ graphs for $i \geq 2$ and $j \geq 3$.

Proposition 3.3.1: For a grid $G = P_2 \times P_j, j \geq 3$,

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{8} \rceil, \text{ if } p \equiv 2, 4 (mod\ 8) \\ \lceil \frac{p}{8} \rceil + 1, \text{ if } p \equiv 0, 6 (mod\ 8). \end{cases}$$

Proof: Let $G = P_2 \times P_j, j \geq 3$. Let $\{v_{11}, v_{12}, v_{13}, \ldots, v_{1j}\}$ and $\{v_{21}, v_{22}, v_{23}, \ldots, v_{2j}\}$ be the vertex sets in first and second row respectively and $\chi(G) = 2$.

Case (i): When $p \equiv 2, 4 (mod\ 8)$. Let $S_1 = \{v_{12}, v_{13}, v_{16}, \ldots, v_{1t_1}\} \subseteq V(G)$ such that $d(v_{13}, v_{ij}) \geq 3$, for $3 \leq j \leq t_1$ with $|S_1| = |t_1| = \lceil \frac{p}{8} \rceil$. Let $p = 8k + 2$. Since every vertex of S has degree 3, $|N[S_1]| = 4t_1 - 2 = 4\lceil \frac{p}{8} \rceil - 2 = 4\lceil \frac{8k+2}{8} \rceil - 2 = 4k + 2 = 4\left(\frac{p-2}{8}\right) + 2 \geq \lceil \frac{p}{2} \rceil$. Let

$p = 8k + 4$. Then $|N[S_1]| = 4t_1 - 2 = 4\lceil \frac{p}{8} \rceil - 2 = 4\lceil \frac{8k+4}{8} \rceil - 2 = 4\left(\frac{p-4}{8}\right) + 2 = \geq \lceil \frac{p}{2} \rceil$. It implies that S_1 is a majority dominating set of G. Since $d(v_{12}, v_{13}) = 1, \chi(\langle S_1 \rangle) = 2 = \chi(G)$. Hence S_1 is a majority dom-chromatic set of G and $\gamma_{M\chi}(G) \leq \lceil \frac{p}{8} \rceil$.

Suppose $S_1' = S_1 - \{v_{1t}\}$. Then $|S_1'| = \lceil \frac{p}{8} \rceil - 1$ and $|N[S_1']| = 4t_1 - 2 = 4\left(\lceil \frac{p}{8} \rceil - 1\right) - 2$. If $p = 8k + 2$ and $p = 8k + 4$ then $|N[S_1']| < \lceil \frac{p}{2} \rceil$. Therefore S_1' would not be a majority dom-chromatic set of G and hence $\gamma_{M\chi}(G) > |S_1'| = \lceil \frac{p}{8} \rceil - 1$. Thus $\gamma_{M\chi}(G) \geq \lceil \frac{p}{8} \rceil$. Hence we obtain, $\gamma_{M\chi}(G) = \lceil \frac{p}{8} \rceil$, if $p \equiv 2, 4(mod\ 8)$.

Case (ii): When $p \equiv 0, 6(mod\ 8)$. Let $S_2 = \{v_{12}, v_{13}, v_{16}, \ldots, v_{1t_2}\}$ be the subset of G such that $d(v_{i3}, v_{ij}) \geq 3$, for $3 \leq j \leq t_2$ with $|S_2| = |t_2| = \lceil \frac{p}{8} \rceil + 1$. If $p = 8k, 8k + 6$ then $|N[S_2]| = 4|t_2| - 2 = 4\left(\lceil \frac{p}{8} \rceil + 1\right) - 2 = 4\left(\lceil \frac{8k}{8} \rceil + 1\right) - 2 = 4k + 2 = 4\left(\frac{p}{8}\right) + 2 = \lceil \frac{p}{2} \rceil + 1$. If $p = 8k+6$ then $|N[S_2]| = 4|t_2| - 2 = 4\left(\lceil \frac{p}{8} \rceil + 1\right) - 2 = 4\left(\lceil \frac{8k+6}{8} \rceil + 1\right) - 2 = 4k + 6 = 4\left(\frac{p-6}{8}\right) + 6 = \lceil \frac{p}{2} \rceil + 2$. Hence $|N[S_2]| \geq \lceil \frac{p}{2} \rceil$. Since $\chi(\langle S_2 \rangle) = 2 = \chi(G)$, the set S_2 is a majority dom-chromatic set of G and $\gamma_{M\chi}(G) \leq \lceil \frac{p}{8} \rceil + 1$.

Applying the same arguments as in case (i), we obtain $\gamma_{M\chi}(G) \geq \lceil \frac{p}{8} \rceil + 1$. Thus, $\gamma_{M\chi}(G) = \lceil \frac{p}{8} \rceil + 1$, if $p \equiv 0, 6(mod\ 8)$. ∎

Proposition 3.3.2: Let $G = P_i \times P_j, i \geq 3, j \geq 2$, be a grid graph. Then

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{10} \rceil + 1, \text{ if } p \equiv 0, 7, 8, 9 (mod\ 10) \\ \lceil \frac{p}{10} \rceil, \text{ if } p \equiv 1, \ldots 6 (mod\ 10). \end{cases}$$

Proof: Let $G = P_i \times P_j$ with $i, j \geq 3$ be a gird graph. Let $\{v_{11}, v_{12}, v_{13}, \ldots, v_{1j}, v_{21}, v_{22}, \ldots, v_{2j}, \ldots, v_{i1}, v_{i2}, v_{i3}, \ldots, v_{ij}\}$, be the vertex sets of the first, second and third row respectively. For the graph $G = P_i \times P_j, \chi(G) = 2$.

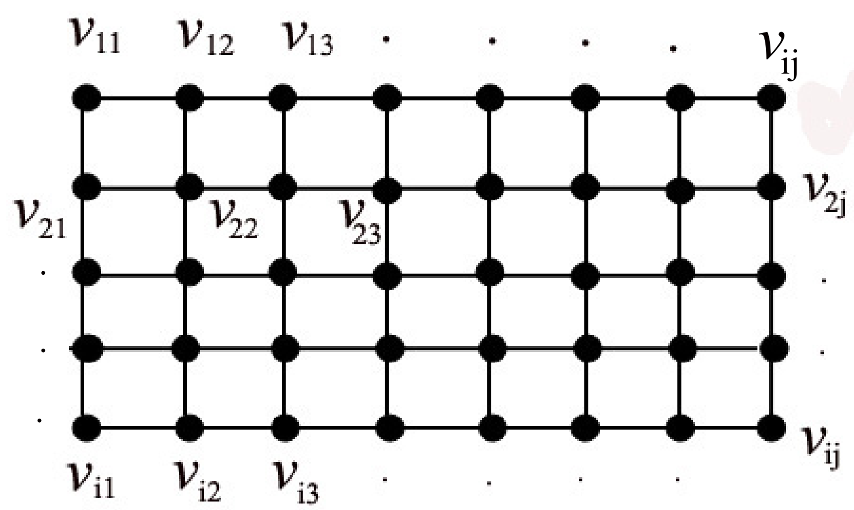

Figure 3.2: $G = P_i \times P_j$

Case (i): When $p \equiv 0, 7, 8, 9 (mod\, 10)$. Let $S = \{v_{22}, v_{23}, v_{26}, \ldots, v_{2t}\} \subseteq V(G)$ such that $d(v_{23}, v_{2j}) \geq 3$, for $3 \leq j \leq t$ with $|S| = |t| = \lceil \frac{p}{10} \rceil + 1$. The degree of each vertex of S is 4. If $p = 10k$ then $|N[S]| = 5t - 2 = 5\left(\lceil \frac{p}{10} \rceil + 1\right) - 2 = 5\lceil \frac{p}{10} \rceil + 3 = 5k + 3 = \frac{p}{2} + 3$. If $p = 10k + 7$

then $|N[S]| = 5\left(\lceil\frac{10k+7}{10}\rceil\right) \mp 3 = 5k+7 = 5\left(\frac{p-7}{10}\right)+8 = \frac{p}{2}+3$. If $p = 10k+8$ then $|N[S]| = 5\left(\lceil\frac{10k+8}{10}\rceil\right)+3 = 5k+7 = 5\left(\frac{p-8}{10}\right)+7 = \frac{p}{2}+3$. If $p = 10k+9$ then $|N[S]| = 5\left(\lceil\frac{10k+9}{10}\rceil\right)+3 = 5k+8 = 5\left(\frac{p-9}{10}\right)+8 = \frac{p}{2}+4$. Therefore in all cases, $|N[S_1]| \geq \lceil\frac{p}{2}\rceil$ and S is a majority dominating set of G. Since $d(v_{22}, v_{23}) = 1, \chi(\langle S\rangle) = 2 = \chi(G)$. Hence S is a majority dom-chromatic set of G and $\gamma_{M\chi}(G) \leq |S| = \lceil\frac{p}{10}\rceil + 1$.

Suppose $S' = S - \{v_{2t}\}$. Then $|S'| = \lceil\frac{p}{10}\rceil$ and $|N[S']| = 5t-2 = 5\left(\lceil\frac{p}{10}\rceil\right) - 2$. If $p = 10k, 10k+7, 10k+8$ and $p = 10k+9$ then $|N[S']| < \lceil\frac{p}{2}\rceil$. Therefore S' not be a majority dom-chromatic set of G and hence $\gamma_{M\chi}(G) > |S'| = \lceil\frac{p}{10}\rceil$. Thus, we obtain, $\gamma_{M\chi}(G) = \lceil\frac{p}{10}\rceil + 1$, if $p \equiv 0, 7, 8, 9 (mod\ 10)$.

Case (ii): Let $p \equiv 1, \ldots, 6 (mod\ 10)$. Applying the same arguments as in case (i), we obtain, $\gamma_{M\chi}(G) = \lceil\frac{p}{10}\rceil$. ■

Proposition 3.3.3: For a cylinder $G = C_3 \times P_j, j \geq 3$,

$$\gamma_{M\chi}(G) = \begin{cases} \lceil\frac{p}{10}\rceil + 2, & \text{if } p \equiv 0, 9 (mod\ 10) \\ \lceil\frac{p}{10}\rceil + 1, & \text{otherwise} \end{cases}.$$

Proof: Let $G = C_3 \times P_j$ be a cylinder with $j \geq 3$. Let $V(G) = \{v_{11}, v_{12}, v_{13}, \ldots, v_{1j}, v_{21}, v_{22}, v_{23}, \ldots, v_{2j}, v_{31}, v_{32}, v_{33}, \ldots, v_{3j}\}$ be the vertex set of G and since G contains a triangle, $\chi(G) = 3$.

Case (i): when $p \equiv 0, 9(mod\ 10)$. Consider the set $S = \{v_{12}, v_{22}, v_{32}, v_{25}, \ldots, v_{2t}\}$ a subset of G with $|S| = |t| = \lceil \frac{p}{10} \rceil + 2$ such that $d(v_{22}, v_{2j}) \geq 3, 2 \leq t \leq j$ and $\{v_{12}, v_{22}, v_{32}\} \in S$ be the vertices of a triangle. Since the degree of each vertex of S is 4, $|N[S]| = 5t - 6 = 5\left(\lceil \frac{p}{10} \rceil + 2\right) - 6 = \lceil \frac{p}{2} \rceil + 4$. If $p = 10k$ then $|N[S]| = 5k + 4 = \frac{p}{2} + 4$. If $p = 10k + 9$ then $|N[S]| = 5k + 8 = 5\left(\frac{p-9}{10}\right) + 8 = \frac{p}{2} + 4$. In these two cases, $|N[S]| > \lceil \frac{p}{2} \rceil$ and S is a majority dominating set of G. Since S contains a triangle, $\chi(\langle S \rangle) = 3 = \chi(G)$. Hence S is a $\gamma_{M\chi}$-set of G and $\gamma_{M\chi}(G) \leq \lceil \frac{p}{10} \rceil + 2$. Suppose $S' = S - \{v_{2t}\}$ with $|S'| = |S| - 1 = \lceil \frac{p}{10} \rceil + 1$. Then $|N[S']| = 5t - 6 = 5\left(\lceil \frac{p}{10} \rceil + 1\right) - 6 < \lceil \frac{p}{2} \rceil$, if $p = 10k$ and $10k + 9$. It implies that, S' would not be a $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G) > \lceil \frac{p}{10} \rceil + 1$. Hence $\gamma_{M\chi}(G) \geq \lceil \frac{p}{10} \rceil + 2$. Thus, we obtain, $\gamma_{M\chi}(G) = \lceil \frac{p}{10} \rceil + 2, p \equiv 0, 9(mod\ 10)$.

Case(ii): when $p \equiv 1, \ldots, 8(mod\ 10)$. Consider the set $S = \{v_{12}, v_{22}, v_{32}, v_{25}, \ldots, v_{2t}\}$ a subset of G with $|S| = |t| = \lceil \frac{p}{10} \rceil + 1$ such that $d(v_{22}, v_{2j}) \geq 3, 2 \leq t \leq j$ and $\{v_{12}, v_{22}, v_{32}\} \in S$ be the vertices of a triangle. Applying the same arguments as in case (i), we get the result. ∎

Proposition 3.3.4: Let $G = C_i \times P_j, i \geq 4, j \geq 2$ be a cylinder. If i is even then

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{10} \rceil, & \text{if } p \equiv 2, 4, 6 (mod\ 10) \\ \lceil \frac{p}{10} \rceil + 1, & \text{if } p \equiv 0, 8 (mod\ 10). \end{cases}$$

Proof: Let $G = C_i \times P_j, i \geq 4, j \geq 2$ be a cylinder. Let $\{v_{11}, v_{12}, v_{13}, \ldots, v_{1j}, v_{21}, v_{22}, \ldots, v_{2j}, \ldots, v_{i1}, v_{i2}, v_{i3}, \ldots, v_{ij}\}$ be the vertex set of G.

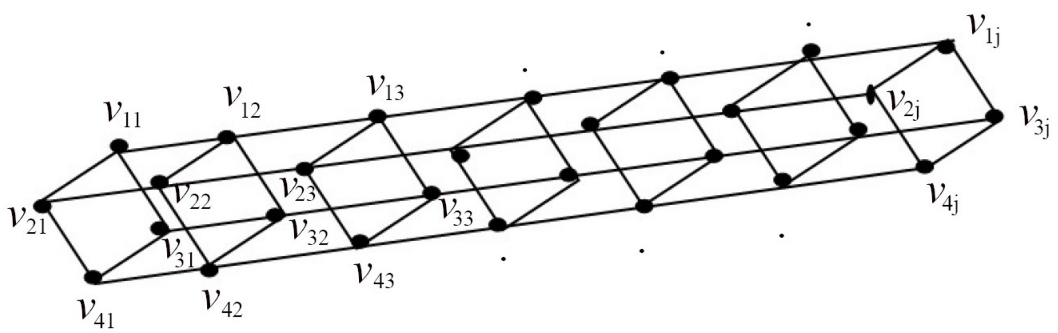

Figure 3.3: $G = C_4 \times P_j$

For $G = C_i \times P_j, \chi(G) = 2$ when i is even. Let $S = \{v_{12}, v_{13}, v_{15}, \ldots, v_{2t}\} \subseteq V(G)$ such that $d(v_{ij}, v_{ij}) \geq 3, i \neq j, 1 \leq i, j \leq t$ with $|S| = |t| = \lceil \frac{p}{10} \rceil + 1$. Applying the same arguments as in proposition (3.3.2) we obtain the result. ∎

Proposition 3.3.5: Let $G = C_3 \times C_j, j \geq 3$ be a torus. Then

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{10} \rceil + 2, & \text{if } p\ \equiv 0, 9 (mod\ 10) \\ \lceil \frac{p}{10} \rceil + 1, & \text{otherwise.} \end{cases}$$

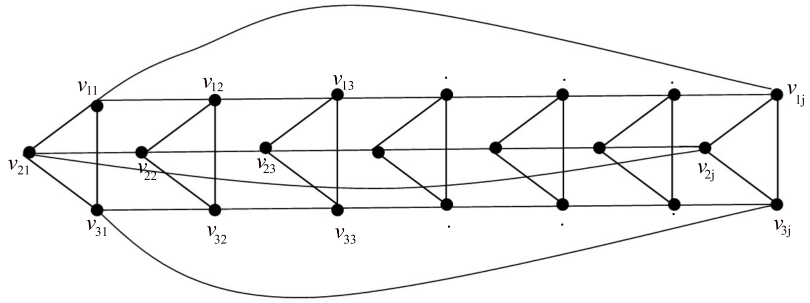

Figure 3.4: $G = C_3 \times C_j$

Proof: By proposition (3.3.3), we obtain the result. ∎

Proposition 3.3.6: Let $G = C_4 \times C_j, j \geq 3$ be a torus. Then

(i) $\gamma_{M\chi}(G) = j$, if j is odd

(ii) If j is even then $\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{10} \rceil + 1, \text{ if } p \equiv 0, 8(mod\ 10) \\ \lceil \frac{p}{10} \rceil, \text{ if } p \equiv 2, 4, 6(mod\ 10). \end{cases}$

Proof: Let $G = C_4 \times C_j, j \geq 3$ be a torus. Let $\{v_{11}, v_{12}, v_{13}, \ldots, v_{1j}\}$, $\{v_{21}, v_{22}, v_{23}, \ldots, v_{2j}\}$, $\{v_{31}, v_{32}, v_{33}, \ldots, v_{3j}\}$ and $\{v_{41}, v_{42}, v_{43}, \ldots, v_{4j}\}$ be the vertex sets of the first, second, third and fourth row respectively.

Case (i): when j is odd. Then C_j becomes a vertex color critical graph and therefore by proposition (2.3.3), $\gamma_{M\chi}(G) = j$.

Case (ii): when j is even. Then $\chi(G) = 2$.

Subcase (i): Let $p \equiv 0, 8(mod\ 10)$. Let $S = \{v_{11}, v_{12}, v_{15}, \ldots, v_{1t}\}$ be a subset of G with $|S| = |t| = \lceil \frac{p}{10} \rceil + 1$ such that $d(v_{12}, v_{1j}) \geq 3$, for $2 \leq t \leq j$. Since the degree of vertices of S is 4, if $p = 10k$ then $|N[S]| = 5t - 2 = 5\left(\lceil \frac{p}{10} \rceil + 1\right) - 2 = \frac{p}{2} + 3$. If $p = 10k + 8$ then $|N[S]| = 5t - 2 = 5\left(\lceil \frac{p}{10} \rceil + 1\right) - 2 = \frac{p}{2} + 4$. Therefore, $|N[S]| > \frac{p}{2}$ and the set S is the γ_M- set of G. Since $d(v_{11}, v_{12}) = 1$, where $\{v_{11}, v_{12}\} \in S, \chi(\langle S \rangle) = 2 = \chi(G)$. Hence S is a $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G) \leq \lceil \frac{p}{10} \rceil + 1$.

Suppose the set $S' = S - \{v_{1t}\}$ and $|S'| = |S| - 1 = \lceil \frac{p}{10} \rceil$. Then $|N[S']| = 5t - 2 = 5\left(\lceil \frac{p}{10} \rceil\right) - 2 = \frac{p}{2} - 2$. If $p = 10k + 8$ and $10k$ then $|N[S']| = 5t - 2 = 5\left(\lceil \frac{p}{10} \rceil\right) - 2 < \lceil \frac{p}{2} \rceil$. Therefore, the set S' wouldn't be a γ_M- set of G and $\gamma_{M\chi}(G) > \lceil \frac{p}{10} \rceil$. Hence $\gamma_{M\chi}(G) \geq \lceil \frac{p}{10} \rceil + 1$. Thus, $\gamma_{M\chi}(G) = \lceil \frac{p}{10} \rceil + 1$, if $p \equiv 0, 8 (mod\ 10)$.

Subcase (ii): Let $p \equiv 2, 4, 6 (mod\ 10)$. Let $S = \{v_{11}, v_{12}, v_{15}, \ldots, v_{1t}\}$ a subset of G with $|S| = |t| = \lceil \frac{p}{10} \rceil$ such that $d(v_{12}, v_{1j}) \geq 3, 2 \leq t \leq j$. Since $d(v_i j) = 4, |N[S]| = 5t - 2 = 5\lceil \frac{p}{10} \rceil - 2$. If $p = 10k + 2$ then $|N[S]| = 5\left(\lceil \frac{p}{10} \rceil\right) - 2 = 5(\lceil \frac{10k+2}{10} \rceil) - 2 = 5k + 3 = 5\left(\frac{p-2}{10}\right) + 3 \geq \lceil \frac{p}{2} \rceil$. If $p = 10k + 4$ then $|N[S]| = 5t - 2 = 5\left(\lceil \frac{p}{10} \rceil\right) - 2 = 5k + 3 = 5\left(\frac{p-4}{10}\right) + 3 \geq \lceil \frac{p}{2} \rceil$. If $p = 10k + 6$ then $|N[S]| = 5t - 2 = 5\left(\lceil \frac{p}{10} \rceil\right) - 2 = 5k + 3 = 5\left(\frac{p-6}{10}\right) + 3 \geq \lceil \frac{p}{2} \rceil$. Hence the set S is the γ_M- set of G. Since $d(v_{11}, v_{12}) = 1$, where $v_{11}, v_{12} \in S, \chi(\langle S \rangle) = 2 = \chi(G)$. Therefore S is a $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G) \leq \lceil \frac{p}{10} \rceil$.

Applying the same arguments as in subcase(i), we get $\gamma_{M\chi}(G) \geq \lceil \frac{p}{10} \rceil$. Hence $\gamma_{M\chi}(G) = \lceil \frac{p}{10} \rceil$, if $p \equiv 2, 4, 6(mod\ 10)$. ∎

3.4 $\gamma_{M\chi}$ for Generalized Petersen Graphs

In this section, the Majority dom-chromatic number $\gamma_{M\chi}$ is investigated for the generalized Petersen graph $P(n, k)$.

Definition 3.4.1: For each $n \geq 3$ and $0 < k < n$, $P(n, k)$ denotes the Generalized Petersen graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and the edge set $E(G) = \{u_i u_{i+1(mod\ n)}, u_i v_i, v_{i+k(mod\ n)}\}, 1 \leq i \leq n$.

Proposition 3.4.2: Let $G = P(n, k), k$ is odd, be a generalized Petersen graph. Then

$$\gamma_{M\chi}(G) = \begin{cases} \frac{p}{2}, \text{ if } p \equiv 2, 6(mod\ 8) \\ \lceil \frac{p}{8} \rceil, \text{ if } p \equiv 4(mod\ 8) \\ \lceil \frac{p}{8} \rceil + 1, \text{ if } p \equiv 0(mod\ 8). \end{cases}$$

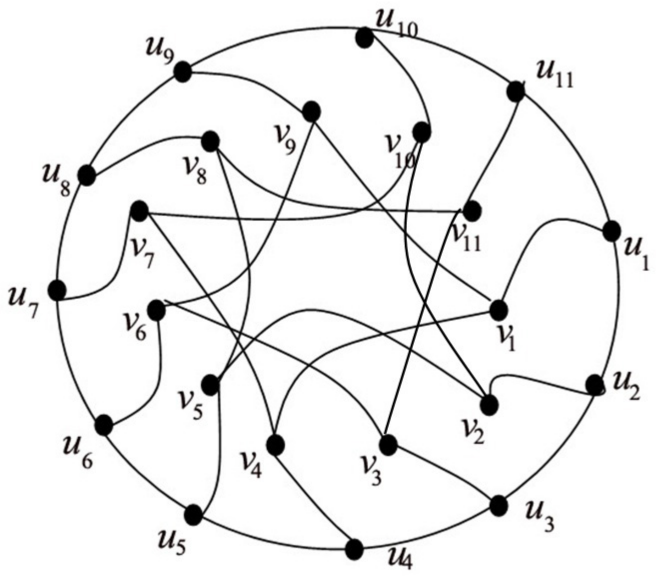

Figure 3.5: $G = p(11, 3)$

Proof: Let $G = P(n, k), k$ is odd, be a generalized Petersen graph. The vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that $V = V_1 \cup V_2$, where the inner polygon has the vertex set as $V_1(G) = \{v_1, v_2, \ldots, v_n\}$ and the outer polygon has the vertex set as $V_2(G) = \{u_1, u_2, \ldots, u_n\}$ with $p = 2n$.

Case (i): when $p \equiv 2, 6(mod\ 8)$. i. e, $\frac{p}{2} = n$ is odd. Then G contains two odd cycles C_1 and C_2 with $|V(C_1)| = |V(C_2)| = \frac{p}{2}$ and hence $\chi(C_1) = \chi(C_2) = 3 = \chi(G)$. Any $\gamma_{M\chi}$ set must contain the full vertex set of any one odd cycle. Let $S = \{u_1, u_2, u_3, \ldots, u_n\} \in V(C_1)$ be the subset of $V(G)$. Clearly $|N[S]| > \lceil \frac{p}{2} \rceil$ and $\chi(\langle S \rangle) = 3 = \chi(G)$. Hence S is a $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G) = \frac{p}{2}$, if $\frac{p}{2}$ is odd.

Case (ii): Let n be even. Then G contains only even cycles. Then $\chi(G) = 2$.

Subcase (i): When $p \equiv 4(mod\ 8)$. Consider the set $S = \{u_1, u_2, u_5, \ldots, u_t\}$ be the subset of $V(G)$ such that $|S| = |t| = \lceil \frac{p}{8} \rceil$ with $d(u_1, u_2) = 1$ and $d(u_i, u_j) \geq 3, i \neq j$. Since G is a 3-regular graph, $|N[S]| = 4t-2 = 4\left(\lceil \frac{p}{8} \rceil\right)-2$. If $p = 8r+4$ then $|N[S]| = 4\lceil \frac{8r+4}{8} \rceil - 2 = 4\left(\frac{p-4}{8}\right) + 2 \geq \lceil \frac{p}{2} \rceil$. It implies that S is a majority dominating set of G. Since $d(u_1, u_2) = 1, \chi(\langle S \rangle) = 3 = \chi(G)$. Hence S is a $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G) \leq \lceil \frac{p}{8} \rceil$.

Suppose, consider the set $S' = S - \{u_i\}$. Then $|S'| = |S| - 1 = \lceil \frac{p}{8} \rceil - 1$. Now, if $p = 8r + 4$ then $|N[S']| = 4t - 2 = 4\left(\lceil \frac{p}{8} \rceil - 1\right) - 2 = 4\left(\frac{p-4}{8}\right) - 2 < \lceil \frac{p}{2} \rceil$. Therefore S' wouldn't be a γ_M- set of G and $\gamma_{M\chi}(G) > \lceil \frac{p}{8} \rceil - 1$. It implies that, $\gamma_{M\chi}(G) \geq \lceil \frac{p}{8} \rceil$. Thus, $\gamma_{M\chi}(G) = \lceil \frac{p}{8} \rceil$, if $p \equiv 4(mod\ 8)$.

Subcase (ii): when $p \equiv 0(mod\ 8)$. Let $S = \{u_1, u_2, u_3, \ldots, u_t\}$ be the subset of $V(G)$ such that $|S| = |t| = \lceil \frac{p}{8} \rceil + 1$ with $d(u_1, u_2) = 1$ and $d(u_i, u_j) \geq 3, i \neq j$. If $p = 8r$ then $|N[S]| = 4t-2 = 4\left(\lceil \frac{p}{8} \rceil + 1\right) - 2 = 4\left(\lceil \frac{8r}{8} \rceil + 1\right) - 2 == 4\left(\frac{p}{8}\right) + 2 > \lceil \frac{p}{2} \rceil$. It implies that S is a majority dominating set of G. Since $d(u_1, u_2) = 1, \chi(\langle S \rangle) = 3 = \chi(G)$. Hence S is a $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G) \leq \lceil \frac{p}{8} \rceil + 1$.

Applying the same arguments as in case (i), we obtain $\gamma_{M\chi}(G) \geq \lceil \frac{p}{8} \rceil + 1$. Therefore $\gamma_{M\chi}(G) = \lceil \frac{p}{8} \rceil + 1$, if $p \equiv 0(mod\ 8)$. ∎

Proposition 3.4.3: Let G be a generalized Petersen graph $G = P(n,k), k = 2$. Then

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{8} \rceil + 2, & \text{if } p \equiv 2,4(mod\ 8) \\ \lceil \frac{p}{8} \rceil + 3, & \text{if } p \equiv 0,4(mod\ 8) \end{cases}$$

Proof: Consider V_1 and V_2 be the vertex partition of inner and outer polygon of the generalized Petersen graph $G = P(n,2)$ such that $V = V_1 \cup V_2$, where $V_1(G) = \{v_1, v_2, \ldots, v_n\}$ and $V_2(G) = \{u_1, u_2, \ldots, u_n\}$ with $p = 2n$. Since G contain 5 - cycles with the vertex set $\{v_i, v_{i+2}, u_i, u_{i+1}, u_{i+2}\}, \chi(G) = 3$, any $\gamma_{M\chi}$- set must contain 5 - cycle.

Case (i): when $p \equiv 2,4\ (mod\ 8)$. Let $S = \{u_1, u_2, u_3, v_1, v_3, u_6 \ldots, u_t\}$ be the subset of $V(G)$ such that $|S| = |t| = \lceil \frac{p}{8} \rceil + 2$ with $d(u_3, u_6) \geq 3$ and the vertex set $\{u_1, u_2, u_3, v_1, v_3\}$ forms a 5-cycle and the degree of each vertex of S is 3. If $p = 8r + 2$ then $|N[S]| = 4t - 10 = 4\left(\lceil \frac{8r+2}{8} \rceil + 2\right) - 10 = 4r + 2 = 4\left(\frac{p-2}{8}\right) + 2 \geq \lceil \frac{p}{2} \rceil$. If $p = 8r + 4$ then $|N[S]| = 4\left(\lceil \frac{8r+4}{8} \rceil + 2\right) - 10 = 4\left(\frac{p-4}{8}\right) + 2 > \lceil \frac{p}{2} \rceil$. It implies that S is a majority dominating set of G. Since S contains 5-cycle $\{u_1, u_2, u_3, v_1, v_3\}, \chi(\langle S\rangle) = 3 = \chi(G)$. Hence S is a $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G) \leq \lceil \frac{p}{8} \rceil + 2$.

Suppose, consider the set $S' = S - \{u_i\}$. Then $|S'| = |S| - 1 = \lceil \frac{p}{8} \rceil + 1$. Now, if $p = 8r + 2$ then $|N[S']| = 4t - 10 = 4\left(\lceil \frac{p}{8} \rceil + 1\right) - 10 = 4r - 2 = 4\left(\frac{p-2}{8}\right) - 2 < \lceil \frac{p}{2} \rceil$. If $p = 8r + 4$ then $|N[S']| = 4t - 10 = 4\left(\lceil \frac{p}{8} \rceil + 1\right) - 10 = 4r - 2 = 4\left(\frac{p-4}{8}\right) - 2 < \lceil \frac{p}{2} \rceil$. Therefore S' wouldn't be a γ_M- set of G and $\gamma_{M\chi}(G) > \lceil \frac{p}{8} \rceil + 1$. It implies that, $\gamma_{M\chi}(G) \geq \lceil \frac{p}{8} \rceil + 2$. Thus, $\gamma_{M\chi}(G) = \lceil \frac{p}{8} \rceil + 2$, if $p \equiv 2, 4 (mod\ 8)$.

Case (ii): when $p \equiv 0, 6 (mod\, 8)$. Let $S = \{u_1, u_2, u_3, v_1, v_3, u_6 \ldots, u_t\}$ be the subset of $V(G)$ such that $|S| = |t| = \lceil \frac{p}{8} \rceil + 3$. If $p = 8r$ then $|N[S]| = 4t - 10 = 4\left(\lceil \frac{p}{8} \rceil + 3\right) - 10 = 4\left(\lceil \frac{8r}{8} \rceil + 3\right) - 10 = 4r + 2 = 4\left(\frac{p}{8}\right) + 2 \geq \lceil \frac{p}{2} \rceil$. If $p = 8r + 6$ then $|N[S]| = 4t - 10 = 4\left(\lceil \frac{p}{8} \rceil + 3\right) - 10 = 4\left(\lceil \frac{8r+6}{8} \rceil + 3\right) - 10 = 4r + 6 = 4\left(\frac{p-6}{8}\right) + 6 \geq \lceil \frac{p}{2} \rceil$. It implies that S is a majority dominating set of G. Since S contains 5-cycle $\{u_1, u_2, u_3, v_1, v_3\}, \chi(\langle S\rangle) = 3 = \chi(G)$. Hence S is a $\gamma_{M\chi} - set of G$ and $\gamma_{M\chi}(G) \leq \lceil \frac{p}{8} \rceil + 3$. Applying the same arguments as in case (i), we obtain $\gamma_{M\chi}(G) \geq \lceil \frac{p}{8} \rceil + 3$. Therefore $\gamma_{M\chi}(G) = \lceil \frac{p}{8} \rceil + 3$, if $p \equiv 0, 6\ (mod\ 8)$. ∎

3.5 $\gamma_{M\chi}$ for Rooted Product Graphs

In this section, the particular values of $\gamma_{M\chi}$ for rooted product graph and some results on $\gamma_{M\chi}$ with respect to $cpn(G)$ are discussed.

Definition 3.5.1: Given a graph G of order $n(G)$ and a graph H with a root vertex v, the rooted product graph $G \circ_v H$ is defined as the graph obtained from G and H by taking one copy of G and $n(G)$ copies of H and identifying the i^th vertex of G with the root vertex v in the ith copy of the H for every the $i \in \{1, 2, \ldots, n(G)\}$.

Example 3.5.2: Let $G = G_1 \circ_v G_2$ be a rooted product graph where $G_1 = C_5$ and $G_2 = K_4$ with $p = 20$ vertices.

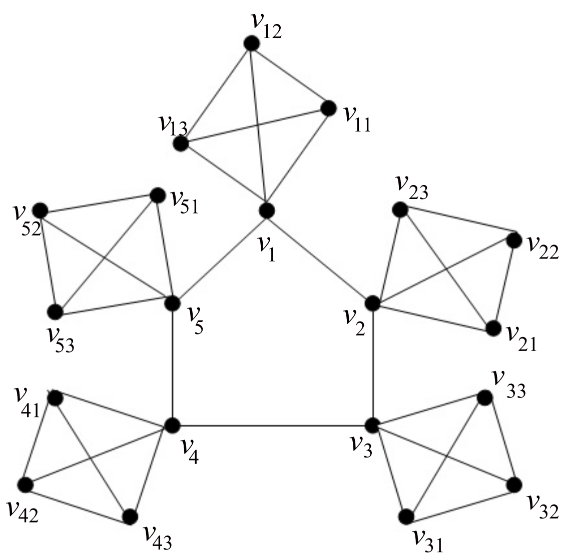

Figure 3.6: $G = C_5 \circ_v K_4$

For the graph G, $\chi(G) = 4$. The $\gamma_{M\chi}$- set of G is $\{v_1, v_{11}, v_{12}, v_{13}, v_4\}$ and $\gamma_{M\chi}(G) = 5$.

Theorem 3.5.3: Let $G = G_1 \circ_v G_2$ be a rooted product graph where G_1 and G_2 are cycles. Then $\gamma_{M\chi}(G) \geq cpn(G)$.

Proof: Since the graph $G = G_1 \circ_v G_2$ contains cycles, $\chi(G) = 2$ or $\chi(G) = 3$. Since G is connected, $\gamma_{M\chi}(G) \geq 2$. If G contains odd cycle $cpn(G) \geq 3$. If G contains even cycle $cpn(G) \geq 2$. Hence $\gamma_{M\chi}(G) \geq cpn(G)$. ∎

Theorem 3.5.4: Let G_1 and G_2 be any two vertex color critical which are complete graphs and $G = G_1 \circ_v G_2$ be a rooted product graph. Let v be any root vertex in G_2. Then (i) $\gamma_{M\chi}(G) = cpn(G_1)$, if $cpn(G_1) > cpn(G_2)$ (ii) $\gamma_{M\chi}(G) > cpn(G)$, if $cpn(G_1) \leq cpn(G_2)$.

Proof: Let $G = G_1 \circ_v G_2$ be a rooted product graph where G_1 and G_2 are any two complete graphs with order m and n. Let S_1 and S_2 be the cp-sets of G_1 and G_2. Since G_1 and G_2 are complete graphs, $S_1 = V(G_1)$ and $S_2 = V(G_2)$ are cp- sets of G_1 and G_2. Then $|S_1| = cpn(G_1) = m$ and $|S_2| = cpn(G_2) = n$.

Case (i): When $cpn(G_1) \geq cpn(G_2)$. Then $\chi(G_1) = \chi(G)$ and any $\gamma_{M\chi}$- set S contain the full vertex set of S_1. Let $S = \{v_1, v_2, \ldots, v_m\} \subseteq V(G_1)$ with $|S| = m$. Since v be any root vertex in G_2, all vertices of G_1 are adjacent to the vertices of G_2. It implies that $|N[S]| > \lceil \frac{p}{2} \rceil$ and $\chi(\langle S \rangle) = \chi(G)$. Hence the set S is the $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G) = m = cpn(G_1)$.

Case (ii): When $cpn(G_1) < cpn(G_2)$. Then $\chi(G_2) = \chi(G)$ and any $\gamma_{M\chi}$- set S contain the full vertex set of G_2. Let $S = \{v_1, v_{11}, \ldots, v_{1n}, v_4, \ldots, v_t\} \subseteq V(G)$ with $|S| = n + t$. Since v be any root vertex in G_2, all vertices of G_1 are adjacent to the vertices of G_2. It implies that $|N[S]| > \lceil \frac{p}{2} \rceil$ and $\chi(\langle S \rangle) = \chi(G)$. Hence the set S is the $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G) = n + t > cpn(G)$. ∎

Theorem 3.5.5: Let $G = G_1 \circ_v G_2$ be a rooted product graph where G_1 and G_2 are cycles. Then $\gamma_{M\chi}(G) \geq cpn(G)$.

Proof: Since the graph $G = G_1 \circ_v G_2$ contains cycles, $\chi(G) = 2$ or $\chi(G) = 3$. Since G is connected, $\gamma_{M\chi}(G) \geq 2$. If G contains odd cycle $cpn(G) \geq 3$. If G contains even cycle $cpn(G) \geq 2$. Hence $\gamma_{M\chi}(G) \geq cpn(G)$. ∎

3.6 $\gamma_{M\chi}$ for Disconnected Graphs

In this section, the characterization on MDC number is determined-for disconnected graphs with isolated and without isolates. Also some inequalities between $|V - S|$ *and* $\sum\limits_{u_i \in S} d(u_i)$ *for disconnected graphs are investigated.*

Proposition 3.6.1: Let G be a disconnected graph of order p. Then $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$ if and only if the graph G is totally disconnected $\overline{K_p}$.

Proof: Let G be a disconnected graph with p vertices. Assume that $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$. It implies that $\gamma_M(G) = \lceil \frac{p}{2} \rceil$ and $\chi(G) \geq 1$. Let $S = \left\{v_1, v_2, \ldots, v_{\lceil \frac{p}{2} \rceil}\right\}$ be a majority dom-chromatic set of G with $|S| = \lceil \frac{p}{2} \rceil$ and $|N[S]| \geq \lceil \frac{p}{2} \rceil$. Then $\chi(\langle S \rangle) \leq \lceil \frac{p}{2} \rceil$. Since G contains n components say $G_1, G_2, \ldots, G_n$, and $\gamma_M(G) = \lceil \frac{p}{2} \rceil$, the majority dominating set S consists of only $\lceil \frac{p}{2} \rceil$ isolates and the maximum color used for this induced subgraph $\chi(\langle S \rangle) = 1 = \chi(G)$. Therefore, if $\chi(\langle S \rangle) = \chi(G) = 1$ and $\gamma_M(G) = \lceil \frac{p}{2} \rceil$ then the resulting graph is totally disconnected graph $G = \overline{K_p}$.

Conversely, suppose $G = \overline{K_p}$. Then $\gamma_M(G) = \lceil \frac{p}{2} \rceil$ and $\xi(G) = 1$. Therefore $\gamma_{M\chi}(G) = max \left\{\lceil \frac{p}{2} \rceil, 1\right\} = \lceil \frac{p}{2} \rceil$. Hence the result. ∎

Theorem 3.6.2: Let G be a disconnected graph. Then $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$ if and only if $G = \overline{K_p}$ or $G = g_t \cup \overline{K_{p-t}}, p \geq 2$, where g_t is a vertex color critical component with $|t| \leq \lceil \frac{p}{2} \rceil$.

Proof: Let G be a disconnected graph with p vertices. Assume $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$. To prove that $G = \overline{K_p}$ or $G = g_t \cup \overline{K_{p-t}}$.

Case (i): Suppose $G \neq \overline{K_p}, p \geq 2$ then G has atleast one edge between a pair of vertices. It implies that G is a disconnected graph without isolates. By result (2.3.1) (i), $\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil + 1$. It is a contradiction to the assumption $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$. Hence $G = \overline{K_p}$.

Case (ii): Suppose $G \neq g_t \cup \overline{K_{p-t}}$ where g_t is not a vertex color critical graph with $|t| \leq \lceil \frac{p}{2} \rceil$. Then the graph G contains a path, an even cycle or any other component g_t with $|t| \leq \lceil \frac{p}{2} \rceil$. Since $\chi(g_t) \geq 2$ and $\gamma_M(g_t) \geq \lceil \frac{p}{6} \rceil$.

Subcase (i): Suppose $|t| = \lceil \frac{p}{2} \rceil$. Then $S = \{u_1, u_2, \ldots, u_{\lceil \frac{p}{6} \rceil}\}$, is a MDC set of G, where $u_i \in V(g_t)$. It implies that $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil$, it condradicts the assumption.

Subcase (ii): Suppose $|t| < \lceil \frac{p}{2} \rceil$. Then $S = \{u_1, u_2, \left(\lceil \frac{p}{2} \rceil - t\right) K_1\}$ is a MDC set of G where $u_i \in V(g_t)$. It follows that $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{2} \rceil - |t| + 2 = \lceil \frac{p}{2} \rceil - \lceil \frac{p}{2} \rceil + 1 + 3$. It implies that $\gamma_{M\chi}(G) = 4 < \lceil \frac{p}{2} \rceil$.

It is a contradiction. Hence g_t is a vertex color critical component in G with $|t| \leq \lceil \frac{p}{2} \rceil$.

Case (iii): Suppose g_t with $|t| \leq \lceil \frac{p}{2} \rceil$. Since g_t is a vertex color critical component of G, g_t is a complete graph or an odd cycle. If g_t is an odd cycle with $|t| \leq \lceil \frac{p}{2} \rceil + 1$ then $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil + 1$. It contradicts the assumption. If g_t is a complete graph with $|t| \leq \lceil \frac{p}{2} + 1 \rceil$ then $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil + 1$ is a contradiction to the assumption. Hence g_t is a vertex color critical component of G with $|t| \leq \lceil \frac{p}{2} \rceil$. Therefore G must be $\overline{K_p}$ or $(g_t \cup \overline{K_{p-t}})$.

Conversely, let $G = \overline{K_p}$ or $(g_t \cup \overline{K_{p-t}})$. To prove that $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$. Suppose $G = \overline{K_p}$ then $\gamma_M(G) = \lceil \frac{p}{2} \rceil$ and $\chi(G) = 1 \Rightarrow \gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$. Suppose $G = (g_t \cup \overline{K_{p-t}})$. Since g_t is a vertex critical component with $|t| = \lceil \frac{p}{2} \rceil$, $\chi(g_t) = \lceil \frac{p}{2} \rceil$ and $\gamma(g_t) \geq 1$ It implies that $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$. Suppose g_t is a vertex critical component with $|t| = \lceil \frac{p}{2} \rceil$. Then $S = \left\{u_1, u_2, \ldots, u_t, v_1, v_2, \ldots v_{\lceil \frac{p}{2} \rceil - t}\right\}$ is a MDC set of G where $u_i \in V(g_t)$ and $v_i \in V(\overline{K_{p-t}})$. Now, $|S| = t + \lceil \frac{p}{2} \rceil - t = \lceil \frac{p}{2} \rceil$. Thus, $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{2} \rceil$. ∎

Observation 3.6.3: For a disconnected graph $G, \chi(G), \gamma_M(G)$ are not comparable

(i) $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

(ii) $\gamma_M(G) < \chi(G) < \gamma_{M\chi}(G)$.

Example: Consider the disconnected graph with isolates with $p = 16$. Let $G = P_{11} \cup \overline{K_5}$. Let $|V(G)| = |\{v_1, v_2, \ldots, v_{11}, u_1, \ldots, u_5\}| = 16$. Then $\gamma_M(G) = |\{v_2, v_5, v_7\}| = 3$ and $\gamma_{M\chi}(G) = |\{v_2, v_5, v_7, v_8\}| = 4$. Since P_{11} is a tree, $\chi(G) = 2$. Therefore $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

(ii) For a disconnected graph G with isolates, $\gamma_M(G) < \chi(G) < \gamma_{M\chi}(G)$.

Example: Let $G = C_3 \cup \overline{K_5}$ and $V(G) = \{v_1, v_2, v_3, u_1, \ldots, u_5\}$. Since C_3 is an odd cycle, $\chi(G) = 3$ and $\gamma_M(G) = |\{v_1, u_1\}| = 2$. Then $S = \{v_1, v_2, v_3, u_1\}$ be the MDC set of G where $v_i \in V(C_3)$ and $u_i \in V(\overline{K_5})$. $\Rightarrow \gamma_{M\chi}(G) = |S| = 4$. Therefore $\gamma_M(G) < \chi(G) < \gamma_{M\chi}(G)$.

Theorem 3.6.4: Let G be a disconnected graph with any vertex critical component then $|V - S| < \sum\limits_{u_i \in S} d(u_i)$.

Proof: Let $G = G_t \cup G_r$ be a disconnected graph with p vertices. Since G has a vertex color critical component, $\chi(G) \geq 3$. Consider $S = \{G_t, u_1, \ldots\}$ be the MDC set of G, where G_t is the vertex color critical component, such that $|t| \geq 3$ and $u_1 \in G_r$. If $|N[G_(t)]| = \lceil \frac{p}{2} \rceil$ then $|S| \geq 3$. If $|N[G_(t)]| < \lceil \frac{p}{2} \rceil$ then $|S| \geq 4$. It implies that $|S| = 3$ or 4 and $|V - S| \leq p - 3$ or $p - 4$. Let $V(G_(t)) = \{u_1, u_2, \ldots, u_t\}$, Then

$$\sum_{u_i \in S} d(u_i) = d(u_1) + d(u_2), \ldots, \geq 3(t-2) + 1 \geq 3t - 5, \text{ if } |t| \geq 3.$$

Then, certainly we get $|V - S| < \sum_{u_i \in S} d(u_i)$. ■

Theorem 3.6.5: For a disconnected graph G without any vertex color critical components, $|V - S| > \sum_{u_i \in S} d(u_i)$ where S is the MDC set of G.

Proof: Let G be a disconnected graph with not vertex color critical component. Let S be a majority dom-chromatic set of G.

Case (i): The graph G is totally disconnected.

Then $S = \{u_1, u_2, \ldots, u_{\lceil \frac{p}{2} \rceil}\}$ be the MDC set of G and $deg(u_i) = 0$, for each $u_i \in S$. It implies that $\sum_{u_i \in S} d(u_i) = 0$. Hence, $|V - S| > \sum_{u_i \in S} d(u_i)$.

Case (ii): The graph G is disconnected with isolates.

Then G contains some connected component 'g' along with isolates.

Subcase (i): If the component 'g' such that $|N[g]| \geq \lceil \frac{p}{2} \rceil$ then S is a MDC set of G with $1 \leq |S| = \lceil \frac{p}{6} \rceil$. Suppose $|S| = 1 \Rightarrow S = \{u\}$ such that $|N[S]| = \lceil \frac{p}{2} \rceil - 1$. Then $|V - S| = p - 1 > \sum\limits_{u_i \in S} d(u_i) = \lceil \frac{p}{2} \rceil - 1$.

Suppose $|S| = \lceil \frac{p}{6} \rceil$.

Then $d(u_i) \leq 2$, for all $u_i \in V(g)$. Now,

$\sum\limits_{u_i \in S} d(u_i) = 2\lceil \frac{p}{6} \rceil = \frac{p}{3}$ or $\frac{p}{3} + 2$ and $|V - S| = p - \lceil \frac{p}{6} \rceil = \frac{5p}{6} - 1$.

Therefore, $|V - S| > \sum\limits_{u_i \in S} d(u_i)$.

Subcase (ii): If the component 'g' such that $|N[S]| < \lceil \frac{p}{2} \rceil$ then S is a MDC set with isolates. Then $\sum\limits_{u_i \in S} d(u_i) \leq \frac{p}{3}$. Since S contains more isolates, the value $\sum\limits_{u_i \in S} d(u_i)$ will be reduced. Then $|V - S| > \sum\limits_{u_i \in S} d(u_i)$.

Case (iii): The graph G is a disconnected graph without isolates.

Then G contains only connected components. Suppose $G = mK_2$. Then by the proposition (2.3.1)(i), $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{4} \rceil + 1$. It implies that $\sum\limits_{u_i \in S} d(u_i) = \lceil \frac{p}{4} \rceil + 1$. But $|V - S| = \left|p - \left(\lceil \frac{p}{4} \rceil + 1\right)\right| = \frac{3p}{4} - 1$.

If the size of the component g increases such as $= mK_4, mK_{1,t}, \ldots$ then $|S|$ will be decreased. i.e., $|S| < \lceil \frac{p}{4} \rceil + 1$ and $\sum\limits_{u_i \in S} d(u_i) > \lceil \frac{p}{4} \rceil + 1$. But in all structures, We obtain, $|V - S| > \sum\limits_{u_i \in S} d(u_i)$. ∎

Theorem 3.6.6: Let G be a disconnected graph without any vertex critical components then $|V - S| = \lfloor \frac{p}{2} \rfloor$ if and only if $G = \overline{K_p}$.

Proof: Let G has no vertex color critical subgraph. Let $G = \overline{K_p}, p$ is odd. Then $S = \{u_1, u_2, \ldots, u_{\lceil \frac{p}{2} \rceil}\}$ is a MDC set of G and $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{2} \rceil$. Hence $|V - S| = \lfloor \frac{p}{2} \rfloor$, if p is odd. When p is even , $S = \{u_1, u_2, \ldots, u_{\frac{p}{2}}\}$ is the MDC set and $\gamma_{M\chi}(G) = |S| = \frac{p}{2}$ and $|V - S| = \frac{p}{2}$. Hence $|V - S| = \lfloor \frac{p}{2} \rfloor$.

Conversely, suppose $G \neq \overline{K_p}$. Then either G is disconnected graph without isolates or G contains atleast one component which is not a vertex color critical with some isolates. Let $|V - S| = \lfloor \frac{p}{2} \rfloor$.

Case (i): If G has components which is not vertex color critical with no isolates then the structure like $G = mK_2$. By the proposition (2.3.1)(i), we have $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{4} \rceil + 1$. If $|S| = \lceil \frac{p}{4} \rceil + 1 \Rightarrow |V - S| = |p - \lceil \frac{p}{4} \rceil + 1| > \lfloor \frac{p}{2} \rfloor$. It is a contradiction to the assumption.

Case (ii): Suppose $G = C_6 \cup \overline{K_{P-6}}$, where C_6 is not a vertex color critical. Then $S = \{u_2, u_5, \left(\lceil \frac{p}{2} \rceil - 6\right) K_1\}$, where $u_2, u_5 \in V(C_6)$. It implies that $|S| = \lceil \frac{p}{2} \rceil - 6 + 2 = |\lceil \frac{p}{2} \rceil - 4|$. Therefore $|V - S| = |p - \lceil \frac{p}{2} \rceil + 4| = \lfloor \frac{p}{2} \rfloor + 4 > \lfloor \frac{p}{2} \rfloor$. It is acontradiction. Hence $G = \overline{K_p}$ if and only if $|V - S| = \lfloor \frac{p}{2} \rfloor$. ∎

Chapter 4

Majority Dom-Chromatic Partition Number of Graphs

Abstract

This chapter introduces a new notion majority dom-chromatic partition of a graph G. The majority dom-chromatic partition number $d_{M\chi}(G)$ is investigated for some families of graphs. Bounds on $d_{M\chi}(G)$ and its relationship with other graph theoretic parameters are studied. Some inequalities on $d_{M\chi}(G)$ are determined. Also characterization theorems on $d_{M\chi}(G)$ are established.

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4.1 Introduction

In 1977, Cockayane and Hedetniemi [14] introduced a concept domatic number in their seminal paper "Towards a Theory of Domination in Graphs". This paper became the point of interest for many researchers to step into domatic number. Then in 2010 Swaminathan and Joseline Manora [37] introduced the concept "Majority domatic number $d_M(G)$" as the maximum number of elements in a partition of $V(G)$ into majority dominating sets. They elucidated the parameter in various levels by establishing many results. They produced the exact values of $d_M(G)$ for some standard graphs, characterisation theorems on $d_{M\chi}(G)$ and some inequalities for $d_M(G)$.

In recent years, several graph-theoretic parameters that combine the concepts of domination and coloring have been investigated and studied by many mathematicians effectively. Dom-chromatic partition was introduced by Janakiraman and Poobalaranjani [31]. Its number $d_{ch}(G)$ was defined and the exact values for various classes of graphs were determined. They established more results on $d_{ch}(G)$ with other parameters for connected and disconnected graphs. Lower and upper bounds of $d_{ch}(G)$ are also found interms of p and $\Delta(G)$.

These two parameters $d_M(G)$ and $d_{ch}(G)$ gave the motivation to introduce a graph theoretical parameter "Majority Dom-Chromatic Partition (MDC Partition) of a graph" and its number $d_{M\chi}(G)$ on graphs.

Organization of this Chapter is as follows. Section 4.1, contains an introduction and of the defined parameters. In section 4.2, the concept of majority dom-chromatic partition of a graph G and its number $d_{M\chi}(G)$ is defined with examples. The exact value of $d_{M\chi}(G)$ for various families of graphs is determined in section 4.3. In sections 4.4 and 4.5, Bounds on $d_{M\chi}(G)$, the relationship of $d_{M\chi}(G)$ with other domatic number such as $d_M(G), d_{ch}(G)$ and $d(G)$ and characterization theorems on $d_{M\chi}(G)$ are also determined.

4.2 Majority Dom-Chromatic Partition

In this section, the concept of Majority Dom-Chromatic Partition (MDC Partition)- set of a graph and its number defined with some examples.

Definition 4.2.1: Let G be a simple, finite and undirected graph with p vertices. A Majority Dom-Chromatic Partition (MDC - Partition) of a graph G is a partition of the vertex set $V(G)$ into majority dom-chromatic sets of G.

Definition 4.2.2: The maximum cardinality of a partition of $V(G)$ into majority dom-chromatic sets is the majority dom-chromatic partition number and is denoted by $d_{M\chi}(G)$.

Example 4.2.3: Consider the following graph with $p = 16$.

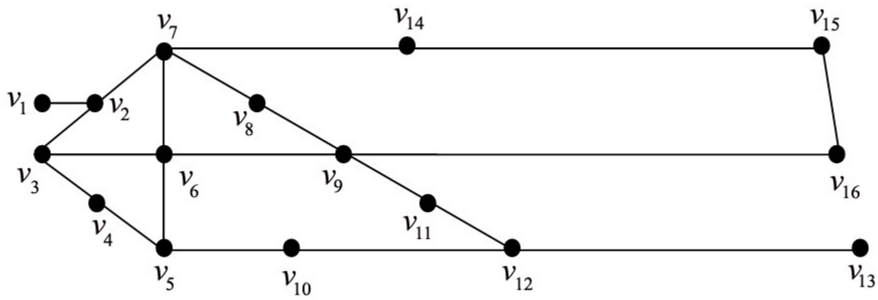

Figure 4.1: G

In the above graph G, $\chi(G) = 2$ and $S_1 = \{v_2, v_7, v_{13}\}, S_2 = \{v_1, v_5, v_6\}, S_3 = \{v_3, v_8, v_9\}, S_4 = \{v_{10}, v_{12}, v_{14}\}$ and $S_5 = \{v_4, v_{11}, v_{15}, v_{16}\}$ are the minimal majority dominating chromatic sets. Hence $\gamma_M\chi(G) = 3$. Also, all the sets are only disjoint majority dominating chromatic sets of graph G. Therefore $d_M\chi(G) = 5$ and $d_{ch}(G) = 2$.

Example 4.2.4: For the unicyclic graph $G = C_9 \circ K_1$ with $p = 18, \chi(G) = 3$. Since C_9 is the vertex color critical graph, the set

$S = \{v_1, v_2, \cdots, v_9\}$ is the only majority dom-chromatic set of G and $\gamma_M\chi(G) = 9 = \gamma_{ch}(G)$. Hence $d_M\chi(G) = 1 = d_{ch}(G)$.

Example 4.2.5: Consider the graph $G = \overline{K_6} + C_6 + C_6 + \overline{K_6}$.

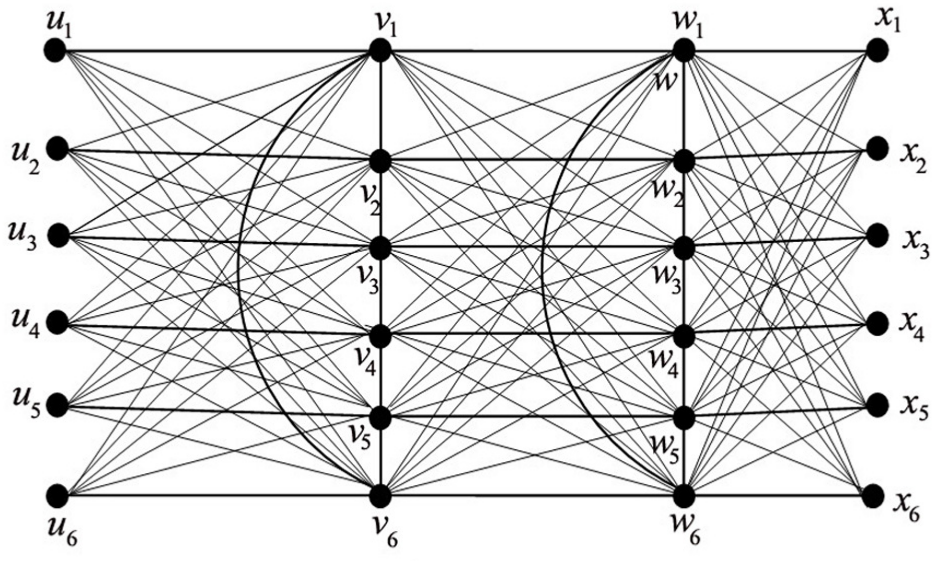

Figure 4.2: G

Let $V_1(\overline{K_6}) = \{u_1, u_2, \cdots, u_6\}$ with $d(u_i) = 6, V_2(C_6) = \{v_1, v_2, \cdots, v_6\}$ with $d(v_i) = 14, V_3(C_6) = \{w_1, w_2, \cdots, w_6\}$ with $d(w_i) = 14$ and $V_4(\overline{K_6}) = \{x_1, x_2, \cdots, x_6\}$ with $d(x_i) = 6$. For the graph $\chi(G) = 3, \gamma_{ch}(G) = 3 = \gamma_M\chi(G)$. The dom- chromatic sets are $S_1 = \{v_1, v_2, w_1\}$, $S_2 = \{w_2, w_3, v_3\}, S_3 = \{v_4, v_5, w_4\}$ and $S_4 = \{w_5, w_6, v_6\}$ and the remaining vertex set $R = \{u_1, u_2, \cdots, u_6, x_1, x_2, \cdots, x_6\}$ will be the dominating set but the set R does not satisfies $\chi(\langle R\rangle) = \chi(G)$. Hence there is no other disjoint dom- chromatic set exists. It implies that $d_{ch}(G) = 4$.

The majority dom-chromatic sets are $S_1 = \{u_1, v_1, v_2\}, S_2 = \{u_2, u_3, v_3\}, S_3 = \{u_4, v_4, v_5\}, S_4 = \{u_5, u_6, v_6\}, S_5 = \{w_1, x_1, x_2\}, S_6 = \{w_2, w_3, x_3\}, S_7 = \{w_4, x_4, x_5\}$ and $S_8 = \{w_5, w_6, x_6\}$. Therefore the vertex set $V(G)$ partitioned into eight majority dom-chromatic sets for G.

Let $S' = \{S_1, S_2, \cdots, S_8\}$ be the majority dom-chromatic partition for G and $|N[S']| = p$ and since $|N[S_i]| = p$, there exists no other disjoint majority dom-chromatic set for G. Hence $d_M\chi(G) = 8$. Therefore, $d_{ch}(G) < d_{M\chi}(G)$.

Proposition 4.2.6: For any graph G, (i) $d_{M\chi}(G) \leq d_M(G)$ and (ii) $d_{ch}(G) \leq d_{M\chi}(G)$.

Proof: (i) Since every majority dom-chromatic set of a graph G is a majority dominating set of $G, \gamma_{M\chi}(G) \geq \gamma_M(G)$. Then $d_{M\chi}(G) \leq d_M(G)$.

(ii) Since every dom-chromatic set of a graph G is a majority dom-chromatic set of $G, \gamma_{ch}(G) \geq \gamma_{M\chi}(G)$. Hence $d_{ch}(G) \leq d_{M\chi}(G)$.

∎

4.3 $d_{M\chi}$ for Various Families of Graphs

In this section, the exact value of $d_{M\chi}$ is determined for some classes of graphs.

Proposition 4.3.1: Let the graph $G = K_{1,p-1}$, a star, $G = F_p$, a Fan and $G = W_p, p \geq 5$, a wheel. Then $d_{M\chi(G)} = 1$.

Proof: Since the graphs $K_{1,p-1}, F_p$ and W_p contains the central vertex $\{v\}$ is of degree $d(v) = p - 1$, any majority dom-chromatic set of G must include the central vertex v. Hence $V(G)$ wouldn't be partitioned into many disjoint majority dom-chromatic sets of G. Hence $d_{M\chi}(G) = 1$. ∎

Proposition 4.3.2: For a complete graph $G = K_p, d_{M\chi}(G) = 1$.

Proof: Since the graph G is vertex color critical, by proposition (2.3.2)(i), $\gamma_{M\chi}(G) = p$. Hence $d_{M\chi}(G) = 1$. ∎

Proposition 4.3.3: Let $G = C_p$ be a cycle with $p \geq 3$. Then

$$d_M\chi(G) = \begin{cases} 1, \text{ if } p \text{ is odd} \\ 2, \text{ if } p = 4 \\ 3, \text{ if } p \equiv 6, 10 \\ 4, \text{ if } p = 8, 12, 14, 16, 18, 22, 24, 28, 34 \\ 5, \text{ if } p = 20, 26, 30, 32 \text{ and } p \geq 36. \end{cases}$$

Proof: Let $V(G) = \{v_1, v_2, \cdots, v_p\}$be the vertex set of G. For the graph G, $\chi(G) = 3,$ if p is odd and $2,$ if p is even and by the proposition (2.3.3),

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{6} \rceil, \text{ if } p \equiv 2 \ (mod\ 6) \\ \lceil \frac{p}{6} \rceil + 1, \text{ if } p \equiv 0, 4 \ (mod\ 6) \\ p, \text{ if } p \text{ is odd.} \end{cases} \tag{4.1}$$

Case (i): Suppose p is odd. Then all the odd cycles $C_p, p \geq 3$ are vertex color critical graphs. By the condition (4.1), $\gamma_{M\chi}(G) = p$ and hence $d_M\chi(G) = 1$.

Case (ii): Let $p = 4$. Then $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_3, v_4\}$ be the only majority dom-chromatic partition set of G. Hence $d_{M\chi}(G) = 2$.

Case (iii): Let $p = 6, 10$. For $p = 6, S = \{(v_1, v_2), (v_3, v_4), (v_5, v_6)\}$ and for $p = 10, S = \{(v_1, v_2, v_7), (v_3, v_4, v_8), (v_5, v_6, v_9)\}$. Therefore S is the only majority dom-chromatic partition set of G for $p = 6, 10$. Hence $d_{M\chi}(G) = 3$.

Case (iv): Suppose $p = 8, 12, 14, 16, 18, 22, 24, 28, 34$. By the condition (4.1), when $p = 8, 14,$ (i.e) $p = 6k + 2$, $\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor = 4$ if $k = 1, 2$. When $p = 12, 18, 24,$ (i.e) $p = 6k$, $\lfloor \frac{p}{\gamma_M\chi(G)} \rfloor = 4$ if $k = 2, 3, 4$. When

$p = 16, 22, 28, 34$, (i.e.) $p = 6k + 4$, $\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor = 4$ if $k = 2, 3, 4, 5$. Let

$S_1 = \{v_1, v_2, \cdots, v_{4(\gamma_{M\chi}(G)-2)+1}, v_{4(\gamma_{M\chi}(G)-1)+1}\}$,

$S_2 = \{v_3, v_4, \cdots, v_{4(\gamma_{M\chi}(G)-2)+2}, v_{4(\gamma_{M\chi}(G)-1)+2}\}$,

$S_3 = \{v_5, v_6, \cdots, v_{4(\gamma_{M\chi}(G)-2)+3}, v_{4(\gamma_{M\chi}(G)-1)+3}\}$ and

$S_4 = \{v_7, v_8, \cdots, v_{4(\gamma_{M\chi}(G)-2)+4}, v_{4(\gamma_{M\chi}(G)-1)+4}\}$.

Now, S_1, S_2, S_3 and S_4 are majority dom-chromatic sets of G such that the first two vertices v_i and v_j are adjacent in all sets S_t and $d(v_j, v_k) \geq 4, v_j \neq v_k, v_j, v_k \in S_t, t = 1, 2, 3, 4$. Therefore in all the sets, the last vertex is $v_{4(\gamma_{M\chi}(G)-1)+i}, i = 1, 2, 3, 4$. Then $\{S_1, S_2, S_3, S_4(V(G) - \cup_{t=1}^{4} S_t)\}$ is a majority dom-chromatic partition of $V(G)$ and therefore $d_M\chi(G) \geq 4$. Since $d_M\chi(G) \leq \lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor, d_{M\chi}(G) \leq 4$. Then $d_M\chi(G) = 4$ when $p = 8, 12, 14, 16, 18, 22, 24, 28, 34$.

Case (v): Let $p = 20, 26, 30, 32$ and $p \geq 36$.

Subcase (i): Suppose $p = 20, 26, 30, 32$. By the condition (4.1), When $p = 20, 26, 32$, (i.e.) $p = 6k + 2$, $\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor = 5$ if $k = 3, 4, 5$. When $p = 30$, (i.e.) $p = 6k$, $\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor = 5$ if $k = 5$. Let

$S_1 = \{v_1, v_2, \cdots, v_{5(\gamma_{M\chi}(G)-2)+1}, v_{5(\gamma_{M\chi}(G)-1)+1}\}$,

$S_2 = \{v_3, v_4, \cdots, v_{5(\gamma_{M\chi}(G)-2)+2}, v_{5(\gamma_{M\chi}(G)-1)+2}\}$,

$S_3 = \{v_5, v_6, \cdots, v_{5(\gamma_{M\chi}(G)-2)+3}, v_{5(\gamma_{M\chi}(G)-1)+3}\}$,

$S_4 = \{v_7, v_8, \cdots, v_{5(\gamma_{M\chi}(G)-2)+4}, v_{5(\gamma_{M\chi}(G)-1)+4}\}$ and

$S_5 = \{v_9, v_{10}, \cdots, v_{5(\gamma_{M\chi}(G)-2)+5}, v_{5(\gamma_{M\chi}(G)-1)+5}\}$

Now, the sets $S_t, t = 1, 2, 3, 4, 5$ are majority dom-chromatic sets of G such that the first two vertices v_i and v_j are adjacent in all sets S_t and $d(v_j, v_k) \geq 4, v_j \neq v_k, v_j, v_k \in S_t, t = 1, 2, 3, 4, 5$. Observe that in all five sets, the last vertex is $v_{5(\gamma_{M\chi}(G)-1)+i}, i = 1, 2, 3, 4, 5$. Then $\{S_1, S_2, S_3, S_4 \cup (V(G) - \cup_{t=1}^{5} S_t)\}$ is a majority dom-chromatic partition of $V(G)$ and therefore $d_M\chi(G) \geq 5$. Since $d_{M\chi}(G) \leq \lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor, d_{M\chi}(G) \leq 5$. Hence $d_{M\chi}(G) = 5$.

Subcase (ii): Let $p \geq 36$. Let $p = 0, 2, 4(mod\ 6)$. By the condition (4.1), $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil$ and $\lceil \frac{p}{6} \rceil + 1$. When $p \geq 36, p = 6k, \lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor = 5$ if $k \geq 6$. When $p \geq 38, p = 6k + 2, \lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor = 5$ if $k \geq 6$. When $p \geq 36, p = 6k + 4, \lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor = 5$ if $k \geq 6$. Then S_1, S_2, S_3, S_4 and S_5 are taken as in the subcase (i) and applying the same arguments, we get $d_{M\chi}(G) = 5$. Therefore $d_M\chi(G) = 5$ if $p = 20, 26, 30, 32$ and $p \geq 36$. ∎

Proposition 4.3.4: For a graph $G = P_p$, a Path with $p \geq 3$ vertices,

$$d_{M\chi}(G) = \begin{cases} 1, \text{ if } p = 3 \\ 2, \text{ if } p = 4, 5 \\ 3, \text{ if } 6 \leq p \leq 11, 15 \\ 4, \text{ if } p = 12, 13, 14, 33, 34 \text{ and } 16 \leq p \leq 29 \\ 5, \text{ if } p = 30, 31, 32 \text{ and } p \geq 35. \end{cases}$$

Proof: Let $G = P_p, p \geq 3$ and $V(G) = \{v_1, v_2, \cdots, v_p\}$. For $G = P_p, \chi(G) = 2$. By corollary (2.3.4),

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{6} \rceil, \text{ if, if } p \equiv 1, 2 \\ \lceil \frac{p}{6} \rceil + 1, \text{ if } p \equiv 0, 3, 4, 5. \end{cases} \tag{4.2}$$

Case (i): Let $p = 3$. Then $\{(v_1, v_2)\}$ is the only disjoint majority dom-chromatic set of G. Hence $d_{M\chi}(G) = 1$.

Case (ii): Let $p = 4, 5$. Then $\{(v_1, v_2), (v_3, v_4)\}$ is the only disjoint majority dom-chromatic partition of G. Hence $d_{M\chi}(G) = 2$.

Case (iii): When $6 \leq p \leq 11$ and $p = 15$. Then $\{(v_1, v_2), (v_3, v_4), (v_5, v_6)\}$ is the disjoint majority dom-chromatic partition of G for $p = 6$. When $p = 7, 8, \{(v_2, v_3), (v_4, v_5), (v_6, v_7)\}$ is the only disjoint majority dom-chromatic partition of G. Hence $d_{M\chi}(G) = 3$. If $p = 9, 10, 11, 15$ then $d_{M\chi}(G) \leq \lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor = 3$. Let

$S_1 = \{v_1, v_2, \cdots, v_{3(\gamma_{M\chi}(G)-2)+1}, v_{3(\gamma_{M\chi}(G)-1)+1}\}$,

$S_2 = \{v_3, v_4, \cdots, v_{3(\gamma_{M\chi}(G)-2)+2}, v_{3(\gamma_{M\chi}(G)-1)+2}\}$ and

$S_3 = \{v_5, v_6, \cdots, v_{3(\gamma_{M\chi}(G)-2)+3}, v_{3(\gamma_{M\chi}(G)-1)+3}\}$

be the MDC sets of G such that $d(v_i, v_j) = 1$ and $d(v_j, v_k) \geq 3, v_i \neq v_j$ and $|N[S_r]| \geq \lceil \frac{p}{2} \rceil$ for all $r = 1, 2, 3$ and $\chi(\langle S_r \rangle) = 2 = \chi(G)$. Then $\{S_1 \cup S_2 \cup S_3 \cup (V(G) - \cup S_r)\}, r = 1, 2, 3$ is the majority dom-chromatic partition of G, $d_{M\chi}(G) \geq 3$. Since $d_{M\chi}(G) \leq 3$,$d_{M\chi}(G) = 3$.

Case (iv): When $p = 12, 13, 14, 16 \leq p \leq 29, 33, 34$. By the condition (4.2), if $p = 12, 16, 17, 18$ then $d_{M\chi}(G) \leq \lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor = 4$. Let

$S_1 = \{v_1, v_2, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-2)+1}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-1)+1}\}$,

$S_2 = \{v_3, v_4, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-2)+2}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-1)+2}\}$,

$S_3 = \{v_5, v_6, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-2)+3}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-1)+3}\}$ and

$S_4 = \{v_7, v_8, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-2)+4}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-1)+4}\}$

be the MDC sets of G for $p = 12, 16, 17, 18$. Also, if $p = 13, 14, 19 \leq p \leq 29, 33, 34$ then by condition (4.2), $d_{M\chi} \leq \lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor = 4$ and the MDC sets are $S_1 = \{v_2, v_3, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-2)+2}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-1)+2}\}$,

$S_2 = \{v_4, v_5, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-2)+3}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-1)+3}\}$,

$S_3 = \{v_6, v_7, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-2)+4}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-1)+4}\}$ and

$S_4 = \{v_8, v_9, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-2)+5}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor(\gamma_{M\chi}(G)-1)+5}\}$.

In the above two classifications of $p, d(v_j, v_k) \geq 4, v_j \neq v_k$, for all $v_j, v_k \in S_r, r = 1, 2, 3, 4$. Then $\{S_1, S_2, S_3, S_4(V(G) - \cup_{r=1}^{4} S_r)\}$ is a majority dominating chromatic partition of $V(G)$ and therefore $d_{M\chi}(G) \geq 4$. Since $d_{M\chi}(G) \leq 4$. Hence $d_{M\chi}(G) = 4$ when $p = 12, 13, 14, 16 \leq p \leq 29, 33, 34$.

Case (v): Let $p = 30, 31, 32$ and $p \geq 35$.

Subcase (i): Suppose $p = 30, 35, 36, 40, 41, 42$. By the condition (4.2), When $p = 30, 35, 36, 40, 41, 42, d_{M\chi}(G) \leq \lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor = 5$. Let

$S_1 = \{v_1, v_2, \cdots, v_{5(\gamma_{M\chi}(G)-2)+1}, v_{5(\gamma_{M\chi}(G)-1)+1}\}$,

$S_2 = \{v_3, v_4, \cdots, v_{5(\gamma_{M\chi}(G)-2)+2}, v_{5(\gamma_{M\chi}(G)-1)+2}\}$,

$S_3 = \{v_5, v_6, \cdots, v_{5(\gamma_{M\chi}(G)-2)+3}, v_{5(\gamma_{M\chi}(G)-1)+3}\}$,

$S_4 = \{v_7, v_8, \cdots, v_{5(\gamma_{M\chi}(G)-2)+4}, v_{5(\gamma_{M\chi}(G)-1)+4}\}$ and

$S_5 = \{v_9, v_{10}, \cdots, v_{5(\gamma_{M\chi}(G)-2)+5}, v_{5(\gamma_{M\chi}(G)-1)+5}\}$.

Now, the sets $S_r, r = 1, 2, 3, 4, 5$ are majority dom-chromatic sets of G such that the first two vertices v_i and v_j are adjacent for all S_r and $d(v_j, v_k) \geq 5, v_j \neq v_k, v_j, v_k \in S_r, r = 1, 2, 3, 4, 5$. Therefore in all five sets the last vertex is $v_{5(\gamma_{M\chi}(G)-1)+i}, i = 1, 2, 3, 4, 5$. Then $\{S_1, S_2, S_3, S_4 \cup (V(G) - \cup_{r=1}^{5} S_r)\}$ is a majority dom-chromatic partition of $V(G)$ and therefore $d_{M\chi}(G) \geq 5$. Since $d_{M\chi}(G) \leq 5, d_{M\chi}(G) = 5$.

Subcase (ii): Let $p = 31, 32, 37, 38, 39$ and $p \geq 43$. By the result (4.2), $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil$ and $\lceil \frac{p}{6} \rceil + 1, d_{M\chi}(G) \leq \lfloor \frac{p}{\gamma_{M\chi}(G))} \rfloor = 5$. Then the MDC sets are, $S_1 = \{v_2, v_3, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor (\gamma_{M\chi}(G)-2)+2}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor (\gamma_{M\chi}(G)-1)+2}\}$,

$S_2 = \{v_4, v_5, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor (\gamma_{M\chi}(G)-2)+3}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor (\gamma_{M\chi}(G)-1)+3}\}$,

$S_3 = \{v_6, v_7, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor (\gamma_{M\chi}(G)-2)+4}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor (\gamma_{M\chi}(G)-1)+4}\}$,

$S_4 = \{v_8, v_9, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor (\gamma_{M\chi}(G)-2)+5}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor (\gamma_{M\chi}(G)-1)+5}\}$ and

$S_5 = \{v_{10}, v_{11}, \cdots, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor (\gamma_{M\chi}(G)-2)+6}, v_{\lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor (\gamma_{M\chi}(G)-1)+6}\}$.

Applying the same arguments as in subcase (i), we get

$d_{M\chi}(G) = 5$. ∎

Proposition 4.3.5: For the graph $G = \overline{K_p}, d_{M\chi}(G) = 1,$ if p is odd, and $d_{M\chi}(G) = 2,$ if p is even.

Proof: Let $G = \overline{K_p}$. By the result (2.3.1)(ii), $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$ and $\chi(G) = 1$. Let $V(G) = \{v_1, v_2, \cdots, v_p\}$ be the vertex set of G.

Case (i): Suppose $p = 2m$. Let $S_1 = \{v_1, v_2, \cdots, v_m\}, S_2 = \{v_{m+1}, v_{m+2}, \cdots, v_{2m}\}$ be the two disjoint majority dominating chromatic subsets of $V(G)$. Hence $d_{M\chi}(G) = 2$.

Case (ii): Let $p = 2m + 1$. Let $S_1 = \{v_1, v_2, \cdots, v_{m+1}\}$ and $S_2 = \{v_{m+2}, v_{m+3}, \cdots, v_{2m+1}\}$ be the two vertex subsets of $V(G)$. Then $|S_1| = \lceil \frac{p}{2} \rceil$, $|S_2| < \frac{p}{2}$ and S_1 is the majority dom-chromatic set of G

and S_2 couldn't form the majority dom-chromatic set of G. Hence $d_{M\chi}(G) = 1$. ∎

Proposition 4.3.6: Let $G = K_{m,n}, m \leq n$ be a complete bipartite graph. Then $d_{M\chi}(G) = \min\{m, n\}$ and $d_{M\chi}(G) = \frac{p}{2}$, if $m = n$.

Proof: Let $V_1(G) = \{u_1, u_2, \cdots, u_m\}$ and $V_2(G) = \{v_1, v_2, \cdots, v_n\}$ be the two vertex sets of G with $p = m + n$.

Case (i): If $m = n$ and then $d(u_i) \geq \lceil \frac{p}{2} \rceil$ and $d(v_i) \geq \lceil \frac{p}{2} \rceil$. Therefore each $\{u_i\}, i = 1, 2, \cdots, m$ and $\{v_i\}, i = 1, 2, \cdots, n$ are the majority dominating sets of G. Since $\chi(G) = 2$, each $\{u_i, v_i\}, i = 1, 2, \ldots, m = n$ is the majority dom-chromatic set of G. Hence $d_{M\chi}(G) = \frac{m+n}{2} = \frac{p}{2}$, if $m = n$.

Case (ii): If $m < n$ then $d(u_i) = n$ and $d(v_i) = m$. Since $n \geq m + 1, |N(u_i)| \geq m + 2 \geq \lceil \frac{p}{2} \rceil$ and $|N(v_i)| < \lceil \frac{p}{2} \rceil$. Hence each $\{u_i\}$ is only a majority dominating set of G. Since $\chi(G) = 2$, choose dominating edges of G such as $\{(u_1, v_1), (u_2, v_2), \cdots, (u_m, v_m)\}$. These subsets of G become the disjoint majority dom-chromatic sets of $V(G)$. Therefore $d_{M\chi}(G) = m = \min(m, n)$, if $m < n$. ∎

Proposition 4.3.7: If $G = D_{r,s}$, a double star then $d_{M\chi}(G) = 2$.

Proof: Let u and v be the central vertices of the graph G. Let $|\{u_1, u_2, \cdots, u_r\}|$ and $|\{v_1, v_2, \cdots, v_s\}|$ be the number of pendants at u and v with $p = r + s + 2$.

Case (i): Suppose $s = r, r+1, r+2$. Then $d(u) \geq \lceil \frac{p}{2} \rceil$ and $d(v) \geq \lceil \frac{p}{2} \rceil$. It implies that the graph G has two majority dominating vertices at the centre. Since $\chi(G) = 2, S_1 = \{u, u_1\}$ and $S_2 = \{v, v_1\}$ are the majority dom-chromatic sets for G. Hence $d_{M\chi}(G) = 2$.

Case (ii): Let $r < s$ and $s \geq r + 3$. Then $d(u) < \lfloor \frac{p}{2} \rfloor - 1$ and $d(v) \geq \lceil \frac{p}{2} \rceil + 1$. Hence the graph G has only one majority dominating vertex v. Since $\chi(G) = 2, S_1 = \{v, v_1\}$ and $S_2 = \{u, u_1, v_2, \cdots, v_t\}$ with $t = \lceil \frac{p}{2} \rceil - (r+2) - 1$. Then $|N[S_1]| \geq \lceil \frac{p}{2} \rceil$ and $|N[S_2]| = d(u) + 1 + t = r + 2 + \lceil \frac{p}{2} \rceil - (r+2) = \lceil \frac{p}{2} \rceil$. Hence, S_1 and S_2 are majority dom-chromatic sets of G and majority dom-chromatic partition is $\{S_1, S_2 \cup R\}$ where R is the remaining pendants. Thus, $d_{M\chi}(G) = 2$. ∎

Proposition 4.3.8: Let $G = G_n^{(m)}$ be a graph which contains m copies of the complete graph K_n. Then $d_{M\chi}(G) = 1$.

Proof: Let $V(G) = \{v, v_{11}, v_{12}, \cdots, v_{1(n-1)}, v_{21}, v_{22}, \cdots, v_{2(n-1)}, \cdots, v_{m1}, v_{m2}, \cdots, v_{m(n-1)}\}$ be the vertex set of G and $p = m(n-1) +$

1. In the structure, all mcopies of K_n meet at a central vertex v. Since the degree of a vertex v is $d(v) = p - 1, \{v\}$ is the majority dominating set of G. The graph G contains a complete subgraph K_n. Since the graph K_n is a vertex color critical, $\chi(G) = n$. Let $S = \{v, v_{11}, v_{12}, \cdots, v_{1(n-1)}\}$ be the subset of G such that $|N[S]| \geq p-1 > \lceil \frac{p}{2} \rceil$ and $\chi(\langle S \rangle) = n = \chi(G)$. Hence S is a majority dom-chromatic set of G. Since there is no other disjoint majority dom-chromatic set without having the central vertex v. Hence the majority dom-chromatic partition of $V(G)$ is one and $d_{M\chi}(G) = 1$. ∎

Corollary 4.3.9: Let $G = D_3^{(m)}, m \geq 2$ be a friendship graph. Then $d_{M\chi}(G) = 1$.

Proof: The graph G contains 'm' triangles attached at a single central vertex 'v' and $d(v) = p-1$. Since $\gamma_{M\chi}(G) = 3, D = \{v, u_1, u_2\}$ is a majority dom-chromatic set of G with a central vertex v. So $V(G)$ would not be partitioned into many sets including v and $d_{M\chi}(G) = 1$.

∎

Definition 4.3.10: Let G be a graph with p vertices and the maximum degree $\Delta(G)$. If $d_{M\chi}(G) = 2\Delta(G) + 1$ then the graph G is called majority domatically chromatic full.

For example, let $G = C_{20}$. By proposition (4.3.3), $d_{M\chi}(G) = 5$ and $\Delta(G) = 2$. Hence $d_{M\chi}(G) = 2\Delta(G) + 1 = 5$.

4.4 Bounds on $d_{M\chi}(G)$

In this section, bounds on $d_{M\chi}(G)$ with respect to $\gamma_{M\chi}(G), p$ and $\Delta(G)$ are investigataed.

Theorem 4.4.1. *Let G be any graph. Then $d_{M\chi}(G) \leq \lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor$.*

Proof: Let $\{V_1, V_2, \cdots, V_k\}$ be the majority dom-chromatic partitions of G. Then, $p = |V_1| + |V_2| + \cdots + |V_k| = \sum_{i=1}^{k} |V_i|$. Let $d_{M\chi}(G) = k$. Therefore $|V_i| \geq \gamma_{M\chi}(G)$, for each i. Then $p = |V_1| + |V_2| + \cdots + |V_k| \geq k\gamma_{M\chi}(G)$and $p \geq k\gamma_{M\chi}(G) \geq d_{M\chi}(G)\gamma_{M\chi}(G)$. Hence $d_{M\chi}(G) \leq \lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor$. ∎

Corollary 4.4.2: For any graph $G, d_{M\chi}(G)\gamma_{M\chi}(G) \leq p$.

Theorem 4.4.3: For any graph G,

(i) $d_{M\chi}(G) \leq 2\Delta(G) + 1$, if $\Delta(G) < \lceil \frac{p}{2} \rceil - 1$

(ii) $\left(\frac{p}{\Delta(G)+1} \right) \leq d_{M\chi}(G) \leq \left(\frac{p}{2} \right)$, if $\Delta(G) \geq \lceil \frac{p}{2} \rceil - 1$.

These bounds are sharp.

Proof: (i) If $\Delta(G) < \lceil \frac{p}{2} \rceil - 1$, the majority domination number satisfies $\gamma_M(G) \geq \lceil \frac{p}{2(\Delta(G)+1)} \rceil$. Since $\gamma_M(G) \leq \gamma_{M\chi}(G)$, $\gamma_{M\chi}(G) \geq \lceil \frac{p}{2(\Delta(G)+1)} \rceil \geq \lceil \frac{p}{2\Delta(G)+1} \rceil$. It implies that $2\Delta(G) + 1 \geq \lfloor \frac{p}{\gamma_{M\chi}(G)} \rfloor \geq d_{M\chi}(G)$. Hence, $d_{M\chi}(G) \leq 2\Delta(G) + 1$, if $\Delta(G) < \lceil \frac{p}{2} \rceil - 1$. This bound is sharp if $G = C_p$, a cycle with $p = 20$. By the proposition (4.3.3), $d_{M\chi}(G) = 5 = 2\Delta(G) + 1$.

(ii) Suppose $\Delta(G) \geq \lceil \frac{p}{2} \rceil - 1$. Then $\gamma_M(G) = 1$. But for any graph $G, \gamma_{M\chi}(G) \geq 2$ and $d_{M\chi}(G) \leq \lfloor \frac{p}{\gamma_{M\chi}(G)} \rceil = \frac{p}{\gamma_{M\chi}(G)}$. It implies that $\gamma_{M\chi}(G) \leq \frac{p}{d_{M\chi}(G)}$ and $2 \leq \gamma_{M\chi}(G) \leq \frac{p}{d_{M\chi}(G)}$. Hence

$$2d_{M\chi}(G) \leq p \text{ and } d_{M\chi}(G) \leq \left(\frac{p}{2}\right). \tag{4.3}$$

This bound is sharp if $G = D_{2,2}, P_4, K_{1,3}$. Since $d_{M\chi}(G) \leq \frac{p}{\gamma_{M\chi}(G)}$ and $\gamma_{M\chi}(G) \leq \Delta(G) + 1, \gamma_{M\chi}(G) \leq \frac{p}{d_{M\chi}(G)} \leq \Delta(G) + 1$. Then

$$\frac{p}{\Delta(G) + 1} \leq d_{M\chi}(G). \tag{4.4}$$

Hence from (4.3) and (4.4), $\left(\frac{p}{\Delta(G)+1}\right) \leq d_{M\chi}(G) \leq \left(\frac{p}{2}\right)$. If $G = K_p$ then $d_{M\chi}(G) = 1 = \left(\frac{p}{\Delta(G)+1}\right)$. Therefore the lower bound is sharp. If $G = P_4$ then $d_{M\chi}(G) = 2 = \left(\frac{p}{2}\right)$ and the upper bound is sharp. ∎

Proposition 4.4.4: For a graph $G = \bar{K}_p, p$ is odd, $d_{M\chi}(G) = d_{ch}(G) = d_M(G) = d(G)$.

Proof: Let p be odd. Let $|S_1| = |\{v_1, v_2, \cdots, v_{\lceil \frac{p}{2} \rceil}\}| \geq \lceil \frac{p}{2} \rceil$ and $|S_2| = |\{v_{\lceil \frac{p}{2} \rceil + 1}, v_{\lceil \frac{p}{2} \rceil + 2}, \cdots, v_p\}| < \lceil \frac{p}{2} \rceil$ be two subsets of G. Clearly S_1 be the majority dominating set of G and S_2 couldn't be the majority dominating set of G. Hence $d_M(G) = 1$. Since $\chi(G) = 1, d_{ch}(G) = 1$ and $d_{M\chi}(G) = 1$. Also since the graph G is totally disconnected, $V(G)$ is the only dominating set of G and $d(G) = 1$. Hence $d_{M\chi}(G) = d_{ch}(G) = d_M(G) = d(G)$. ∎

Result 4.4.5: For given any positive integer $p \geq 4$, there exists always a connected graph for which $d_{M\chi}(G) - d_{ch}(G) = 1$.

Proof: For $p \geq 4$, there exists a graph $G = D_{r,s}, r \leq s$ be a double star with $p = r + s + 2$. Then $d_{ch}(G) = 1$ and by proposition (4.3.7, $d_{M\chi}(G) = 2$. Hence, $d_{M\chi}(G) - d_{ch}(G) = 1$. ∎

Result 4.4.6: For given any positive integer k, there exists a graph which is not complete for which $d_{M\chi}(G) = d_{ch}(G)$.

Proof: For any positive integer $k \geq 1$, there exists a graph $G = K_{1,p-1}$, a star which is not complete graph. By the proposition (4.3.1), $d_{M\chi}(G) = 1$. Since $\gamma_{ch}(G) = 2, d_{ch}(G) = 1 = d_{M\chi}(G)$. ∎

Result 4.4.7: Given any positive integer k, there exists a graph G which is not complete for which $d_{M\chi}(G) = k$.

Proof: Suppose $G = K_{k,k}, k \geq 1$ with $p = 2k$. Since $\gamma_{M\chi}(G) = 2$, the vertex set $V(G)$ can be partitioned into $\frac{p}{2}$ majority dom-chromatic sets. Hence $d_{M\chi}(G) = \frac{p}{2} = k$. ∎

Proposition 4.4.8: Let P be a Petersen graph. Then $d_{M\chi}(P) = 2$.

Proof: The graph P contains two pentagons C_5. Since C_5 is a vertex color critical graph and C_5 is a subgraph of $P, \gamma_{M\chi}(P) = 5$. Therefore the vertex set $V(G)$ can be partitioned into only two majority dom-chromatic sets S_1 and S_2. Hence $d_{M\chi}(P) = 2$. ∎

4.5 Characterization Theorems on $d_{M\chi}(G)$

The necessary and sufficient conditions on $d_{M\chi}(G)$ with respect to diameter of the graph are discussed in this section.

Theorem 4.5.1: Let G be a connected bipartite graph. Then $d_{M\chi}(G) \leq \frac{p}{2}$ and $d_{M\chi}(G) = \frac{p}{2}$ if and only if $G = P_4, C_4, K_{m,n}, m = n$ and K_2.

Proof: The theorem is proved by induction on $diam(G)$. If $diam(G) = 1$ then the graph structures become $G = K_p$ and $K_{m,n}$. If $G = K_p$

then G is vertex color critical and $\gamma_{M\chi}(G) = p$. Hence $d_{M\chi}(G) = 1 < \frac{p}{2}$. If $G = K_{m,n}$, by proposition (4.3.6), $d_{M\chi}(G) = \frac{p}{2}$, if $m = n$. When $diam(G) = 2$, the graph G becomes P_3 and C_4. By the proposition (4.3.5), $d_{M\chi}(P_3) = 1 < \frac{p}{2}$ and by the proposition (4.3.3), $d_{M\chi}(C_4) = 2 = \frac{p}{2}$. If $diam(G) = 3$, the graph structures become $G = P_4$ and $D_{r,s}$ a double star with $p = r + s + 2$. By the proposition (4.3.5), $d_{M\chi}(P_4) = 2 = \frac{p}{2}$ and by the proposition (4.3.7), $d_{M\chi}(D_{r,s}) = 2 < \frac{p}{2}$.

Therefore the result is true for $diam(G) = 1, 2, 3, \cdots, (p - 2)$. Suppose $diam(G) = p - 1$. Then the graph $G = P_p$. By proposition (4.3.5), $d_{M\chi}(P_p) < \frac{p}{2}$. Hence, $d_{M\chi}(G) \le \frac{p}{2}$.

Now, assume that $d_{M\chi}(G) = \frac{p}{2}$. From the above arguments, if $diam(G) \le 2$, $d_{M\chi} = \frac{p}{2}$ holds for $G = K_{m,n}, m = n, C_4$ and K_2. If $diam(G) = 3$, the graph has two central vertices u and v with the degree $d(u) \ge 2$ and $d(v) \ge 2$. Let $\{u_1, u_2, \cdots, u_m\}$ and $\{v_1, v_2, \cdots, v_n\}$ be the pendants at u and v and $p = m+n+2$, where $m, n \ge 1$. Suppose $m = n$. Then $|N[u]| \ge \frac{p}{2}$ and $|N[v]| \ge \frac{p}{2}$. It implies that $\{u\}$ and $\{v\}$ are the majority dominating sets of G. Since $\chi(G) = 2$, the sets $S_1 = \{(u, u_i)\}$ and $S_2 = \{(v, v_i)\}$ where u_i and v_i are the only majority dom-chromatic set for G. Hence $d_{M\chi}(G) = 2$. By assumption, $d_{M\chi}(G) = \frac{p}{2}$ and implies that $\frac{p}{2} = 2, \frac{m+n+2}{2} = 2 \Rightarrow m = n = 1$. Therefore the graph G has one pendant vertex at both u and v. Hence $G = P_4$. The converse is obvious. ∎

Theorem 4.5.2: If the graph G is vertex color critical then $d_{M\chi}(G) = 1$ and $d_{M\chi}(G)\gamma_{M\chi}(G) = p$.

Proof: Since the graph G is vertex color critical, $\chi(G-u) < \chi(G)$ for all $u \in V(G)$ Let S be the majority dom-chromatic set of G. By the definition of majority dom-chromatic set, $\chi(\langle S \rangle) = \chi(G)$. Then the majority dom-chromatic set S contains all vertices of G. Therefore $S = \{v_1, v_2, \cdots, v_p\}$ and $\gamma_{M\chi}(G) = |S| = p$. Also, since $\gamma_{M\chi}(G) = p, d_{M\chi}(G) = 1 = d_{ch}(G)$. Hence $d_{M\chi}(G)\gamma_{M\chi}(G) = p$. ∎

Theorem 4.5.3: Let G be any connected graph of p vertices. Then $d_{M\chi}(G) = 1$ if and only if

(i) The graph G contains a full degree vertex,

(ii) The graph G is vertex color critical and

(iii) The graph G contains an induced vertex color critical subgraphs which are not disjoint.

Proof: Let G be any connected graph of p and $d_{M\chi}(G) = 1$.

Case (i): Then the vertex set $V(G)$ is partitioned into only one majority dom-chromatic set S. Therefore $S = \{v_1, v_2, \cdots, v_p\}$ and

$\gamma_{M\chi}(G)|S| = p$. It implies that the graph G with full vertex set is a vertex color critical graph. Hence condition (ii) holds.

Case (ii): Suppose the majority dom-chromatic set $S = S_1 \cup S_2$ such that $|N[S_1]| \geq \lceil \frac{p}{2} \rceil$ and $|N[S_2]| < \lceil \frac{p}{2} \rceil$. It implies that S_1 is the only majority dom-chromatic set of G. Since for any connected graph G, $\chi(G) \geq 2, |S_1| \geq 2$ and $|N[S_1]| \geq \lceil \frac{p}{2} \rceil$.

Subcase (i): Suppose $|S_1| = 2$. If G is a tree, $\chi(G) = 2$. Therefore $S_1 = \{u_1, u_2\}$, where u_1 is of degree $d(u_1) \leq p-1$ and u_2 is of degree $d(u_2) \geq 1$ such that $|N[S_1]| = p$. Hence the graph G contains a full degree vertex u_1. If $d(u_1) = p-1$ and $d(u_2) = 1$ then $|N[S_1]| = p > \lceil \frac{p}{2} \rceil$. Therefore G contains a full degree vertex u_1. If $d(u_1) < p-1$ and $d(u_2) \geq$ 1then there are two disjoint majority dom-chromatic sets and $d_{M\chi}(G) = 2$, which is a contradiction to to the assumption. Hence G contains a full degree vertex and then condition (i) holds.

Subcase (ii): If $|S_1| = 3$, the graph G is a tree or it contains a triangle. If G is a tree, $S_1 = \{u_1, u_2, u_3\}$ is the majority dom-chromatic set of G. Suppose $d(u_1) < p-1$ and $d(u_i) \geq 1, i = 2, 3$. Then there exists atleast two disjoint majority dom-chromatic set in G. Hence $d_{M\chi}(G) \geq 2$, which is a contradiction to the assumption.

Suppose G contains a triangle, $\chi(G) = 3$ and $\gamma_{M\chi}(G) \geq 3$. Since $d(u_1) \leq p-1$ and $d(u_i) \geq 2, i = 2, 3, S_1$ is a majority dom-chromatic set of G. By the above arguments, $d_{M\chi}(G) \geq 2$, which is also a contradiction. Hence the set S_1 with $d(u_1) = p - 1$ and $d(u_i) \geq 2$. Therefore G contains a full degree vertex u_1 and the condition (i) holds.

Subcase (iii) Suppose $|S_1| \geq 4$. Then the graph G is a tree or it contains a vertex color critical graph as an induced subgraph. If G is a tree then $S_1 = \{u_1, u_2, u_3, u_4\}$, where $d(u_1) < p - 1$ and by the similar arguments as in the above case, $d_{M\chi}(G) \geq 2$, which is a contradiction to the assumption. Suppose the MDC set $S = S_1 \cup S_2$ such that $|N[S_1]| < \lceil \frac{p}{2} \rceil$ and $|N[S_2]| \geq \lceil \frac{p}{2} \rceil$. Then S_2 is the majority dom-chromatic set of G and apply the same argument as in case (i). Hence the condition (ii) holds.

Subcase (iv) Suppose G contains a vertex color critical subgraphs g_1 and g_2 and let the induced subgraphs g_1 and g_2 such that $|v(g_1)| = |v(g_2)|$ are disjoint. If $|N[g_1]| \geq \lceil \frac{p}{2} \rceil$ and $|N[g_2]| \geq \lceil \frac{p}{2} \rceil$ then there are atleast two disjoint majority dom-chromatic sets in G and $d_{M\chi}(G) \geq 2$, which is a contradiction to $d_{M\chi}(G) = 1$. If $|N[g_1]| < \lceil \frac{p}{2} \rceil$ and $|N[g_2]| \geq \lceil \frac{p}{2} \rceil$ and vice versa then there exists atleast two majority dom-chromatic sets in G and $d_{M\chi}(G) \geq 2$. If $|N[g_1]| < \lceil \frac{p}{2} \rceil$ and $|N[g_2]| < \lceil \frac{p}{2} \rceil$ then there exists atleast two majority dom-chromatic

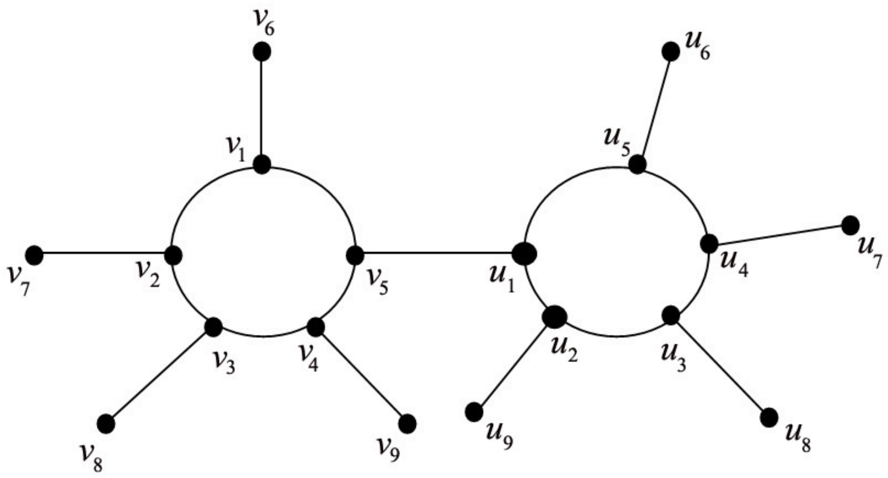

Figure 4.3: G

sets in G and $d_{M\chi}(G) \geq 2$. Hence from the above cases, if the vertex color critical induced subgraphs are disjoint then $d_{M\chi}(G) \geq 2$, which is a contradiction. Thus, the vertex color critical induced subgraphs are not disjoint. Thus the condition (iii) holds.

Case (iii): Suppose $S = S_1 \cup S_2$ such that $|N[S_1]| = \lceil \frac{p}{2} \rceil = |N[S_2]|$. It implies that S_1 and S_2 are majority dominating chromatic partition sets of G and $d_{M\chi}(G) = 2$, which is a contradiction to $d_{M\chi}(G) = 1$.

Hence by propositions (4.3.1) and (4.3.2), the converse is true. ∎

Proposition 4.5.4: Let G be a cycle on p vertices. Then $d_{M\chi}(G) = \frac{p}{\gamma_{M\chi}(G)}$ if and only if (i) p is odd (ii) $p = 4, 6, 8, 12, 16, 20, 30, 40$.

Proof: Let $G = C_p$ be a cycle. By the proposition (2.3.3),

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{6} \rceil, \text{ if } p \equiv 2(mod\ 6) \\ \lceil \frac{p}{6} \rceil + 1, \text{ if } p \equiv 0, 4(mod\ 6) \\ p, \text{ if } p \text{ is odd.} \end{cases} \tag{4.5}$$

Assume that $d_{M\chi}(G) = \frac{p}{\gamma_{M\chi}(G)}$, (ie), $d_{M\chi}(G)\gamma_{M\chi}(G) = p$.

Case (i): Suppose $d_{M\chi}(G) = 1$. Then $\gamma_{M\chi}(G) = p$. Then the majority dom- chromatic set contains the whole vertex set $V(G)$. It implies that the graph G is vertex color critical. By proposition (4.5.2) $d_{M\chi}(G) = 1$ if p is odd. Hence the condition (i) holds.

Case (ii): Let $d_{M\chi}(G) = 2$. Then by proposition (4.3.5), if $d_{M\chi}(G) = 2$ then $p = 4$ and $\gamma_{M\chi}(G) = 2$. Therefore, $d_{M\chi}(G)\gamma_{M\chi}(G) = 2(2) = 4 = p$. Hence the result is true for $p = 4$.

Case (iii): If $d_{M\chi}(G) = 3$ then by proposition (4.3.5), $p = 6, 10$. By the result (4.5), $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil + 1$ if $p = 6, 10$. Then $\gamma_{M\chi}(G) = 2$ and 3. From the assumption, $d_{M\chi}(G)\gamma_{M\chi}(G) = 3(2) = 6 = p$ and $d_{M\chi}(G)\gamma_{M\chi}(G) = 3(3) = 9 < p$. Hence if $d_{M\chi}(G) = \frac{p}{\gamma_{M\chi}(G)}$ then $p = 6$ only.

Case (iv): Let $d_{M\chi}(G) = 4$. Then by proposition (4.3.5), $p = 8, 12, 14, 16, 18, 22, 24, 28, 34$. When $p = 6k + 2$, $\gamma_{M\chi}(G) = 2$ and 3, if $k = 1$ and 2. When $p = 6k$, $\gamma_{M\chi}(G) = 3, 4, 5$, if $k = 2, 3, 4$. When $p = 6k + 4$, $\gamma_{M\chi}(G) = 4, 5, 6, 7$ if $k = 2, 3, 4, 5$. Then $d_{M\chi}(G)\gamma_{M\chi}(G) = p$

if $p = 8, 12, 16$. For all other vertices, $d_{M\chi}(G)\gamma_{M\chi}(G) < p$. Hence if $d_{M\chi}(G) = \frac{p}{\gamma_{M\chi}(G)}$ then $p = 8, 12, 16$.

Case (v): Let $d_{M\chi}(G) = 5$. Then by proposition (4.3.5), $p = 20, 26, 30, 32$ and $p \geq 36$.

Subcase (i): Let $p = 6k + 2$. Then $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil$. When $p = 20, \gamma_{M\chi}(G) = 4$ and $\frac{p}{\gamma_{M\chi}(G)} = 5$. When $p = 26$, then $\gamma_{M\chi}(G) = 5$ and $\frac{p}{\gamma_{M\chi}(G)} = 5$. When $p = 32$, then $\gamma_{M\chi}(G) = 6$ and $\frac{p}{\gamma_{M\chi}(G)} = 5$. When $p = 6k$ and $p = 30, \gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil + 1 = 6$ and $\frac{p}{\gamma_{M\chi}(G)} = 5$ if $k = 5$. Therefore in all sets, $\frac{p}{\gamma_{M\chi}(G)} = 5 = d_{M\chi}(G)$ if $p = 20, 30$.

Subcase (ii): Let $p \geq 36$. Then by proposition (4.3.5), $\frac{p}{\gamma_{M\chi}(G)} = 5$. By the (4.5), if $p = 40$, then $\gamma_{M\chi}(G) = 8$ and $\frac{p}{\gamma_{M\chi}(G)} = 5 = d_{M\chi}(G)$. When $p = 6k, \gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil + 1$ and $\frac{p}{\gamma_{M\chi}(G)} = d_{M\chi}(G)$ if $k \geq 6$. When $p = 6k + 2, \gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil$ and $\frac{p}{\gamma_{M\chi}(G)} = d_{M\chi}(G)$ if $k \geq 6$. When $p = 6k + 4, \gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil + 1$ and $\frac{p}{\gamma_{M\chi}(G)} = d_{M\chi}(G)$ if $k \geq 7$. Therefore in all cases, $\frac{p}{\gamma_{M\chi}(G)} = 5 = d_{M\chi}(G)$ if $p = 20, 30, 40$. ∎

Proposition 4.5.5: Let G be a Path on p vertices. Then $d_{M\chi}(G) = \frac{p}{\gamma_{M\chi}(G)}$ if and only if $p = 4, 6, 9, 12, 16, 30, 35, 40, 45$.

Proof: Applying the same arguments as in Proposition (4.5.3), we obtain the result. ∎

Chapter 5

Changing and Unchanging Properties of Majority Dom-Chromatic Number

Abstract

In this chapter, the effects of majority dom-chromatic number $\gamma_{M\chi}(G)$ when removing any vertex, edge and adding any edge in the graph G are investigated. Nine classification of the vertex set and the edge set are discussed accordingly the vertex sets namely $V^0_{M\chi}(G), V^-_{M\chi}(G)$ and $V^+_{M\chi}(G)$ by vertex removal and the edge sets $E^0_{M\chi}(G), E^-_{M\chi}(G)$ and $E^+_{M\chi}(G)$ by edge removal and $\xi^\circ_{M\chi}(G), \xi^+_{M\chi}(G)$ and $\xi^-_{M\chi}(G)$ when adding any edge in G. Also results on these classifications and some characterization theorems are determined.

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5.1 Introduction

In 1982, Harary [21] and Haynes [25] surveyed the classification of graphs as (i) domination number changes when any vertex is removed (ii) domination number changes when any edge is removed (iii) domination number changes when new edge is added (iv) domination number unchanged when any vertex is removed (v) domination number unchanged when any edge is removed (vi) domination number unchanged when new edge is added. They established many results on these six types. In 2012, Janakiraman and Poobalaranjani [31] were studied the changing and unchanging properties with respect to dom-chromatic number of a graph. They produced more results with the property such as degree, diameter and chromatic number on the effects of removing vertex, edge and adding any edge in the graph G.

In 2011, Joseline Manora and Swaminathan [38] were surveyed the effects of majority domination number when removal of any edge from the graph. Also, in 2013 [39] they studied vertex critical on majority domination with respect to the deletion of a vertex from the graph G. They established many results about these two effects on graphs.These concepts gave the motivation to study this concept, changing and unchanging properties of majority dom-chromatic number when removal of a vertex, an edge and adding an edge in the graph G.

5.2 Changing and Unchanging of MDC Number by Vertex Removal

In this section, changing and unchanging of Majority Dom-Chromatic (MDC) number $\gamma_{M\chi}(G)$ *is defined for the graphs by vertex removal with some examples.*

Definition 5.2.1: For any graph G, the vertex set $V(G)$ can be partitioned into three sets $V^o_{M\chi}(G)$, $V^-_{M\chi}(G)$ and $V^+_{M\chi}(G)$ with respect to MDC sets by a vertex deletion and is defined by,
$V^o_{M\chi}(G) = \{v \in V(G)/\gamma_{M\chi}(G-v) = \gamma_{M\chi}(G)\}$,
$V^-_{M\chi}(G) = \{v \in V(G)/\gamma_{M\chi}(G-v) < \gamma_{M\chi}(G)\}$ and
$V^+_{M\chi}(G) = \{v \in V(G)/\gamma_{M\chi}(G-v) > \gamma_{M\chi}(G)\}$.

Definition 5.2.2: A graph G is said to be a CVR$_{M\chi}$- graph if $\gamma_{M\chi}(G-v) \neq \gamma_{M\chi}(G)$, for every $v \in V(G)$. A graph G is said to be a UVR$_{M\chi}$- graph if $\gamma_{M\chi}(G-v) = \gamma_{M\chi}(G)$, for every $v \in V(G)$.

Example 5.2.3: Consider the graph G with $p = 16$.

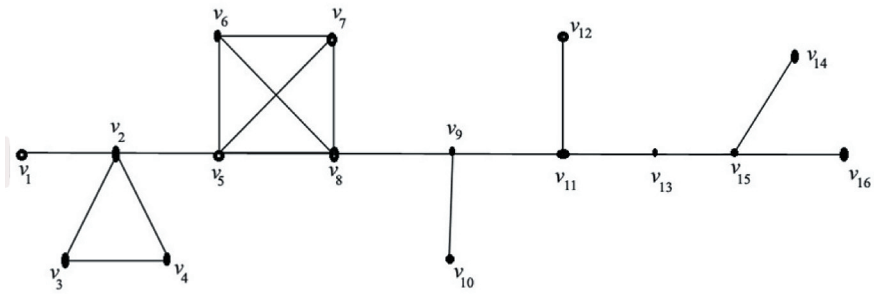

Figure 5.1: G

In this graph $G, S = \{v_5, v_6, v_7, v_8, v_{15}\}$ is the $\gamma_{M\chi}$- set of G. Then $\gamma_{M\chi}(G) = 5$. For the graph $G - \{v_5\}, \gamma_{M\chi}(G - \{v_5\}) = |\{v_2, v_3, v_4, v_8\}| = 4$. Therefore $\gamma_{M\chi}(G - v_5) < \gamma_{M\chi}(G)$. Hence $v_5 \in V^{-}_{M\chi}(G)$. For the graph $G - \{v_8\}, \gamma_{M\chi}(G - v_2) = |\{v_5, v_6, v_7, v_8, v_{15}\}| = 5$. Therefore $\gamma_{M\chi}(G - v_8) = \gamma_{M\chi}(G)$. It implies that $v_8 \in V^{0}_{M\chi}(G)$.

Example 5.2.4: Consider the graph $G = F_p, p = 17$ a Fan.

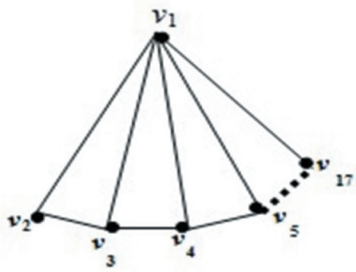

Figure 5.2: G

In this graph $G, \gamma_{M\chi}(G) = |\{v_1, v_2, v_3\}| = 3$. For $G - \{v_2\}, \gamma_{M\chi}(G - v_2) = |\{v_1, v_3, v_4\}| = 3$. Therefore $\gamma_{M\chi}(G - v_2) = \gamma_{M\chi}(G)$ and $v_2 \in V^{0}_{M\chi}(G)$. For the graph $\{G - v_1\}, \gamma_{M\chi}(G - v_1) = |\{v_3, v_4, v_7, v_{10}\}| = 4$. Hence $\gamma_{M\chi}(G - v_2) > \gamma_{M\chi}(G)$ and $v_1 \in V^{+}_{M\chi}(G)$.

Theorem 5.2.5: If a graph G is a vertex color critical then $G \in CVR_{M\chi}$.

Proof: Since the graph G is vertex color critical, $\gamma_{M\chi}(G) = p$. If the removal of any vertex v from $V(G), \chi(G - v) \neq \chi(G)$. It implies that $\gamma_{M\chi}(G - v) < \gamma_{M\chi}(G)$, for every vertex $v \in V(G)$. Hence $G \in CVR_{M\chi}$. ∎

Corollary 5.2.6: Let $G = K_p, p \geq 2$. Then $G \in CVR_{M\chi}$.

Proof: By the proposition (2.3.2)(i), $\gamma_{M\chi}(G) = p$. For the graph $\gamma_{M\chi}(G - v_1) = p - 1$. Hence $\gamma_{M\chi}(G - v_1) < \gamma_{M\chi}(G)$. Therefore $v_1 \in V^{-}_{M\chi}(G)$. For every vertex $v \in V(G), \gamma_{M\chi}(G - v) < \gamma_{M\chi}(G)$. and $G \in CVR_{M\chi}$. ∎

Proposition 5.2.7: Any Path $P_p,\ p \equiv 3(mod\ 6)$ is a $CVR_{M\chi}$ graph.

Proof: Let $G = P_p, p = 6k+3, k \leq 1$. Then by the corollary (2.3.4), $\gamma_{M\chi}(G) = k + 2$. For each vertex $v \in V(G), \gamma_{M\chi}(G - v) = k + 1 < k + 2$, where $p \equiv 6k + 2$. Hence $P_p \in CVR_{M\chi}$ if $p \equiv 3(mod\ 6)$. ∎

Proposition 5.2.8: A Wheel graph $G = W_p, p > 5$ is a $CVR_{M\chi}$ graph when p is even.

Proof: Let $G = W_p = C_{p-1} \vee K_1$. By the proposition (2.3.6), $\gamma_{M\chi}(G) = p$, when p is even. Let $V(G) = \{v_1, v_2, \ldots, v_{p-1}, v_p\}$ where $v_i \in C_{p-1}, i = 1, 2, \cdots, p-1$ and $v_p \in K_1$ such that $d(v_p) = p - 1$. Suppose $G' = G - \{v_p\}$.

Case (i): Let $\{v_p\}$ be the central vertex of G. Then $G-\{v_p\} = G' = C_{p-1}$. Since p is even, C_{p-1} is an odd cycle. By the proposition (2.3.3), $\gamma_{M\chi}(G') = p-1$. Therefore $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$, for $v_p \in V(G)$.
Case (ii): Suppose $\{v_t\}$ be any vertex in C_{p-1}. Then the graph G becomes a Fan $G' = (G - \{v_t\}) = P_{p-2} \vee K_1$. By the proposition (2.3.7), $\gamma_{M\chi}(G') = 3$. Hence $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$.

In these two cases, the removal of any vertex $\{v_i\}$ in $V(G)$, $\gamma_{M\chi}(G - v_i) < \gamma_{M\chi}(G)$. Hence $G \in CVR_{M\chi}$. ∎

5.3 Results on $V^{\circ}_{M\chi}(G)$ and $V^{+}_{M\chi}(G)$

In this section, the vertex sets $V^{\circ}_{M\chi}(G)$ and $V^{+}_{M\chi}(G)$ are discussed for the graph G with the property such as majority dominating vertex, private neighborhood and cp-set.

Proposition 5.3.1: Let $G = K_{1,p-1}$. Then $v_1 \in V^{+}_{M\chi}(G)$ and $v_i \in V^{0}_{M\chi}(G)$ where v_1 is a central vertex and v_i's are pendants.

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_p\}$, where v_1 is the central vertex and others are pendants. The set $S = \{v_1, v_2\}$ is the MDC set of G and $\gamma_{M\chi}(G) = 2$. For a graph $G - \{v_1\}, \gamma_{M\chi}(G - v_1) = \lceil \frac{p-1}{2} \rceil$. Therefore $\gamma_{M\chi}(G - v_1) > \gamma_{M\chi}(G)$. Hence $v_1 \in V^{+}_{M\chi}(G)$. Suppose any pendant $v_i, i = 2, \cdots, p, \gamma_{M\chi}(G - v_i) = 2 = \gamma_{M\chi}(G)$. Therefore $v_i \in V^{0}_{M\chi}(G)$, where v_i's are pendants. ∎

Theorem 5.3.2: If G has exactly one full degree vertex and other vertices are of degree $d(v_i) < \frac{p-1}{2}$ then $|V^{+}_{M\chi}(G)| = 1$.

Proof: Let G be a graph which contains a full degree vertex 'v' and S be a MDC set of G v must be in a majority dominating set S and a minimal cp - set of G. Then $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $\chi(\langle S \rangle) = \chi(G)$. Let S' be the $\gamma_{M\chi}$- set of $G' = \{G - v\}$ and $\{G - v\}$ contains isolates, then $\gamma_{M\chi}(G') > |S| = \gamma_{M\chi}(G)$. It implies that $v \in V^0_{M\chi}(G)$. If $\{G - v\}$ contains the vertices v_i with $d(v_i) < \lceil \frac{p-1}{2} \rceil$ then $|S'| \geq 2$. Therefore $|S'| \geq |S| + 1$. It implies that $\gamma_{M\chi}(G - v) > \gamma_{M\chi}(G)$ and $v \in V^+_{M\chi}(T)$. Thus, all other vertices are $V^0_{M\chi}(G)$. Hence $|V^+_{M\chi}(G)| = 1$. ∎

Theorem 5.3.3: Let T be a tree with p vertices. If a vertex $v \in V(T)$ satisfies one of the following conditions.

(i) v is in a dominating edge $e = \{uv\}$ with $d(v) \geq \lceil \frac{p}{2} \rceil - 1$ and $d(u) < \lceil \frac{p}{2} \rceil - 1$

(ii) v is a vertex with degree $d(v) = p - 1$ and others are pendants

(iii) v is in every $\gamma_{M\chi}$- set of T.

then $v \in V^+_{M\chi}(T)$.

Proof: Let T be a tree with p vertices and $v \in V(T)$.

Case (i): Let $e = \{uv\}$ is a dominating edge with $d(v) \geq \lceil \frac{p}{2} \rceil - 1$ and $d(u) < \lceil \frac{p}{2} \rceil - 1$. Since $\chi(G) = 2, S = \{u, v\}$ be a $\gamma_{M\chi}$- set of T. Let $S_1 = \{u, u_1, v_i\}$ be a set of $T - \{v\}$, where u and u_1 are adjacent and v_i's are isolates such that $|N[S_1]| \geq \lceil \frac{p}{2} \rceil$ with $|S_1| > |S|$. Then $\chi(T) = \chi(\langle S_1 \rangle) = \chi(T - v)$. Thus S_1 is a MDC set of $T - \{v\}$ and $\gamma_{M\chi}(T - v) \leq |S_1|$. Since $|S_1| > |S|, \gamma_{M\chi}(T - v) > |S| = \gamma_{M\chi}(T)$. Hence $v \in V^+_{M\chi}(T)$.

Case (ii): Let $d(v) = p - 1$ and $d(v_i) = 1$, for all $v_i \in V(T)$. Then $\gamma_{M\chi}(T) = |\{v, v_1\}| = 2$, for some v_1 such that $d(v_1) = 1$. Since v is adjacent to all vertices v_i of $T, \langle T-\{v\}\rangle$ is disconnected with only isolates. Now, there exists a MDC set S in $T-\{v\}$ with only isolates and $|S| = \lceil\frac{p-1}{2}\rceil$. It implies that $|S| = \gamma_{M\chi}(T-\{v\}) > \gamma_{M\chi}(T)$ and $v \in V^+_{M\chi}(T)$.

Case (iii): If the vertex v is in every minimum MDC set of T, then v is in a dominating edge $e = uv$ or v is a full degree vertex of T. It implies that $d(v) \geq \lceil\frac{p}{2}\rceil - 1, d(u) < \lceil\frac{p}{2}\rceil - 1$ and other vertices v_i's are of degree with $d(v_i) < \lceil\frac{p}{2}\rceil - 1$. By Case (i), the vertex $v \in V^+_{M\chi}(T)$. ■

Theorem 5.3.4: For any graph $G, |V^+_{M\chi}(G)| \leq \gamma_{M\chi}(G)$.

Proof: Let S be a $\gamma_{M\chi}$- set of G. Let $v \in V^+_{M\chi}(G)$. It implies that v is in every $\gamma_{M\chi}$- set S of G. Then $v \in S$ and $V^+_{M\chi}(G) \subseteq S$. Hence $|V^+_{M\chi}(G)| \leq |S| = \gamma_{M\chi}(G)$. ■

Theorem 5.3.5: If $v \in V^+_{M\chi}(G)$ and v is in every minimal cp- set of G then $|Pn[v, S]| \geq 2$, for all $\gamma_{M\chi}$ set S of G.

Proof: Let S be a $\gamma_{M\chi}$- set of G. Let v be a vertex in every minimal cp-set of G. Then $\chi(\langle S-v\rangle) = \chi(G-v) < \chi(G)$. Let $Pn[v, S] = \phi$. Then $\{S-v\}$ is a $\gamma_{M\chi}$- set of $\{G-v\}$. It is a contradiction to $v \in V^+_{M\chi}(G)$. Suppose $|Pn[v, S]| = \{v\}$. Then v is an isolated vertex in S and hence $v \in V^0_{M\chi}(G)$, which is a contradiction. If $|Pn[v, S]| = \{u\}$ then $\{S-v\} \cup \{u\}$ is a $\gamma_{M\chi}$- set of $\{G-v\}$. Thus $\gamma_{M\chi}(G-v) \leq |S| = \gamma_{M\chi}(G)$, which is a contradiction to $v \in V^+_{M\chi}(G)$. Hence, $|Pn[v, S]| \geq 2$. ■

Theorem 5.3.6: If v is an isolated of G then $v \in V^0_{M\chi}(G)$.

Proof: Let v be an isolated vertex of G. Then v is not in minimal cp-set of G. Let S be a $\gamma_{M\chi}$- set of G and not containing the vertex v. Then $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $\chi(\langle S \rangle) = \chi(G)$. Then $\gamma_{M\chi}(G) = |S|$. For the graph $\{G - v\}, \chi(\langle G - v \rangle) = \chi(G)$ and S is again the $\gamma_{M\chi}$- set of $\{G - v\}$. Therefore $\gamma_{M\chi}(G - v) = \gamma_{M\chi}(G)$ and $v \in V^0_{M\chi}(G)$. ∎

Theorem 5.3.7: If a vertex $v \in V(G)$ is not in any minimal cp-set of G then $v \in V^0_{M\chi}(G)$.

Proof: Let S be a $\gamma_{M\chi}$- set of G. If a vertex v which is not in any minimal cp-set of G then $\chi(\langle S - v \rangle) = \chi(G)$. Hence $Pn[v, S] \neq \phi$. Let $|Pn[v, S]| = 1$. If $Pn[v, S] = \{v\}$ then v is an isolated vertex in S. By the theorem (5.3.6), $v \in V^0_{M\chi}(G)$. ∎

Proposition 5.3.8: Let $G = W_p = C_{p-1} \vee K_1, p$ is odd be a wheel. Then

(i) $v_i \in V^0_{M\chi}(G)$, if $v_i \in C_{p-1}$.

(ii) $v_i \in V^+_{M\chi}(G)$, if v is a central vertex of G and $p \leq 17$.

Proof: For $G = W_p = C_{p-1} \vee K_1, p$ is odd, $V(G) = \{v_1, v_2, \cdots, v_{p-1}, v_p\}$. By the proposition (2.3.6), $\gamma_{M\chi}(G) = 3$. The removal of any vertex v from $V(G)$, there exists two cases.

Case (i): Suppose any vertex $v_i \in C_{p-1}$. Then $G' = G - \{v_i\}$ and $G' = F_{p-1} = P_{p-2} \vee K_1$, where $(p - 1)$ is even. By the proposition (2.3.7), $\gamma_{M\chi}(G') = 3$. Hence $\gamma_{M\chi}(G') = \gamma_{M\chi}(G)$ and $v_i \in V^0_{M\chi}(G)$.

Case (ii): Suppose v_p is a central vertex and $p \leq 17$. The $\gamma_{M\chi}$- set of G is $S = \{v_1, v_2, v_p\}$. Then $\gamma_{M\chi}(G) = |S| = 3$. If the removal of a central vertex $v_p, G' = G - \{v_p\}$ and G' becomes C_{p-1} even cycle. By the proposition (2.3.3),

$$\gamma_{M\chi}(G') = \begin{cases} p, \text{ if } p \; isodd \\ \lceil \frac{p}{6} \rceil, \text{ if } p \equiv 2(mod\ 6) \\ \lceil \frac{p}{6} \rceil + 1, \text{ if } p \equiv 0, 4(mod\ 6). \end{cases} \tag{5.1}$$

For $p \leq 16$, by the result (5.1), $\gamma_{M\chi}(G') = |S'| = 3$. If $p \leq 17$, by the result (5.1), $\gamma_{M\chi}(G') = |S'| \leq 4$. Therefore, $\gamma_{M\chi}(G') > \gamma_{M\chi}(G)$ and $v_p \in V^+_{M\chi}(G)$. Hence $v_p \in V^+_{M\chi}(G)$, if $p \leq 17$. ∎

Theorem 5.3.9: Let v be a vertex of G with $v \in V^+_{M\chi}(G)$. Then there exists a vertex $u \in V(G)$ such that $\gamma_{M\chi}(G - u) = \gamma_{M\chi}(G)$.

Proof: Let S be the $\gamma_{M\chi}$- set of G. Then $|N[S]| \geq \lceil \frac{p}{2} \rceil$.

Case (i): Suppose $|N[S]| \neq V(G)$. Then there exists a vertex $u \notin N[S]$ and implies that $u \notin S, u \in V - N[S]$. Then $S \subseteq V - u$ and $|N[S]| \leq \lceil \frac{p}{2} \rceil$ and $|N_{G-u}[S]| \leq \lceil \frac{p}{2} \rceil$. It implies that $|N_{G-u}[S]| = \lceil \frac{p-1}{2} \rceil$. Therefore S is a MDC set of $\{G - u\}$. Then $\gamma_{M\chi}(G - u) \leq |S| = \gamma_{M\chi}(G)$. If $\gamma_{M\chi}(G - u) < \gamma_{M\chi}(G)$ then $u \in V^-_{M\chi}(G)$, which is a contradiction to $v \in V^+_{M\chi}(G)$. Hence $\gamma_{M\chi}(G - u) = \gamma_{M\chi}(G)$.

Case (ii): Suppose $N[S] = V(G)$. Let $u \notin S$ and $u \in N[S]$. Then $|N_{G-u}[S]| = p - 1 \leq \lceil \frac{p-1}{2} \rceil$. Therefore S is a majority dom-chromatic set of $\{G - u\}$. Then $\gamma_{M\chi}(G - u) \leq |S| = \gamma_{M\chi}(G)$. If $\gamma_{M\chi}(G - u) < \gamma_{M\chi}(G)$ then $u \in V^-_{M\chi}(G)$ and $V(G) = V^-_{M\chi}(G)$, which is a contradiction to $v \in V^+_{M\chi}(G)$. Hence $\gamma_{M\chi}(G - u) = \gamma_{M\chi}(G)$.

Case (iii): Suppose $|N[S]| \leq V(G)$. Then there exists a vertex $u \in S$ and $|N[S]| \leq \lceil \frac{p}{2} \rceil$. For $S - \{u\}, \chi(\langle S - u \rangle) < \chi(\langle S \rangle) = \chi(G)$

and S is not a $\gamma_{M\chi}$- set of G. Therefore choose $S_1 = S - \{u\} \cup \{w\}$ where $w \in V - S$ such that $|N[S_1]| \leq \lceil \frac{p}{2} \rceil$ and w is adjacent to any vertex of S with $|S_1| = |S|$. Hence S_1 is a $\gamma_{M\chi}$- set of $\{G - u\}$ and $\gamma_{M\chi}(G - u) = |S_1| = |S| = \gamma_{M\chi}(G)$. ∎

5.4 Results on $V^{-}_{M\chi}(G)$ and $CVR_{M\chi}$

In this section, the vertex set $V^{-}_{M\chi}(G)$ is investigated when a vertex is removed from the graph G and $CVR_{M\chi}$ graphs are also discussed.

Theorem 5.4.1: If G is a vertex color critical graph then $V(G) = V^{-}_{M\chi}(G)$ but the converse is not true.

Proof: Let G be vertex color critical graph with p vertices. Then by observation (2.2.4)(ii), $\gamma_{M\chi}(G) = p$ and for all $v, \gamma_{M\chi}(G - v) < \gamma_{M\chi}(G)$. It implies that $v \in V^{-}_{M\chi}(G)$ for all $v \in (G)$ and $V^{-}_{M\chi}(G) = V(G)$. For the converse, Let $G = P_p, p = 9$. Then $\gamma_{M\chi}(G) = 3$. For any vertex $v, P_9 - \{v\} = P_8$ and $\gamma_{M\chi}(P_8) = 2 < \gamma_{M\chi}(G)$. Hence $V(G) = V^{-}_{M\chi}(G)$ and $G \in CVR_{M\chi}$ but $G = P_9$ is not a vertex color critical graph. ∎

Proposition 5.4.2: If G is a $CVR_{M\chi}$ - graph then $V^{-}_{M\chi}(G) \neq \phi$.

Proof: Since G is a $CVR_{M\chi}$- graph, $V = V^{+}_{M\chi} \cup V^{-}_{M\chi}$. Suppose $V^{-}_{M\chi}(G) = \phi$. Then $V(G) = V^{+}_{M\chi}(G)$ and $\gamma_{M\chi}(G - v) > \gamma_{M\chi}(G)$, for all $v \in V(G)$. Let S be a $\gamma_{M\chi}$- set of G with $|S| = p - 1$. Then $V - S \neq \phi$. Let $u \in V - S$ and $\{u\} \subseteq V(G) - S$. It implies that $S \subseteq V(G) - \{u\} = G - u$. Since G is a $CVR_{M\chi}$- graph, $\chi(\langle S \rangle) = \chi(G)$

and $\chi(\langle S\rangle) = \chi(\langle G-u\rangle)$. It implies that S is a $\gamma_{M\chi}$- set of $(G-u)$ and $\gamma_{M\chi}(G-u) \leq |S| = \gamma_{M\chi}(G)$. Therefore $u \in V^{-}_{M\chi}(G)$, which is a contradiction to the assumption. Hence $V^{-}_{M\chi}(G) \neq \phi$, for any $CVR_{M\chi}$ graph G. ∎

Theorem 5.4.3: Let G be a $CVR_{M\chi}$ graph with p vertices. Then $|V^{-}_{M\chi}(G)| \geq p - \gamma_{M\chi}(G)$.

Proof: Let S be a $\gamma_{M\chi}$- set of G. If G is a $CVR_{M\chi}$- graph then $\gamma_{M\chi}(G{-}v) < \gamma_{M\chi}(G)$. Suppose $|S| = \gamma_{M\chi}(G) = p$. Then $|V^{-}_{M\chi}(G)| \leq p - \gamma_{M\chi}(G)$ holds. Suppose $|S| = \gamma_{M\chi}(G) < p$. Then $V - S \neq \phi$. Now choose any vertex $v \in V - S$. Since $\gamma_{M\chi}(G-v) < \gamma_{M\chi}(G), v \in V^{-}_{M\chi}(G)$. Therefore $V - S \subseteq V^{-}_{M\chi}(G)$. It implies that $|V - S| \leq |V^{-}_{M\chi}(G)|$. Hence $|V^{-}_{M\chi}(G)| \geq p - \gamma_{M\chi}$. ∎

Theorem 5.4.4: Let $\gamma_{M\chi}(G)$ be the MDC number of a graph G and $\gamma_{M\chi}(G) = |V(G)|$. Then $|V^{-}_{M\chi}(G)| = |V(G)|$.

Proof: Let S be a $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G) = |V(G)| = p$. Then G is a vertex color critical graph. For any $v \in V(G), \chi(G-v) < \chi(G)$ and it implies that $\gamma_{M\chi}(G-v) < \gamma_{M\chi}(G)$. Hence $v \in V^{-}_{M\chi}(G)$, for all $v \in V(G)$. For every $v \in S, \gamma_{M\chi}(G-v) < \gamma_{M\chi}(G)$ is true. Hence $|V^{-}_{M\chi}(G)| = |V(G)|$. ∎

Theorem 5.4.5: If G is a graph with $\gamma_{M\chi}(G) = |V(G)|$ then $G \in CVR_{M\chi}$.

Proof: Let G be a graph with p vertices and $\gamma_{M\chi}(G) = |V(G)| = p$.

Then G is a vertex color critical graph. Therefore, for any vertex $v \in V(G)$, the graph $G' = G - \{v\}$ has the value $\gamma_{M\chi}(G') < p$. It implies that $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$, for every $v \in V(G)$. Hence $G \in CVR_{M\chi}$. ∎

The following theorem establishes the characterization on $CVR_{M\chi}(G)$

Theorem 5.4.6: Let G be a connected $CVR_{M\chi}$- graph with $\chi(G) \leq 3$. Then G has a unique $\gamma_{M\chi}$- set of G if and only if $\gamma_{M\chi}(G) = |V(G)|$.

Proof: Let the graph G have a unique $\gamma_{M\chi}$- set S.Then we claim that $V(G) - S = \phi$. Suppose $V - S \neq \phi$. Since G is a $CVR_{M\chi}$-graph, $\gamma_{M\chi}(G - v) < \gamma_{M\chi}(G)$, for every $v \in V - S$. Then for each $v \in V - S, \chi(\langle s - v\rangle) < \chi(\langle s\rangle)$ and the induced subgraph $\langle s\rangle$ is a vertex color critical. Hence for any $u \in V - S, S$ is a MDC set of $G - \{u\}$, which is a contradiction to the assumption. Therefore there exist $v \in V - S$ such that $\chi(\langle s - v\rangle) < \chi(\langle s\rangle)$. Then $Pn[u, S] \neq \phi$, for any $u \in S$.

Case (i): Let $|Pn[u, S]| = 1$. If $Pn[u, S] = \{u\}$ then u is an isolate in $\langle S\rangle$. Since G is connected, $N(u) \neq \phi$ and $N(u) \subseteq V - S$. Also some vertex $w \in V - S$ is adjacent to any vertex in S. Let $w \in N(u)$. Then $(S - u) \cup \{w\}$ is a $\gamma_{M\chi}$- set of G, which is a contradiction to the assumption. So $Pn[u, s] = \{v\}$. Then $(S - u) \cup \{v\}$ is $\gamma_{M\chi}$- set of G, which is a contradiction to the assumption. Hence $V - S = \phi$. Thus $|V(G)| = \gamma_{M\chi}(G)$.

Case (ii): Suppose $|Pn[v, S]| \leq 2$. Let $v \in Pn[v, S]$. Then there exists a vertex $w \neq v$ such that $w \in Pn[v, S]$. It implies that $(S - v) \cup$

$\{w\}$ is a $\gamma_{M\chi}$- set of G, which is a contradiction to the assumption. Let $x, w \in Pn[v, S]$. Then $(S-u)\cup\{w\}$ is a $\gamma_{M\chi}$- set of $G-x$. Thus, $|V(G)| = |S| = \gamma_{M\chi}(G)$.

Conversely, $\gamma_{M\chi}(G) = |V(G)| = p$. It implies that the graph G have a unique MDC set of G. ∎

5.5 Changing and Unchanging of MDC Number by Edge Deletion

In this section, Changing Edge Removal and Unchanging Edge Removal with respect to the MDC number of $\gamma_{M\chi}$ graphs are investigated.

Definition 5.5.1: The edge set $E(G)$ is partitioned into three sets, each depending on the effect of the removal of an edge on $\gamma_M(G)$ and $\chi(G)$.

$E^0_{M\chi}(G) = \{e \in E(G)/\gamma_{M\chi}(G-e) = \gamma_{M\chi}(G)\}$

$E^-_{M\chi}(G) = \{e \in E(G)/\gamma_{M\chi}(G-e) < \gamma_{M\chi}(G)\}$

$E^+_{M\chi}(G) = \{e \in E(G)/\gamma_{M\chi}(G-e) > \gamma_{M\chi}(G)\}$.

Example 5.5.2: Consider the graph G with $p = 15$ vertices.

In this graph G, the $\gamma_{M\chi}$- set of G is $S_1 = \{v_1, v_3, v_4, v_{12}\}$ and $\gamma_{M\chi}(G) = |S_1| = 4$. The dominating set is $S_2 = \{v_1, v_7, v_{10}, v_{12}\}$ and

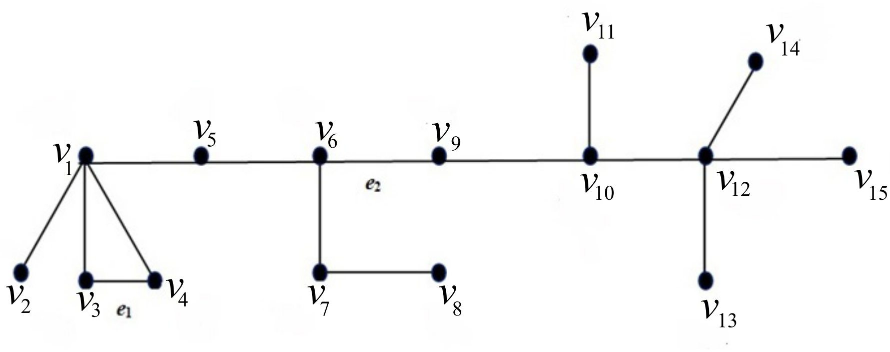

Figure 5.3: G

$\gamma(G) = |S_2| = 4$. The γ_{ch}- set and γ_M- sets are $S_3 = \{v_1, v_3, v_4, v_7, v_{10}, v_{12}\}$ and $S_4 = \{v_1, v_{12}\}$ respectively. Therefore $\gamma_{ch}(G) = |S_3| = 6$ and $\gamma_M(G) = |S_4| = 2$. Thus for the graph $\{G - e_1\}$, the $\gamma_{M\chi}$- set is $\{v_1, v_5, v_{10}\}$ and $\gamma_{M\chi}(G - e_1) = 3$. Therefore $\gamma_{M\chi}(G - e_1) < \gamma_{M\chi}(G)$ and $e_1 \in E^-_{M\chi}(G)$. Again for the graph $\{G - e_2\}, \gamma_{M\chi}(G - e_2) = 4$. Hence $\gamma_{M\chi}(G - e_2) = \gamma_{M\chi}(G)$ and $e_2 \in E^0_{M\chi}(G)$.

Example 5.5.3: Consider the following graph G with $p = 13$ vertices.

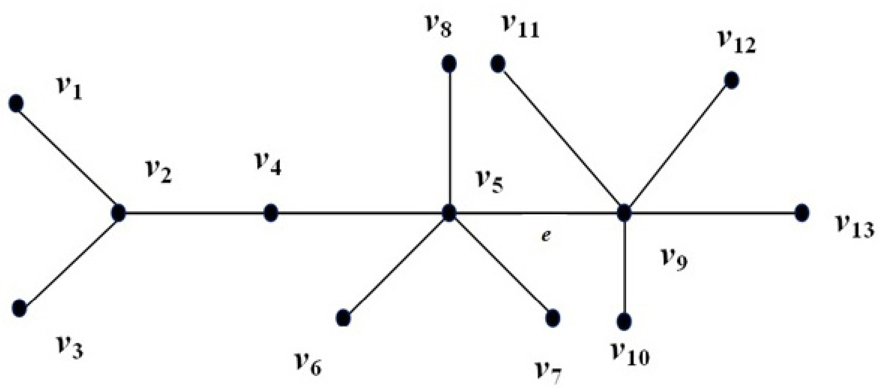

Figure 5.4: G

For the graph G, the $\gamma_{M\chi}$- set is $S_1 = \{v_5, v_9\}$ and $\gamma_{M\chi}(G) = |S_1| = 2$. Then for the graph $\{G-e\}$, the MDC set is $S_2 = \{v_4, v_5, v_9\}$ and $\gamma_{M\chi}(G-e) = |S_2| = 3$. Therefore $\gamma_{M\chi}(G-e) > \gamma_{M\chi}(G)$. Hence $e \in E^{+}_{M\chi}(G)$.

Observations 5.5.4:

(i) Let $G = K_{1,P-1}$ be a Star graph. Then $e \in E^{0}_{M\chi}(G)$, for all $e \in E(G)$.

(ii) For $G = D_{r,s}$ a Double star, $e \in E^{0}_{M\chi}(G)$, for all $e \in E(G)$.

(iii) If $G = W_p = C_{P-1} \vee K_1$ is a Wheel graph then $e \in E^{0}_{M\chi}(G)$, for all $e \in E(G), p$ is odd and $e \in E^{-}_{M\chi}(G)$, for all $e \in E(C_{P-1}), p$ is even.

(iv) Let $G = F_p$ be a Fan graph. Then $e \in E^{0}_{M\chi}(G)$, for all $e \in E(G)$.

(v) For $G = K_p$ a complete graph, $e \in E^{-}_{M\chi}(G)$, for all $e \in E(G)$.

(vi) Let $G = K_p - \{e\}$ be a graph. Then $e \in E^{0}_{M\chi}(G)$, for all $e \in E(G)$.

(vii) If G is a caterpillar graph then $e \in E^{0}_{M\chi}(G)$, for all $e \in E(G)$.

(viii) Let G be a Petersen graph. Then $e \in E^0_{M\chi}(G)$, for all $e \in E(G)$.

(ix) Let $G = K_{m,n}$ be a complete bipartite graph. Then $e \in E^0_{M\chi}(G)$, for all $e \in E(G)$.

Proposition 5.5.5: Let G be any Cycle C_p with p vertices. Then

(i) $e \in E^-_{M\chi}(G)$, if p is odd

(ii) $e \in E^0_{M\chi}(G)$, if p is even.

Proof: Let $G = C_p$ be a Cycle. Then by proposition (2.3.3),

$$\gamma_{M\chi}(G) = \begin{cases} p, \text{ if } p \text{ is odd} \\ \lceil \frac{p}{6} \rceil, \text{ if } p \equiv 2(mod\ 6) \\ \lceil \frac{p}{6} \rceil + 1, \text{if } p \equiv 0, 4(mod\ 6). \end{cases} \tag{5.2}$$

Case (i): When p is odd. Then the graph G becomes an edge color critical. i.e. $\chi(G-e) < \chi(G)$. If the removal of any edge $e = v_iv_j$, the graph $(G-e)$ becomes P_p, a Path. By corollary (2.3.4),

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{6} \rceil, \text{ if } p \equiv 1, 2(mod\ 6) \\ \lceil \frac{p}{6} \rceil + 1, \text{ if } p \equiv 0, 3, 4, 5(mod\ 6) \end{cases} \tag{5.3}$$

Therefore, $\gamma_{M\chi}(G-e) = \gamma_{M\chi}(P_p) < p = \gamma_{M\chi}(G)$. Hence $\gamma_{M\chi}(G-e) < \gamma_{M\chi}(G)$ and $e \in E^-_{M\chi}(G)$. Thus (i) holds.

Case (ii): When p is even. Let $e = v_i v_j$ be an edge in G and the removal of the edge e from $G, (G - e)$ becomes P_p, a Path and p is even. By the result (5.3), $\gamma_{M\chi}(G - e) = \gamma_{M\chi}(P_p), p$ is even. Hence from (5.2) and (5.3), $\gamma_{M\chi}(G - e) = \gamma_{M\chi}(G)$ and $e \in E^0_{M\chi}(G)$, if p is even. ∎

5.6 Results on $E^0_{M\chi}(G), E^-_{M\chi}(G)$ and $E^+_{M\chi}(G)$

In this section, the edge set $E(G)$ is classified into three sets namely $E^0_{M\chi}(G), E^-_{M\chi}(G)$ and $E^+_{M\chi}(G)$.

Proposition 5.6.1: Let $G = P_p$ be a path. Then $e \in E^0_{M\chi}(G)$, for any edge $e \in E(G)$.

Proof: Let $G = P_p$ be a path of p vertices with $d(v_i) = 2$, for all $i = 2, 3, \cdots, (p-1)$. By the corollary (2.3.4),

$$\gamma_{M\chi}(G) = \begin{cases} \lceil \frac{p}{6} \rceil, \text{ if } p \equiv 1, 2 (mod\ 6) \\ \lceil \frac{p}{6} \rceil + 1, \text{ if } p \equiv 0, 3, 4, 5 (mod\ 6). \end{cases} \tag{5.4}$$

Let S be a $\gamma_{M\chi}$- set of G with this cardinality of (5.4). Then the removal of any edge $e = v_i v_j$ from G, it creates two paths P_1 and P_2 in $(G-e)$ and $\chi(G-e) = 2 = \chi(G)$. Then there exists another $\gamma_{M\chi}$-set S_1 for $(G-e)$ with the same cardinality of $\gamma_{M\chi}$ of G in any of these Paths P_1 or P_2. Therefore $\gamma_{M\chi}(G-e) = |S_1| = |S| = \gamma_{M\chi}(G)$ and $e \in E^0_{M\chi}(G)$, for any edge $e \in E(G)$. ∎

Theorem 5.6.2: Let T be any tree with $p \geq 3$ vertices and S be a $\gamma_{M\chi}$- set of T. If $e \notin (\langle N[S] \rangle)$ then $e \in E^0_{M\chi}(T)$.

Proof: Let T be any tree with $p \geq 3$. Then $\chi(T) = 2$. Let S be a $\gamma_{M\chi}$- set of T. Then $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $\chi(\langle S \rangle) = \chi(T)$. Let $e \notin (\langle N[S] \rangle)$. Then the removal of such edge e, would not affect its MDC – number and S is also a MDC – set of $(T-e)$. Thus $\gamma_{M\chi}(T-e) = \gamma_{M\chi}(T)$ and $e \in E^0_{M\chi}(T)$. ∎

Theorem 5.6.3: Let $e \in E(G)$ be any edge of a graph G. If $\chi(G - e) = \chi(G)$ then $e \in E^0_{M\chi}(G)$.

Proof: Let $\chi(G-e) = \chi(G)$, for any $e \in E(G)$. Suppose $e \neq E^0_{M\chi(G)}$. Then either $e \in E^-_{M\chi}(G)$ or $e \in E^+_{M}\chi(G)$. If $e \in E^-_{M\chi}(G)$ then $\gamma_{M\chi}(G-e) < \gamma_{M\chi}(G)$. It implies that $\chi(G-e) < \chi(G)$, which is a contradiction to the assumption. If $e \in E^+_{M\chi}(G)$ then $\gamma_{M\chi}(G-e) > \gamma_{M\chi}(G)$. It implies that $\chi(G-e) > \chi(G)$, which is a contradiction to the assumption and $e \in E^0_{M\chi}(G)$. ∎

Theorem 5.6.4: If an edge $e \in E(G)$ is not in every $\gamma_{M\chi}$- set of G then $e \in E^0_{M\chi}(G)$.

Proof: Let $e = uv$ be any edge in G and S be a $\gamma_{M\chi}$- set of G. Let $e \notin S$, for every $\gamma_{M\chi}$- set S of G. Suppose $e \notin E^0_{M\chi}(G)$. Then either $e \in E^-_{M\chi}(G)$ or $e \in E^+_{M\chi}(G)$. If $e \in E^-_{M\chi}(G)$ then $\gamma_{M\chi}(G-e) < \gamma_{M\chi}(G)$. It implies that e is in every $\gamma_{M\chi}$- set of G. It is a contradiction to the assumption. If $e \in E^+_{M\chi}(G)$ then $\gamma_{M\chi}(G-e) > \gamma_{M\chi}(G)$. It implies that $\gamma_M(G-e) > \gamma_M(G)$. Hence e is in every $\gamma_{M\chi}$- set of G, which is a contradiction to the assumption. Thus, $e \in E^0_{M\chi}(G)$. ∎

Theorem 5.6.5: Let G be any graph. If $\chi(G-e) < \chi(G)$ then $e \in E^-_{M\chi}(G)$, for any edge $e \in E(G)$.

Proof: Let $\chi(G-e) < \chi(G)$.To prove that $e \in E^-_{M\chi}(G)$. Suppose that $\gamma_{M\chi}(G-e) \not< \gamma_{M\chi}(G)$. Then $\gamma_{M\chi}(G-e) \geq \gamma_{M\chi}(G)$. Let S and S' be the $\gamma_{M\chi}$- sets of G and $(G-e)$. If $\gamma_{M\chi}(G-e) = \gamma_{M\chi}(G)$ then $|S| = |S'|$. It implies that $\chi(\langle S' \rangle) = \chi(\langle S \rangle) = \chi(G)$, which is a contradiction to the assumption. Also if $\gamma_{M\chi}(G-e) > \gamma_{M\chi}(G)$, then $|S'| > |S|$ and $\chi(\langle S' \rangle) > \chi(\langle S \rangle) = \chi(G)$. It follows that $\chi(G-e) > \chi(G)$ which is a contradiction to the assumption. Hence $e \in E^-_{M\chi}(G)$. ∎

Theorem 5.6.6: If the graph G is vertex color critical then $e \in E^{-}_{M\chi}(G)$, for any edge $e \in E(G)$.

Proof: Let the graph G be vertex color critical. Then $\chi(G-e) < \chi(G)$ and $\gamma_{M\chi}(G) = p$. Let $e \in E(G)$. Then $\gamma_{M\chi}(G-e) \leq \gamma_{M\chi}(G)$, for any edge $e = uv$. Since $\chi(G-e) < \chi(G), \gamma_{M\chi}(G-e) < \gamma_{M\chi}(G)$. Hence $e \in E^{-}_{M\chi}(G)$, for any edge $e \in E(G)$.

For example, let $G = C_9$ be a vertex color critical graph. For any edge $e \in E(G), (G-e) = P_9$. By proposition (2.3.3), $\gamma_{M\chi}(C_9) = 9$ and by the corollary (2.3.4), $\gamma_{M\chi}(P_9) = 2$. Therefore $\gamma_{M\chi}(G-e) < \gamma_{M\chi}(G)$, for any edge e and $e \in E^{-}_{M\chi}(G)$. ∎

Theorem 5.6.7: If $e = uv$ be an edge of a graph G and both u and v are in every $\gamma_{M\chi}$- set of G then $e \in E^{-}_{M\chi}(G)$.

Proof: Let G be any graph with p vertices and S be the $\gamma_{M\chi}$- set of G. Then $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $\chi(\langle S \rangle) = \chi(G)$. Let $e = uv$ be an edge in G. Since u and v are in every $\gamma_{M\chi}$- set of G, the edge e must be in every cp- set of G. Hence the removal of an edge e from G, it affects the chromatic number of G. Then $\chi(G-e) < \chi(G)$. Hence by theorem (5.6.6), $e \in E^{-}_{M\chi}(G)$. ∎

The next theorem gives the necessary and sufficient condition for an edge 'e' belongs to the set $E^-_{M\chi}(G)$.

Theorem 5.6.8: Let G be any graph with p vertices and q edges. Then an edge $e \in E^-_{M\chi}(G)$ if and only if either the condition (i) or the condition (ii) is true.

(i) The graph G is vertex color critical

(ii) The graph G contains a clique H with $|V(H)| \geq 3$.

Proof: Let $e \in E^-_{M\chi}(G)$. Then $\gamma_{M\chi}(G-e) < \gamma_{M\chi}(G)$. To prove the conditions (i) and (ii) are true. Since $\gamma_{M\chi}(G-e) < \gamma_{M\chi}(G), \chi(G-e) < \chi(G)$. It implies that the graph G is edge color critical and therefore the graph G is vertex color critical. Hence (i) is true. If $\chi(G-e) < \chi(G)$ then the end vertices u and v of an edge $e = uv$ must be in cp- set of G. Then the graph G contains a complete subgraph H as a clique. Suppose $|V(H)| = 2$ then there will be another $\gamma_{M\chi}$- set of $(G-e)$ with the same cardinality of $\gamma_{M\chi}$ of G and $\gamma_{M\chi}(G-e) = \gamma_{M\chi}(G)$, which is a contradiction to the assumption. Hence $|V(H)| \geq 3$. Therefore condition (ii) is true.

Conversely, suppose the conditions (i) and (ii) are true. By theorem (5.6.7), if G is a vertex color critical then $e \in E^-_{M\chi}(G)$. Let H be a clique of G such that $|V(H)| \geq 3$. If the removal of any edge e in the subgraph $\langle H \rangle$ then $\chi(G-e) < \chi(G)$ and $\gamma_{M\chi}(G-e) < \gamma_{M\chi}(G)$. Hence $e \in E^{-}_{M\chi}(G)$. ∎

5.7 Results on $CER_{M\chi}(G)$ and $UER_{M\chi}(G)$

In this section, some conditions for changing Edge trmoval graphs and vertex Edge removal graph with respect to the MDC number of $\gamma_{M\chi}$ *of a graph are discussed and the characterization on* $CER_{M\chi}$ *is also determined.*

Theorem 5.7.1: If the graph G has a unique $\gamma_{M\chi}$- set then G is a $CER_{M\chi}$- graph.

Proof: Let the graph G has a unique $\gamma_{M\chi}$- set S. Suppose the graph G is $UER_{M\chi}$. Then $\gamma_{M\chi}(G-e) = \gamma_{M\chi}(G)$, for all $e \in E(G)$. Now let S_1 and S_2 be any two subsets of G such that $S_1 \cap S_2 = \phi$ with $|N[S_1|] \geq \lceil \frac{p}{2} \rceil$ and $|N[S_2|] \geq \lceil \frac{p}{2} \rceil$. Consider an edge $e = uv$ with $u \in V(S_1)$ and $v \in V(S_2)$. Since $\gamma_{M\chi}(G-e) = \gamma_{M\chi}(G)$, the removal of any edge e would not affect the chromatic number of G. Therefore $\chi(\langle S_1 \rangle) = \chi(\langle S_2 \rangle) = \chi(G)$. Thus, the two sets S_1 and S_2 are $\gamma_{M\chi}$-

sets of G. It is a contradiction to the assumption that G has a unique $\gamma_{M\chi}$- set.Hence $G \in CER_{M\chi}$. ∎

Theorem 5.7.2: If a graph G has a unique cpn-set with $\chi(G) \geq 3$ then G is a connected $CER_{M\chi}$– graph.

Proof: Let G has a unique cpn-set with $\chi(G) \geq 3$. Suppose $G \in UER_{M\chi}$. Then S is a $\gamma_{M\chi}$- set of G and $\gamma_{M\chi}(G-e) = \gamma_{M\chi}(G)$, for every $e \in E(G)$. Therefore, the graph $(G-e)$ has two or more $\gamma_{M\chi}$- sets. Let S_1 and S_2 be the $\gamma_{M\chi}$- sets of the graph $(G-e)$ with $\chi(\langle S_1 \rangle) = \chi(G)$ and $\chi(\langle S_2 \rangle) = \chi(G)$. Hence the graph G has atleast two cpn-sets, which is a contradiction to the assumption. Thus, $G \in CER_{M\chi}$. ∎

Theorem 5.7.3: If every $\gamma_{M\chi}$- set S of a graph G induces a color critical graph then the graph G is $CER_{M\chi}$.

Proof: Let S be a $\gamma_{M\chi}$- set of G and it induces a color critical graph. Then there exists an edge $e = uv$ such that $\chi(G-e) < \chi(G)$, for all edges $e \in E\langle S \rangle$. Then $\gamma_{M\chi}(G-e) < \gamma_{M\chi}(G)$, for all $e \in E(G)$. ∎

Theorem 5.7.4: If the graph G is a connected $CER_{M\chi}$- graph then $d_{M\chi}(G) = 1$.

Proof: Let $G \in CER_{M\chi}$. This result is proved by the induction on $diam(G)$. When $diam(G) = 1$. Then G becomes K_p and $\gamma_{M\chi}(G) = cpn(G) = p$. Hence $d_{M\chi}(G) = 1$. When $diam(G) = 2$. Then $= K_{1,n}$. Since G has no independent edges, $\gamma_{M\chi}(G - e) = \gamma_{M\chi}(G)$, for all $e \in E(G)$ and $e \in E^0_{M\chi}(G)$. Then the graph $\in UER_{M\chi}$, which is a contradiction to the assumption. When $diam(G) \geq 3$. Then the graph G has atleast two cpn-sets. It implies that G has two or more $\gamma_{M\chi}$- sets and $\gamma_{M\chi}(G - e) = \gamma_{M\chi}(G)$, for any $e \in E(G)$. Therefore $G \in UER_{M\chi}$, which is a contradiction to the assumption. Hence the graph G has a unique cpn- set. Thus $d_{M\chi}(G) = 1$. ∎

The following theorem establishes the characterization of $CER_{M\chi}$ *for a connected graph* G*.*

Theorem 5.7.5: Let G be any graph and S be a $\gamma_{M\chi}$- set of G. Then $G \in CER_{M\chi}$ if and only if

(i) Each edge $e = uv$ joins either S and $V - S$ or lies in S itself

(ii) The graph G is vertex color critical

(iii) The graph G contains a Clique H with $V(H) \geq 3$.

Proof: Let G be a $CER_{M\chi}$- graph and S be its $\gamma_{M\chi}$- set. Then $\gamma_{M\chi}(G - e) \neq \gamma_{M\chi}(G)$. It implies that either $\gamma_{M\chi}(G - e) < \gamma_{M\chi}(G)$

or $\gamma_{M\chi}(G-e) > \gamma_{M\chi}(G)$, for any edge $e \in E(G)$. If $\gamma_{M\chi}(G-e) < \gamma_{M\chi}(G)$ then $e \in E^{-}_{M\chi}(G)$. Hence $\chi(G-e) < \chi(G)$. It implies that $e \in E(\langle S \rangle)$ and the end vertices of $e = uv$ both are in S. Otherwise $e \in (\langle N[S] \rangle)$ and $e = uv$ joins S and $V - S$. Hence (i) holds. Also suppose $\chi(G-e) < \chi(G)$ then the graph G becomes edge color critical and $e \in E^{-}_{M\chi}(G)$. If $e \in E^{-}_{M\chi}(G)$, by theorem (5.6.8), conditions (ii) and (iii) are true.

Conversely, conditions (i), (ii) and (iii) holds. Then by theorem (5.7.3) and (5.6.8), the graph $G \in CER_{M\chi}$. ∎

Theorem 5.7.6: All trees are $UER_{M\chi}$- graph.

Proof: Let T be a tree with e pendants and p vertices. The result is proved by induction on pendants e of T. Since each tree has pendants $e \geq 2$, tree is a path if $e = 2$. If $T = P_p$, by corollary (2.3.4),

$$\gamma_{M\chi}(T) = \begin{cases} \lceil \frac{p}{6} \rceil, \text{ if } p \equiv 1, 2 (mod\ 6) \\ \lceil \frac{p}{6} \rceil + 1, \text{ if } p \equiv 0, 3, 4, 5 (mod 6). \end{cases} \tag{5.5}$$

Therefore any edge in $T, (T-e)$ is also a path with 2 components. By the result (5.5), $\gamma_{M\chi}(T-e) = \gamma_{M\chi}(T)$, for any $e \in E(T)$. Hence

$T \in UER_{M\chi}$ if $e = 2$. If $e = 3$ then $T = K_{1,3}$ or $D_{1,2}$ or caterpillar structure. For $T = K_{1,3}$ and $D_{1,2}, \gamma_{M\chi}(T) = 2$ and $\gamma_{M\chi}(T - e) = 2$. By result (2.3.1)(v), $\gamma_{M\chi}(T) = \lceil \frac{p}{8} \rceil + 1 = 2 = \gamma_{M\chi}(T - e)$. Hence $T \in UER_{M\chi}$, if $e = 3$. This result is true if $e = 2, 3, \cdots, (p - 2)$. When $e = p - 1$ then the tree $T = K_{1,p-1}$. By proposition (2.3.2)(ii), $\gamma_{M\chi}(T) = 2$. In $(T - e), \chi(T - e) = 2 = \chi(T)$ and $\gamma_{M\chi}(T - e) = 2 = \gamma_{M\chi}(T)$. Hence $T \in UER_{M\chi}$. In all cases, all trees are belonging to the class $UER_{M\chi}$. ∎

Theorem 5.7.7: If a graph G contains an induced subgraph as an even cycle then $G \in UER_{M\chi}$.

Proof: Let G be a graph which contains an induced subgraph as an even cycle. Then $\chi(G) = 2$. Let $S = \{u_1, u_2, u_5, \cdots, u_t\}$ be the $\gamma_{M\chi}$-set of G such that $|N[S]| \geq \lceil \frac{p}{2} \rceil$ with $|S| = t$. Now delete an edge $e = u_1u_2 \in E(G)$ and form the $\gamma_{M\chi}-$set $S' = \{u_3, u_4, u_7, \cdots, u_t\}$ for $(G - e)$ such that $|N[S']| \geq \lceil \frac{p}{2} \rceil$ with $|S'| = t$. Since u_3 and u_4 are adjacent, $\chi(G - 2) = 2 = \chi(G)$. Hence $\gamma_{M\chi}(G - e) = |S'| = |S| = \gamma_{M\chi}(G)$ and $e \in E^0_{M\chi}(G)$, for all $e \in E(G)$. Thus, $G \in UER_{M\chi}$. Suppose the graph G itself is an even cycle. Then by proposition (5.5.5), $e \in E^0_{M\chi}(G)$, for all edge $e \in E(G)$. Hence $G \in UER_{M\chi}$. ∎

Theorem 5.7.8: If the graph G is a complete bipartite then $G \in UER_{M\chi}$.

Proof: For the complete bipartite graph, $\chi(G) = 2$. By proposition (2.3.5), $\gamma_{M\chi}(G) = 2$. Now the deletion of any edge $e = u_i u_j$, where $u_i \in V_1(G)$ and $v_j \in V_2(G), \gamma_{M\chi}(G-e) = 2$. Therefore $\gamma_{M\chi}(G-e) = \gamma_{M\chi}(G)$ and $G \in UER_{M\chi}$. ∎

5.8 Results on $E^0_{M\chi}, E^-_{M\chi}$ and $E^+_{M\chi}$ for Disconnected graphs

In this section, the effects of an edge removal from G and its three classifications namely $E^0_{M\chi}, E^-_{M\chi}$ and $E^+_{M\chi}$ are studied with respect to the chromatic preserving property for disconnected graphs.

Observations 5.8.1:

(i) Let $G = K_2 \cup \overline{K_{p-2}}$ be a graph. Then $e \in E^0_{M\chi}(G)$, for $e \in E(K_2)$ and $\gamma_{M\chi}(G-e) = \lceil \frac{p}{2} \rceil = \gamma_{M\chi}(G)$.

(ii) Let $G = K_3 \cup \overline{K_{p-3}}$ be a graph. Then $e \in E^-_{M\chi}(G)$, for all $e \in E(K_3)$. Also $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$ and $\gamma_{M\chi}(G-e) = \lceil \frac{p}{2} \rceil - 1$.

(iii) Let $G = C_8 \cup \overline{K_{p-8}}$. Since C_8 is an even cycle, $e \in E^0_{M\chi}(G)$, for all $e \in E(C_8)$.

(iv) Let $G = C_9 \cup \overline{K_{p-9}}$. Since C_9 is vertex color critical component of G, $e \in E^-_{M\chi}(G)$, for all $e \in E(C_9)$.

(v) Let $G = P_4 \cup \overline{K_{p-4}}$ be a graph. Then $e \in E^+_{M\chi}(G)$, for all $e \in E(P_4)$.

(vi) Let $G = C_7 \cup C_9$ be a graph. Then $e \in E^+_{M\chi}(G)$, for all $e \in E(C_7)$ and $e \in E^0_{M\chi}(G)$, for all $e \in E(C_9)$.

Theorem 5.8.2: Let G be a disconnected graph with atleast two color critical components g_1 and g_2 and $cpn(g_1) \leq cpn(g_2)$. If $\chi(g_1) > \chi(g_2)$ then (i) $e \in E^-_{M\chi(G)}$, for all $e \in E(g_1)$ (ii) $e \in E^0_{M\chi}(G)$, for all $e \in E(g_2)$.

Proof: Let g_1 and g_2 be the color critical components of a disconnected graph G. Then $\chi(g_1 - e) < \chi(g_1)$ and $\chi(g_2 - e) < \chi(g_2)$. Let $\chi(g_1) > \chi(g_2)$. Then $\chi(g_1) = \chi(G)$. Hence any $\gamma_{M\chi}$- set of G must contain the full vertex set of g_1. Let $S = \{v_1, v_2, \cdots, v_r\}$ be a subset of $V(G)$, where $\{v_1, v_2, \cdots, v_r\} \subseteq V(g_1)$. Since S contains the full vertex set of g_1. It implies that $\chi(\langle S \rangle) = \chi(g_1) = \chi(G)$. Since $cpn(g_1) \leq cpn(g_2), |N[S]| < \lceil \frac{p}{2} \rceil$. Hence S wouldn't be a MDC set of G and S will be a MDC set by adding some vertices u_i from other components such that $|N[S]| \geq \lceil \frac{p}{2} \rceil$. Suppose that, the deletion of any edge e in g_1 then $\chi(g_1 - e) < \chi(g_1) \geq \chi(g_2)$. If $\chi(g_1 - e) = \chi(G)$ then the set $S' = \{v_1, v_2, \cdots, v_{r-1}\}$ is the MDC set of $G - e$. Since $|S'| < |S|, \gamma_{M\chi}(G - e) = |S'| < |S| = \gamma_{M\chi}(G)$. It implies that $e \in E^-_{M\chi}(G)$, for all $e \in E(g_1)$. Hence (i) holds.

Suppose that the deletion of any edge e in g_2 then it does not affect the cp-set of G. Hence the MDC set of G will be the MDC set of $G - e$. Therefore, $\gamma_{M\chi}(G - e) = |S'| = |S| = \gamma_{M\chi}(G)$ and $e \in E^0_{M\chi}(G)$, for all $e \in E(g_2)$. Hence (ii) holds. ■

Theorem 5.8.3: Let G be a disconnected graph with color critical components g_1 and g_2, such that $cpn(g_1) < cpn(g_2)$. If $\chi(g_1) = \chi(g_2)$ then (i) $e \in E^+_{M\chi}(G)$, for all $e \in E(g_1)$ (ii) $e \in E^0_{M\chi}(G)$, for all $e \in E(g_2)$.

Proof: Let G be a disconnected graph with the components $g_1, g_2, g_3, \cdots, g_k$ such that g_1 and g_2 are color critical. Then $\chi(g_1 - e) < \chi(g_1)$ and $\chi(g_2 - e) < \chi(g_2)$. If $\chi(g_1) = \chi(g_2)$ then either $\chi(g_1) = \chi(G)$ or $\chi(g_2) = \chi(G)$. Since $cpn(g_1) < cpn(g_2)$, the $\gamma_{M\chi}$- set of G contain the full vertex set of g_1. Let $S = \{v_1, v_2, \cdots, v_r, u_i\}$ be a $\gamma_{M\chi}$- set of G, where $\{v_1, v_2, \cdots, v_r\} \subseteq V(g_1)$ and u_i's are the vertices of other component with $t = |u_i|$. Then $\gamma_{M\chi}(G) = |S| = r + t$. Now the deletion of any edge $e \in E(g_1), \chi(g_1 - e) < \chi(G)$. Hence $\chi(g_2) = \chi(G)$. In $\{G - e\}, S' = \{u_1, u_2, \cdots, u_s, v_i\}$ be the $\gamma_{M\chi}$- set, where $\{v_1, v_2, \cdots, v_s\} \subseteq V(g_2)$ and u_i's are the vertices of other components with $t = |u_i|$. Then $\gamma_{M\chi}(G - e) = |S'| = s + t$. Since $cpn(g_1) < cpn(g_2), |r| < |s|$. Hence $\gamma_{M\chi}(G - e) = |S'| > |S| = \gamma_{M\chi}(G)$ and $e \in E_M^+\chi(G)$, if $e \in E(g_1)$. Thus, condition (i) holds.

Suppose that the deletion of any edge $e \in E(g_2), \chi(g_2 - e) < \chi(G)$. Hence $\chi(g_1) = \chi(G).InG - e, S' = \{v_1, v_2, \cdots, v_r, u_i\}$ be a $\gamma_{M\chi}$- set, where $\{v_1, v_2, \cdots, v_r\} \subseteq V(g_1)$ and u_i's are the vertices of other component with $t = |u_i|$. Then $\gamma_{M\chi}(G - e) = |S'| = r + t$. Hence $\gamma_{M\chi}(G) = \gamma_{M\chi}(G - e)$ and $e \in E^0_{M\chi}(G)$, if $e \in E(g_2)$. Thus, condition (ii) holds. ∎

Corollary 5.8.4: If $cpn(g_1) < cpn(g_2)$ and $\chi(g_1) < \chi(g_2)$ then (i) $e \in E^0_{M\chi}(G)$, for all $e \in E(g_1)$ (ii) $e \in E^-_{M\chi}(G)$, for all $e \in E(g_2)$.

5.9 Changing and Unchanging of MDC Number by Edge Addition

In this section, the effects of changing and unchanging of MDC number by edge addition are introduced and defined the three classifications for this parameter.

Definition 5.9.1: Let G be a simple graph without parallel edges. Let $\gamma_{M\chi}(G)$ be the Majority Dom-Chromatic Number (MDC number) of G. A graph G is said to be a $CEA_{M\chi}$- graph if $\gamma_{M\chi}(G+e) \neq \gamma_{M\chi}(G)$, for each $e \in E(G^c)$ and a graph G is said to be $UEA_{M\chi}$-graph if $\gamma_{M\chi}(G+e) = \gamma_{M\chi}(G)$, for each $e \in E(G^c)$ where G^c is the complement of G.

Definition 5.9.2: The following are the notations of changing and unchanging of $\gamma_{M\chi}(G)$ when an edge is added to the given graph G (from the complement G^c of G).

(i) $\xi^{\circ}_{M\chi}(G) = \{e \in E(G^c) / \gamma_{M\chi}(G+e) = \gamma_{M\chi}(G)\}$

(ii) $\xi^{+}_{M\chi}(G) = \{e \in E(G^c) / \gamma_{M\chi}(G+e) > \gamma_{M\chi}(G)\}$

(iii) $\xi^{-}_{M\chi}(G) = \{e \in E(G^c) / \gamma_{M\chi}(G+e) < \gamma_{M\chi}(G)\}$.

Example 5.9.3: Consider the following graph G with $p = 21$.

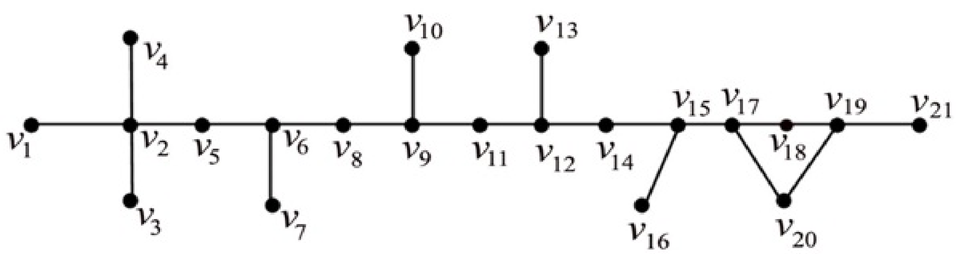

Figure 5.5: G :

In this graph, $S_1 = \{v_2, v_5, v_{12}, v_{19}\}$ is the minimal MDC-set of G. Hence $\gamma_{M\chi}(G) = 4$. The set $S_2 = \{v_2, v_9, v_{17}\}$ is the minimal γ_M- set of G and $\gamma_M(G) = 3$. Also the sets $S_3 = \{v_2, v_5, v_7, v_9, v_{12}, v_{15}, v_{19}, v_{21}\}$ and $S_4 = \{v_2, v_6, v_9, v_{12}, v_{15}, v_{19}\}$ are the γ_{ch}- set and γ- set of G respectively. Therefore $\gamma_{ch} = 8$ and $\gamma(G) = 6$. In the above graph G, add an edge $e = (v_{10}, v_{13})$ and $G' = \{G + e\}$. Here, $S = \{v_2, v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$ is the $\gamma_{M\chi}-$ set of G'. Hence $\gamma_{M\chi}(G') = 6$. Therefore $\gamma_{M\chi}(G') > \gamma_{M\chi}(G)$ and $e \in \xi^{+}_{M\chi}(G)$.

5.10 Results on $\xi^{\circ}_{M\chi}(G), \xi^{+}_{M\chi}(G)$ and $\xi^{-}_{M\chi}(G)$

In this section, the effects of an edge addition $e \in E(G^c)$ are classified into three cases with respect to MDC number of the graphs, where G^c is the complement of G. Also some results are established on these classifications $\xi^{\circ}_{M\chi}(G), \xi^{+}_{M\chi}(G)$ and $\xi^{-}_{M\chi}(G)$.

Proposition 5.10.1: Let G be an even Cycle with p vertices and $p \equiv 0(mod\ 6)$. If $d(v_i, v_j) = n$ and $e = v_iv_j \in E(G^c)$ then (i) $e \in \xi^{-}_{M\chi}(G)$, n is even (ii) $e \in \xi^{+}_{M\chi}(G)$, n is odd (iii) $G \in CEA_M\chi,\ p > 6$.

Proof: Let $p \equiv 0(mod\ 6)$ and $p = 6k$. By the proposition (2.3.3),

$\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil + 1 = k + 1$. If $k = 1$, $\gamma_{M\chi}(G) = 2$. Adding an edge $e = v_1v_4$ and it creates two even cycles. It implies that $\gamma_{M\chi}(C_6+e) = 2$. Thus $\gamma_{M\chi}(C_6+e) = \gamma_{M\chi}(G)$. It is a contradiction to $\gamma_{M\chi}(G)$. Let $k \geq 2$. By adding an edge $e = v_iv_j \in E(G^c)$ such that $d(v_i, v_j) = n$, we obtain $G' = G + e$.

Case (i): If $d(v_i, v_j) = n, n$ is odd then G' constitutes two even cycle C_1 and C_2 with the common edge $e = v_iv_j$ and $|N[v_i]| = |N[v_j]| = 3 = \Delta(G)$. Hence any $\gamma_{M\chi}$ - set of G' must contain the end vertices of an edge e. Then $\gamma_{M\chi}(G') = \lceil \frac{p}{6} \rceil + 1 - 1 = \lceil \frac{6k}{6} \rceil = k$. Therefore, $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$ and $e \in \xi^-_{M\chi}(G)$.

Case (ii): Let $d(v_i, v_j) = n$ and n is even. Then G' creates two odd cycles with C_1 and C_2 with $|V(C_1)| = |V(C_2)| < p$ and $\chi(G') = 3$. Each cycles are vertex color critical, any $\gamma_{M\chi}$- set of $\{G + e\}$ must contain any one cycle with the edge e. Let $S' = \{v_1, v_2, v_3, \cdots, v_t\}$ be the $\gamma_{M\chi}$-set of G' such that $|N[S']]| \geq \lceil \frac{p}{2} \rceil$ and $\chi(\langle S' \rangle) = 3 = \chi(G')$. Hence, $\gamma_{M\chi}(G') = |V(C_1)| + \lceil \frac{p}{6} \rceil + 1 - 2 \geq 3 + \lceil \frac{6k}{6} \rceil + 1 - 2 \geq k + 2$. But $\gamma_{M\chi}(G) = k + 1$. Therefore, $\gamma_{M\chi}(G') > \gamma_{M\chi}(G)$ and $e \in \xi^+_{M\chi}(G), p > 6$. In both cases, for all $e = v_iv_j \in E(G^c)$ of C_p, $p \equiv 0 (mod 6)$, we obtain $e \in \xi^-_{M\chi}(G)$ and $e \in \xi^+_{M\chi}(G)$. Hence the even cycle C_p, $p \equiv 0 \ (mod\ 6)$ is a $CEA_{M\chi}$- graph. ∎

Proposition 5.10.2: Let $G = C_p$ be an even cycle. If $p \equiv 2(mod\ 6)$ and $d(v_i, v_j) = n$ and $e = v_iv_j \in E(G^c)$ then (i) $e \in \xi^{\circ}_{M\chi}(G)$, n is odd (ii) $e \in \xi^{+}_{M\chi}(G), n$ is even.

Proof: Let $p \equiv 2(mod\ 6)$ and $p = 6k+2$. Then $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil = k+1, k \geq 1$. By adding an edge $e = v_iv_j \in E(G^c)$ such that $d(v_i, v_j) = n$ in G, we obtain $G' = \{G + e\}$.

Case (i): Let n be odd. Then the graph G' constitutes two even cycles C_1 and C_2 with the common edge $e = v_iv_j$ and $\chi(G') = 2$. If the vertices of $e = v_iv_j$ has the maximum degree of G, any $\gamma_{M\chi}$- set of G' will contain the end vertices v_i and v_j. Therefore $\gamma_M\chi(G') = 3 + \lceil \frac{p}{6} \rceil - 3 = 3 + \lceil \frac{6k+2}{6} \rceil - 3 = k + 1$. Hence $\gamma_{M\chi}(G') = \gamma_{M\chi}(G)$ and $e \in \xi^{\circ}_{M\chi}(G)$.

Case(ii): Let n be even. Then G' creates two odd cycles C_1 and C_2 with $|V(C_1)| = |V(C_2)| < p$ and $\chi(G') = 3$. Then $|V(C_1)|+|V(C_2)| = \lceil \frac{p}{2} \rceil + \left(\lceil \frac{p}{2} \rceil + 2\right)$. Since the odd cycles are vertex color critical, $\gamma_{M\chi}(G')$ must be the minimum value of $V(C_1)$ or $V(C_2)$. Hence $\gamma_{M\chi}(G') = \lceil \frac{p}{2} \rceil = \frac{6k+2}{2} = 3k + 1 > \gamma_{M\chi}(G)$.Thus, $e \in \xi^{+}_{M\chi}(G)$. ∎

Proposition 5.10.3: Let G be an even cycle with p vertices. If $p \equiv 4(mod\ 6)$ and $d(v_i, v_j) = n$ and $e = v_iv_j \in E(G^C)$. Then (i)

$e \in \xi^{\circ}_{M\chi}(G)$, if $e = v_1v_3$ (ii) $e \in \xi^{-}_{\bar{M}\chi}(G)$, if n is odd (iii) $e \in \xi_M\chi^{+}(G)$, if n is even.

Proof: Let $p \equiv 4(mod\ 6)$ and $p = 6k+4, k = \lceil \frac{p-4}{6} \rceil$. Then by proposition (2.3.3), $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil + 1 = k+2, k \geq 1$. By adding an edge $e = v_iv_j \in E(G^C)$ such that $d(v_i, v_j) = n$ we obtain $G' = \{G + e\}$.

Case (i): If $d(v_i, v_j) = 2$ then $e = v_1v_3$ such that G' contains a triangle and a $(p-1)$- cycle. It implies that $\chi(G') = 3$ and any $\gamma_{M\chi}$- set of G' contains the triangle. Let $S_1 = \{v_1, v_2, v_3, v_5, v_8, \cdots, v_t\}$ with $(v_1, v_2, v_3) = a$ triangle and other vertices are of $d(v_iv_j) \geq 2$ be the $\gamma_{M\chi}$- set of G' such that $|N[S_1]| \geq \lceil \frac{p}{2} \rceil$ and $\chi(\langle S_1 \rangle) = 3 = \chi(G')$. Therefore $\gamma_{M\chi}(G') = \lceil \frac{p}{6} \rceil + 1 = \lceil \frac{6k+4}{6} \rceil + 1 = k+2$. Hence, $\gamma_{M\chi}(G') = k + 2 = \gamma_{M\chi}(G)$ and $e \in \xi^{\circ}_{M\chi}(G)$, if $e = v_1v_3$.

Case (ii): If $d(v_i, v_j) = n$ and n is odd then G' constitutes two even cycles C_1 and C_2 with $|V(C_1)| + |V(C_2)| = p + 2$. Then $\chi(G') = 2$. Both the cycles contain the edge $e = (v_iv_j)$ and any $\gamma_{M\chi}$- set of G' contains the end vertices of e. Let $S_2 = \{v_i, v_j, v_1, v_4, \cdots, v_t\}$ be the $\gamma_{M\chi}$- set of G' with $d(v_r, v_s) \geq 3$ for $r, s \neq i, j$ such that $|N[S_2]| \geq \lceil \frac{p}{2} \rceil$. Then $\chi(\langle S_2 \rangle) = 2 = \chi(G')$. Hence $\gamma_{M\chi}(G') = \lceil \frac{p}{6} \rceil + 1 - 2 = \lceil \frac{6k+4}{6} \rceil -$

$1 = k + 1 - 1 = k$. Therefore, $\gamma_{M\chi}(G') = k < \gamma_{M\chi}(G) = k + 2$ and $e \in \xi^{-}_{M\chi}(G)$.

Case (iii): Suppose $d(v_i, v_j) = n, n \geq 2$, is even then $\{G\}$ consists of two odd Cycles C_1 and C_2 with $|V(C_1)| = |V(C_2)| < p$. Then $\chi(G') = 3$. Both the Cycles are vertex color critical. Thus any $\gamma_{M\chi}$-set S_3 of G' contains at least one cycle. Hence, $\lceil \frac{p}{2} \rceil - 3 \leq \gamma_{M\chi}(G') \leq p - 1$. It implies that $\lceil \frac{6k+4}{6} \rceil - 3 \leq \gamma_{M\chi}(G') \leq 6k + 3$. Therefore $\gamma_{M\chi}(G') > \gamma_{M\chi}(G)$ and $e \in \xi_M\chi^{+}(G)$. ∎

Proposition 5.10.4: Let G be an odd cycle with p vertices and $p \equiv 1, 3, 5 (mod 6)$. Then $e \in \xi^{-}_{M\chi}(G)$ and G is $CEA_{M\chi}$- graph.

Proof: Let $G = C_p$ and $p \equiv 1, 3, 5 (mod\, 6)$. Since p is odd and vertex color critical, $\gamma_{M\chi}(G) = p$. If adding any edge $e \in E(G^c)$ between any two vertices in G then $G' = G + e$ contains either two odd cycles or one odd and one even cycle.

Case (i): Let G' contains only two odd cycles C_1 and C_2 with $|V(C_1)| = |V(C_2)| < p$. Let S_1 and S_2 be the $\gamma_{M\chi}$- sets of C_1 and C_2. Since the two cycles are vertex color critical, $\gamma_{M\chi}(C_1) = |S_1| < p$ and $\gamma_{M\chi}(C_2) = |S_2| < p$. Therefore $\gamma_{M\chi}(G') < p = \gamma_{M\chi}(G)$. Hence, $e \in \xi^{-}_{M\chi}(G)$.

Case (ii): If G' contains one odd cycle C_1 and one even cycle C_2 with $|V(C_1)| = |V(C_2)| < p$. Since C_1 is vertex color critical graph, $\gamma_{M\chi}(C_1) < p$. Since C_2 is even cycle, $\chi(C_2) = 2$. Then by proposition (2.3.3), $\gamma_{M\chi}(C_2) = \lceil \frac{p}{6} \rceil + 1 < p$. Therefore $\gamma_{M\chi}(G') < p = \gamma_{M\chi}(G)$ and $e \in \xi^-_{M\chi}(G^c)$. In both cases, $e \in \xi^-_{M\chi}(G)$ or $e \in \xi^+_{M\chi}(G)$, for all $e \in E(G^c)$. Hence $G \in CEA_{M\chi}$. ∎

Proposition 5.10.5: For the wheel graph $G = W_p, \gamma_{M\chi}(G+e) = 4$ if and only if $e = v_iv_j \in E(G^c)$ such that $d(v_i, v_j) = 2$ and $e \in \xi^+_{M\chi}(G)$.

Proof: Let $G = W_p$ be a wheel graph. By the proposition (2.3.6),

$$\gamma_{M\chi}(G) = \begin{cases} 3, & \text{if } p \text{ is odd} \\ p, & \text{if } p \text{ is even.} \end{cases} \tag{5.6}$$

The graph G contains $(p-1)$ triangle. Since $\gamma_{M\chi}(G+e) = 4$, there is a clique K_4 in $(G+e)$ where $e \in E(G^c)$ and $e = v_iv_j$. The clique K_4 in $(G+e)$ will be obtained only by adding any edge $e = v_iv_j$ between any two of adjacent triangles. By the condition (5.6), $\gamma_{M\chi}(G+e) > \gamma_{M\chi}(G)$, for any $e = v_iv_j$ with $d(v_iv_j) = 2$ and $e \in \xi^+_{M\chi}(G)$. Hence, for K_4 in $(G+e)$ such that $d(v_i, v_j) = 2$. Conversely, if an edge

$e = v_i v_j \in E(G^c)$ such that $d(v_i, v_j) = 2$ is added to G, then there exist a clique K_4 in $(G + e)$. Therefore $\gamma_{M\chi}(G + e) = 4$. ■

Proposition 5.10.6: Let $G = P_p$ be a Path with p vertices. If $e = v_i v_j \in E(G^c)$ with $d(v_i, v_j) = n$ is odd then (i) $e \in \xi^0_{M\chi}(G)$, if $p \equiv 1, 2(mod\ 6)$ (ii) $e \in \xi^-_{M\chi}(G)$, if $p \equiv 0, 3, 4, 5(mod\ 6)$.

Proof: Let $G = P_p$ a Path. If adding any edge $e = v_i v_j \in E(G^c)$ with $d(v_i, v_j) = n, n$ being odd between any two internal vertices v_i and v_j in a Path, the degree of the end vertices of e is increased by one and the resultant graph is $G' = G + e$. Since n is odd, G' creates even cycle C_m as an induced subgraph and $\chi(G') = 2$.

Case(i): when $p \equiv 1, 2(mod\ 6)$. Let $S = \{v_i, v_j, v_3, v_6, \cdots, v_t\} \subseteq V(G')$ with $|t| = \lceil \frac{p}{6} \rceil$ such that $d(v_i) = d(v_i) = 3, d(v_i, v_j) = 1$ and $d(v_r, v_s) \geq 3$, for $r, s \neq i, j$ and $\chi(\langle S' \rangle) = 2 = \chi(G')$. Let $p = 6k + 1$. Then $|N[S]| = 3t = 3\lceil \frac{p}{6} \rceil = 3\lceil \frac{6k+1}{6} \rceil = 3\left(\frac{p-1}{6}\right) + 3 = \lceil \frac{p}{2} \rceil + 1$. Let $p = 6k + 2$. Then $|N[S]| = 3t = 3\lceil \frac{p}{6} \rceil = 3\lceil \frac{6k+2}{6} \rceil = 3\left(\frac{p-2}{6}\right) + 3 = \lceil \frac{p}{2} \rceil + 1$. It implies that S would be a $\gamma_{M\chi}$- set of G'. Therefore $\gamma_{M\chi}(G') = \lceil \frac{p}{6} \rceil$. By corollary (2.3.4), $\gamma_{M\chi}(G') = \gamma_{M\chi}(G)$ and hence $e \in \xi^0_{M\chi}(G)$.

Case(ii): when $p \equiv 0, 3, 4, 5(mod\ 6)$. Let $S = \{v_i, v_j, v_3, v_6, \cdots, v_t\} \subseteq$

$V(G')$ with $|t| = |S| = \lceil \frac{p}{6} \rceil + 1$ such that $d(v_i) = d(v_i) = 3, |N(S)| = 3(t-2)+4 = 3t-6+4 = 3t-2, d(v_i, v_j) = 1$ and $\chi(\langle S' \rangle) = 2 = \chi(G')$. For $p = 6k, |N[S]| = 3t = 3\left(\lceil \frac{p}{6} \rceil + 1\right) = 3\left(\lceil \frac{6k}{6} \rceil + 1\right) = 3\left(\frac{p}{6}\right) + 3 \geq \lceil \frac{p}{2} \rceil + 3$. If $p = 6k+3$ then $|N[S]| = 3t = 3\left(\lceil \frac{p}{6} \rceil + 1\right) = 3\left(\lceil \frac{6k+3}{6} \rceil + 1\right) = 3\left(\frac{p-3}{6}\right) + 6 \geq \lceil \frac{p}{2} \rceil + 3$. If $6k+4$ then $|N[S]| = 3t = 3\left(\lceil \frac{p}{6} \rceil + 1\right) = 3\left(\lceil \frac{6k+4}{6} \rceil + 1\right) = 3\left(\frac{p-4}{6}\right) + 6 \geq \lceil \frac{p}{2} \rceil + 3$. For $6k+5, |N[S]| = 3t = 3\left(\lceil \frac{p}{6} \rceil + 1\right) = 3\left(\lceil \frac{6k+5}{6} \rceil + 1\rceil\right) = 3\left(\frac{p-5}{6}\right) + 6 \geq \lceil \frac{p}{2} \rceil + 3$. It implies that S would be a $\gamma_{M\chi}$- set of G' but not minimal. If $|S| - 1 = \lceil \frac{p}{6} \rceil$ then $|N[S]| = \lceil \frac{p}{2} \rceil$. Hence the set S will be a $\gamma_{M\chi}$- set of G'. Therefore $\gamma_{M\chi}(G') = \lceil \frac{p}{6} \rceil$. By corollary (2.3.4), $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$ and hence, $e \in \xi^-_{M\chi}(G)$. ∎

Theorem 5.10.7: Let G be any graph with p vertices and $e \in E(G^c)$. If a graph $(G+e)$ contains an even cycle then $\gamma_{M\chi}(G+e) < \gamma_{M\chi}(G)$ and an added edge $e \in \xi^-_{M\chi}(G)$.

Proof: Let $G = C_p$ be a cycle. Suppose G is an odd cycle, $\gamma_{M\chi}(G) = p$. Let $e \in E(G^c)$ and adding in G it creates an odd cycle and an even cycle. Since both cycles containing an edge e, the degree of end vertices of e increased by three. Suppose the odd cycle contains $\lceil \frac{p}{2} \rceil$ vertices, $\gamma_{M\chi}(G+e) = \lceil \frac{p}{2} \rceil < \gamma_{M\chi}(G)$. If even cycle contains the remaining $\lceil \frac{p}{2} \rceil + 1$ vertices, then by result (3.2)[5], $\gamma_{M\chi}(G+e) = \lceil \frac{p}{6} \rceil + 1 = \left\lceil \frac{\frac{p}{2}+2}{2} \right\rceil + 1 = \lceil \frac{p}{12} \rceil < \gamma_{M\chi}(G)$. Hence $\gamma_{M\chi}(G+e) < \gamma_{M\chi}(G)$ and $e \in \xi^{-}_{M\chi}(G)$. ∎

Theorem 5.10.8: If a graph G consists of exactly two pendants with p vertices then (i) $\gamma_{M\chi}(G+e) > \gamma_{M\chi}(G)$, if p is odd (ii) $\gamma_{M\chi}(G+e) = \gamma_{M\chi}(G)$, if p is even.

Proof: Let G be any graph which consists of two pendants. For any tree $G, \chi(G) = 2$. Suppose joining the two pendants by an edge e, it creates a cycle.

Case (i): Let p be odd. Then $(G+e)$ becomes an odd cycle. Hence $\gamma_{M\chi}(G+e) = p$. Since $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil$ or $\lceil \frac{p}{6} \rceil + 1, \gamma_{M\chi}(G+e) > \gamma_{M\chi}(G)$, if p is odd.

Case (ii): Let p be even. Then $(G+e)$ becomes an even cycle. By proposition (2.3.3), $\gamma_{M\chi}(G+e) = \gamma_{M\chi}(G)$, if p is even. ∎

Theorem 5.10.9: Let T be a tree with $diam(T) = 3$. Then

(i) $\gamma_{M\chi}(T+e) = \gamma_{M\chi}(T)$, if $e = v_i v_j$ such that $d(v_i, v_j) = 3$

(ii) $\gamma_{M\chi}(T+e) > \gamma_{M\chi}(T)$, if $e = v_i v_j$ such that $d(v_i, v_j) = 2$.

Proof: Let T be a tree and $\chi(T) = 2$. Since $diam(T) = 3$, it has two central u and v vertices with $(p-2)$ vertices and $e = uv$ is a dominating edge of T. Let S be the $\gamma_{M\chi}$- set of T which consists of u and v. Then $\gamma_{M\chi}(T) = |S| = 2$.

Case (i): Let $e = v_iv_j \in E(T^c)$. Adding the edge e in T with $d(v_i, v_j) = 2$, $\{T+e\}$ constitutes a triangle. It implies that $\chi\{T+e\} = 3$. Hence any $\gamma_{M\chi}$- set of $\{T+e\}$ must contain the triangle and $\gamma_{M\chi}(T+e) = 3$. Thus $\gamma_{M\chi}(T+e) > \gamma_{M\chi}(T)$ and $e \in \xi^{+}_{M\chi}(T)$.

Case (ii): If $d(v_i, v_j) = 3$ then $\{T+e\}$ constitutes a 4 - Cycle and $\chi(T+e) = 2$. It does not affect the value of $\chi(T)$. Hence $\gamma_{M\chi}(T+e) = 2 = \gamma_{M\chi}(T)$ and $e \in \xi^{\circ}_{M\chi}(T)$. ∎

Theorem 5.10.10: Let T be a tree with atleast one vertex v such that $d(v) \geq \lceil \frac{p}{2} \rceil - 1$ and $d(v_i) \leq 2$. If $e = v_iv_j$ is any edge of T^X such that $d(v_i, v_j) = n$ then (i) $e \in \xi^{0}_{M\chi}(T)$, if $d(v_i, v_j) = n$ is odd (ii) $e \in \xi^{+}_{M\chi}(T)$, if $d(v_i, v_j) = n$ is even.

Proof: Let T be a tree and $\chi(T) = 2$. Since T has a vertex v such that $d(v) \geq \lceil \frac{p}{2} \rceil - 1$, any $\gamma_{M\chi}$ set of T contains the vertex v. Since $\chi\langle(T)\rangle = 2, S = \{v, u\}$ is the subset of T with $d(v, u) = 1$. Then $|N[S]| \geq \lceil \frac{p}{2} \rceil + 1$ and $\chi(\langle S \rangle) = 2 = \chi(T)$. Therefore S is the $\gamma_{M\chi}$ set of T and $\gamma_{M\chi}(T) = 2$. Suppose adding an edge $e = v_iv_j \in T^C$ to a tree T such that $d(v_i, v_j) = n$, then two cases arise.

Case (i): Let $d(v_i, v_j) = n$ be odd. Then $(T + e)$ contains an even cycle and $\chi(T + e) = 2 = \chi(T)$. Then $\gamma_{M\chi}$ of $(T + e)$ is same as the $\gamma_{M\chi}$ set S of T. It implies that $\gamma_{M\chi}(T + e) = 2 = \gamma_{M\chi}(T)$ and hence $e \in \xi^0_{M\chi}(T)$, if n is odd.

Case (ii): Let $d(v_i, v_j) = n$ be even. Then $(T + e)$ constitutes an odd cycle and $\chi(T + e) = 3$. Hence any $\gamma_{M\chi}$ set S' of $(T + e)$ must contain the odd cycle. Therefore $\gamma_{M\chi}(T + e) = |S'| \geq 3$. Since $\gamma_{M\chi}(T) = 2, \gamma_{M\chi}(T + e) > \gamma_{M\chi}(T)$. It implies that $e \in \xi^+_{M\chi}(T)$, if n is even. ∎

Theorem 5.10.11: If there exists a clique K_r in $(G+e)$ then $\gamma_{M\chi}(G+e) \neq \gamma_{M\chi}(G)$ for some edge $e \in \xi^+_{M\chi}(G)$.

Proof: Let G be a graph with $\chi(G) = t$. Then any $\gamma_{M\chi}$ set of G must contains the cpn-set of G with $cpn(G) = k$. Suppose there is any clique K_r in $(G + e)$ when adding any edge e in $G, \chi(G + e) = s > t$. Then $cpn(G + e) = m > k$. Hence any $\gamma_{M\chi}$ set of $(G + e)$ must contains the cpn-set of $(G + e)$. Therefore, $\gamma_{M\chi}(G + e) > \gamma_{M\chi}(G)$ and $e \in \xi^+_{M\chi}(G)$. ∎

5.11 $CEA_{M\chi}$ and $UEA_{M\chi}$ Graphs

In this section, $CEA_{M\chi}$ and $UEA_{M\chi}$- graphs are investigated for the graphs.

Theorem 5.11.1: Let G be a connected graph of diameter 2. Then $G \in CEA_{M\chi}$.

Proof: Since $diam(G) = 2$, the graph G contains a cycle or pendants. The graph structure becomes like $G = K_{1,p-1}$ a star and C_4, an even cycle. By proposition (2.3.2)(ii) and (2.3.3), $\gamma_{M\chi}(G) = 2$. By adding an edge e to G, it creates at least a triangle. Then $\gamma_{M\chi}(G+e) = 3 > \gamma_{M\chi}(G)$, for all $e \in E(G^c)$. Hence $G \in CEA_{M\chi}$. ■

Theorem 5.11.2: Let G be a cycle on p vertices. Then $G \in CEA_{M\chi}$ if and only if one of the following holds. (i) G is an odd cycle, $p \geq 5$ (ii) $G = C_4$ and $G = C_p, p \equiv 0 \ (mod\ 6), p \geq 7$.

Proof: Let $G = C_p$ be a cycle with p vertices and $G \in CEA_{M\chi}$. Then to prove the conditions (i) and (ii) are true. Let $G' = G + e$. Since the graph G is $CEA_{M\chi}$, either $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$ or $\gamma_{M\chi}(G') > \gamma_{M\chi}(G)$. Suppose $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$, for any $e \in E(G^c)$. Then there exists either two odd cycles or one odd and one even cycles in G'. These two odd cycles are color critical. Now, let $\gamma_{M\chi}(G') > \gamma_{M\chi}(G)$. Then G' must contain either odd cycles or one odd and one even cycles. Therefore, we get $p \equiv 0(mod\ 6)$, and $G = C_4$.

Conversely, if G is an odd cycle and $G = C_4, G = C_p, p \equiv 0(mod\ 6)$, by proposition (5.10.1), we obtain $G \in CEA_{M\chi}$. ∎

Theorem 5.11.3: Let G_1 and G_2 be any two connected graphs. If either the graph G_1 or G_2 is not complete then $(G_1 \cup G_2) \in CEA_{M\chi}$.

Proof: Suppose G_1 and G_2 both are complete graphs with p_1 and p_2 vertices. Then both are vertex color critical graphs. Let $G = G_1 \cup G_2$. Let $p_1 = p_2$ and $p = p_1 + p_2$ then $\chi(G_1) = \chi(G_2)$ and $\gamma_{M\chi}(G) = \chi(G_1)$. If an edge $e = uv \in E(G^c)$ is added to G where $u \in V(G_1)$ and $v \in V(G_2)$ then $\gamma_{M\chi}(G+e) = \gamma_{M\chi}(G)$. It implies that $G \in UEA_{M\chi}$. It is a contradiction to the assumption. If $p_1 < p_2$ then $\chi(G_1) < \chi(G_2)$ and $\gamma_{M\chi}(G) = \chi(G_2)$. If $e = uv \in E(G^c)$ is added to G where $u \in V(G_1)$ and $v \in V(G_2)$ then $\gamma_{M\chi}(G + e) = \gamma_{M\chi}(G)$. It implies that $G \in UEA_{M\chi}$, which is a contradiction. Hence, $G \in CEA_{M\chi}$. ∎

Proposition 5.11.4: Let $G = W_p$ be a wheel graph on $p \geq 5$ vertices. Then $W_p \in UEA_{M\chi}$, for an edge $e = v_iv_j \in E(G^C)$ such that $d(v_i, v_j) > 2$.

Proof: Let $G = W_p = C_{p-1}VK_1, p \geq 5$ with $(v_1, v_2, \cdots, v_{p-1}) \in V(C_{p-1})$ and v_p is a central vertex.

Case (i): When p is odd. Then $\chi(W_p) = 3$. By the proposition (2.3.6), $\gamma_{M\chi}(G) = 3$. If adding any edge $e = v_iv_j \in E(G^c)$ such that $d(v_i, v_j) > 2$, it creates another triangle and it does not affect the chromatic number of G. Hence $\chi(G + e) = \chi(G) = 3$. Thus $\gamma_{M\chi}(G + e) = \gamma_{M\chi}(G)$ for any $e \in E(G^C)$ and $W_p \in UER_{M\chi}(G)$.

Case (ii): When p is even. Then by the proposition (2.3.6), $\gamma_{M\chi}(G) = p$. Let $e = v_iv_j \in E(G^c)$ such that $d(v_i, v_j) > 2$, it creates another triangle and it does not affect the chromatic number of G. Hence $\chi(G+e) = \chi(G) = 4$ and $(G+e)$ is vertex color critical graph. Thus $\gamma_{M\chi}(G + e) = p$. Hence $\gamma_{M\chi}(G + e) = \gamma_{M\chi}(G)$. Thus $W_p \in UEA_{M\chi}$ when $d(v_i, v_j) > 2$ and p is even. ■

Proposition 5.11.5: Let $G = W_p$ be a wheel graph with $p \geq 5$. Then $W_p \in CEA_{M\chi}$, for an edge $e = v_iv_j$ such that $d(v_i, v_j) = 2$.

Proof: For $G = W_p$, a wheel, by the proposition (2.3.6),

$$\gamma_{M\chi}(G) = \begin{cases} 3, & \text{if } p \text{ is odd} \\ p, & \text{if } p \text{ is even.} \end{cases}$$

Case (i): When p is odd. Let $e = v_i v_j$ such that $d(v_i, v_j) = 2$. If add an edge $e \in E(G^C)$ in G then $\{G+e\}$ contains a clique K_4. It implies that $\chi\{G+e\} = 4$ and hence any $\gamma_{M\chi}$- set of $\{G+e\}$ must contain K_4. Therefore $\gamma_{M\chi}(G+e) = 4$. Since $\gamma_{M\chi}(G) = 3, \gamma_{M\chi}(G+e) > \gamma_{M\chi}(G)$. It implies that $e \in \xi^{+}_{M\chi}(G)$ and $G \in CEA_{M\chi}$ if p is odd.

Case (ii): When p is even. By adding an edge $e = v_i v_j \in E(G^C)$ such that $d(v_i, v_j) = 2$ in $G, \{G+e\}$ contains a clique K_4. Hence any $\gamma_{M\chi}$- set of $\{G+e\}$ contains K_4 and it satisfies $\chi(G+e) = \chi(G) = 4$. Therefore $\gamma_{M\chi}\{G+e\} = 4$. Since $\gamma_{M\chi}(G) = p, \gamma_{M\chi}\{G+e\} < \gamma_{M\chi}(G)$. Hence, $e \in \xi^{-}_{M\chi}(G)$ and $G \in CEA_{M\chi}$, if p is even. ∎

The following theorem gives the characterization of $CEA_{M\chi}$ of graph G. When adding an edge 'e' from the complement of a graph G.

Theorem 5.11.6: Let G be any connected graph and $e \in E(G^c)$. Then $G \in CEA_{M\chi}$ if and only if (i) $\{G + e\}$ contains a clique (ii) G is a vertex color critical graph.

Proof: Let $G \in CEA_{M\chi}$. Then the graph G satisfies the condition either $\gamma_{M\chi}(G+e) < \gamma_{M\chi}(G)$ or $\gamma_{M\chi}(G+e) > \gamma_{M\chi}(G)$.

Case (i): If $\gamma_{M\chi}(G+e) < \gamma_{M\chi}(G)$ then $\chi(G+e) < \chi(G)$. It implies that the graph G is vertex color critical graph. Hence the condition (ii) holds.

Case(ii): If $\gamma_{M\chi}(G+e) > \gamma_{M\chi}(G)$ then $\chi(G+e) > \chi(G)$. It implies that the graph $(G+e)$ contains a vertex color critical subgraph as aclique. Hence the condition (i) holds.

Conversely, if the conditions (ii) then by theorem (5.10.11), $G \in CEA_{M\chi}$. Let the graph G be vertex color critical. Then $cpn(G) = p$ and $\gamma_{M\chi}(G) = p$. If adding any edge in $G, \chi(G+e) < \chi(G)$ and $cpn(G+e) < p$. Hence $\gamma_{M\chi}(G+e) < p$ and $G \in CEA_{M\chi}$. ∎

Chapter 6

Connected Majority Dom-chromatic Number of a Graph

Abstract

This chapter introduces a new notion connected majority dom-chromatic set of a graph G. For a graph G, the connected majority dom-chromatic number $\gamma_{CM\chi}(G)$ is determined for some standard graphs. The exact values of $\gamma_{CM\chi}(G)$ is investigated for product graphs. Bounds and characterization theorem on $\gamma_{CM\chi}(G)$ for connected and disconnected graphs are also studied.

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6.1 Introduction

In 1979, the concept "Connected Domination Number in Graphs" was introduced by Sampathkumar and Walikar [53] and they produced many results in their article. In 2012, the parameter "connected dom-chromatic number" was studied by Janakiraman and Poobalaranjani [31]. They established more results on $\gamma_{ch}(G)$ with other parameters for connected and disconnected graphs. In 2017, Joseline Manora and Muthukani Vairavel [34] introduced "Connected majority dominating set of a graph" and its number $\gamma_{CM}(G)$. They elucidated the parameter $\gamma_{CM}(G)$ in various levels by establishing many results and inequalities. They produced the exact values of γ_{CM} for some standard graphs and particularly product graphs. Also they developed some inequalities for $\gamma_{CM}(G)$ with other parameters.

These two parameters $\gamma_{ch}(G)$ and $\gamma_{CM}(G)$ gave the motivation to define a new graph theoretical parameter "Connected Majority Dom-Chromatic set of a graph" and its number $\gamma_{CM\chi}(G)$ on graphs.

Organization of this chapter is as follows. In section 6.1, the introduction of this chapter is given and in section 6.2, the concept of connected majority dom-chromatic set of a graph G and its number $\gamma_{CM_\chi}(G)$ are defined with examples. The particular values of $\gamma_{CM_\chi}(G)$ for various structures such as some standard graphs, grid, cylinder, Torus, corona graphs are determined in section 6.3. In section 6.4, characterization theorems and bounds on $\gamma_{CM_\chi}(G)$ are discussed.

6.2 Connected Majority Dom-Chromatic Set of a Graph

In this section, the concept of Connected Majority Dom-Chromatic set of a graph and its number are defined with an example.

Definition 6.2.1: A Majority Dom-Chromatic (MDC) set S is said to be a connected Majority Dom-Chromatic (connected MDC) set if the induced subgraph $\langle S \rangle$ is connected in G. The connected MDC set is minimal if no proper subset of S is a connected MDC set.

Definition 6.2.2: The minimum cardinality of a minimal connected MDC set is called the connected MDC number and is denoted by $\gamma_{CM_\chi}(G)$. The maximum cardinality of a minimal connected MDC set is called the upper connected MDC number of G and it is denoted by $\Gamma_{CM_\chi}(G)$.

Example 6.2.3: Consider the graph G with $p = 21$ vertices.

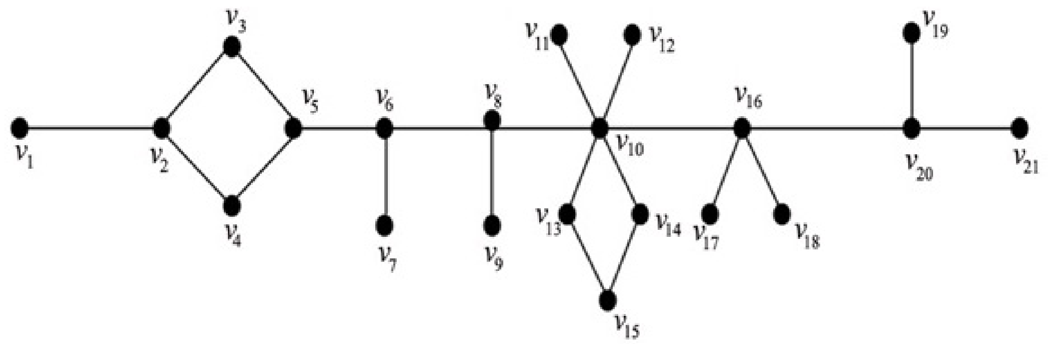

Figure 6.1: G

For the above graph, $S_1 = \{v_2, v_3, v_5, v_6, v_8, v_{10}\}, S_2 = \{v_8, v_{10}, v_{16}\}$ are the minimal connected MDC sets of G. Hence $\gamma_{CM\chi}(G) = 3$ and $\Gamma_{CM\chi}(G) = 6$. For the graph $G, \gamma_{M\chi}(G) = 3, \gamma_{ch}(G) = 7$ and $\gamma_M(G) = 2$.

Proposition 6.2.4: For any connected graph G, (i) $\gamma_{M\chi}(G) \leq \gamma_{CM\chi}(G)$ (ii) $\gamma_{CM\chi}(G) \leq \gamma_{ch}(G) \leq \gamma_{cch}(G)$ and (iii) $\gamma_c(G) \leq \gamma_{CM\chi}(G)$.

Proof: (i) Since any connected MDC set is a MDC set of G, $\gamma_{M\chi}(G) \leq \gamma_{CM\chi}(G)$.

(ii) Since any connected dom-chromatic set of G is dom-chromatic set, $\gamma_{cch}(G) \geq \gamma_{ch}(G)$. Also since every dom-chromatic set contains a connected MDC set of G, $\gamma_{CM\chi}(G) \leq \gamma_{ch}(G)$. Hence $\gamma_{CM\chi}(G) \leq \gamma_{ch}(G) \leq \gamma_{cch}(G)$.

(iii) Since any γ_c- set dominates the full vertex set of G and any $\gamma_{CM\chi}$-set dominates half of the vertices and it preserving the chromatic set, $\gamma_c(G) \leq \gamma_{CM\chi}(G)$. ∎

Observations 6.2.5: (i) If the graph G is vertex color critical graph then $\gamma_{CM\chi}(G) = \gamma_{M\chi}(G) = p$.

(ii) If G is a triangle free graph with $\chi(G) \geq 5$, $\gamma_{CM\chi}(G) \geq 5$.

(iii) For any bipartite graph with dominating edge, $\gamma_{CM\chi}(G) = 2$.

(iv) If a connected graph G has at least one full degree vertex then $\gamma_C(G) < \gamma_{CM\chi}(G)$. For example, $G = K_{1,p-1}, \gamma_C(G) = 1$ and $\gamma_{CM\chi}(G) = 2$.

(v) For any vertex color critical graph G, $\gamma_C(G) < \gamma_{CM\chi}(G)$.

(vii) If a connected graph G with at least one majority dominating vertex v then $\gamma_{CM\chi}(G) = \gamma_C(G)$.

For example, $G = D_{r,s}, r \leq s, \gamma_C(G) = 2$ and $\gamma_{CM\chi}(G) = 2$.

Results 6.2.6: (i) For $G = D_{r,s}, K_{1,p-1}, p \geq 2$, $\gamma_{CM\chi}(G) = 2$.

(ii) Let $G = K_{m,n}$. Then $\gamma_{CM\chi}(G) = 2$.

(iii) Let $G = K_p$ be a complete graph. Then $\gamma_{CM\chi}(G) = p$.

(iv) For any caterpillar graph $G, \gamma_{CM\chi}(G) = \lceil \frac{p}{4} \rceil - 1$.

Proposition 6.2.7: For any cycle $G = C_p, p \geq 8$,

$$\gamma_{CM\chi}(G) = \begin{cases} p, \text{ if } p \text{ is odd} \\ \lceil \frac{p}{2} \rceil - 2, \text{ if } p \text{ is even.} \end{cases}$$

Proof: Let $V(G) = \{v_1, v_2, \cdots, v_p\}$ be the vertex set of G. For a cycle $G = C_p$, $\chi(G) = 2$, if p is even and $\chi(G) = 3$, if p is odd.

Case (i): Let p be odd. Then the graph $G = C_p$ becomes an odd cycle. Since the graph G is vertex color critical, by the propsition (2.3.5), $\gamma_{M\chi}(G) = p$. It implies that $\gamma_{M\chi}(G) = |S| = |\{v_1, v_2, \cdots, v_p\}|$ where S is a MDC set of G and the induced subgraph $\langle S \rangle$ is connected. Therefore S is a connected MDC set of G. Thus $\gamma_{CM\chi}(G) = p$.

Case (ii): Let p be an even. Let $S = \{v_1, v_2, \cdots, v_t\}$ be any set with $|S| = t = \lceil \frac{p}{2} \rceil - 2$ and $d(v_i, v_{i+1}) = 1, i = 1, 2, \cdots, (t-1)$. Then $|N[S]| = \lceil \frac{p}{2} \rceil - 2 + 2 = \lceil \frac{p}{2} \rceil$. Since $\chi(G) = 2, \chi(\langle S \rangle) = \chi(G)$. Therefore S is a $\gamma_{M\chi}$- set of G. Since $d(v_i, v_{i+1}) = 1$, the vertices of S are in consecutive. Thus, the induced subgraph $\langle S \rangle$ is connected. Hence S is a $\gamma_{CM\chi}$- set of G and $\gamma_{CM\chi}(G) \leq \lceil \frac{p}{2} \rceil - 2$.

Let $S' = S - \{v\}$ with $S' = \lceil \frac{p}{2} \rceil - 3$. Then $|N[S']| = \lceil \frac{p}{2} \rceil - 3 + 2 = \lceil \frac{p}{2} \rceil - 1 < \lceil \frac{p}{2} \rceil$. Hence the set S' will not be a γ_M-set of G. Therefore $\gamma_{CM\chi}(G) > |S| = \lceil \frac{p}{2} \rceil - 3$ and $\gamma_{CM\chi}(G) \geq \lceil \frac{p}{2} \rceil - 2$. Hence, $\gamma_{CM\chi}(G) = \lceil \frac{p}{2} \rceil - 2$. ∎

Proposition 6.2.8: Let G be a path $P_p, p \geq 7$. Then $\gamma_{CM_\chi}(G) = \lceil \frac{p}{2} \rceil - 2$.

Proof: From the similar arguments as in case (ii) of proposition (6.2.7), $\gamma_{CM_\chi}(G) = \lceil \frac{p}{2} \rceil - 2$. ∎

6.3 γ_{CM_χ} for Product and Corona Graphs

The particular value of γ_{CM_χ} for Corona graph and product graph such as Grid, Torus and Cylinder are discussed in this section.

Proposition 6.3.1: Let the product graph $G = P_2 \times P_j, j \geq 5$, be a Grid. Then $\gamma_{CM_\chi}(G) = \lceil \frac{p}{4} \rceil - 1$.

Proof: Let $G = P_2 \times P_j, j \geq 5$. Let $\{v_1, v_2, \ldots, v_j, u_1, u_2, \ldots, u_j\}$ be the vertex set of $V(G)$ in the first and second row respectively and $\chi(G) = 2$. Consider the set $S = \{v_2, v_3, \ldots, v_t\}$ with $|S| = \lceil \frac{p}{4} \rceil - 1$ such that $d(v_i, v_j) = 1, i \neq j$. Then $|N[S]| = 2t + 2 = 2\left(\lceil \frac{p}{4} \rceil - 1\right) + 2 = 2\lceil \frac{p}{4} \rceil = \lceil \frac{p}{2} \rceil \geq \lceil \frac{p}{2} \rceil$. It implies that the set S is the majority dominating set of G. Since $d(v_i, v_j) = 1, \chi(\langle S \rangle) = 2 = \chi(G)$ and the induced subgraph $\langle S \rangle$ is connected. Hence S is connected MDC set of G and $\gamma_{CM_\chi}(G) \leq |S| = \lceil \frac{p}{4} \rceil - 1$.

Suppose the set $|S'| = |S| - 1 = \lceil \frac{p}{4} \rceil - 2$. Then $|N[S']| = 2t + 2 = 2(\lceil \frac{p}{4} \rceil - 2) + 2 = 2\lceil \frac{p}{4} \rceil - 2 < \lceil \frac{p}{2} \rceil$. It implies that the set S' would not be a connected majority dom-chromatic set for G. Hence $\gamma_{CM\chi}(G) > |S'| = \lceil \frac{p}{4} \rceil - 2$ or $\gamma_{CM\chi}(G) \geq \lceil \frac{p}{4} \rceil - 1$. Thus, $\gamma_{CM\chi}(G) = \lceil \frac{p}{4} \rceil - 1$. ■

Remark 6.3.2: For the graph $G = P_2 \times P_j, j < 5, \gamma_{CM\chi}(G) = 2$.

Proposition 6.3.3: For a grid graph $G = P_3 \times P_j, j \geq 4$,

$$\gamma_{CM\chi}(G) = \begin{cases} \lfloor \frac{p}{6} \rfloor, \text{ if } j \text{ is odd} \\ \frac{p}{6}, \text{ if } j \text{ is even.} \end{cases}$$

Proof: Let $G = P_3 \times P_j, j \geq 4$ and $V(G) = \{v_{11}, v_{12}, \ldots, v_{1j}, v_{21}, v_{22}, \ldots, v_{2j}, v_{31}, v_{32}, \ldots, v_{3j}\}$ be the vertex set of first, second and third row respectively and $|V(G)| = p = 3j, j \geq 4$. Then $\chi(G) = 2$.

Case (i): Let j be odd. Consider the set $S \subseteq V(G), S = \{v_{22}, v_{23}, \ldots, v_{2t}\}$ with $|S| = \lfloor \frac{p}{6} \rfloor$. Now, $|N[S]| = 3t + 2 = 3\lfloor \frac{p}{6} \rfloor + 2 = \frac{p}{2} + 2 > \lceil \frac{p}{2} \rceil$. It implies that $\frac{p}{2} = \frac{3j}{2}$ and S is a majority dominating set of G. Since every vertex in S is of distance one, $\chi(\langle S \rangle) = 2 = \chi(G)$ and the induced subgraph $\langle S \rangle$ of G is connected. Therefore, the set S is a $\gamma_{CM\chi}$- set of G. Hence $\gamma_{CM\chi}(G) \leq |S| = \lfloor \frac{p}{6} \rfloor$.

Let, $S' = S - \{v_{2j}\}$ with $|S'| = \lfloor \frac{p}{6} \rfloor - 1$. Then $|N[S']| = 3(t+2) = 3\left(\lfloor \frac{p}{6} \rfloor\right) + 2 = 3\lfloor \frac{p}{6} \rfloor - 1 = \frac{p}{2} - 1 < \lceil \frac{p}{2} \rceil$. Hence S' would not be a majority dominating set of G. Therefore $\gamma_{CM_\chi}(G) > |S'| = \lfloor \frac{p}{6} \rfloor - 1$. Then $\gamma_{CM_\chi}(G) \geq \lfloor \frac{p}{6} \rfloor$. Hence, $\gamma_{CM_\chi}(G) = \lfloor \frac{p}{6} \rfloor$, if j is odd.

Case (ii): Let j be even. Let $S = \{v_{21}, v_{22}, \ldots, v_{2t}\} \subseteq V(G)$ with $|S| = t = \frac{p}{6}$. Now, $|N[S]| = 3t + 2 = 3\left(\frac{p}{6}\right) + 2 > \lceil \frac{p}{2} \rceil$. It implies that S is a majority dominating set of G and the induced subgraph $\langle S \rangle$ of G is connected. Therefore, the set S is a γ_{CM_χ}- set of G and $\gamma_{CM_\chi}(G) \leq |S| = \frac{p}{6}$. Applying the same arguments as in case (i), we get, $\gamma_{CM_\chi}(G) \geq \frac{p}{6}$. Thus $\gamma_{CM_\chi}(G) = \frac{p}{6}$, if j is even. ∎

Corollary 6.3.4: For a grid graph $G = P_4 \times P_j, j \geq 3$,

$$\gamma_{CM_\chi}(G) = \begin{cases} \lceil \frac{p}{6} \rceil, & \text{if } p \equiv 0 \ (mod\ 6) \\ \lceil \frac{p}{6} \rceil - 1, & \text{if } p \equiv 2, 4 \ (mod\ 6). \end{cases}$$

Proof: By similar arguments as in proposition (6.3.3), we obtain the result. ∎

Proposition 6.3.5: Let the product graph $G = C_3 \times P_j, j \geq 3$ be a cylinder. Then $\gamma_{CM_\chi}(G) = \lceil \frac{p}{6} \rceil + 1$.

Proof: Consider the graph $G = C_3 \times P_j, j \geq 3$. Let $V(G) = \{v_{11}, v_{12}, \ldots, v_{1j}, v_{21}, v_{22}, \ldots, v_{2j}, v_{31}, v_{32}, \ldots, v_{3j}\}$ be the vertex set of G and $|V(G)| = p = 3j$. Since the graph G contains a triangle,

$\chi(G) = 3$. Consider the subset $S = \{v_{12}, v_{22}, v_{32}, v_{13}, \ldots, v_{1j}\} \subseteq V(G)$ with $|S| = \lceil \frac{p}{6} \rceil + 1$ such that the vertices $\{v_{12}, v_{22}, v_{32}\}$ forms a triangle and $d(v_{1i}, v_{1j}) = 1, i \neq j$. Then $|N[S]| = 3(t-2) + 4 = 3\left(\lceil \frac{p}{6} \rceil - 1\right) + 4 = \lceil \frac{p}{2} \rceil + 1 \geq \lceil \frac{p}{2} \rceil$. It implies that the set S is the majority dominating set of G. Since the set S contains a triangle, $\chi(\langle S \rangle) = 3 = \chi(G)$ and since $d(v_{1i}, v_{1j}) = 1$, the induced subgraph $\langle S \rangle$ is connected. Hence S is the connected MDC set of G and $\gamma_{CM\chi}(G) \leq \lceil \frac{p}{6} \rceil + 1$.

Suppose the set $|S'| = |S| - 1 = \lceil \frac{p}{6} \rceil$. Then $|N[S']| = 3\left(\lceil \frac{p}{6} \rceil - 2\right) + 4 = \lceil \frac{p}{2} \rceil - 2 < \lceil \frac{p}{2} \rceil$. Hence the set S' will not be a connected MDC set for G. Therefore, $\gamma_{CM\chi}(G) > \lceil \frac{p}{6} \rceil$ or $\gamma_{CM\chi}(G) \geq \lceil \frac{p}{6} \rceil + 1$. Thus, $\gamma_{CM\chi}(G) = \lceil \frac{p}{6} \rceil + 1$. ∎

Proposition 6.3.6: Let $G = C_4 \times P_j, j \geq 4$ be a cylinder. Then

$$\gamma_{CM\chi}(G) = \begin{cases} \frac{p}{6}, \text{ if } j \equiv 0 \ (mod\ 3) \\ \lceil \frac{p}{6} \rceil - 1, \text{ if } j \equiv 1, 2 \ (mod\ 3). \end{cases}$$

Proof: Let $G = C_4 \times P_j, j \geq 4$. Let $V(G) = \{v_{11}, v_{12}, \ldots, v_{1j}, v_{21}, v_{22}, \ldots, v_{2j}, v_{31}, v_{32}, \ldots, v_{3j}, v_{41}, v_{42}, \ldots, v_{4j}\}$ be the vertex set of first, second, third and fourth rows of G respectively and $\chi(G) = 2$.

Case (i): Let $j \equiv 0(mod\ 3)$. Then $p = 0(mod\ 6)$ and $p = 6r$. Let $S = \{v_{11}, v_{12}, \ldots, v_{1(\frac{p}{6})}\} \subseteq V(G)$ with $|S| = \frac{p}{6}$. Now, $|N[S]| = 3|S| + 1 = 3\left(\frac{p}{6}\right) + 1 = \frac{p}{2} + 1 \geq \lceil\frac{p}{2}\rceil$. Therefore S is a majority dominating set of G. Since every vertex of S is of distance one, $\chi(\langle S\rangle) = 2 = \chi(G)$ and the induced subgraph $\langle S\rangle$ of G is connected. Therefore, the set S is a $\gamma_{CM\chi}$- set of G. Hence, $\gamma_{CM\chi}(G) \leq |S| = \frac{p}{6}$. Suppose, let $S' = S - \{v_1 j\}$ with $|S'| = \frac{p}{6} - 1$. Then $|N[S']| = 3|S'| - 3 = 3(\frac{p}{6} - 1) - 3 < \lceil\frac{p}{2}\rceil$. It implies that S' could not be majority dominating set of G and $\gamma_{CM\chi}(G) > |S'| = \frac{p}{6} - 1$. Therefore $\gamma_{CM\chi}(G) \geq \frac{p}{6}$. Thus, $\gamma_{CM\chi}(G) = \frac{p}{6}$, if $j \equiv 0(mod\ 3)$.

Case (ii): Let $j \equiv 1, 2(mod\ 3)$. Then $p = 2(mod\ 6)$ such that p is divided by 4. Let $S = \{v_{11}, v_{12}, \ldots, v_{1t}\} \subseteq V(G)$ with $|S| = t = \lceil\frac{p}{6}\rceil - 1$. Now, $|N[S]| = 3|S| + 1 = 3\{\lceil\frac{p}{6}\rceil - 1\} + 1$. Let $p = 6r + 2$. Then $|N[S]| = 3\left[\frac{6r+2}{6} - 1\right] + 1 = \frac{6r+2}{2} - 2 = 3r - 1 = 3\left(\frac{p-2}{6}\right) - 1 \geq \lceil\frac{p}{2}\rceil$.

Let $p = 4(mod\ 6)$ such that p is divided by 4. Let $S = \{v_{12}, v_{13}, \ldots, v_{1t}\} \subseteq V(G)$ with $|S| = t = \lceil\frac{p}{6}\rceil - 1$. Now, $|N[S]| = 3|S| + 1 = 3\{\lceil\frac{p}{6}\rceil - 1\} + 2$. Let $p = 6r + 4$. Then $|N[S]| = 3\left[\frac{6r+4}{6} - 1\right] + 2 = \frac{6r+4}{2} - 1 = 3r + 1 = 3\left(\frac{p-4}{6}\right) - 1 > \lceil\frac{p}{2}\rceil$. It implies that S be a majority dominating set of G. Since all vertices of S are of distance one, the vertex set of S is connected. Hence S is connected MDC set of G.

Therefore $\gamma_{CM\chi}(G) \leq \lceil \frac{p}{6} \rceil - 1$.

Now, suppose $S' = S - \{v_{1j}\}$ with $|S'| = \lceil \frac{p}{6} \rceil - 2$. Then $|N[S']| = 3|S'| - 3 = 3(\lceil \frac{p}{6} \rceil - 1) - 3 < \lceil \frac{p}{2} \rceil$. It implies that S' could not be a majority dominating set of G and $\gamma_{CM\chi}(G) > |S'| = \lceil \frac{p}{6} \rceil - 2$. Therefore $\gamma_{CM\chi}(G) \geq \lceil \frac{p}{6} \rceil - 1$. Hence, $\gamma_{CM\chi}(G) = \lceil \frac{p}{6} \rceil - 1$ if $j = 1, 2(mod\ 6)$. ∎

Corollary 6.3.7: For the graph $G = C_5 \times P_j, j \geq 6$ be a cylinder,

$$\gamma_{CM\chi}(G) = \begin{cases} \lceil \frac{p}{6} \rceil, & \text{if } p \equiv 1, 2(mod\ 6) \\ \lceil \frac{p}{6} \rceil + 1, & \text{if } p \equiv 0.3, 4, 5(mod\ 6). \end{cases}$$

The following results are concerned with the corona graph structure for various two graphs.

Proposition 6.3.8: Let $G = C_t \circ K_j, t = 6$ and $j \geq 2$ be a corona graph with a cycle C_t and a complete graph K_j. Then $\gamma_{CM\chi}(G) = \lceil \frac{p}{6} \rceil + 2$.

Proof: Let $V(G) = \{v_1, v_{11}, v_{12}, \ldots, v_{1j}, v_2, v_{21}, v_{22}, \ldots, v_{2j}, \ldots, v_6, v_{61}, \ldots, v_{6j}\}$ with $|V(G)| = p = 6j+6$, where $v_i \in C_t$ and $v_{ij} \in K_j, i = 1, \ldots, 6, j \geq 2$. Let $S = \{v_1, v_{11}, v_{12}, \ldots, v_{1(\frac{p}{6}-1)}, v_2, \ldots, v_{\frac{t}{2}}\} \subseteq V(G)$ with $|S| = \frac{p}{6} - 1 + 3 = \lceil \frac{p}{6} \rceil + 2$. Now, $|N[S]| \geq \sum_{i=1}^{6} d(v_{ij}) + \sum_{i=1}^{\frac{t}{2}} d(v_i) \geq$

$3\lceil\frac{p}{6}\rceil + 2 \geq \lceil\frac{p}{2}\rceil + 2 \geq \lceil\frac{p}{2}\rceil = 3(j+1)$. It implies that S is a majority dominating set of G. Since G contains a complete graph $K_j, j = \frac{p}{6} - 1, \chi(\langle S\rangle) = \frac{p}{6} - 1 = \chi(G)$. Since all the vertices of S are connected, the S is a $\gamma_{CM\chi}$- set of G. Hence $\gamma_{CM\chi}(G) \leq \lceil\frac{p}{6}\rceil + 2$. Now, suppose $S' = S - \{v_i\}$ with $|S'| = \lceil\frac{p}{6}\rceil + 2 - 1$. Then $|N[S']| = 3d(v_i) + 2 - d(v_i) = \frac{p}{3} + 2 < \lceil\frac{p}{2}\rceil$. It implies that S' could not be a majority dominating set of G. Hence $\gamma_{CM\chi}(G) < |S'| = \lceil\frac{p}{6}\rceil + 1$. Therefore $\gamma_{CM\chi}(G) \geq \lceil\frac{p}{6}\rceil + 2$. Thus, we get $\gamma_{CM\chi}(G) = \lceil\frac{p}{6}\rceil + 2$. ■

Proposition 6.3.9: Let G be any vertex color critical graph of t vertices and H be any graph with order $s \leq t$ Let $G' = G \circ H$ be any corona graph. Then $\gamma_{CM\chi}(G') = \gamma_{CM\chi}(G) = t$.

Corollary 6.3.10: For a Corona graph $G = K_t \circ K_{m,n}, m \leq n$, $\gamma_{CM\chi}(G) = t$, where K_t and $K_{m,n}$ are complete and complete bipartite graph respectively.

Theorem 6.3.11: Let $G = K_m \circ K_n$ be a Corona graph with Complete graphs of order m and n. Then $\gamma_{CM\chi}(G) = m$, if $m > n$ and $\gamma_{CM\chi}(G) = \lceil\frac{p}{m}\rceil + 2$, if $m \leq n$.

Proof: Consider the graph $G = K_m \circ K_n$ with $p = m(n+1)$. Let

$\{v_{11}, v_{12}, \ldots, v_{1n}\}, \{v_{21}, v_{22}, \ldots, v_{2n}\}, \ldots, \{v_{m1}, v_{m2}, \ldots, v_{mn}\}$ be the vertex set of G.

Case (i): When $m > n$. Then $\chi(K_m) > \chi(K_n)$ and $\chi(K_m) = cpn(K_m) = \chi(G)$. Hence any $\gamma_{M\chi}$- set must contain the full vertex set of K_m. Let $S = \{v_{11}, v_{21}, \ldots, v_{m1}\} \subseteq V(K_m)$ with $|S| = m$. Then $\chi(\langle S \rangle) = \chi(G)$. Since the graph G contains K_m as a central graph, it dominates all vertices in m copies of K_n. Then $|N[S]| > \lceil \frac{p}{2} \rceil$ and since the set S contains the vertices of K_m, the induced subgraph $\langle S \rangle$ is connected and S is connected MDC set of G. Therefore $\gamma_{CM\chi}(G) = m$, if $m > n$.

Case (ii): When $m \leq n$. Then $\chi(K_m) < \chi(K_n)$ and $\chi(K_n) = cpn(K_n) = \chi(G)$. Hence any $\gamma_{M\chi}$- set must contain the full vertex set of any one copy of K_n. Let $S = \{v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, \ldots, v_{t1}\}$, where $\{v_{11}, v_{12}, \ldots, v_{1n}\} \subseteq V(K_n)$ and other vertices are from K_m such that $d(v_{i1}, v_{j1}) = 1, i \neq j$ with $|S| = \lceil \frac{p}{m} \rceil + 2$. Then $\chi(\langle S \rangle) = \chi(G)$. Now, $|N[S]| = n + \left(\lceil \frac{p}{m} \rceil + 2 - n\right) n = n + (n + 3 - n)n = 4n \geq \lceil \frac{p}{2} \rceil$. Since $d(v_{i1}, v_{j1}) = 1, i \neq j$, the induced subgraph $\langle S \rangle$ is connected and S is connected MDC set of G. Therefore $\gamma_{CM\chi}(G) \leq \lceil \frac{p}{m} \rceil + 2$.

Suppose the set $|S'| = |S| - 1 = \lceil \frac{p}{m} \rceil + 1$. Then $|N[S']| = n + (\lceil \frac{p}{m} \rceil + 1 - n)n = n + (n + 1 + 1 - n)n = 3n < \lceil \frac{p}{2} \rceil$. It implies that the set S' wouldn't be a connected MDC set of G. Therefore $\gamma_{CM\chi}(G) > \lceil \frac{p}{m} \rceil + 1$ or $\gamma_{CM\chi}(G) \geq \lceil \frac{p}{m} \rceil + 2$. Thus, $\gamma_{CM\chi}(G) \leq \lceil \frac{p}{m} \rceil + 2$, if $m \leq n$. ∎

Theorem 6.3.12: Let $G = C_m \circ C_n$ with $m \geq 3, n \geq 4$, be a Corona graph with two cycles pf order m and n. Then $\gamma_{CM\chi}(G) = \lceil \frac{m}{2} \rceil + 2$.

Proof: Let $G = C_m \circ C_n$ be a graph with $p = m(n + 1)$. Let $\{v_{11}, v_{12}, \ldots, v_{1n}\}, \{v_{21}, v_{22}, \ldots, v_{2n}\}, \ldots, \{v_{m1}, v_{m2}, \ldots, v_{mn}\}$ be the vertex set of G. Since G contains a triangle, $\chi(G) = 3$. Let $S = \{v_{11}, v_{12}, v_{13}, v_{21}, \ldots, v_{t1}\} \subseteq V(G)$, where $\langle v_{11}, v_{12}, v_{13} \rangle$ a triangle and $d(v_{1i}, v_{j1}) = 1, i \neq j$ with $|S| = t = \lceil \frac{m}{2} \rceil + 2$. Then $\chi(\langle S \rangle) = \chi(G)$. Now, $|N[S]| = \left(\lceil \frac{m}{2} \rceil + 2\right)(n+1) - 2(n+1) + 2 = \lceil \frac{p}{2}(n+1) \rceil + 2 \geq \lceil \frac{p}{2} \rceil$. S is connected MDC set of G. Therefore $\gamma_{CM\chi}(G) \leq \lceil \frac{m}{2} \rceil + 2$.

Suppose the set $|S'| = |S| - 1 = \lceil \frac{m}{2} \rceil + 2$. Then $|N[S']| = \left(\lceil \frac{m}{2} \rceil + 1\right)(n + 1) - 2(n + 1) + 2 = \lceil \frac{p}{2}(n + 1) \rceil - (n + 1) + 2 < \lceil \frac{p}{2} \rceil$. It implies that the set S' wouldn't be a connected MDC set of G. Therefore $\gamma_{CM\chi}(G) > \lceil \frac{m}{2} \rceil + 1$ or $\gamma_{CM\chi}(G) \geq \lceil \frac{m}{2} \rceil + 2$. Thus, $\gamma_{CM\chi}(G) = \lceil \frac{m}{2} \rceil + 2$. ∎

Corollary 6.3.13: Let $G = C_m \circ C_n$ with $m \geq 3, n = 3$ be a graph with Cycles of order m and n. Then

(i) $\gamma_{CM\chi}(G) = \lceil \frac{m}{2} \rceil + 2$, if m is odd,

(ii) $\gamma_{CM\chi}(G) = \lceil \frac{m}{2} \rceil + 3$, if m is even.

Theorem 6.3.14: Let G be a connected graph with $\chi(G) = \gamma_{M\chi}(G)$. Then $\gamma_{CM\chi}(G) = \gamma_{M\chi}(G)$.

Proof: Let $\chi(G) = \gamma_{M\chi}(G)$ and S be a $\gamma_{M\chi}$- set of G. Suppose $\chi(G) = \chi(\langle S \rangle) = k$ then $\langle S \rangle = K_k$. It implies that $\gamma_{M\chi}(G) = |S| = k$. The induced subgraph $\langle S \rangle$ is connected and $\gamma_{CM\chi}(G) = k$. Hence $\gamma_{CM\chi}(G) = \gamma_{M\chi}(G)$. ∎

Proposition 6.3.15: Let G be a connected graph which contains all its vertices of degree $d(v_i) < \lceil \frac{p}{2} \rceil - 1$. Then $\gamma_{CM\chi}(G) < \gamma_C(G)$.

6.4 Characterization Theorems on $\gamma_{CM\chi}(G)$

In this section, the necessary and sufficient conditions for the connected MDC number $\gamma_{CM\chi}(G)$ and bounds on $\gamma_{CM\chi}(G)$ are investigated for the graphs.

Theorem 6.4.1: Let G be a connected graph $p \geq 2$. Then G is vertex color critical if and only if $\gamma_{CM_\chi}(G) = p$.

Proof: Let G be a vertex color critical graph. Then by the observation [2.2.4](ii), $\gamma_{M_\chi}(G) = |S| = p$, where S is a γ_{M_χ}- set of G. Since $\gamma_{M_\chi}(G) = p$, all vertices are in consecutive and $\langle S \rangle$ is connected. It implies that S is a connected MDC set of G and $\gamma_{CM_\chi}(G) = p$. The converse is obvious. ∎

Theorem 6.4.2: Let G be a tree with p vertices. Then $\gamma_{CM_\chi}(G) = \gamma_C(G)$ if and only if $diam(G) = 3$, where $\gamma_C(G)$ is the connected domination number of a graph G.

Proof: Let $\gamma_{CM_\chi}(G) = \gamma_C(G)$. Let S and S' be the γ_{CM_χ}-set and γ_C-set of G.

Case (i): Suppose $diam(G) = 1$ then the graph structures become $G = K_2$. Then by result (6.2.6)(iii), $\gamma_{CM_\chi}(G) = 2$. The γ_C-set of G is $S' = \{v\}$. It implies that $\gamma_C(G) = |S'| = 1 < \gamma_{CM_\chi}(G)$, which is a contradiction to the assumption.

Case (ii): Suppose $diam(G) = 2$ then the graphs are like $G = P_3$ and $G = K_{1,p-1}$. For $G = P_3$, the γ_{CM_χ}-set is $S = \{v_1, v_2\}$ and

$\gamma_{CM_\chi}(G) = 2$. The γ_C-set is $S' = \{v_2\}$ and $\gamma_C(G) = 1 < \gamma_{CM_\chi}(G)$. It is a contradiction to the assumption. From cases (i) and (ii) we obtain then $diam \geq 3$.

Case (iii): Suppose $diam(G) \geq 4$. Then the graph structures being $G = P_p, p \geq 5$. By the result (6.2.8), (1.3.43)(iv), $\gamma_{CM_\chi}(G) = \lceil \frac{p}{2} \rceil - 2 < \gamma_C(G) = p - 2$. Hence the condition $\gamma_{CM_\chi}(G) < \gamma_C(G)$ gives the contradicton to our assumption. Hence the graph G with $diam = 3$ is true if $\gamma_{CM_\chi}(G) = \gamma_C G$.

Conversely, if $diam(G) = 3$ then the graph G has a dominating edge $e = uv$ and both u and v have some pendants. Let $S = \{u, v\} \subseteq V(G)$ with $d(u, v) = 1$. Then $\chi(\langle S \rangle) = 2 = \chi(G)$ and $\langle S \rangle$ is connected. Clearly $|N[S]| = p > \lceil \frac{p}{2} \rceil$, then S is both γ_C- set and γ_{CM_χ}-set of G. Hence, $\gamma_{CM_\chi}(G) = 2 = \gamma_C(G)$. ∎

Theorem 6.4.3: For even cycle $G = C_p, \gamma_{CM_\chi}(G) = \gamma_{M_\chi}(G)$ if and only if $G = C_p, p \leq 10$.

Proof: Let $\gamma_{CM_\chi}(G) = \gamma_{M_\chi}(G)$. For even cycle, $\chi(G) = 2$. By the proposition (6.2.7), $\gamma_{CM_\chi}(G) = \lceil \frac{p}{2} \rceil - 2$, if p is even. The γ_{CM_χ}-of even cycles with $p \geq 4$ are $2, 2, 2, 3, 4, 5, \ldots,$ By the proposition (2.3.3), $\gamma_{M_\chi}(G) = 2$, if $4 \leq p \leq 8$ and $\gamma_{CM}(G) = 3$, if $p = 9, 10$.

Also it gives $\gamma_{CM\chi}(G) = \gamma_{M\chi}(G)$ if $p \leq 10$. Suppose $p \geq 11$. Let G be an even cycle and $\gamma_{CM\chi} = \gamma_{M\chi}$. Then $\chi(G) = 2$. By the proposition(6.2.8) and (2.3.4) $\gamma_{CM\chi}(G) = \lceil \frac{p}{2} \rceil - 2$ and $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil$. It implies that $\gamma_{M\chi} < \gamma_{CM\chi}(G)$, which is a contridiction to the assumption. Hence $G = C_p, p \leq 10$.

Conversely for $G = C_p, \gamma_{CM\chi}(G) = 2, 2, 2, 3, \ldots, 3 \leq p \leq 10$ and $\gamma_{M\chi}(G) = 2, 2, \ldots, 3 \leq p \leq 10$. Hence $\gamma_{CM\chi}(G) = \gamma_{M\chi}(G)$. ∎

Remark 6.4.4: For any path, $\gamma_{CM\chi}(G) = \gamma_{cch}(G)$ if and only if $G = P_p, p = 3, 4$ where $\gamma_{Cch}(G)$ is connected dom-chromatic number of G.

Theorem 6.4.5: Let T be a tree with p vertices. Then $\gamma_{CM\chi}(T) = \gamma_{M\chi}(T)$ if and only if one of the following conditions holds.

(i) T has a vertex of degree $d(u) \geq \lceil \frac{p}{2} \rceil - 1$,

(ii) Each non-pendant vertex is adjacent to a pendant vertex and

(iii) $diam(T) \leq 9$.

Proof: Let $\gamma_{CM\chi}(T) = \gamma_{M\chi}(T)$. Then S is a $\gamma_{M\chi}$ and $\gamma_{CM\chi}$- set of T with same cardinality. If $diam(T) = 1$ then T becomes K_2 and $S = \{u, v\}$ is the $\gamma_{M\chi}$ and $\gamma_{CM\chi}$- set of T where $d(u) \geq \lceil \frac{p}{2} \rceil - 1$. If

$diam(T) = 2$ then the graph structures like $P_3, K_{1,p-1}$. Let $S = \{u, v\}$ be set with $d(u, v) = 1$ where u is non-pendent and v is pendent of G. Then $d(u) = p - 1 \geq \lceil \frac{p}{2} \rceil - 1$. Therefore the tree T satisfies the conditions (i) and (ii). Suppose $diam(T) = 3$. Then T has a dominating edge $e = uv$ and $\chi(T) = 2$. Let $S = \{u, v\}$ be the $\gamma_{M\chi}$ and $\gamma_{CM\chi}$- set of T with $d(u) \geq \lceil \frac{p}{2} \rceil - 1$ and $d(v) = 1$. Hence both vertices u and v are adjacent and the non pendent vertex u is alwaus adjacent to a pendent v and vice versa. Hence condition (i) and (ii) holds.

Next consider a tree T with $diam \geq 4$. Then T path P_p with $4 \leq diam(T) \leq 7$. By the corollary (2.3.4), $\gamma_{M\chi}(T) = 2$ and by proposition (6.2.8), $\gamma_{CM\chi}(T) = 2$. Hence this result is true only if $p = 5, 6, 7, 8$ and $diam(T) = 4, 5, 6, 7$. Thus $4 \leq diam(T) \leq 7 \leq 9$ and the condition (iii) holds. Let $S = \{v_i, v_j, v_k\}$ be the $\gamma_{M\chi}$- set of T such that $d(v_i, v_j) = 1$. Then $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $\chi(\langle S \rangle) = 2 = \chi(T)$. It implies that $\gamma_{M\chi}(T) = |S| = 3$. Since $d(v_i, v_j) = 1 = d(v_j, v_k)$, the induced subgraph $\langle S \rangle$ is connected and S is $\gamma_{CM\chi}$- set of T and $\gamma_{CM\chi}(T) = |S| = 3$. Then the tree T has $diam(T) = 8$ or 9. Hence the condition (iii) holds. Now, suppose $diam(T) \geq 10$. Then by corollary (2.3.4), $\gamma_{M\chi}(T) = \lceil \frac{p}{6} \rceil$ or $\lceil \frac{p}{6} \rceil + 1$ and by proposition (6.2.8),

$\gamma_{CM_\chi}(T) = \lceil \frac{p}{2} \rceil - 2$. It implies that $\gamma_{CM_\chi}(T) > \gamma_{M_\chi}(T)$, which is a contradiction to the assumption. Hence T has $diam \leq 9$ and the condtion (iii) holds. The converse is obvious. ∎

Theorem 6.4.6: For any tree $T, \gamma_{CM_\chi}(T) = 2$ if and only if T has for at least two vertices v_i with $d(v_i) \geq \lceil \frac{p}{2} \rceil - 2$.

Proof: Let $\gamma_{CM_\chi}(T) = 2$. Let S be a γ_{CM_χ}- set of T and $\gamma_{CM_\chi}(T) = |S| = 2$. Then $S = \{v_i, v_j\}$ with $d(v_i, v_j) = 1$. To prove that T has at least two vertices v_i with $d(v_i) \geq \lceil \frac{p}{2} \rceil - 2$. Suppose T has vertices v_i with $d(v_i) \leq \lceil \frac{p}{2} \rceil - 3$. Then $|N[S]| = d(v_i) + d(v_j) \leq \lceil \frac{p}{2} \rceil - 3 + \lceil \frac{p}{2} \rceil - 3 \leq p - 6 < \lceil \frac{p}{2} \rceil$. It implies that S is not be a majority dominating set of T with $|S| = 2$. It is a contradiction to the assumption that S is a γ_{CM_χ}- set of T. Hence T has at least two vertices v_i with $d(v_i) \geq \lceil \frac{p}{2} \rceil - 2$.

Conversely, suppose T has at least two vertices v_i and v_j with $d(v_i) \geq \lceil \frac{p}{2} \rceil - 2$, for i and j. To prove $\gamma_{CM_\chi}(T) = 2$. Let $S = \{v_i, v_j\} \subseteq V(T)$ with $d(v_i, v_j) = 1$. If $d(v_i) = \lceil \frac{p}{2} \rceil - 2 = d(v_j)$ then $|N[S]| = |N[v_i]| + |N[v_j]| = \lceil \frac{p}{2} \rceil - 2 + \lceil \frac{p}{2} \rceil - 2 = p - 4 = \lceil \frac{p}{2} \rceil$. It implies that S is a majority dominating set of T. If $d(v_i) = \lceil \frac{p}{2} \rceil - 2$ and $d(v_j) \geq 2$ then $|N[S]| = \lceil \frac{p}{2} \rceil$. Hence S is a majority dominating set of T. Since $\chi(T) = 2, \chi(\langle S \rangle) = 2$ and $\langle S \rangle$ is connected. Hence S is a connected MDC set of T and $\gamma_{CM\chi}(T) \leq |S| = 2$. Suppose $S' = \{v_i\}$ and $|S'| < |S|$. Then $|N[S']| < \lceil \frac{p}{2} \rceil$ and S' is not a majority dominating set of T. Since $\chi(T) = 2, \chi(\langle S \rangle) = 1 \neq \chi(T)$. Therefore S is not a connected MDC set of T. Hence $\gamma_{CM\chi}(T) > |S'|$ and $\gamma_{CM\chi}(T) \geq |S| = 2$. Thus, $\gamma_{CM\chi}(T) = 2$. ∎

Theorem 6.4.7: If G is a vertex color critical graph with p vertices and q edges then $\lceil \frac{p-\Delta(G)}{2} \rceil + 1 \leq \gamma_{CM\chi}(G) \leq 2q - p$.

Proof: The theorem is proved by induction on $\Delta(G)$. Since the graph G is vertex color critical, $\gamma_{CM\chi}(G) = p$. When $\Delta(G) = 1$, then the graph $G = K_2$ and $\gamma_{CM\chi}(G) = 2 = \lceil \frac{2-\Delta(G)}{2} \rceil + 1 = \lceil \frac{p-\Delta(G)}{2} \rceil + 1$. When $\Delta(G) = 2$, then the graphs are cycles C_p. Since odd cycle C_p is a vertex color critical, $\gamma_{CM\chi}(G) = p, p \geq 3$. Since $|V(G)| = p = |E(G)| = q, \gamma_{CM\chi}(G) = 2q - p$. Therefore the upper bound exists for odd cycle. Since $\Delta(G) = 2, G = K_3$ is a complete graph and $\gamma_{CM\chi}(G) = 3 = 2q - p$. When $\Delta(G) \geq 3$ and G is a vertex color critical graph then $G = K_p, p \geq 4$ is a complete graphs. Since $|E(G)| = q = \frac{p(p-1)}{2}, \gamma_{CM\chi}(G) = p < 2q - p = 2\left(\frac{p(p-1)}{2}\right) - p = (p^2 - 2p)$. Hence $\gamma_{CM\chi}(G) < 2q - p$, if $\Delta(G) \geq 3$. The lower bound is sharp for $G = K_2$ and the upper bound is sharp for $G = C_p$, odd cycle. ∎

6.5 $\gamma_{CM\chi}$ for Disconnected Graphs

In this section, results on $\gamma_{CM\chi}(G)$ are investigated for disconnected graphs.

Theorem 6.5.1: Let G be a disconnected graph with $G_1, G_2, \ldots, G_m$ components which are all vertex color critical. If any one component G_i such that $cpn(G_i) \geq \lceil \frac{p}{2} \rceil$ then $\gamma_{CM\chi}(G) = |V(G_i)|$, for any $1 \leq i \leq m$.

Proof: Let $G_1, G_2, \ldots, G_m$ be the m components which are vertex color critical. Then $\chi(\langle G_i - v \rangle) < \chi(G_i)$, for all $i = 1, 2, \ldots, m$. Let $r_1, r_2, \ldots, r_m$ be the chromatic preserving numbers of $G_1, G_2, \ldots, G_m$. Then $cpn(G) = \max\{r_1, r_2, \ldots, r_m\}$. Suppose $cpn(G) = r_i$ for any $i = 1, 2, \ldots, m$ such that $|r_i| \geq \lceil \frac{p}{2} \rceil$. Hence any $\gamma_{CM\chi}$- set S must contain the full vertex set of G_i such that $cpn(G_i) = r_i$, for any i.

Case (i): Consider $|V(G_i)| < \lceil \frac{p}{2} \rceil$. Then the set S wouldn't be a majority dominating set of G. It implies that S would be a majority dominating set by adding some vertices from different components. But the induced subgraph $\langle S \rangle$ is not connected. Hence $\gamma_{CM\chi}$ does not exist for $|V(G_i)| < \lceil \frac{p}{2} \rceil$.

Case (ii): Let $|V(G_i)| \geq \lceil \frac{p}{2} \rceil$. Since the set S contains the full vertex set of G_i, the induced subgraph $\langle S \rangle$ is connected. Also, if $|V(G_i)| \geq \lceil \frac{p}{2} \rceil$ and $cpn(G_i) = cpn(G), \chi(\langle S \rangle) = \chi(G)$. Therefore $\gamma_{CM\chi}(G) = |V(G_i)|$, for any $i \leq i \leq m$. ∎

Theorem 6.5.2: Let G be a disconnected graph with m components. If all components G_i such that $|V(G_i)| < \lceil \frac{p}{2} \rceil, i = 1, 2, \ldots, m$ then $\gamma_{CM\chi}(G)$ does not exist.

Proof: Let $S = \{v_1, v_2, \ldots, v_r\} \subseteq V(G_i)$ for $i = 1, 2, \ldots, r$ and $r \leq m$. Since each component G_i has the vertex set $|V(G_i)| < \lceil \frac{p}{2} \rceil, i = 1, 2, \ldots, m$, the set S contains vertex set of any one component G_i and some vertices from other components to satisfies the condition $|N[S]| \geq \lceil \frac{p}{2} \rceil$. Hence the induced subgraph $\langle S \rangle$ would not be connected. Therefore $\gamma_{CM\chi}(G)$ does not exist for the disconnected graph such that $|V(G_i)| < \lceil \frac{p}{2} \rceil$, for all i. ∎

Theorem 6.5.3: Let G be a disconnected graph with m components which are not vertex color critical. If any one component G_i such that $|V(G_i)| \geq \lceil \frac{p}{2} \rceil$ then $2 \leq \gamma_{CM\chi}(G) \leq \lceil \frac{p}{2} \rceil - 2$.

Proof: Let $G_1, G_2, \ldots, G_m$ be the components of a disconnected graph G. Let $\chi(G) \geq 2$ and S be the subset of any one component G_i of G such that $\chi(\langle S \rangle) = \chi(G)$ and $|V(G_i)| \geq \lceil \frac{p}{2} \rceil$. Since the components of G are not vertex color critical, $G_i's$ are not either K_p or C_p, p is odd. If $diam(G_i) = 1$ then the component G_i is a path P_2 and $\gamma_{CM\chi}(G) = 2$. If $diam(G_i) = 2$ then $G_i = P_3$ or $K_{1,2}$ and $\gamma_{CM\chi}(G) = 2$. If $diam(G_i) = 3$ then $G_i = D_{r,s}$, a double star and $G_i = P_4$ and by the corollary (2.3.4) and (2.3.1)(iv), $\gamma_{CM\chi}(G) = 2$.

Suppose $diam(G_i) \geq 4$ and G_i' are all paths. Then $G = G_1 \cup G_2 \cup \ldots \cup G_m$ where $|V(G_1)| \geq \lceil \frac{p}{2} \rceil$ and $|V(G_i)| < \lceil \frac{p}{2} \rceil$ for $i \geq 2$. Since the component G_1 has $\lceil \frac{p}{2} \rceil$ vertices and $d(u_i) \leq 2$ for $u_i \in V(G_1)$, the set $S \subseteq V(G_1)$ such that $S = \{u_2, u_3, \ldots, u_{\lceil \frac{p}{2} \rceil - 1}\}$ is a majority dom-chromatic set with $|S| = \lceil \frac{p}{2} \rceil - 2$ and $|N[S]| \geq \lceil \frac{p}{2} \rceil$. Since $d(u_i, u_j) = 1$ and G_1 is a path, the induced subgraphs $\langle S \rangle$ is connected and $\chi(\langle S \rangle) = \chi(G)$. Hence S is a connected MDC set of G and $\gamma_{CM\chi}(G) \leq \lceil \frac{p}{2} \rceil - 2$. Therefore the upper bound exists.

Suppose any one component G_1 of G which is not tree. Since $|V(G_1)| \geq \lceil \frac{p}{2} \rceil$, the set S would be a connected majority dom-chromatic set of G such that $|S| > 2$ and $\gamma_{CM\chi}(G) > 2$. Hence the bounds of $\gamma_{CM\chi}(G)$ lies between 2 and $\lceil \frac{p}{2} \rceil - 2$. Thus $2 \leq \gamma_{CM\chi}(G) \leq \lceil \frac{p}{2} - 2 \rceil$. ∎

Conclusion

The research work primarily concentrates on majority dom-chromatic set of a graph G. The researcher has related the newly defined parameters with other graph theoretical parameters and extensive works were made on this new parameters for a graph G. The exact values of major dom-chromatic number $\gamma_{M\chi}$ were determined. Algorithms and Applications to majority dom-chromatic sets were also discussed. Majority dom-chromatic partition number $d_{M\chi}(G)$, connected majority dom-chromatic number $\gamma_{CM\chi}(G)$, Edge critical, Vertex critical and Edge addition regarding MDC number were studied and many theorems were produced. Further, the researcher has also opened the gateway for doing more work with majority dom-chromatic sets by imposing more conditions on them.

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RESULTS ON MAJORITY DOM-CHROMATIC SETS OF A GRAPH

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Abstract. A majority dominating set $S \subseteq V(G)$ is said to be majority dominating chromatic set if S satisfies the condition $\chi(\langle S \rangle) = \chi(G)$. The majority dom-chromatic number $\gamma_{M\chi}(G)$ is the minimum cardinality of majority dominating chromatic set. In this article we investigated some inequalities on Majority dominating chromatic sets of a connected and disconnected graph G. Also characterization theorems and some results on majority dom-chromatic number $\gamma_{M\chi}(G)$ for a vertex color critical graph and biparte graph are determined. we established the relationship between three parameters namely $\chi(G), \gamma_M(G)$ and $\gamma_{M\chi}(G)$ for some graphs.

Keywords: Majority dominating set, Majority dominating chromatic set, Majority dom-chromatic number.
AMS Subject Classification: 05C15

1. Introduction

All the graphs $G = (V, E)$ considered here are simple, finite and undirected. The concept of domination is early discussed by Ore and Berge in 1962. Then Haynes et.al [2] defined the domination number $\gamma(G)$ as the minimum cardinality of a minimal dominating set $D \subseteq V(G)$ such that each vertex of $(V - D)$ is adjacent to some vertex in D. The majority dominating number $\gamma_M(G)$ was introduced by Joseline Manora and Swaminathan [6] is the smallest cardinality of a minimal majority dominating set $S \subseteq V(G)$ of vertices and S satisfies $|N[S]| \geq \left| \left\lceil \frac{(V(G))}{2} \right\rceil \right|$.

Janakiraman and Poobalaranjani [3] defined the dom-chromatic set as a dominating set $S \subseteq V(G)$ such that the induced subgraph $\langle S \rangle$ satisfies the property $\chi(\langle S \rangle) = \chi(G)$. The minimum cardinality of a dom-chromatic S is called dom-chromatic number and it is denoted by $\gamma_{ch(G)}$ or $\gamma_{\chi(G)}$.

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Definition 1.1. *[4] A majority dom-chromatic number $\gamma_{M\chi}(G)$ is defined as the smallest cardinality of the majority dom-chromatic set (MDC set) S of $V(G)$ if S is a majority dominating set and it satisfies the property $\chi(\langle S\rangle) = \chi(G)$.*

Results 1.2.

(i) *[4] Let $G = mK_2, m \geq 1$ with $p = 2m$. Then $\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil + 1, p \geq 2$.*

(ii) *[4] For any graph $G, \max\{\chi(G), \gamma_M(G)\} \leq \gamma_{M\chi}(G) \leq p$*
.

(iii) *[4] Let G be any graph of order p. Then $\gamma_{M\chi}(G) = p$ if and only if G is vertex χ - critical.*

(iv) *[6] For a cycle $C_p, \gamma_M(C_p) = \lceil \frac{p}{6} \rceil$.*

(v) *[6] For a path $P_p, \gamma_M(P_p) = \lceil \frac{p}{6} \rceil$.*

Definition 1.3. *[5] If a vertex with degree $d(u) \geq \lceil \frac{p}{2} \rceil - 1$ then u is called a majority dominating vertex. A full degree vertex is a majority dominating vertex but a majority dominating vertex is not a full degree vertex.*

2. Some Inequalities On Majority Dom-Chromatic Sets.

In this section, Inequality between the sum of the degrees of all vertices of a MDC set S of G and the complement of S i.e., $(V - S)$ in a graph G is discussed. We determine some inequalities such as

$|V - S| \leq \sum\limits_{v_i \in S} deg\ (v_i)$ and $|V - S| \geq \sum\limits_{v_i \in S} deg\ (v_i)$ with respect to the MDC set S of a connected graph G.

Theorem 2.1. *If S is a MDC set with two majority dominating vertices of a connected graph G then $|V - S| \leq \sum\limits_{v_i \in S} deg(v_i)$.*

Proof: Let $v_i \in V(G)$ be a majority dominating vertex such that $d(v_i) \geq \lceil \frac{p}{2} \rceil - 1$ and $S = \{v_1, v_2\}$ be a MDC set with only two majority dominating vertices of G.

Case 1. *The graph G is a tree.*

Since $d(v_i) \geq \lceil \frac{p}{2} \rceil - 1, i = 1, 2,$ for all $v_i \in S$. It implies that $\chi(G) = 2, \gamma_M(G) = 1$

$$\text{then} \sum_{v_i \in S} deg(v_i) = d(v_1) + d(v_2) \geq \left\lceil \frac{p}{2} \right\rceil - 1 + \left\lceil \frac{p}{2} \right\rceil - 1$$

$$\sum_{v_i \in S} deg(v_i) = p - 2 \text{ or } p \text{ if } \ p \text{ is even or odd}$$

$$\text{Therefore } |V - S| = p - 2 \leq \sum_{v_i \in S} deg(v_i).$$

Case 2. *The graph G is not a tree and G contains two majority dominating vertices. Then G is not complete but G consists of triangles. It implies that $\chi(G) = 3, \gamma_M(G) = 1.$*

Then $S = \{v_1, v_2, v_3\}$ be a majority dominating chromatic set of G where v_3 is joined with a majority dominating vertex v_1 or v_2 of G.

$$Therefore \sum_{v_i \in S} deg(v_i) = d(v_1) + d(v_2) + d(v_3) \geq \left\lceil \frac{p}{2} \right\rceil - 1 + \left\lceil \frac{p}{2} \right\rceil - 1 + 2$$
$$\geq p \text{ or } p + 2$$
$$Hence\ |V - S| = p - 3 < \sum_{(v_i \in S)} deg(v_i).$$

In the above cases, we obtain $|V - S| \leq \sum_{(v_i \in S)} deg(v_i)$. □

Example 2.2. *Consider the following Hajos graph with $p = 10$.*

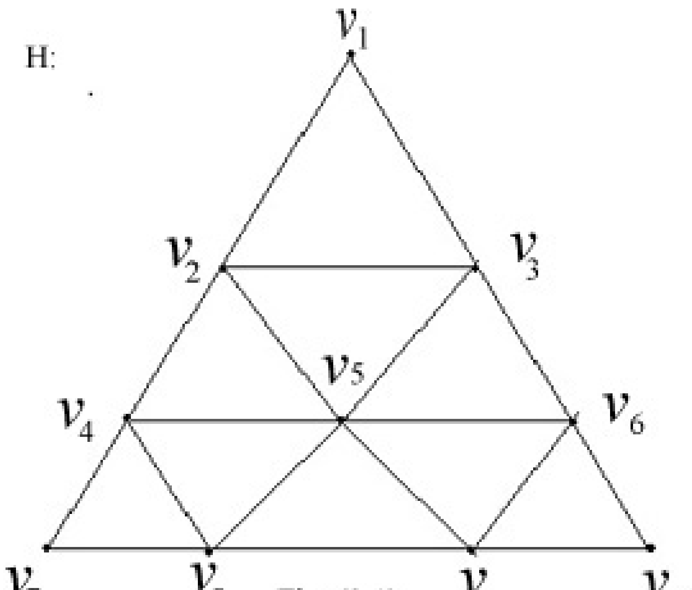

Fig. (2.1)

For the graph $H, \chi(H) = 3, \gamma_M(H) = 1$

Then $S = \{v_2, v_3, v_5\}$ is the MDC set of H.

$$\sum_{v_i \in S} deg(v_i) = d(v_2) + d(v_3) + d(v_5) = 14 \text{ and } |V - S| = 7 < \sum_{v_i \in S} deg(v_i).$$

□

Proposition 2.3. *Let G be a non-trivial connected graph with atleast one full degree vertex. If S is a majority dom-chromatic set of G then*

$$|V - S| < \sum_{u_i \in S} deg(u_i).$$

Proof: The graph G contains atleast one full degree vertex $u_1 \in V(G)$ then $d(u_1) = p - 1$.

Case 1. *The graph G is complete.*

Then the graph G contains all vertices are full degree vertices. Since $\chi(G) = p$, $S = \{u_1, u_2, \cdots, u_p\}$ is a MDC set of G.

$$\text{Therefore } |V - S| = 0 \text{ and } \sum_{u_i \in S} deg(u_i) = p(p-1) \Rightarrow |V - S| < \sum_{u_i \in S} deg(u_i).$$

Case 2. *The graph G is not complete.*

SubCase 1. *If G has only one full degree vertex u and it is not tree then G contains a triangle. Since $\chi(G) = 3, S = \{u, u_1, u_2\}$ is a MDC set of G. It implies that $|V-S| = p-3$.*

$$\sum_{u_i \in S} deg(u_i) = (p-1) + 3 + 3 = p + 5. \text{ Hence, } |V - S| < \sum_{u_i \in S} deg(u_i).$$

SubCase 2. *If G has only one full degree vertex and the graph G is a tree.*

Consider $S = \{u_1, u_2\}$ be the MDC set of G which contains a full degree vertex u_1. Then $\gamma_{M\chi}(G) = 2$. Hence $|V - S| \leq p - 2$.

$$\text{Also } \sum_{u_i \in S} deg(u_i) = d(u_1) + d(u_2) \geq p - 1 + 1 = p. \quad \text{Hence, } |V - S| < \sum_{u_i \in S} deg(u_i).$$

SubCase 3. *Suppose the graph G has two full degree vertices u_1 and u_2, then G contains a triangle. Hence, $\chi(G) = 3$. Let $S = \{u_1, u_2, u_3\}$ be a majority dominating chromatic set of G. Then $|V - S| = p - 3$.*

$$\text{Now, } \sum_{u_i \in S} deg(u_i) = (p-1) + (p-1) + 2 = 2p. \Rightarrow |V - S| < \sum_{u_i \in S} deg(u_i).$$

In all cases, the vertices of S majority dominates the graph G and also addition with its coloring number. Thus $|V - S| < \sum_{u_i \in S} deg(u_i)$. □

Corollary 2.4. *If the graph G is a vertex color critical and S is a MDC set of G then $|V - S| = 0$.*

Proof. Let G be a vertex color critical graph with p vertices. Then $S = \{v_1, v_2, \cdots, v_p\}$ is a MDC set for G. It implies that $\gamma_{M\chi}(G) = |S| = p$. Hence $|V - S| = 0$. □

Proposition 2.5. *If a connected graph G contains all vertices are majority dominating vertices then $|V - S| \leq \sum_{u_i \in S} deg(u_i)$, where S is the MDC set of G.*

Proof: Let G be a connected graph which contains only majority dominating vertices. Then $\gamma_M(G) = 1$ and $\chi(G) \geq 2$. Consider the set $S = \{u_1, u_2, \cdots, u_t\}$ be a MDC set of G. Then $|V - S| \leq p - 2$. Since G contains only majority dominating vertices, $d(u_i) \geq \lceil \frac{p}{2} \rceil - 1$,for each $u_i \in S$.

Case 1. *The graph G has no triangles. Let $S = \{u_1, u_2\}$ be a majority dominating chromatic set of G.*

$$\text{Then } \sum_{u_i \in S} deg(u_i) = d(u_1) + d(u_2) \geq \left\lceil \frac{p}{2} \right\rceil - 1 + \left\lceil \frac{p}{2} \right\rceil - 1$$

$$\sum_{u_i \in S} deg(u_i) \geq p \text{ or } p - 2 \text{ and } |V - S| = p - 2. \text{ Hence } |V - S| \leq \sum_{u_i \in S} deg(u_i).$$

Case 2. *The graph G has triangles.*

Then $\gamma_M(G) = 1$ *and* $\chi(G) \geq 3$. *It implies that* $S = \{u_1, u_2, u_3\}$ *is a MDC set of* G. *Hence* $|V - S| = p - 3$.

$$Then \sum_{u_i \in S} deg(u_i) = 3\left(\left\lceil \frac{p}{2} \right\rceil - 1\right) \geq \frac{3p}{2} \; or \; \left(\frac{3p}{2} - 3\right). Hence \; |V - S| \leq \sum_{u_i \in S} deg(u_i).$$

□

Proposition 2.6. *If a connected graph G has no majority dominating vertices then* $|V - S| \geq \sum\limits_{u_i \in S} deg(u_i)$, *where S is the MDC set of G.*

Proof: Let S be the MDC set of a connected graph G of p vertices and q edges. Since the graph G has no majority dominating vertices, it has no full degree vertex and it contains all vertices with degree of $d(u_i) < \lceil \frac{p}{2} \rceil - 1$. Assume that $S = \{u_1, u_2, \cdots\}$ be the MDC set of G. Then $|V - S| \leq p - 2, p > 6$.

$$Also, \sum_{u_i \in S} deg(u_i) = d(u_1) + d(u_2) + \cdots \leq \left\lceil \frac{p}{2} \right\rceil - 2 + \left\lceil \frac{p}{2} \right\rceil - 2 + \cdots \leq 2\left\lceil \frac{p}{2} \right\rceil - 4$$

$$\sum_{u_i \in S} deg(u_i) \leq (p-2) \; or \; (p-4), \; if \; p \; is \; odd \; or \; even.$$

$$Hence \; we \; obtain, \; |V - S| \geq \sum_{u_i \in S} deg(u_i).$$

□

Proposition 2.7. *If a MDC set S contains a majority dominating vertex v and other vertices* u_i *such that* $d(u_i) \leq \lceil \frac{p}{2} \rceil - 3$ *then*

$$|V - S| > \sum_{u_i \in S} deg(u_i).$$

Proof: Let u be the majority dominating vertex such that $d(u) = \lceil \frac{p}{2} \rceil - 1$ and other vertices u_i with degree $d(u_i) \leq \lceil \frac{p}{2} \rceil - 3$ in G. Then $\gamma_M(G) = |\{u\}| = 1$ and $\chi(G) = 2$. Therefore $S = \{u, u_1\}$ is a MDC set of G and $|V - S| \leq p - 2$.

$$Then \sum_{u_i \in S} deg(u_i) = d(u) + d(u_1) \leq \left\lceil \frac{p}{2} \right\rceil - 1 + \left\lceil \frac{p}{2} \right\rceil - 3$$

$$\leq \begin{cases} \frac{p}{2} - 1 + \frac{p}{2} - 3 = p - 4, & if \; p \; is \; even \\ \frac{p}{2} + \frac{p}{2} + 1 - 4 = p - 3, & if \; p \; is \; odd \end{cases}$$

$$Therefore \sum_{u_i \in S} deg(u_i) \leq (p-4) \; or \; (p-3). \; Hence \; |V - S| > \sum_{u_i \in S} deg(u_i).$$

□

Theorem 2.8. *Let G be a connected graph with exactly one vertex v such that* $\lceil \frac{p}{2} \rceil - 1 \leq d(v) \leq \lceil \frac{p}{2} \rceil + 2$ *and* $d(u_i) \leq 3$, *for all* $u_i \in V(G)$. *Then*

$$|V - S| > \sum_{v_i \in S} deg(v_i), \; where \; S \; is \; MDC \; set \; such \; that \; v \in S.$$

Proof: Let $v \in V(G)$ with the condition $\lceil \frac{p}{2} \rceil - 1 \leq d(v) \leq \lceil \frac{p}{2} \rceil + 2$. (1)

Case 1. *The graph G is a tree. Let $S = \{v, u_1\}$ be a MDC set in which u_1 is a pendant or $d(u_1) = 3$. Then by (1), $d(v) = \lceil \frac{p}{2} \rceil - 1$ and $|V - S| = p - 2$.*

$$Then \sum_{v_i \in S} deg(v_i) = d(v) + d(u_1) = \left\lceil \frac{p}{2} \right\rceil - 1 + 1 = \left\lceil \frac{p}{2} \right\rceil \ or \ \left\lceil \frac{p}{2} \right\rceil + 1$$

$$It\ implies\ that\ |V - S| = p - 2 > \sum_{v_i \in S} deg(v_i).$$

Suppose $d(v) = \lceil \frac{p}{2} \rceil + 2$.

$$Then, \sum_{v_i \in S} deg(v_i) = d(v) + d(u_1) = \left\lceil \frac{p}{2} \right\rceil + 2 + 1 = \left\lceil \frac{p}{2} \right\rceil + 3 \ or \ \left\lceil \frac{p}{2} \right\rceil + 4.$$

$$Therefore\ by\ (1), \sum_{v_i \in S} deg(v_i)\ takes\ the\ value\ from\ \left\lceil \frac{p}{2} \right\rceil\ to \left\lceil \frac{p}{2} \right\rceil + 4.$$

$$Hence\ |V - S| > \sum_{v_i \in S} deg(v_i).$$

Case 2. *The graph G is not a tree.*

Let S be a MDC set of G and $S = \{v, v_1\}$ where v is a majority dominating vertex and v_1 is not a pendant of G. Then $|V - S| \leq p - 2$.

$$Then \sum_{v_i \in S} deg(v_i) = d(v) + d(v_1) \geq \left\lceil \frac{p}{2} \right\rceil - 1 + 3$$

$$Therefore \sum_{v_i \in S} deg(v_i) = \left\lceil \frac{p}{2} \right\rceil + 2,\ if\ d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1\ and$$

$$\sum_{v_i \in S} deg(v_i) = \left\lceil \frac{p}{2} \right\rceil + 5,\ if\ d(v) \leq \left\lceil \frac{p}{2} \right\rceil + 2$$

$$Hence,\ |V - S| = p - 2 > \sum_{v_i \in S} deg(v_i).$$

□

3. Results on $\gamma_{M\chi}(G)$

Proposition 3.1. *Let G be any bipartite graph with a majority dominating vertex. Then $\gamma_{M\chi}(G) = 2$ and $\gamma_M(G) < \gamma_{M\chi}(G)$.*

Proof: Let $G = K_{m,n}, m \leq n$, be a complete bipartite graph.

Case 1. *Since G has a majority dominating vertex, $\gamma_M(G) = 1$ and $\chi(G) = 2$. Then $S = \{u_1, v_1\}$ is a MDC set of G, where $u_1 \in V_1(G)$ and $v_1 \in V_2(G)$.*

$$\Rightarrow \gamma_{M\chi}(G) = 2\ and\ \gamma_M(G) < \gamma_{M\chi}(G).$$

Case 2. *If G is not a complete bipartite graph then G may contains pendants. Since G has a majority dominating vertex $u_1 \in V(G), S = \{u_1, u_2\}$ is a MDC set of G where $u_1 \in V_1(G)$ and $v_1 \in V_2(G)$.*

$\Rightarrow \gamma_{M\chi}(G) = 2$ and $\gamma_M(G) = 1$. Hence $\gamma_M(G) < \gamma_{M\chi}(G)$. □

The following theorem gives the characterization of $\gamma_{M\chi}(G) = p - q$, where G is any graph with p vertices and q edges.

Theorem 3.2. *Let G be any graph with p vertices and q edges. Then $\gamma_{M\chi}(G) = p - q$ if and only if $G = K_p, p = 1$.*

Proof: Let $\gamma_{M\chi}(G) = p - q$. Since $\gamma_{M\chi}(G) \geq 1, (p-q) \geq 1$. (1)

Case 1. *The graph G is connected.*

Then $q \geq p - 1 \Rightarrow (p - q) \leq 1$. Hence by (1) we obtain $p - q = 1 = \gamma_{M\chi}(G)$. (2)
It implies that G is a tree. If G is a tree then $\chi(G) = 2$ and for any connected graph, $1 \leq \gamma_M(G) \leq \lceil \frac{p}{6} \rceil$.

By (2), since $p - q = 1 = \gamma_{M\chi}(G)$, the two numbers $\gamma(G)$ and $\gamma_M(G)$ must be one. In a tree, suppose $\chi(G) = 2$ and $\gamma_M(G) = 1$, then the graph becomes $G = K_2$. By the result (ii) of (1.2), $\gamma_{M\chi}(G) \geq \max\{\chi(G), \gamma_M(G)\}$. We have $\gamma_{M\chi}(G) = 2$. But it is contradiction to the result (2). Hence $G \neq K_2$ and $G = K_2$.

Case 2. *Suppose G is disconnected. If G is disconnected with isolates and without isolates. Then by the result (i) of (1.2), $\lceil \frac{p}{4} \rceil + 1 \leq \gamma_{M\chi}(G) \leq \lceil \frac{p}{2} \rceil$. The lower bound is attained for $G = mK_2$. If $m = 1, \gamma_{M\chi}(K_2) = 2 \neq p - q = 1$. Also the upper bound is attained for $G = \overline{k_p}$, when $p = 2$ then $\gamma_{M\chi}(\overline{K_2}) = 1 \neq p - q = 2$. Hence $G \neq \overline{K_2}$ or K_2. It follows that the graph must be $G = K_1$. The converse is obvious.*

□

Next result is the characterization of $|V - S| = 0$, where S is a MDC set of vertex color critical graph G.

Proposition 3.3. *A MDC set S belongs to a vertex color critical graph if and only if $|V - S| = 0$.*

Proof: Suppose $|V - S| = 0. \Rightarrow |V(G)| = |S| = p$. Then the set $S = \{u_1, u_2 \cdots, u_p\}$ is a MDC set for G. Suppose we remove one vertex from S then S may not be a MDC set of G. Hence G is vertex color critical graph.

Conversely by the definition (iv) in (1.1), if G is vertex color critical graph with p vertices then $\gamma_{M\chi}(G) = p$. Hence $|V - S| = 0$. □

Proposition 3.4. *Let G be any graph with p vertices. Then $\gamma_{M\chi}(G) \leq \gamma_\chi(G)$, where $\gamma_\chi(G)$ is the dom-chromatic number of G.*

Proof: Let $\gamma_{M\chi}(G)$ be the majority dom-chromatic number of graph G. Since every dom-chromatic set of a graph G is a majority dom-chromatic set of a graph G, $\gamma_{M\chi}(G) \leq \gamma_\chi(G)$.

Case 1. *When G is vertex color critical graph.*
By the known results (3.2.6) of [3] and (ii) of (1.2), $\gamma_\chi(G) = p = \gamma_{M\chi}(G)$.

Case 2. *The graph G is a tree.*
If diam $(G) \leq 3$, then $\gamma_\chi(G) = \gamma_{M\chi}(G) = 2$.
Suppose diam$(G) \geq 4$, then the graph structures like $P_p, p \geq 5$, Caterpillar, etc. By the known results, $\gamma_\chi(G) \leq \frac{p+3}{3}$ and $\gamma_{M\chi}(G) \leq \lceil \frac{p}{6} \rceil + 1$.
Hence $\gamma_{M\chi}(G) < \gamma_\chi(G)$.

Case 3. *When the graph G is not a tree and not a vertex color critical graph.*
Then the graph structures like C_p (cycle, p is even), F_p (Fan), W_p (wheel), etc. By the known results, $\gamma_{M\chi}(G) \leq \lceil \frac{p}{6} \rceil + 1$ and $\gamma_\chi(G) \leq \frac{p+4}{3}$.
Hence $\gamma_{M\chi}(G) \leq \gamma_\chi(G)$. □

Corollary 3.5.
(i) *If the graph G is a sub division of a star, then* $\gamma_{M\chi} < \lceil \frac{\gamma_\chi}{2} \rceil$.
(ii) *If G is a path or cycle then,*
a) $\gamma_{M\chi} \leq \lceil \frac{\gamma_\chi(G)}{2} \rceil ; p \equiv 0,1,2,5(\mod 6)$,
b) $\gamma_{M\chi} \leq \lceil \frac{\gamma_\chi(G)}{2} \rceil + 1 ; p \equiv 3,4(\mod 6)$.

Example 3.6.
(i) *Let P_p be a path with $p \equiv 0(mod6)$. Consider $G = P_{18}$ then $\gamma_{M\chi}(G) = 4$ and $\gamma_\chi(G) = 7$. Now, $\left\lceil \frac{\gamma_\chi(G)}{2} \right\rceil = \lceil \frac{7}{2} \rceil = 4$. Hence $\gamma_{M\chi}(G) = \left\lceil \frac{\gamma_\chi(G)}{2} \right\rceil$.*
(ii) *Let $G = S(K_{1,t})$. Then $S_1 = \{u, u_1, u_2, \cdots, u_t\}$ is a dom-chromatic set which contains a central vertex u of G. $\Rightarrow \gamma_\chi(G) = |S_1| = t+1$ and $S_2 = \{u, u_1\}$ is a MDC set of G. $\Rightarrow \gamma_{M\chi}(G) = 2$. Hence $\gamma_{M\chi}(G) < \left\lceil \frac{\gamma_\chi(G)}{2} \right\rceil$.*

Construction 3.7. *For every integer $k \geq 0$, there exist a graph G such that $\left\lceil \frac{\gamma_\chi(G)}{2} \right\rceil - \gamma_{M\chi}(G) = k$.*

Proof. Let G be the subdivision of a star $K_{1,2k+2}$ by dividing each edge exactly once. Then $|V(G)| = 2(2k+2)+1, \gamma_\chi(G) = 2k+2+1$ and $\gamma_{M\chi}(G) = 2$.
Then $\left\lceil \frac{\gamma_\chi(G)}{2} \right\rceil - \gamma_{M\chi}(G) = k+2-2 = k$. □

Observation 3.8. *Let G be any connected graph with p vertices. Let $\chi(G), \gamma_M(G)$ and $\gamma_{M\chi}(G)$ be the chromatic number, majority domination number and majority dom-chromatic number respectively. Then $\chi(G)$ and $\gamma_M(G)$ are not comparable.*
i.e., $\gamma_M(G) < \chi(G) < \gamma_{M\chi}(G)$ and $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

For Example:-

(i) Let $G = C_p, p \leq 11$ and p is odd. Since C_p is vertex χ-critical, by the result (iv) of (1.2), $\gamma_M(G) = \lceil \frac{p}{6} \rceil, \chi(G) = 3$ and $\gamma_{M\chi}(G) = 5$.
Hence, $\gamma_M(G) < \chi(G) < \gamma_{M\chi}(G)$.
(ii) Let $G = C_p, p$ is odd and $p \geq 19$. By the result (iv) of (1.2),
$\gamma_M(G) = \lceil \frac{p}{6} \rceil, \chi(G) = 3$ and $\gamma_{M\chi}(G) = p$. Hence, $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.
(iii) If $p = 13, 15, 17$ for $G = C_p$ then $\chi(G) = \gamma_M(G) < \gamma_{M\chi}(G)$.

4. Results of $\gamma_{M\chi}(G)$ for a Disconnected Graph

Theorem 4.1. *Let G be a disconnected graph then $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$ if and only if $G = \overline{K_p}$ or $G = g_t \cup \overline{K_{p-t}}$, $p \geq 2$, where g_t is a vertex color critical component with $|t| \leq \lceil \frac{p}{2} \rceil$.*

Proof: Let G be a disconnected graph with p vertices.

Assume that, $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$. (1)

Case 1. *Suppose $G \neq \overline{K_p}, p \geq 2$ then G has atleast one edge between a pair of vertices. It implies that G is a disconnected graph without isolates or $G = K_2 \cup \overline{K_{p-2}}$. By known result (i) of (1.2),$\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil + 1$ or $\gamma_{M\chi}(G) = \lceil \frac{p}{4} \rceil - 1$. But it is a contradiction to (1). Therefore $G = \overline{K_p}, p \geq 2$.*

Case 2. *Suppose $G = g_t \cup \overline{K_{p-t}}$, where g_t is not a vertex color critical component with $|t| \leq \lceil \frac{p}{2} \rceil$. Then the graph G contains a path, an even cycle or any other component g_t with $|t| \leq \lceil \frac{p}{2} \rceil$. Since $\chi(g_t) \geq 2$ and $\gamma_M(g_t) \geq \lceil \frac{p}{6} \rceil$,*

SubCase 1. *Suppose* $|t| = \lceil \frac{p}{2} \rceil$. *Then* $S = \left\{u_1, u_2, \ldots, u_{\lceil \frac{p}{6} \rceil}\right\}$, *is a MDC set of* G, *where* $u_i \in V(g_t)$. *It implies that* $\gamma_{M\chi}(G) = \lceil \frac{p}{6} \rceil$, *it condradicts the condition (1).*

SubCase 2. *Suppose* $|t| < \lceil \frac{p}{2} \rceil$. *Then* $S = \left\{u_1, u_2, \left(\lceil \frac{p}{2} \rceil - t\right) K_1\right\}$ *is a MDC set of* G *where* $u_i \in V(g_t)$.

Therefore $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{2} \rceil - |t| + 2 = \lceil \frac{p}{2} \rceil - \lceil \frac{p}{2} \rceil + 1 + 3$ $(if\ |t| = \lceil \frac{p}{2} \rceil - 1)$.

$\Rightarrow \gamma_{M\chi}(G) = 4 < \lceil \frac{p}{2} \rceil$. *It is a contradiction to (1). Hence* g_t *is a vertex color critical component in* G *with* $|t| \leq \lceil \frac{p}{2} \rceil$.

Case 3. *Suppose* g_t *with* $|t| > \lceil \frac{p}{2} \rceil$. *Since* g_t *is a vertex color critical component of* G, g_t *is a complete graph or an odd cycle. If* g_t *is an odd cycle with* $|t| = \lceil \frac{p}{2} \rceil + 1$ *then* $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil + 1$. *It contradicts our assumption.*

If g_t *is a complete graph with* $|t| = \lceil \frac{p}{2} \rceil + 1$ *then* $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil + 1$, *it is a contradiction to (1). Hence,* g_t *is a vertex color critical component of* G *with* $|t| \leq \lceil \frac{p}{2} \rceil$. *Therefore* G *must be* $\overline{K_p}$ *or* $(g_t \cup \overline{K_{p-t}})$ *with* $|t| \leq \lceil \frac{p}{2} \rceil$. *In all the three cases if* $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$, *then* $G = \overline{K_p}$ *or* $(g_t \cup \overline{K_{p-t}})$.

Conversely, let $G = \overline{K_p}$ *or* $(g_t \cup \overline{K_{p-t}})$. *Suppose* $G = \overline{K_p}$ *then* $\gamma_M(G) = \lceil \frac{p}{2} \rceil$ *and* $\chi(G) = 1 \Rightarrow \gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$.*Suppose* $G = (g_t \cup \overline{K_{p-t}})$. *Since* g_t *is a vertex critical component with* $|t| = \lceil \frac{p}{2} \rceil, \chi(g_t) = \lceil \frac{p}{2} \rceil$ *and* $\gamma_M(g_t) \geq 1$. *It implies that* $\gamma_{M\chi}(G) = \lceil \frac{p}{2} \rceil$.*Suppose* g_t *is a vertex critical component with* $|t| < \lceil \frac{p}{2} \rceil$. *Then* $S = \{u_1, u_2, ..., u_t, v_1, v_2, ..., v_{\lceil \frac{p}{2} \rceil - t}\}$ *is a MDC set of* G *where* $u_i \in V(g_t)$ *and* $v_i \in V(\overline{K_{p-t}})$. *Now,* $|S| = t + \lceil \frac{p}{2} \rceil - t = \lceil \frac{p}{2} \rceil$. *Hence* $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{2} \rceil$.

□

Observation 4.2. *(i) For a disconnected graph* $G, \chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

Example: Consider the disconnected graph with isolates with $p = 16$.

Let $G = P_{11} \cup \overline{K_5}$. Let $|V(G)| = |\{v_1, v_2, \cdots, v_{11}, u_1, \cdots, u_5\}| = 16$. Then $\gamma_M(G) = |\{v_2, v_5, v_7\}| = 3$ and $\gamma_{M\chi}(G) = |\{v_2, v_5, v_7, v_8\}| = 4$. Since P_{11} is a tree, $\chi(G) = 2$. Therefore $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

(ii) For a disconnected graph G with isolates, $\gamma_M(G) < \chi(G) < \gamma_{M\chi}(G)$.

Example: Let $G = C_3 \cup \overline{K_5}$ and $V(G) = \{v_1, v_2, v_3, u_1, \cdots, u_5\}$. Since C_3 is an odd cycle, $\chi(G) = 3$ and $\gamma_M(G) = |\{v_1, u_1\}| = 2$. Then $S = \{v_1, v_2, v_3, u_1\}$ be the MDC set of G where $v_i \in V(C_3)$ and $u_i \in V(\overline{K_5})$. $\Rightarrow \gamma_{M\chi}(G) = |S| = 4$. Therefore $\gamma_M(G) < \chi(G) < \gamma_{M\chi}(G)$.

(iii) Let G be a disconnected graph without isolates. Then $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

Example: Consider the graph $G = P_7 \cup C_6 \cup K_{1,3}$. For a tree with $p = 17$ and an even cycle, $\chi(G) = 2$.

$V(G) = \{u_1, \cdots, u_7, v_1, \cdots, v_6, w, w_1, w_2, w_3\}$. Then $\gamma_M(G) = |\{w, u_2, u_4\}| = 3$ and $\gamma_{M\chi}(G) = |\{w, u_2, u_4, u_5\}| = 4$. Hence $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

(iv) For a disconnected graph G with vertex color critical component, $\chi(G) < \gamma_M(G) < \gamma_{M\chi}(G)$.

Example: Let $G = C_{13} \cup \overline{K_6}$ be a graph with $p = 19$.

And $V(G) = \{u_1, \cdots, u_{13}, v_1, \cdots, v_6\}$. Since C_{13} is an odd cycle, $\chi(G) = 3$. The set $\{u_2, u_5, u_8\}$ be the γ_M-set of G and $\gamma_M(G) = 3$. By the result (iii) of (1.2), C_{13} is a vertex color critical component , $\gamma_{M\chi}(G) = 13$. Therefore $\chi(G) \leq \gamma_M(G) < \gamma_{M\chi}(G)$.

Proposition 4.3. *G be a disconnected graph with any vertex color critical*

$$\textit{component then } |V - S| < \sum_{u_i \in S} deg(u_i).$$

Proof: Let $G = G_t \cup G_r$ be a disconnected graph with p vertices . Since G has a vertex color critical component , $\chi(G) \geq 3$. Consider $S = \{G_t, u_1, \cdots\}$ be the MDC set of G, where G_t is the vertex color critical component, such that $|t| \geq 3$ and $u_1 \in G_r$. If $|N[G_t]| = \lceil \frac{p}{2} \rceil$ then $|S| \geq 3$. If $|N[G_t]| < \lceil \frac{p}{2} \rceil$ then $|S| \geq 4$. It implies that $|S| = 3$ or 4 and $|V - S| \leq p - 3$ or $p - 4$. Let $V(G_t) = \{u_1, u_2, \cdots, u_t\}$, then

$$\sum_{u_i \in S} deg(u_i) = d(u_1) + d(u_2) \cdots \geq 3(t-2) + 1 \geq 3t - 5, \; if \; |t| \geq 3.$$

$$\textit{Then, certainly we get } |V - S| < \sum_{u_i \in S} deg(u_i)$$

□

Proposition 4.4. *For a disconnected graph G without any vertex critical component,* $|V - S| > \sum_{u_i \in S} deg(u_i)$.

Proof: Let G be a disconnected graph with not vertex color critical component. Let S be a MDC set of G.

Case 1. *The graph G is totally disconnected.*

Then $S = \{u_1, u_2, \cdots, u_{\lceil \frac{p}{2} \rceil}\}$ be the MDC set of G and $deg(u_i) = 0$, for each $u_i \in S$. It implies that $\sum_{u_i \in S} deg(u_i) = 0$. Hence, $|V - S| > \sum_{u_i \in S} deg(u_i)$.

Case 2. *The graph G is disconnected with isolates.*

Then G contains some connected component 'g' along with isolates.

SubCase 1. *If the component 'g' such that $|N[g]| \geq \lceil \frac{p}{2} \rceil$ then S is a MDC set of G with $1 \leq |S| = \lceil \frac{p}{6} \rceil$. Suppose $|S| = 1 \Rightarrow S = \{u\}$ such that $|N[S]| = \lceil \frac{p}{2} \rceil - 1$.*

Then $|V - S| = p - 1 > \sum_{u_i \in S} deg(u_i) = \lceil \frac{p}{2} \rceil - 1$. Suppose $|S| = \lceil \frac{p}{6} \rceil$.

Then $d(u_i) \leq 2$, for all $u_i \in V(g)$. Now, $\sum_{u_i \in S} deg(u_i) = 2\lceil \frac{p}{6} \rceil = \frac{p}{3}$ or $\frac{p}{3} + 2$ and $|V - S| = p - \lceil \frac{p}{6} \rceil = \frac{5p}{6} - 1$.

Therefore, $|V - S| > \sum_{u_i \in S} deg(u_i)$.

SubCase 2. *If the component 'g' such that $|N[S]| < \lceil \frac{p}{2} \rceil$ then S is a MDC set with isolates.*

$\Rightarrow \sum\limits_{u_i \in S} deg(u_i) \leq \frac{p}{3}$. *Since S contains more isolates, the value* $\sum\limits_{u_i \in S} deg(u_i)$ *will be reduced. Then* $|V - S| > \sum\limits_{u_i \in S} deg(u_i)$.

Case 3. *G is a disconnected graph without isolates.*

Then G contains only connected components . Suppose $G = mK_2$. Then by the result (i) of (1.2), $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{4} \rceil + 1$. It implies that

$$\sum_{u_i \in S} deg(u_i) = \left\lceil \frac{p}{4} \right\rceil + 1. \textit{ But } |V - S| = \left| p - \left(\left\lceil \frac{p}{4} \right\rceil + 1 \right) \right| = \frac{3p}{4} - 1$$

If the size of the component g increases such as $G = mC_4, mK_{1,t}, \cdots$ then $|S|$ will be decreased. i.e.,

$$|S| < \left\lceil \frac{p}{4} \right\rceil + 1 \textit{ and } \sum_{u_i \in S} deg(u_i) > \left\lceil \frac{p}{4} \right\rceil + 1. \textit{ But in all structures,}$$

$$We \ obtain, \ |V - S| > \sum_{u_i \in S} deg(u_i).$$

□

Proposition 4.5. *Let G be a disconnected graph without any vertex color critical component then $|V - S| = \lfloor \frac{p}{2} \rfloor$ if and only if $G = \overline{K_p}$.*

Proof: Let G has no vertex color critical subgraph. Let $G = \overline{K_p}$, p is odd. Then $S = \{u_1, u_2, \cdots, u_{\lceil \frac{p}{2} \rceil}\}$ is a MDC set of G and $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{2} \rceil$. Hence $|V - S| = \lfloor \frac{p}{2} \rfloor$, if p is odd. When p is even, $S = \{u_1, u_2, \cdots, u_{\frac{p}{2}}\}$ is the MDC set and $\gamma_{M\chi}(G) = |S| = \frac{p}{2}$ and $|V - S| = \frac{p}{2}$. Hence $|V - S| = \lfloor \frac{p}{2} \rfloor$.

Conversely , suppose $G \neq \overline{K_p}$. Then either G is disconnected graph without isolates or G contains atleast one component which is not a vertex color critical with some isolates. Let $|V - S| = \lfloor \frac{p}{2} \rfloor$. (1)

Case 1. *If G has components which is not vertex color critical with no isolates then the structure like $G = mK_2$. By the result (i) of (1.2), we have $\gamma_{M\chi}(G) = |S| = \lceil \frac{p}{4} \rceil + 1$. If $|S| = \lceil \frac{p}{4} \rceil + 1 \Rightarrow |V - S| = |p - \lceil \frac{p}{4} \rceil + 1| > \lfloor \frac{p}{2} \rceil$. It is a contradiction to (1).*

Case 2. *Suppose $G = C_6 \cup \overline{K_{P-6}}$, where C_6 is not a vertex color critical. Then $S = \{u_2, u_5, (\lceil \frac{p}{2} \rceil - 6)K_1\}$, where $u_2, u_5 \in V(C_6)$.*

$\Rightarrow |S| = \lceil \frac{p}{2} \rceil - 6 + 2 = \lceil \frac{p}{2} \rceil - 4$.

Therefore $|V - S| = |p - \lceil \frac{p}{2} \rceil + 4| = \lfloor \frac{p}{2} \rfloor + 4 > \lfloor \frac{p}{2} \rfloor$. It is a contradiction to (1).

Hence $G = \overline{K_p}$ if and only if $|V - S| = \lfloor \frac{p}{2} \rfloor$.

□

5. Conclusion

In this article, we have discussed the inequality between the sum of the degrees of the vertices of majority dominating chromatic set S and its complement $(V-S)$ of a graph. The comparison between the domination parameters $\gamma_M(G), \chi(G)$ and $\gamma_{M\chi}(G)$ are discussed. Also some results of $\gamma_{M\chi}(G)$ of a disconnected graph with isolates and without isolates are studied.

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CHANGING AND UNCHANGING OF MAJORITY DOMINATING CHROMATIC NUMBER WHEN REMOVAL OFA SINGLE VERTEX

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Abstract

In this article, how the removal of a single vertex from a graph G can change the majority dom-chromatic number is determined for any graph. A graph is majority dom-chromatic critical if the removal of any vertex decreases or increases its majority dom-chromatic number. There are two types namely *CVR* and *UVR* with respect to majority dom-chromatic sets of a graph. Also the vertex classification $V^0_{M\chi}(G)$, $V^-_{M\chi}(G)$ and $V^+_{M\chi}(G)$ are studied and its characterisation theorems are determined.

1. Introduction

Let G be a finite and simple graph with p vertices and q edges. A subset D of vertices in a graph $G = (V, E)$ is called a dominating set [1] of G if every vertex in $(V - D)$ is adjacent to some vertex in D. A dominating set D is called a minimal dominating set if no proper subset of D is a dominating set. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a

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minimal dominating set in G. A set $S \subseteq V(G)$ of vertices in a graph $G = (V, E)$ is called a majority dominating set [4] of G if at least half of the vertices of $V(G)$ are either in S or adjacent to the elements of S. A majority dominating set S is minimal if no proper subset of S is a majority dominating set of a graph G. The minimum cardinality of a minimal majority dominating set is called majority domination number of G, is denoted by $\gamma_M(G)$. It is the minimum majority dominating set of G.

A dominating set $S \subseteq V(G)$ such that the induced sub graph $\langle S \rangle$ satisfies the property $\chi(\langle S \rangle) = \chi(G)$ is called as dom-chromatic set [2] of a graph G. The minimum cardinality of a dominating chromatic set is called dom-chromatic number and it is denoted by $\gamma_{ch}(G)$ or $\gamma_\chi(G)$. A dom-chromatic set S of G such that $|S| = \gamma_{ch}(G)$ is the minimum dom-chromatic set of a graph G.

[6] For any graph G, CVR and UVR with respect to domination numbers are defined by, $CVR : \gamma(G - v) \neq \gamma(G)$, for all $v \in V(G)$ and $UVR : \gamma(G - v) = \gamma(G)$, for all $v \in V(G)$.

[5] For any graph G, CVR_M and UVR_M with respect to majority domination numbers are defined by, $CVR_M : \gamma_M(G - v) \neq \gamma_M(G)$, for all $v \in V(G)$ and $UVR_M : \gamma_M(G - v) \neq \gamma_M(G)$, for all $v \in V(G)$.

[2] A graph G said to be a CVR-graph if $\gamma_{ch}(G - u) \neq \gamma_{ch}(G)$, for all $u \in V(G)$ and a graph G said to be a UVR-graph if $\gamma_{ch}(G - u) \neq \gamma_{ch}(G)$, for all $u \in V(G)$.

A set $S \subseteq V(G)$ is said to be a chromatic preserving set or a cp-set if $\chi(\langle S \rangle) = \chi(G)$ and the minimum cardinality of a cp-set in G is called the chromatic preserving number or cp-number of G and is denoted by $cpn(G)$.

[1] The private neighbour set of u with respect to S denoted by $pn[u, S]$ is defined by $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$

2. $CVR_{M\chi}$ and $UVR_{M\chi}$ Graphs

Definition 2.1 [3]. A subset S of $V(G)$ is said to be Majority Dominating Chromatic set (MDC set) if S is a majority dominating set and S satisfies $\chi(\langle S \rangle) = \chi(G)$. The minimum cardinality of a majority dominating chromatic set of G is called a majority dominating chromatic number and is denoted by $\gamma_{M\chi}(G)$.

Definition 2.2. For any graph G, the vertex set can be partitioned with respect to MDC sets into three sets $V^0_{M\chi}(G)$, $V^-_{M\chi}(G)$ and $V^+_{M\chi}(G)$ and is defined by,

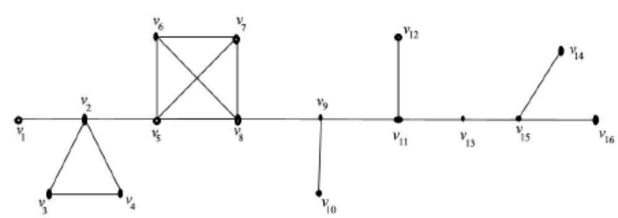
$V^0_{M\chi}(G) = \{v \in V(G) / \gamma_{M\chi}(G - v) = \gamma_{M\chi}(G)\}$,

$V^-_{M\chi}(G) = \{v \in V(G) / \gamma_{M\chi}(G - v) < \gamma_{M\chi}(G)\}$ and

$V^+_{M\chi}(G) = \{v \in V(G) / \gamma_{M\chi}(G - v) > \gamma_{M\chi}(G)\}$.

Definition 2.3. A graph G is said to be a $CVR_{M\chi}$-graph if $\gamma_{M\chi}(G - v) \neq \gamma_{M\chi}(G)$, for every $v \in V(G)$. A graph G is said to be a $UVR_{M\chi}$-graph if $\gamma_{M\chi}(G - v) = (G)$, for every $v \in V(G)$.

Example 2.4. Consider the graph G with $p = 16$.

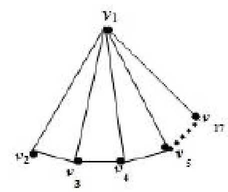


***G*. Figure - (i)**

In the graph G, $S = \{v_5, v_6, v_7, v_8, v_{15}\}$ is the $\gamma_{M\chi}$-set of G. Then $\gamma_{M\chi}(G) = 5$. For the graph $G - \{v_5\}$, $\gamma_{M\chi}(G - \{v_5\}) = |\{v_2, v_3, v_4, v_8\}| = 4$. Therefore $\gamma_{M\chi}(G - v_5) < \gamma_{M\chi}(G)$. Hence $v_5 \in V^-_{M\chi}(G)$. For the graph

$G - \{v_8\}$, $\gamma_{M\chi}(G - v_2) = |\{v_5, v_6, v_7, v_8, v_{15}\}| = 5$. Therefore $\gamma_{M\chi}(G - v_8) = \gamma_{M\chi}(G)$. It implies that $v_8 \in V^0_{M\chi}(G)$.

Example 2.5. Consider the graph $G = F_p$, $p = 17$ a Fan.



***G*. Figure- (ii)**

In this graph G, $\gamma_{M\chi}(G) = |\{v_1, v_2, v_3\}| = 3$. For $G - \{v_2\}$, $\gamma_{M\chi}(G - v_2) = |\{v_1, v_3, v_4\}| = 3$. Therefore $\gamma_{M\chi}(G - v_2) = \gamma_{M\chi}(G)$ and $v_2 \in V^0_{M\chi}(G)$. For the graph $\{G - v_1\}$, $\gamma_{M\chi}(G - v_1) = |\{v_3, v_4, v_7, v_{10}\}| = 4$. Hence $\gamma_{M\chi}(G - v_2) > \gamma_{M\chi}(G)$ and $v_1 \in V^+_{M\chi}(G)$.

Theorem 2.6. *If a graph G is a vertex color critical then* $G \in CVR_{M\chi}$.

Proof. Since the graph G is vertex color critical, $\gamma_{M\chi}(G) = p$. If the removal of any vertex v from $V(G)$, $\chi(G - v) \neq \chi(G)$. It implies that $\gamma_{M\chi}(G - v) < \gamma_{M\chi}(G)$, for every vertex $v \in V(G)$. Hence $G \in CVR_{M\chi}$.

Corollary 2.7. *Let* $G = K_p$, $p \geq 2$. *Then* $G \in CVR_{M\chi}$.

Proof. By the result (3.1) [3], $\gamma_{M\chi}(G) = p$. For the graph $\gamma_{M\chi}(G - v_1) = p - 1$. Hence $\gamma_{M\chi}(G - v_1) < \gamma_{M\chi}(G)$. Therefore $v_1 \in V^-_{M\chi}(G)$. For every vertex $v \in V(G)$, $\gamma_{M\chi}(G - v) < \gamma_{M\chi}(G)$ and $G \in CVR_{M\chi}$.

3. Results on $V^+_{M\chi}(G)$

Proposition 3.1. *Let* $G = K_{1,\, p-1}$. *Then* $v_1 \in V^+_{M\chi}(G)$ *and* $v_i \in V^0_{M\chi}(G)$ *where* v_1 *is a central vertex and* v_i*'s are pendants.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$, where v_1 is the central vertex and others are pendants. The set $S = \{v_1, v_2\}$ is the MDC set of G and $\gamma_{M\chi}(G) = 2$. For a graph $G - \{v_1\}$, $\gamma_{M\chi}(G - v_1) = \left\lceil \frac{p-1}{2} \right\rceil$. Therefore $\gamma_{M\chi}(G - v_1) > \gamma_{M\chi}(G)$. Hence $v_1 \in V^{+}_{M\chi}(G)$. Suppose any pendant v_i, $i = 2, \ldots, p$, $\gamma_{M\chi}(G - v_i) = 2 = \gamma_{M\chi}(G)$. Therefore $v_i \in V^{0}_{M\chi}(G)$, where v_i's are pendants.

Proposition 3.2. *If G has exactly one full degree vertex and other vertices are of degree $d(v_i) < \frac{p-1}{2}$ then $|V^{+}_{M\chi}(G)| = 1$.*

Proof. Let G be a graph which contains a full degree vertex v such that $d(v) = p - 1$. Let S be a MDC set of G. Since $d(v) = p - 1$, v must be in majority dominating set S and minimal cp-set of G. Then $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $\chi(\langle S \rangle) = \chi(G)$. Let S' be the $\gamma_{M\chi}$-set of $G' = \{G - v\}$ and $\{G - v\}$ contains isolates, then $\gamma_{M\chi}(G') > |S| = \gamma_{M\chi}(G)$. It implies that $v \in V^{0}_{M\chi}(G)$. If $\{G - v\}$ contains the vertices v_i with $d(v_i) \leq \left\lceil \frac{p-1}{2} \right\rceil$ then $|S'| \geq 2$. Therefore $|S'| \geq |S| + 1$. It implies that $\gamma_{M\chi}(G - v) > \gamma_{M\chi}(G)$ and $v \in V^{+}_{M\chi}(G)$. Thus, all other vertices are $V^{0}_{M\chi}(G)$. Hence $|V^{+}_{M\chi}(G)| = 1$.

Proposition 3.3. *Let T be a tree with p vertices. If a vertex $v \in V(T)$ satisfies one of the following conditions.*

(i) *v is in a dominating edge $e = \{uv\}$ with $d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1$ and $d(u) < \left\lceil \frac{p}{2} \right\rceil - 1$.*

(ii) *v is a vertex with degree $d(v) = p - 1$ and others pendants.*

(iii) *v is in every $\gamma_{M\chi}$-set of T. Then $V^{+}_{M\chi}(T)$.*

Proof. Let T be a tree with p vertices.

Case (i) Let $e = \{uv\}$ is a dominating edge with $d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1$ and $d(u) < \left\lceil \frac{p}{2} \right\rceil - 1$. Since $\chi(G) = 2$, $S = \{u, v\}$ be a $\gamma_{M\chi}$-set of T. Let $S_1 = \{u, u_1, v_i\}$ be a set of $T - \{v\}$, where u and u_1 are adjacent and v_i's are isolates such that $|N[S_1]| \geq \left\lceil \frac{p}{2} \right\rceil$ with $|S_1| > |S|$. Then $\chi(T) = \chi(\langle S_1 \rangle) = \chi(T - v)$. Thus S_1 is a MDC set of $T - \{v\}$ and $\gamma_{M\chi}(T - v) \leq |S_1|$. Since $|S_1| > |S|$, $\gamma_{M\chi}(T - v) > |S| = \gamma_{M\chi}(T)$. Hence $v \in V^{+}_{M\chi}(T)$.

Case (ii) Let $d(v) = p - 1$ and $d(v_i) = 1$, for all $v_i \in V(T)$. Then $\gamma_{M\chi}(T) = |\{v, v_1\} = 2|$, for some v_1 such that $d(v_1) = 1$. Since v is adjacent to all vertices v_i of T, $\langle T - \{v\} \rangle$ is disconnected with only isolates. Now, there exists a MDC set S in $T - \{v\}$ with only isolates and $|S| = \left\lceil \frac{p-1}{2} \right\rceil$. It implies that $|S| = \gamma_{M\chi}(T - \{v\}) > \gamma_{M\chi}(T)$ and $v \in V^{+}_{M\chi}(T)$.

Case (iii) If the vertex v is in every minimum MDC set of T, then v is in a dominating edge $e = uv$ or v is a full degree vertex of T. It implies that $d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1$, $d(u) < \left\lceil \frac{p}{2} \right\rceil - 1$ and other vertices v_i's are of degree with $d(v_i) < \left\lceil \frac{p}{2} \right\rceil - 1$. By Case (i), the vertex $v \in V^{+}_{M\chi}(T)$.

Proposition 3.4. *For any graph G, $|V^{+}_{M\chi}(G)| \leq \gamma_{M\chi}(G)$.*

Proof. Let S be a $\gamma_{M\chi}$- set of G. Let $v \in V^{+}_{M\chi}(G)$. By proposition (3.3), v is in every $\gamma_{M\chi}$-set S of G. Then $v \in S$ and $V^{+}_{M\chi}(G) \subseteq S$. Hence $|V^{+}_{M\chi}(G)| \leq |S| = \gamma_{M\chi}(G)$.

Theorem 3.5. *If $v \in V^{+}_{M\chi}(G)$ and v is in every minimal cp-set of G then $|p_n[v, S]| \geq 2$, for all $\gamma_{M\chi}$-set of G.*

Proof. Let S be a $\gamma_{M\chi}$-set of G. Let v be a vertex in every minimal cp-set of G. Let v be a vertex in every minimal cp-set of G. Then $\chi(\langle S - v \rangle) = \chi(G - v) < \chi(G)$. Let $Pn[v, S] = \phi$. Then $\{S - v\}$ is a $\gamma_{M\chi}$- set of $\{G - v\}$. It is a contradiction to $v \in V^{+}_{M\chi}(G)$. Suppose $| Pn[v, S] | = \{v\}$. Then v is an isolated vertex in S and hence $v \in V^{0}_{M\chi}(G)$. It is a contradiction to $v \in V^{+}_{M\chi}(G)$. If $| Pn[v, S] | = \{u\}$ then $\{S - v\} \cup \{u\}$ is a $\gamma_{M\chi}$-set of $\{G - v\}$. Thus $\gamma_{M\chi}(G - v) \leq | S | = \gamma_{M\chi}(G)$. It is a contradiction to $v \in V^{+}_{M\chi}(G)$.

Hence, $| Pn[v, S] | \geq 2$.

Proposition 3.6. *For any graph G with an isolate, there exists a $\gamma_{M\chi}$-set of G not containing that isolate.*

Proof. Let v be an isolate of G. If S is a $\gamma_{M\chi}$-set of G containing v then $| N[S] | \geq \left\lceil \frac{p}{2} \right\rceil$ and $\chi(\langle S \rangle) = \chi(G)$.

Case (i) If $| N[S] | > \left\lceil \frac{p}{2} \right\rceil$ then $| N[S - \{v\}] | \geq \left\lceil \frac{p}{2} \right\rceil$ and $\chi(\langle S - v \rangle) = \chi(G)$. It implies that $S - \{v\}$ is a $\gamma_{M\chi}$-set. Hence $S - \{v\} = S'$ is a $\gamma_{M\chi}$-set of G without an isolate v.

Case (ii) If $| N[S] | = \left\lceil \frac{p}{2} \right\rceil$ then $| N[S - \{v\}] | \geq \left\lceil \frac{p}{2} \right\rceil - 1$ and $v \neq N[S]$. Now, if $| N[S - \{v\}] \cup \{v_1\} | \geq \left\lceil \frac{p}{2} \right\rceil$, for any $v_1 \in V(G)$ then $S' = S - \{v\} \cup \{v_1\}$. Also, $\chi(\langle S' \rangle) = \chi(\langle S \rangle)$ and $| S' | = | S | = \gamma_{M\chi}(G)$. Hence S' is a $\gamma_{M\chi}$-set of G without an isolate v.

4. Results on $V^{0}_{M\chi}(G)$ and $V^{-}_{M\chi}(G)$

Proposition 4.1. *If G is a graph with $\gamma_{M\chi}(G) = | V(G) |$ then $\in CVR_{M\chi}$.*

Proof. Let G be a graph with p vertices and $\gamma_{M\chi}(G) = | V(G) | = p$. Then

G is a vertex color critical graph. Therefore, for any vertex $v \in V(G)$, the graph $G' = G - \{v\}$ has the value $\gamma_{M\chi}(G') < p$. It implies that $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$, for every $v \in V(G)$. Hence $G \in CVR_{M\chi}$.

Proposition 4.2. *If G is a vertex color critical graph then* $V(G) = V^-_{M\chi}(G)$ *but the converse is not true.*

Proof. By Proposition (4.1), $\gamma_{M\chi}(G) = p$ and for all v, $\gamma_{M\chi}(G - v) < \gamma_{M\chi}(G) \Rightarrow V^-_{M\chi}(G) = V(G)$. For the converse, Let $G = P_p$, $p = 9$. Then $\gamma_{M\chi}(G) = 3$. For any vertex v, $P_9 - \{v\} = P_8$ and $\gamma_{M\chi}(P_8) = 2$. Hence $V(G) = V^-_{M\chi}(G)$ and $G \in CVR_{M\chi}$ but $G = P_9$ is not a vertex color critical graph.

Proposition 4.3. *Any Path* P_p, $p \equiv 3 (\mathrm{mod}\, 6)$ *is a* $CVR_{M\chi}$ *graph.*

Proof. Let $G = P_p$, $p = 6k + 3$, $k \geq 1$. Then by the result (3.3) [3], $\gamma_{M\chi}(G) = k + 2$. For each vertex $v \in V(G)$, $\gamma_{M\chi}(G - v) = k + 1$, where $p \equiv 6k + 2$. Hence $P_p \in CVR_{M\chi}$ where $p \equiv 3 (\mathrm{mod}\, 6)$.

Proposition 4.4. *If G is a* $CVR_{M\chi}$ *graph then* $V^-_{M\chi}(G) \neq \phi$.

Proof. Since G is a $CVR_{M\chi}$ graph, $V = V^+_{M\chi} \cup V^-_{M\chi}$.

Suppose $V^-_{M\chi}(G) \neq \phi$. (1)

Then $V = V^+_{M\chi}(G)$ and $\gamma_{M\chi}(G - v) > \gamma_{M\chi}(G)$, for all $v \in V(G)$. Let S be a $\gamma_{M\chi}$-set with $|S| = p - 1$ of G. Then $V - S \neq \phi$. Let $u \in V - S$ and $\{u\} \subseteq V(G) - S$. It implies that $S \subseteq V(G) - \{u\} = G - u$. Since G is a $CVR_{M\chi}$ graph, $\chi(\langle S \rangle) = \chi(G)$ and $\chi(\langle S \rangle) = \chi(\langle G - u \rangle)$. It implies that S is a $\gamma_{M\chi}$- set of $(G - u)$ and $\gamma_{M\chi}(G - u) \leq |S| = \gamma_{M\chi}(G)$. Therefore $u \in V^-_{M\chi}(G)$, it is a contradiction to (1). Hence $V^-_{M\chi}(G) \neq \phi$, for any $CVR_{M\chi}$ graph G.

Proposition 4.5. *A Wheel graph* $G = W_p$, $p > 5$ *is a* $CVR_{M\chi}$ *graph when p is even.*

Proof. Let $G = W_p$, $C_{p-1} \vee K_1$. By the result (3.5) [3], $\gamma_{M\chi}(G) = p$, when p is even. Let $V(G) = \{v_1, v_2, \ldots, v_{p-1}, v_p\}$ where $v_i \in C_{p-1}$, $i = 1, 2, \ldots, p-1$ and $v_p \in k_1$. Suppose $G' = G - \{v_k\}$.

Case (i) Let $\{v_k\}$ be the central vertex of G. Then $G - \{v_k\} = G' = C_{p-1}$. Since p is even, C_{p-1} is an odd cycle. By the result (3.2) [3], $\gamma_{M\chi}(G') = p-1$, Therefore $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$.

Case (ii) Suppose $\{v_t\}$ be any vertex in C_{p-1}. Then the graph G becomes a Fan $G' = (G - \{v_t\}) = P_{p-2} \vee K_1$. By the result (3.6) [3], $\gamma_{M\chi}(G') = 3$. Hence $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$.

In these two cases, the removal of any vertex $\{v_i\}$ in G,

$\gamma_{M\chi}(G - v_i) < \gamma_{M\chi}(G)$. Hence $G \in CVR_{M\chi}$.

Proposition 4.6. *Let* $G = W_p$, p *is odd. Then*

(i) $v_i \in V^0_{M\chi}(G)$ *if* $v_i \in C_{p-1}$.

(ii) $v_i \in V^+_{M\chi}(G)$ *if v is a central vertex of G and* $p \geq 17$.

Proof. For $G = W_p$, $C_{p-1} \vee K_1$, p is odd, $V(G) = \{v_1, v_2, \ldots, v_{p-1}, v_p\}$. By the result (3.5) [3], $\gamma_{M\chi}(G) = 3$. (1) The removal of any vertex v from $V(G)$, there exists two cases.

Case (i) Suppose any vertex $v_i \in C_{p-1}$. Then $G' = G - \{v_i\}$ and $G = F_{p-1} = p_{p-2} \vee K_1$, where $(p-1)$ is even. By the result (3.6) [3], $\gamma_{M\chi}(G') = 3$. By the result (1), $\gamma_{M\chi}(G') < \gamma_{M\chi}(G)$ and $v_i \in V^0_{M\chi}(G)$, for any vertex $v_i \in C_{p-1}$.

Case (ii) Suppose v_p is a central vertex and $p \geq 17$. The $\gamma_{M\chi}$-set of G is

$S = \{v_1, v_2, v_p\}$. Then $\gamma_{M\chi}(G) = |S| = 3$. If the removal of a central vertex v_p, $G' = G - \{v_p\}$ and G' becomes C_{p-1} even cycle. By the proposition (3.2) [3],

$$\gamma_{M\chi}(G') = \begin{cases} p, & \text{if } p \text{ is odd} \\ \left\lceil \frac{p}{6} \right\rceil, & \text{if } p \equiv (\text{mod } 6) \\ \left\lceil \frac{p}{6} \right\rceil + 1, & \text{if } p \equiv 0,\ 4 (\text{mod } 6). \end{cases} \quad (2)$$

For $p \leq 16$, by the result (2), $\gamma_{M\chi}(G') = |S'| = 3$. If $p \geq 17$, by the result (2), $\gamma_{M\chi}(G') = |S'| \geq 4$. Therefore, by the result (1), $\gamma_{M\chi}(G') > \gamma_{M\chi}(G)$ and $v_p \in V^{+}_{M\chi}(G)$. Hence $v_p \in V^{+}_{M\chi}(G)$, if $p \geq 17$.

Theorem 4.7. *Let G be a $CVR_{M\chi}$ graph with p vertices. Then $|V^{-}_{M\chi}(G)| \geq p - \gamma_{M\chi}(G)$.*

Proof. Let S be a $\gamma_{M\chi}$-set of G. If G is a $CVR_{M\chi}$-graph then $\gamma_{M\chi}(G-u) < \gamma_{M\chi}(G)$. Suppose $|S| = \gamma_{M\chi}(G) = p$. Then $|V^{-}_{M\chi}(G)| \geq p - \gamma_{M\chi}(G)$ holds. Suppose $|S| = \gamma_{M\chi}(G) < p$. Then $V - S \neq \phi$. Now choose any vertex $v \in V - S$. Since $\gamma_{M\chi}(G-u) < \gamma_{M\chi}(G)$, $v \in V^{-}_{M\chi}(G)$. Therefore $V - S \subseteq V^{-}_{M\chi}(G)$. It implies that $|V - S| \leq |V^{-}_{M\chi}(G)|$. Hence $|V^{-}_{M\chi}(G)| \geq p - \gamma_{M\chi}(G)$.

Theorem 4.8. *Let $\gamma_{M\chi}(G)$ be the $\gamma_{M\chi}$-number of a graph G. If $\gamma_{M\chi}(G) = |V(G)|$ then $V(G) = V^{-}_{M\chi}(G)$.*

Proof. Let S be a $\gamma_{M\chi}$-set of G and $\gamma_{M\chi}(G) = |V(G)| = p$. Then G is a vertex color critical graph. For any $v \in V(G)$, $\chi(G-v) < \chi(G)$ and it implies that $\gamma_{M\chi}(G-u) < \gamma_{M\chi}(G)$. Hence $v \in V^{-}_{M\chi}(G)$. For every $v \in S$,

$\gamma_{M\chi}(G-u) < \gamma_{M\chi}(G)$ is true. Hence $V^{-}_{M\chi}(G) = |V(G)|$.

Theorem 4.9. *Let G be a connected* $CVR_{M\chi}$ *graph with* $\chi(G) \geq 3$. *Then G has a unique* $\gamma_{M\chi}$*-set of G if and only if* $\gamma_{M\chi}(G) = |V(G)|$.

Proof. Let the graph G have a unique $\gamma_{M\chi}$-set S. Then we claim that

$$V(G) - S = \phi. \tag{3}$$

Suppose $V - S = \phi$. Since G is a $CVR_{M\chi}$ graph, $\gamma_{M\chi}(G - u) < \gamma_{M\chi}(G)$, for every $v \in V - S$. Then for each $v \in V - S$, $\chi(\langle S - v \rangle) = \chi(\langle S \rangle)$ and the induced sub graph $\langle S \rangle$ is a vertex color critical. Hence for any $u \in V - S$, S is a MDC set of $G - \{u\}$, which is a contradiction to the assumption (1). Therefore there exist $v \in V - S$ such that $\chi(\langle S - v \rangle) = \chi(\langle S \rangle)$. Then $Pn[u, S] \neq \phi$, for any $u \in S$,

Case (i) Let $|Pn[u, S]| = 1$. If $Pn[u, S] = \{u\}$ then u is an isolate in $\langle S \rangle$. Since G is connected, $N(u) \neq \phi$ and $N(u) \subseteq V - S$. Also some vertex $w \in V - S$ is adjacent to any vertex in S. Let $w \in N(u)$. Then $(S - u) \cup \{w\}$ is a $\gamma_{M\chi}$-set of G, which is a contradiction to the assumption (1). So $Pn[u, S] = \{v\}$. Then $(S - u) \cup \{v\}$ is $\gamma_{M\chi}$-set of G, which is a contradiction to (1). Hence $V - S = \phi$. Thus $|V(G)| = \gamma_{M\chi}(G)$.

Case (ii) Suppose $|Pn[v, S]| \geq 2$. Let $v \in Pn[v, S]$. Then there exists a $w \neq v$ such that $w \in Pn[v, S]$. It implies that $(S - u) \cup \{w\}$ is a $\gamma_{M\chi}$-set of G, which is a contradiction to (1). Let $x, w \in Pn[v, S]$. Then $(S - u) \cup \{w\}$ is a $\gamma_{M\chi}$-set of $G - x$. Thus, $|V(G)| = |S| = \gamma_{M\chi}(G)$.

Conversely, $\gamma_{M\chi}(G) = |V(G)| = p$. It implies that the graph G have a unique MDC set of G.

Theorem 4.10. *If v is an isolated of G then* $v \in V^0_{M\chi}(G)$.

Proof. Let v be an isolated vertex of G. Then v is not in minimal cp-set of G. Let S be a $\gamma_{M\chi}$-set of G and not containing the vertex v. Then

$|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $\chi(\langle S \rangle) = \chi(G)$. Then $\gamma_{M\chi}(G) = |S|$. For the graph $\{G - v\}$, $\chi(\langle G - v \rangle) = \chi(G)$ and S is again the $\gamma_{M\chi}$-set of $\{G - v\}$. Therefore $\gamma_{M\chi}(G - u) = \gamma_{M\chi}(G)$ and $v \in V^0_{M\chi}(G)$.

Theorem 4.11. *If a vertex $v \in V(G)$ is not in any minimal cp-set of G then $v \in V^0_{M\chi}(G)$.*

Proof. Let S be a $\gamma_{M\chi}$-set of G. Let v be a vertex which is not in any minimal cp-set of G. Then $\chi(\langle S - v \rangle) = \chi(G)$. Hence $Pn[v, S] \neq \phi$. Let $Pn[v, S] = 1$. If $Pn[v, S] = \{v\}$ then v is an isolated vertex in S. By the Proposition (4.10), $v \in V^0_{M\chi}(G)$.

Theorem 4.12. *Let v be a vertex of G with $v \in V^+_{M\chi}(G)$. Then there exists a vertex $u \in V(G)$ such that $\gamma_{M\chi}(G - u) = \gamma_{M\chi}(G)$.*

Proof. Let S be as MDC set of G. Then $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$.

Case (i) Suppose $|N[S]| \neq \vee(G)$. Then there exists a vertex $u \in N[S]$ and implies that $u \notin S$, $u \in \vee - N[S]$. Then $S \subseteq V - u$ and $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $|N_{G-u}[S]| \geq \left\lceil \frac{p}{2} \right\rceil$. It implies that $|N_{G-u}[S]| \geq \left\lceil \frac{p-1}{2} \right\rceil$. Therefore S is a MDC set of $\{G - v\}$. Then $\gamma_{M\chi}(G - u) \leq |S| = \gamma_{M\chi}(G)$. If $\gamma_{M\chi}(G - u) < \gamma_{M\chi}(G)$ then $u \in V^-_{M\chi}(G)$, which is a contradiction to $v \in V^+_{M\chi}(G)$.

Hence $\gamma_{M\chi}(G - u) = \gamma_{M\chi}(G)$.

Case (ii) Suppose $N[S] = V(G)$. Let $u \notin S$ and $u \in N[S]$. Then $|N_{G-u}[S]| = p - 1 \geq \left\lceil \frac{p-1}{2} \right\rceil$. Therefore S is a MDC set of $\{G - v\}$. Then $\gamma_{M\chi}(G - u) \leq |S| = \gamma_{M\chi}(G)$. If $\gamma_{M\chi}(G - u) < \gamma_{M\chi}(G)$ then $u \in V^-_{M\chi}(G)$ and

$V(G) \in V^{-}_{M\chi}(G)$, which is a contradiction to $v \in V^{+}_{M\chi}(G)$. Hence $\gamma_{M\chi}(G - u) = \gamma_{M\chi}(G)$.

Case (iii) Suppose $|N[S]| \leq V(G)$. Then there exists a vertex $u \in S$ and $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$. For $S - \{u\}, \chi(\langle S - v \rangle) < \chi(\langle S \rangle) = \chi(G)$ and S is not a $\gamma_{M\chi}$-set of G. Therefore choose $S_1 = S - \{u\} \cup \{w\}$ where $w \in V - S$ such that $|N[S_1]| \geq \left\lceil \frac{p}{2} \right\rceil$ and w is adjacent to any vertex of S with $|S_1| = |S|$. Hence S_1 is a $\gamma_{M\chi}$-set of $\{G - v\}$ and $\gamma_{M\chi}(G - u) = |S_1| = |S| = \gamma_{M\chi}(G)$.

5. Conclusion

In this article, it has been discussed that the removal of any vertex of a graph G how affects the majority dom-chromatic number of G. Also the vertex critical classifications $V^{0}_{M\chi}(G), V^{-}_{M\chi}(G)$ and $V^{+}_{M\chi}(G)$ are discussed. The characterisation theorems are also determined for $V^{0}_{M\chi}(G), V^{-}_{M\chi}(G)$ and $V^{+}_{M\chi}(G)$.

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Majority Dominating Chromatic Partition Number of a Graph

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Abstract
This paper introduces the concept majority dom-chromatic partition number of a graph G and denoted by $d_{Mx}(G)$. Also, $d_{M\chi}(G)$ value for some particular classes of graphs and bounds of $d_{M_X}(G)$ are investigated.

Keywords
Majority dominating chromatic set, Majority dominating chromatic number, Majority dom-chromatic partition number.

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Contents

1. Introduction

By a graph $G=(V,E)$, we mean a finite and undirected graph with neither loops nor multiple edges. This article introduces a new parameter namely majority dom-chromatic partition number of G. A subset D of $V(G)$ is said to be a dominating set [1] and [4] of G if every vertex in $(V-D)$ is adjacent to at least one vertex in D. The minimum cardinality of the minimal dominating set of G is called the domination number of G, denoted by $\gamma(G)$. A domatic partition of a graph G is a partition of $V(G)$ into dominating sets of G. The maximum number of sets of a domatic partition of G is called the domatic number of G, denoted by $d(G)$.

The majority dominating number $\gamma_M(G)$[8] of a graph G is the smallest cardinality of a minimal majority dominating set (MD-set) $S\subseteq V(G)$ of vertices and the set S satisfies $|N[S]|\geq ||\frac{V(G)}{2}||$. A majority domatic partition [6] of a graph G is a partition of the vertex set $V(G)$ into majority dominating sets of G. The maximum number of sets of majority domatic partition of G is called the majority domatic number of G, denoted by $d_M(G)$. A dominating set $S\subseteq V(G)$ is called the dom-chromatic set [2] and [3] such that the induced subgraph set $<S>$ satisfies the property $\chi(<S>)=\chi(G)$. The minimum cardinality of a dom-chromatic set S is called dom-chromatic number and is denoted by $\gamma_{ch}(G)$ or $\gamma_x(G)$. A dom-chromatic- partition [4] of a graph G is a partition of $V(G)$ into dom-chromatic sets. The maximum cardinality of a partition of $V(G)$ into dom-chromatic sets is the dom-chromatic -partition number and denoted by $d_{ch}(G)$.

A subset S of $V(G)$ is majority dominating chromatic set $(MDC-set)$[5] if (i) S is a majority dominating set and (ii) $\chi(<S>)=\chi(G)$. The minimum cardinality of a minimal majority dominating chromatic set is called a majority dom-chromatic number denoted by $\gamma_{MX}(G)$.

1.1 Results of $\gamma_{M\chi}(G)$ [5]

(i) For a graph $G=K_{1,p-1}, \gamma_{M\chi}(G)=2$.

(ii) For $G=W_p$ a wheel,

$$\gamma_{M\chi}(G)=\begin{cases} p & \text{, if } p \text{ is odd} \\ 3, & \text{if } p \text{ is even.}\end{cases}$$

(iii) Let $G=\overline{K_p}$ be a totally disconnected graph of p vertices. Then $\gamma_{M_X}\left(\overline{K_p}\right)=\left[\frac{p}{2}\right]$.

(iv) Let $G=C_p$ be a cycle of p vertice, $p\geq 3$. Then

$$\gamma_{M_X}(G)=\begin{cases} \left[\frac{p}{6}\right], & \text{if } p\equiv 2(\bmod 6) \\ \frac{p}{6}+1, & \text{if } p\equiv 0,4(\bmod 6) \\ p, & \text{if } p \text{ is odd.}\end{cases}$$

(v) Let $G = P_p$ be a path. Then

$$\gamma_{M_X}(G) = \begin{cases} \left\lceil \frac{p}{6} \right\rceil, & \text{if } p \equiv 1,2(\bmod 6) \\ \left\lceil \frac{p}{6} \right\rceil + 1, & \text{if } p \equiv 0,3,4,5(\bmod 6) \end{cases}$$

(vi) For a complete bipartite graph $G = K_{m,n}, m \leq n, \gamma_{M\chi}(G) = 2$

(vii) If a graph G is vertex color critical then $\gamma_{M\chi}(G) = p$.

2. Majority Dom-chromatic Partition Number in a Graph

Definition 2.1. *A majority dom-chromatic partition (MDC - Partition) of a graph G is a partition of $V(G)$ into majority dom-chromatic sets of G. The maximum cardinality of a partition of $V(G)$ into majority dom-chromatic sets is the majority dom-chromatic partition number and denoted by $d_{M\chi}(G)$.*

Example 2.2. *Let $G = \overline{K_6} + C_6 + C_6 + \overline{K_6}$ be a graph. Let $V\left(\overline{K_6}\right) = \{u_1, u_2, \ldots, u_6\}$ with $d(u_i) = 6$, $V(C_6) = \{v_1, v_2, \ldots, v_6\}$ with $d(v_i) = 13$, $V(C_6) = \{w_1, w_2, \ldots, w_6\}$ with $d(w_i) = 13$ and $V\left(\overline{K_6}\right) = \{x_1, x_2, \ldots, x_6\}$ with $d(x_i) = 6$. For the graph $\chi(G) = 3, \gamma_{ch}(G) = 3 = \gamma_{M\chi}(G)$. The dominating chromatic sets are $S_1 = \{v_1, v_2, w_1\}, S_2 = \{w_2, w_3, v_3\}, S_3 = \{v_4, v_5, w_4\}$ and $S_4 = \{w_5, w_6, v_6\}$ and the remaining vertex set*

$$R = \{u_1, u_2, \ldots, u_6, x_1, x_2, \ldots, x_6\}$$

will be the dominating set but the set R does not satisfies $\chi(< R >) = \chi(G)$. Hence there is no other disjoint dominating chromatic set exists. It implies that $d_{ch}(G) = 4$.

3. Main Results

3.1 Results on $d_{M\chi}(G)$.

(i) Let the graph $G = K_{1,p-1}$. Then $d_{M\chi}(G) = 1$.

(ii) If the graph $G = F_p$ is a Fan , $p \geq 3$ then $d_{M\chi}(G) = 1$.

(iii) For a complete graph $G = K_p$, $\quad d_{M\chi}(G) = 1$.

(iv) Let $G = W_p$ be a wheel , $p \geq 5$. Then $\quad d_{M\chi}(G) = 1$.

(v) For the graph $G = \overline{K_p}$, $d_{M_X}(G) = \begin{cases} 1, & \text{if } p \text{ is odd} \\ 2, & \text{if } p \text{ is even.} \end{cases}$

(vi) If the graph $G = D_{rs}$ a double star then $d_{MX}(G) = 2$, if $r \leq s$

(vii) Let $G = K_{m,n}, m \leq n$ be a complete bipartite graph. Then $d_{M\chi}(G) = \begin{cases} \frac{p}{2} & \text{, if } m = n \\ m & \text{, if } m < n \end{cases}$ viii) For a graph $G = P_p$, a path with $p \geq 3$ vertices,

$$d_{M\chi}(G) = \begin{cases} 1, & \text{if } p = 3 \\ 2, & \text{if } p = 4,5 \\ 3, & \text{if } 6 \leq p \leq 11, 15 \\ 4, & \text{if } p = 12, 13, 14, 33, 34 \\ & \text{and } 16 \leq p \leq 29 \\ 5, & \text{if } p = 30, 31, 32 \text{ and } p \geq 35 \end{cases}$$

Proposition 3.1. *Let $G = C_p$ be a cycle with $p \geq 3$. Then*

$$d_{M_X}(G) = \begin{cases} 1, & \textit{if } p \textit{ is odd} \\ 2, & \textit{if } p = 4 \\ 3, & \textit{if } p \equiv 6, 10 \\ 4, & \textit{if } p = 8, 12, 14, 16, 18, 22, 24, 28, 34 \\ 5, & \textit{if } p = 20, 26, 30, 32 \textit{ and } p \geq 36 \end{cases}$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ be the vertex set of G. For the graph G,

$$\chi(G) = \begin{cases} 1, & \text{if } p \text{ is odd} \\ 2, & \text{if } p \text{ is even} \end{cases}$$

and by the result $(1.1)(iv)$,

$$\gamma_{M\chi}(G) = \begin{cases} \left\lceil \frac{p}{6} \right\rceil, & \text{if } p \equiv 2(\bmod 6) \\ \left\lceil \frac{p}{6} \right\rceil + 1, & \text{if } p \equiv 0,4(\bmod 6) \\ p, & \text{if } p \text{ is odd.} \end{cases} \tag{3.1}$$

Case (i): Suppose p is odd. Then the all odd cycles $C_p, p \geq 3$ are vertex color critical graphs. By the result (1.1) (vii), $\gamma_{M\chi}(G) = p$. Hence $d_{M\gamma}(G) = 1$.
Case (ii): Let $p = 4$. Then $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_3, v_4\}$ be the only majority dominating chromatic partition set of G. Hence $d_{M\gamma}(G) = 2$.
Case (iii): Let $p = 6, 10$. For $p = 6$,

$$S = \{(v_1, v_2), (v_3, v_4), (v_5, v_6)\}.$$

For $p = 10$, $S = \{(v_1, v_2, v_7), (v_3, v_4, v_8), (v_5, v_6, v_9)\}$. Therefore S is the only majority dominating chromatic partition set of G for $p = 6, 10$. Hence $d_{M_X}(G) = 3$.
Case (v): Let $p = 20, 26, 30, 32$ and $p \geq 36$.
Subcase (i): Suppose $p = 20, 26, 30, 32$. By the result (3.1), When $p = 20, 26, 32$, (i.e) $p = 6k + 2$, $\left\lfloor \frac{p}{\gamma_{M\chi}(G)} \right\rfloor = 5$ if $k = 3, 4, 5$. When $p = 30$, (i.e.) $p = 6k$, $\left\lfloor \frac{p}{\gamma_{M\chi}(G)} \right\rfloor = 5$ if $k = 5$. Let

$$S_1 = \left\{v_1, v_2, \ldots, v_5\left(\gamma_{M\chi}(G) - 2\right) + 1' v_{5(\gamma_{MX}(G)-1)+1}\right\}$$
$$S_2 = \left\{v_3, v_4, \ldots, v_{5(\gamma_{MX}(G)-2)+2^2} v_{5(\gamma_{MX}(G)-1)+2}\right\}$$
$$S_3 = \left\{v_5, v_6, \ldots, v_5\left(\gamma_{M_X}(G) - 2\right) + \gamma' v_{5\left(\gamma_{M}\chi^{(G)-1}\right)+3}\right\}$$
$$S_4 = \left\{v_7, v_8, \ldots, v_{5\left(\gamma_{M}\chi^{(G)-2}\right)+4^v} v_{5\left(\gamma_{M_X}(G)-1\right)+4}\right\}$$
$$S_5 = \left\{v_9, v_{10}, \ldots, v_{5(\gamma_{MX}(G)-2)+5'} v_{5(\gamma_{MX}(G)-1)+5}\right\}$$

Now the sets $S_t t = 1,2,3,4,5$ are majority dominating chromatic sets of G such that $d(v_i, v_j) = 1$, where the first two vertices v_i and v_j are adjacent for all $S_t t = 1,2,3,4,5$ and $d(v_j, v_k) \geq 4, v_j \neq v_k, v_j, v_k \in S_t, t = 1,2,3,4,5$. Therefore in all five sets the last vertex is $v_5\left(\gamma_{M_X}(G) - 1\right) + i, i = 1,2,3,4,5$. Then $\left\{S_1, S_2, S_3, S_4 \cup \left(V(G) - U_{t=1}^{5} S_t\right)\right\}$ is a majority dominating chromatic partition of $V(G)$ and therefore $d_{M\chi}(G) \geq 5$. Since $d_{M_X}(G) \leq \left| \frac{p}{\gamma_{MX}(G)} \right\rfloor, d_{M\chi}(G) \leq 5$ Hence $d_{M_X}(G) = 5$.
Subcase (ii): Let $p \geq 36$. Let $p = 0,2,4(\bmod 6)$ By the result

$(1), \gamma_{M\chi}(G) = \left\lceil \frac{p}{6} \right\rceil$ and $\left\lceil \frac{p}{6} \right\rceil + 1$. When $p \geq 36, p = 6k, \left\lfloor \frac{p}{\gamma_{M\chi}(G)} \right\rfloor = 5$ if $k \geq 6$. When $p \geq 38, p = 6k+2, \left| \frac{p}{\gamma_M(G)} \right\rfloor = 5$ if $k \geq 6$. When $p \geq 36, p = 6k+4, \left\lfloor \frac{p}{\gamma_{MX}(G)} \right\rfloor = 5$ if $k \geq 6$. Then S_1, S_2, S_3, S_4 and S_5 are taken as in the subcase (i) and applying the same arguments, we get $d_{M\chi}(G) = 5$. Therefore $d_{M\chi}(G) = 5$ if $p = 20, 26, 30, 32$ and $p \geq 36$. □

Definition 3.2. *Let G be any graph with p vertices and the maximum degree $\Delta(G)$. If $d_{M_\chi}(G) = 2\Delta(G)+1$ then the graph G is called majority dogmatically chromatic full.*

Example 3.3. *Let $G = C_{20}$. By proposition (3.2), $d_{M_X}(G) = 5$ and $\Delta(G) = 2$. Hence $d_{M\chi}(G) = 2\Delta(G)+1 = 5$*

4. Bounds on $d_{M\chi}(G)$

Theorem 4.1. *Let G be any graph. Then $d_{M\chi}(G) \leq \left| \frac{p}{\gamma_M(G)} \right|$.*

Proof. Let $\{V_1, V_2, \ldots, V_k\}$ be the majority dominating chromatic partitions of G. Then, $p = |V_1| + |V_2| + \cdots + |V_k| = \sum_{i=1}^{k} |V_i|$. Let $d_{MX}(G) = k$. Therefore $|V_i| \geq \gamma_{MX}(G)$, for each i. $p = |V_1| + |V_2| + \cdots + |V_k| \geq k\gamma_{Mx}$ and $p \geq k\gamma_{M\chi}(G) \geq d_{M\chi}(G)\gamma_{M\chi}(G)$ Hence $d_{M_X}(G) \leq \left| \frac{p}{\gamma_{MX}(G)} \right|$. □

Corollary 4.2. *For any graph $G, d_{M\chi}(G)\gamma_{MX}(G) \leq p$.*

5. Charecterization Theorems on $d_{M\chi}(G)$

Theorem 5.1. *Let G be a cycle on p vertices. Then $d_{M\chi}(G) = \frac{p}{\gamma_{MX}(G)}$ if and only if*

(i) p is odd

(ii) $p = 4, 6, 8, 12, 16, 20, 30, 40$

Proof. Let $G = C_p$ be a cycle. By the result (1.1) (iv),

$$\gamma_{M\chi}(G) = \begin{cases} \left\lceil \frac{p}{6} \right\rceil, & \text{if } p \equiv 2(\bmod 6) \\ \left\lceil \frac{p}{6} \right\rceil + 1, & \text{if } p \equiv 0, 4(\bmod 6) \\ p, & \text{if } p \text{ is odd.} \end{cases} \quad (5.1)$$

Assume that $d_{M_\chi}(G) = \frac{p}{\gamma_{M\chi}(G)}$.

$$(i.e.) d_{M_X}(G)\gamma_{M_\chi}(G) = p \quad (5.2)$$

. **Case (i):** Suppose $d_{M_X}(G) = 1$. Since by (5.2), $\gamma_{M_\chi}(G) = p$. Then the majority dominating chromatic set contains the whole vertex set $V(G)$. It implies that the graph G is vertex color critical. By proposition (3.2), in the graph $C_p, d_{M_X}(G) = 1$ if p is odd. Hence condition (i) holds.

Case (ii): Let $d_{M_\chi}(G) = 2$. Then by proposition (3.2), if $d_{M_\chi}(G) = 2$ then $p = 4$ and $\gamma_{M\chi}(G) = 2$. Substitute in (5.2), $d_{M_X}(G)\gamma_{M_\chi}(G) = 2(2) = 4 = p$. It implies $p = 4$. Hence (2) is true for $p = 4$.

Case (iii): If $d_{M_\chi}(G) = 3$ then by proposition $(3.2), p = 6, 10. By (5.1), \gamma_{M\chi}(G) = \left\lceil \frac{p}{6} \right\rceil + 1$ if $p = 6, 10$. Then $\gamma_{M\chi}(G) = 2$ and 3. From the assumption $(5.2), d_M(G)\gamma_M(G) = 3(2) = 6 = p$ and $d_{M_X}(G)\gamma_{M_X}(G) = 3(3) = 9 < p$. Hence if $d_{M_X}(G) = \frac{p}{\gamma_{MX}(G)}$ then $p = 6$ only.

Case (iv): Let $d_{M\chi}(G) = 4$. Then by proposition $(3.2), p = 8, 12, 14, 16, 18, 22, 24, 28, 34$. When $p = 6k+2$, by (5.1) $\gamma_M(G) = 2$ and 3 if $k = 1$ and 2 . When $p = 6k$ $\gamma_{M_X}(G) = 3, 4, 5$ if $k = 2, 3, 4$. When $p = 6k+4$, by $(1), \gamma_{M\chi}(G) = 4, 5, 6, 7$ if $k = 2, 3, 4, 5$ Then $d_{MX}(G)\gamma_{M\chi}(G) = p$ if $p = 8, 12, 16$. For all other vertices $, d_{M\chi}(G)\gamma_{M\chi}(G) < p$. Hence if

$$d_{M_X}(G) = \frac{p}{\gamma_M(G)}$$

then $p = 8, 12, 16$. □

Theorem 5.2. *Let G be a Path on p vertices. Then $d_{M_X}(G) = \frac{p}{\gamma_M(G)}$ if and only if $p = 4, 6, 9, 12, 16, 30, 35, 40, 45$.*

Proof. Applying the same argument as in the theorem 5.2., we obtain the result.

Subcase (i): Suppose there exists a vertex color critical subgraph in G. Then G contains a complete graph or odd cycles or an induced subgraph in G. Therefore by result (3.1)(iii) and the proposition (3.2), $d_{MX}(G) = 1$. Hence if the graph G contains a full degree vertex then $d_{M\chi}(G) = 1$.

Subcase (ii): Suppose $|S_1| = 2$. If G is a tree $\chi(G) = 2$. Therefore $S_1 = \{u_1, u_2\}$, where u_1 is of degree $d(u_1) \leq p-1$ and u_2 is of degree $d(u_2) \geq 1$ such that $|N[S_1]| = p$. Hence the graph G contains a full degree vertex. If $d(u_1) = p-1$ and $d(u_2) = 1$ then $|N[S_1]| = p > \left\lceil \frac{p}{2} \right\rceil$. Therefore G contains a full degree vertex u_1. If $d(u_1) < p-1$ and $d(u_2) \geq 1$ then there are two disjoint majority dominating chromatic sets and $d_{M\chi}(G) = 2$, which is a contradiction to (5.1). Hence G contains a full degree vertex with $d(u_1) = p-1$. Therefore condition (i) holds.

Subcase (iii): If $|S_1| = 3$, the graph G is a tree or it contains a triangle. If G is a tree, $S_1 = \{u_1, u_2, u_3\}$ is the majority dominating chromatic set of G. Suppose $d(u_1) < p-1$ and $d(u_i) \geq 1, i = 2, 3$. Then there exists atleast two disjoint majority dominating chromatic set in G. Hence $d_{M_X}(G) \geq 2$, which is a contradiction to (5.1). Suppose G contains a triangle, $\chi(G) = 3$ and $\gamma_{M_X}(G) \geq 3$. since $d(u_1) \leq p-1$ and $d(u_i) \geq 2, i = 2, 3$, S_1 is a majority dominating chromatic set of G. By the above arguments, $d_{M\chi}(G) \geq 2$, which is a contradiction to (5.1). Hence the set $S_1, d(u_1) = p-1$ and $d(u_i) \geq 2$. Therefore G contains a full degree vertex. Thus condition (i) holds. □

6. Conclusion

In this article, a new parameter majority dominating chromatic partition for a graph G is introduced. Majority dom-chromatic sets in the partition of the vertex set $V(G)$ is studied. Majority dom-chromatic partition number $d_{M\chi}(G)$ is determined for some families of graphs. Also the relationships of $d_{M\chi}(G)$ among the other domatic partition such as $d_{ch}(G), d_M(G)$ and $d(G)$ established.

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