CONTRIBUTIONS TO THE STUDY ON CIRCULANT POLYNOMIAL MATRICES

A THESIS SUBMITTED TO

BHARATHIDASAN UNIVERSITY, THIRUCHIRAPPALLI

IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE AWARD OF THE DEGREE OF

DOCTOR OF PHILOSOPHY IN MATHEMATICS

By

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(Ref.No.29381/Ph.D.K3/Mathematics/Part- Time/October 2016/Date 18.10.2016)

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FEBRUARY – 2022

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CERTIFICATE

This is to certify that the thesis entitled **"CONTRIBUTIONS TO THE STUDY ON CIRCULANT POLYNOMIAL MATRICES"** submitted to Bharathidasan University, Thiruchirappalli, for the award of the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICS** is a bonafide record of research work carried out by **R.MUTHAMILSELVAM**, under my guidance and supervision in the Ramanujan Research Centre, Post Graduate and Research Department of Mathematics, Government Arts College (Autonomous), Kumbakonam–612 002, Tamil Nadu, India.

I further certify that no part of the thesis has been submitted anywhere else for the award of any degree, diploma, associateship, fellowship or other similar titles to any candidate.

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DECLARATION

I do hereby declare that this work has been originally carried out by me under the guidance and supervision of **Dr. G. RAMESH,** Associate Professor of Mathematics, Ramanujan Research Centre, Post Graduate and Research Department of Mathematics, Government Arts College (Autonomous), Kumbakonam – 612 002, and this work has not been submitted elsewhere for the award of any other degree or diploma.

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Document Information

Analyzed document	R. MUTHAMILSELVAM- Intro & Review.pdf (D126911459)
Submitted	2022-02-03T05:40:00.0000000
Submitted by	Dr.S.Vanitha
Submitter email	vanitha@bdu.ac.in
Similarity	6%
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Document Information

Analyzed document	R. MUTHAMILSELVAM- Chapter II - Conclusion.pdf (D126911458)
Submitted	2022-02-03T05:40:00.0000000
Submitted by	Dr.S.Vanitha
Submitter email	vanitha@bdu.ac.in
Similarity	5%
Analysis address	navitha.bdu@analysis.urkund.com

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ACKNOWLEDGEMENT

The writing of a thesis for Doctor of Philosophy is indeed a significant dream to everyone, especially for those who are engaged in academic ambience. It being the time for me to write a thesis, I would like to take this golden opportunity to express my profound gratitude and deep regards to all who directly or indirectly has helped me to touch this milestone of my life. However, in this regard the following personages deserve special mention indeed.

First and foremost, I would like to express my deepest appreciation to my honorable supervisor **Dr. G. Ramesh,** Associate Professor of Mathematics, Ramanujan Research Centre, Department of Mathematics, Government Arts College (Autonomous), Kumbakonam for the continuous support and encouragement for my Ph. D study and related research, for his patient motivation and immense knowledge. His guidance helped me throughout my research and writing of this thesis. I am greatly indebted to him for making me complete my research.

This thesis would not have been possible without the help, support and patience of our principal **Dr. K. Duraiarasan, M.A., M.Ed., M.Phil., Ph.D.,** Government Arts College (Autonomous), Kumbakonam, who has facilitated every need in my research.

I take this opportunity to sincerely acknowledge my Doctoral committee members **Dr. J. Malarvizhi,** Principal, Government Arts College, Ariyalur, **Dr. B. Vasudevan,** Assistant Professor of Mathematics, Thiru.Vi.Ka. Government Arts College, Thiruvarur. I also want to thank them for letting my defense be an enjoyable moment and for their excellent comments and valuable suggestions.

I am deeply indebted to express my hearty thanks to all the staff members of Department of Mathematics especially to **Dr. R.Vijayakumar**, Assistant Professor, Ramanujan Research Centre, Government Arts College (Autonomous), Kumbakonam.

I am very much thankful to **Sri. R. Thirunavukkarsu,** Founder, **Sri. T. Senthil Kumar,** Chairman, **Prof. S. Kothandapani,** Advisor, **Dr. T. Balamurugan,** Principal, **Dr. Kalaimani Shanmugam,** Vice Principal and **Dr. M. Rukmangathan,** Dean of Academics, Arasu Engineering College, Kumbakonam for their constant support and motivation.

I thank the authorities of Bharathidasan University for providing all infrastructures for carrying out my research work. I thank the EDITORS and REFEREES of various International Journals who have published my research articles.

I would like to express my special thanks to my well-wishers **Dr. Vijayaselvi Ramesh, Miss. R. Varshini B.E** and **Mr. R. Shanmuga Priyan** for their understanding, encouragement and personal attention which have provided good and smooth basis for my Ph.D. tenure.

Last but not the least, a special thanks to my family. Words cannot express how grateful I am to my father **Mr. P. Rajangam**, mother **Mrs. R. Malarkodi**, brother **Mr. Aravazhi**, sister **Miss. R. Jayaruthradevi**, wife **Dr. P. Maduranthaki** and my children master **M. Shethrabaalan** and **M.M.Rupesh**.

I would also like to thank all of my friends who supported me in writing, and incented me to strive towards my goal. Besides this, several people have knowingly and unknowingly helped me in the successful completion of this project.

Kumbakonam (**R. MUTHAMILSELVAM**)

PREFACE

This thesis consisting of five chapters is primarily confined to a study on circulant polynomial matrices.

Review of literature, notations, preliminaries and summary of results obtained in the thesis are given in chapter one.

In chapter two, the concept of k-circulant and (r,s) -pair circulant polynomial matrices is introduced. Some important properties of k-circulant and (r,s) -pair circulant matrices are extended to k-circulant and (r,s)-pair circulant polynomial matrices.

Chapter three is devoted to develop, different kinds of circulant polynomial matrices. Such as hermitian, normal and conjugate normal circulant polynomial matrices. Some characterization of hermitian, normal and conjugate normal circulant polynomial matrices are obtained.

Block circulant and circulant block polynomial matrices are introduced in chapter four. Some important results of block circulant and circulant block polynomial matrices are also given.

In chapter five we have presented, an application of a circulant polynomial matrix, to find the solution for travelling salesman problem.

CONTENTS

Chapter I

Introduction

In this chapter, review of literature, notations, basic definitions, primary results used elsewhere in the thesis and summary of results obtained are given.

1.1 Review of Literature

The introduction and development of the notion of a matrix and the subject of linear algebra followed the development of determinants, which arose from the study of co –efficient of systems of linear equations.

Although the origin of the theory of matrices can be traced back to the 18^{th} century and although it was not until the 20^{th} century that it had become sufficiently absorbed into mathematical mainstream to warrant extensive treatment in text books and monographs, it was truly a creation of the 19^{th} century.

The French mathematician Cayley discovered matrices in the year of 1860. Matrices have been found to be great utility in many branches of applied Mathematics, physics and Engineering.

Matrices at the end of the 19^{th} century were heavily connected with physics issues and for mathematicians, more attention was given to vectors as the proved to be basic mathematical elements. For a time, however, interest in a lot of Linear Algebra showed until the end of world War-II brought on the development of computers. Now instead of having to break down an enormous $n \times n$ matrix, computers could quickly and accurately solve these systems of linear algebra. With the advancement of technology using the methods of Cayley, Gauss, Leibnitz, Euler, and others, determinants and linear algebra moved forward more quickly and more effective.

In matrix theory, we come across many special types of matrices. Let $A \in C_{n \times n}$ be a complex matrix. It is symmetric if $A = A^T$ and is skew symmetric if $A = -A^T$ where A^T is the transpose of A.

Loo-Kong Hna established the simplectic classification of hermitian matrices, which has applications to the geometry of symmetric matrices. A complex matrix of order n is said to be hermitian if the conjugate transpose of A equals to A.

In 1918, the concept of a normal matrix with entries from the complex field was introduced by Toeplitz $[51]$ who gave necessarily and sufficient conditions for a complex matrix to be normal. A complex square matrix A is normal if $AA^* = A^*A,$ where A^* is the conjugate transpose of A. That is, a matrix is normal if it commutes with its conjugate transpose. A matrix A with real entries satisfies $A^* = A^T,$ and is therefore normal if

$A^T A = AA^T$. This concept of normal was introduced as a generalization of hermitian matrices and the singular values of normal matrices were studied in $[5, 24, 31, 48]$. The importance of normal matrices explains the appearance of the survey $[47]$. As put forth by Robert grone.et.al. hoped that it will be useful to a wide audience and presented that, a long list of conditions on an $n \times n$ complex matrix A, equivalent to it's being normal. In most cases, a description of why the condition is equivalent to normality is given in$[47]$. Elsner and Ikramov $[15]$ have a list of 70 conditions on an $n \times n$ complex matrix A equivalent to its being normal, published in 1987 by Robert Grone.et.al. in $[47]$ and it has been proved to be very useful.

In multidimensional system theory, problem related to multivariable control system invertibility require the use of generalized inverse of matrices whose elements are polynomials in several variable with coefficients over a real field (or) a rational field (or) on integral domain of integers polynomials and polynomial matrices arise naturally as modeling tools in several arrears of applied mathematics, science and engineering specially in systems theory $[7, 21, 26-30, 35, 53, 55]$.

In this thesis the concept of circulant polynomial matrices is introduced as a generalization of circulant matrices. A $n \times n$ circulant polynomial matrix is formed by cycling its entries until (n-1) new rows are formed $[13,22,25,32]$. Circulant polynomial matrices share a relationship with a special permutation polynomial matrix $\pi(\lambda)$ (with polynomial of degree zero). Knowing the relationship that these polynomial matrices hold is enough to write $A(\lambda)$ as a linear combination of the powers of $\pi(\lambda)$. That is, a circulant polynomial matrix $A(\lambda)=circ(a_1(\lambda),a_2(\lambda),a_3(\lambda),...,a_n(\lambda))$ can be expressed as $A(\lambda)=a_1(\lambda)I_n(\lambda)+a_2(\lambda)\pi_n(\lambda)+a_3(\lambda)\pi_n^2(\lambda)+...+a_n(\lambda)\pi_n^{n-1}(\lambda)$. It is easy to observe that raising the powers of $\pi(\lambda)$ to 0 to n provides the matrices necessary to produce $A(\lambda)$ as a linear combination of the powers of $\pi(\lambda)$. Each $a_j(\lambda)$ component can be multiplied by its corresponding $\pi(\lambda)$ polynomial matrix always one less degree than its subscript. We proved the basic properties of circulant polynomial matrix. Also, we obtained that circulant polynomial matrices commutes under multiplication.

We found the concept of k-circulant polynomial matrix. We have given the important characterization of k- circulant polynomial matrix such as $A(\lambda)$ is a k-circulant polynomial matrix if and only if $\pi(\lambda)A(\lambda)=A(\lambda)\pi^k(\lambda)$. We have introduced a new type of polynomial matrix called (r, s)-pair circulant polynomial matrix, which is a generalization of the k- circulant polynomial matrix. Some properties such as sum, difference, product, inverse and adjoint of (r, s)- pair circulant polynomial matrices are investigated. Moreover, we give some necessary and sufficient conditions for a matrix to be an (r, s)-pair circulant polynomial matrix. That is, a polynomial matrix $A(\lambda) \in C^{n\times n}(\lambda)$ is an (r,s) pair circulant polynomial matrix if and only if $A(\lambda)\mathfrak{B}(\lambda) = \mathfrak{B}(\lambda)A(\lambda)$. Also, we discuss about the Hermitian, normal and conjugate normal circulant polynomial matrices. We have derived the properties of hermitian, normal and conjugate normal circulant polynomial matrices.

A representation of block circulant polynomial matrices can be developed as b circ $\left(A_1(\lambda), A_2(\lambda), ..., A_n(\lambda)\right) = \sum_{k=0}^{m-1}\left[\pi_m^k(\lambda) \otimes A_{K+1}(\lambda)\right]$. Also, some characterizations for block circulant, circulant block and block circulant matrices: where the blocks are circulant polynomial matrices are discussed.

Finally, an application of circulant polynomial matrices in travelling salesman problems is given.

1.2 Notations and Preliminaries

In this section, the notations, definitions and theorems used in the thesis are given.

$C_{n\times n}$	:	The space of $n\times n$ complex matrices of order n
$C_{n\times n}(\lambda)$	:	The space of $n\times n$ complex polynomial matrices of order n
C_n	:	The space of complex n-tuples
$I_n(\lambda)$	:	identify polynomial matrix of order n
$O(\lambda)$	:	Zero Polynomial Matrix
$A^T(\lambda)$	:	The transpose of $A(\lambda)$
$\overline{A(\lambda)}$	:	The Conjugate of $A(\lambda)$
$A^*(\lambda)$	:	The Conjugate transpose of $A(\lambda)$
adj $A(\lambda)$	:	Adjoint of $A(\lambda)$
det $A(\lambda)$	:	Determinant of $A(\lambda)$
$A^{-1}(\lambda)$	:	Inverse of $A(\lambda)$
$A^{\dagger}(\lambda)$	:	Generalized inverse `of $A(\lambda)$
$A(\lambda)\otimes B(\lambda)$	:	Kronecker product of $A(\lambda)$ and $B(\lambda)$
tr $A(\lambda)$	:	Trace of $A(\lambda)$

$circ\left(a_1(\lambda),a_2(\lambda),\ldots,a_n(\lambda)\right)$: Circulant polynomial matrix

$\pi(\lambda)$: Permutation matrix (with polynomial of degree zero)

$C_{(r,s)}\left(a_0(\lambda),a_1(\lambda),\ldots,a_{n-1}(\lambda)\right)$: (r, s) -pair circulant polynomial matrix

$\mathcal{B}(\lambda)$: Basic $(r,s)-$pair circulant polynomial matrix.

$F(\lambda)$: Fourier Polynomial matrix

$F^*(\lambda)$: The Conjugate transpose of $F(\lambda)$

$\Lambda_k(\lambda)$: diag (0,0,0,1, 0,1, …, 0), 1 is in the k^{th} position

$\Omega(\lambda)$: diag $\left(1,w(\lambda),w^2(\lambda),\ldots,w^{n-1}(\lambda)\right)$

$\mathscr{BC}_{\mathrm{m,n}}(\lambda)$: Block circulant polynomial matrix

$\mathscr{CB}_{\mathrm{m,n}}(\lambda)$: Circulant block polynomial matrix

$\mathscr{CPBC}_{m,\mathrm{n}}(\lambda)$: Block circulant matrix with circulant polynomial matrices as its blocks.

$V(G)$: Vertex set of G

$E(G)$: Edge set of G

$\delta(G)$: Minimum degree of G

$\Delta(G)$: Maximum degree of G

ν : Number of vertices

ε : Number of edges

Definition 1.2.1

A polynomial matrix $A(\lambda)$ of order n is called circulant polynomial matrix if it is of the form

$$A(\lambda)=\begin{bmatrix} a_1(\lambda) & a_2(\lambda) & a_3(\lambda) & \cdots & a_n(\lambda) \\ a_n(\lambda) & a_1(\lambda) & a_2(\lambda) & \cdots & a_{n-1}(\lambda) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_2(\lambda) & a_3(\lambda) & a_4(\lambda) & \ldots & a_1(\lambda) \end{bmatrix}$$

Which is denoted by $A(\lambda)=circ\left(a_1(\lambda),a_2(\lambda),\ldots,a_n(\lambda)\right)$.

Example 1.2.2

A 3×3 circulant polynomial matrix of degree 2 is as follows

$$A(\lambda)=\begin{pmatrix} 1+2\lambda+\lambda^2 & 2+6\lambda+3\lambda^2 & 3+4\lambda+4\lambda^2 \\ 3+4\lambda+4\lambda^2 & 1+2\lambda+\lambda^2 & 2+6\lambda+3\lambda^2 \\ 2+6\lambda+3\lambda^2 & 3+4\lambda+4\lambda^2 & 1+2\lambda+\lambda^2 \end{pmatrix}$$

$$=A_0+A_1\lambda+A_2\lambda^2$$

Where $A_0=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$, $A_1=\begin{pmatrix} 2 & 6 & 4 \\ 4 & 2 & 6 \\ 6 & 4 & 2 \end{pmatrix}$, $A_2=\begin{pmatrix} 1 & 3 & 4 \\ 4 & 1 & 3 \\ 3 & 4 & 1 \end{pmatrix}$

Definition 1.2.3

The identity circulant polynomial matrix of order n, denoted by $\left[I_n(\lambda)\right]$ is the identity circulant polynomial matrix which is of degree zero.

Definition 1.2 4

A circulant polynomial matrix whose elements are all equal to 0, is the null (or) zero circulant polynomial matrix which is denoted by $O(\lambda)$.

Definition 1.2.5

Let $A(\lambda)=\left[a_{ij}(\lambda)\right]$ and $B(\lambda)=\left[b_{ij}(\lambda)\right]$ be two $n\times n$ circulant polynomial matrices. Then the sum $A(\lambda)+B(\lambda)$ is defined to be the circulant polynomial matrix $C(\lambda)=\left[C_{ij}(\lambda)\right]$ with $C_{ij}(\lambda)=a_{ij}(\lambda)+b_{ij}(\lambda)$.

1.2.6 Properties of Circulant Polynomial Matrix Addition

(i) Circulant polynomial matrix addition is commutative.

If $A(\lambda)$ and $B(\lambda)$ are $n\times n$ circulant polynomial matrices then $A(\lambda)+B(\lambda)=B(\lambda)+A(\lambda)$.

(ii) Circulant polynomial matrix addition is associative.

If $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are $n\times n$ circulant polynomial matrices then $\left[A(\lambda)+B(\lambda)\right]+C(\lambda)=A(\lambda)+\left[B(\lambda)+C(\lambda)\right]$.

(iii) Existence of additive identity.

If $O(\lambda)$ is the $n\times n$ zero circulant polynomial matrix each of whose elements is zero then $A(\lambda)+O(\lambda)=O(\lambda)+A(\lambda)=A(\lambda)$ for every $n\times n$ circulant polynomial matrix $A(\lambda)$.

(iv) Existence of additive inverse.

Let $A(\lambda)$ be $n\times n$ circulant polynomial matrix. Then the negative of the circulant polynomial matrix $A(\lambda)$ is denoted by $-A(\lambda)$. Then $-A(\lambda)$ is the additive inverse of the circulant polynomial matrix $A(\lambda)$.

That is, $-A(\lambda)+A(\lambda)=A(\lambda)-A(\lambda)=O(\lambda)$.

(v) $k\left[A(\lambda)+B(\lambda)\right]=kA(\lambda)+kB(\lambda)$ for all scalars k and all $n\times n$ circulant polynomial matrices $A(\lambda)$ and $B(\lambda)$.

(vi) $(k_1+k_2)A(\lambda)=k_1A(\lambda)+k_2A(\lambda)$, for all scalars k_1 and k_2.

Proposition 1.2.7

Let $A(\lambda)$ and $B(\lambda)$ be $n\times n$ circulant polynomial matrices. Then for any scalars α and β, $\alpha A(\lambda)+\beta B(\lambda)$ is a circulant polynomial matrix.

Proof

Let $A(\lambda) = circ(a_1(\lambda),\, a_2(\lambda),\ldots,\, a_n(\lambda))$ and $B(\lambda) = circ(b_1(\lambda),\, b_2(\lambda),\ldots,\, b_n(\lambda))$ be two circulant polynomial matrices and α and β be any scalars.

Now

$$\alpha A(\lambda)+\beta B(\lambda)=\alpha\, circ(a_1(\lambda),a_1(\lambda),\ldots,a_n(\lambda))+\beta\, circ(b_1(\lambda),b_1(\lambda),\ldots,b_n(\lambda))$$

$$= circ(\alpha a_1(\lambda),\alpha a_2(\lambda),\alpha a_3(\lambda),\ldots,\alpha a_n(\lambda))+ circ(\beta b_1(\lambda),\beta b_2(\lambda),\beta b_3(\lambda),\ldots,\beta b_n(\lambda))$$

$$= circ(\alpha a_1(\lambda)+\beta b_1(\lambda),\alpha a_2(\lambda)+\beta b_2(\lambda),\ldots,\alpha a_n(\lambda)+\beta b_n(\lambda))$$

Thus, $\alpha A(\lambda)+\beta b(\lambda)$ is a circulant polynomial matrix.

Definition 1.2.8

Two circulant polynomial matrices $A(\lambda)$ and $B(\lambda)$ are said to be equal if

(i) They have the same size.

(ii) Corresponding entries are equal polynomials.

Theorem 1.2.9

The $n\times n$ polynomial matrix $A(\lambda)$ is a circulant polynomial matrix if and only if $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{'}(\lambda)$.

Theorem 1.2.10

The circulant polynomial matrix $A(\lambda)=circ\left(a_1(\lambda),a_2(\lambda),a_3(\lambda),...,a_n(\lambda)\right)$ can be expressed as $A(\lambda)=a_1(\lambda)I_n(\lambda)+a_2(\lambda)\pi_n(\lambda)+a_3(\lambda)\pi_n^2(\lambda)+...+a_n(\lambda)\pi_n^{n-1}(\lambda).$

Proof

Given that $A(\lambda)=circ\left(a_1(\lambda),a_2(\lambda),a_3(\lambda),...,a_n(\lambda)\right)$ is a circulant polynomial matrix.

That is, $A(\lambda)=\begin{pmatrix} a_1(\lambda) & a_2(\lambda) & a_3(\lambda) & \cdots & a_n(\lambda) \\ a_n(\lambda) & a_1(\lambda) & a_2(\lambda) & \cdots & a_{n-1}(\lambda) \\ a_{n-1}(\lambda) & a_n(\lambda) & a_1(\lambda) & \cdots & a_{n-2}(\lambda) \\ & & \vdots & & \\ a_2(\lambda) & a_3(\lambda) & a_4(\lambda) & \cdots & a_1(\lambda) \end{pmatrix}$

Now

$$\begin{aligned} A(\lambda)=a_1(\lambda)I_n(\lambda)+a_2(\lambda)\left(e_n(\lambda), e_1(\lambda), e_2(\lambda),..., e_{n-1}(\lambda)\right)+ \\ a_3(\lambda)\left(e_{n-1}(\lambda), e_n(\lambda), e_1(\lambda), ..., e_{n-2}(\lambda)\right)+... \\ +a_n(\lambda)\left(e_2(\lambda), e_3(\lambda), e_4(\lambda), ..., e_n(\lambda), e_1(\lambda)\right) \end{aligned}$$

Here $\pi_n(\lambda)=\left(e_n(\lambda), e_1(\lambda), e_2(\lambda),..., e_{n-1}(\lambda)\right)$

$$=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ & & \vdots & & \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\pi_n^2(\lambda) = \pi_n(\lambda)\pi_n(\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ & & \vdots & & & \\ 0 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \left(e_{n-1}(\lambda), e_n(\lambda), e_1(\lambda), ..., e_{n-2}(\lambda)\right)$$

Since the post multiplication of any $n \times n$ polynomial matrix by $\pi_n(\lambda)$ shifts the columns of that matrix one place to the right.

Therefore, we find that

$$\pi_n^2(\lambda) = \left(e_{n-1}(\lambda), e_n(\lambda), e_1(\lambda), ..., e_{n-2}(\lambda)\right)$$
$$\vdots$$
$$\pi_n^{n-1}(\lambda) = \left(e_2(\lambda), e_3(\lambda), e_4(\lambda), ..., e_n(\lambda), e_1(\lambda)\right)$$

Hence,

$$A(\lambda) = a_1(\lambda) I_n(\lambda) + a_2(\lambda)\pi_n(\lambda) + a_3(\lambda)\pi_n^2(\lambda) + ... + a_n(\lambda)\pi_n^{n-1}(\lambda).$$

Example 1.2.11

Let $A(\lambda) = \begin{pmatrix} 1+3\lambda+5\lambda^2 & 7\lambda+2\lambda^2 & 2+4\lambda^2 \\ 2+4\lambda^2 & 1+3\lambda+5\lambda^2 & 7\lambda+2\lambda^2 \\ 7\lambda+2\lambda^2 & 2+4\lambda^2 & 1+3\lambda+5\lambda^2 \end{pmatrix}$

$$= A_0 + A_1\lambda + A_2\lambda^2$$

Where $A_0 = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 3 & 7 & 0 \\ 0 & 3 & 7 \\ 7 & 0 & 3 \end{pmatrix}$, $A_2 = \begin{pmatrix} 5 & 2 & 4 \\ 4 & 5 & 2 \\ 2 & 4 & 5 \end{pmatrix}$

Now $a_1(\lambda)I_3(\lambda)+a_2(\lambda)\pi_3(\lambda)+a_3(\lambda)\pi_3^2(\lambda)$

$$=\left(1+3\lambda+5\lambda^2\right)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\left(7\lambda+2\lambda^2\right)\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}+\left(2+4\lambda^2\right)\begin{pmatrix}0&0&1\\1&0&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1+3\lambda+5\lambda^2&0&0\\0&1+3\lambda+5\lambda^2&0\\0&0&1+3\lambda+5\lambda^2\end{pmatrix}+\begin{pmatrix}0&7\lambda+2\lambda^2&0\\0&0&7\lambda+2\lambda^2\\7\lambda+2\lambda^2&0&0\end{pmatrix}$$
$$+\begin{pmatrix}0&0&2+4\lambda^2\\2+4\lambda^2&0&0\\0&2+4\lambda^2&0\end{pmatrix}$$

$$=\begin{pmatrix}1+3\lambda+5\lambda^2&7\lambda+2\lambda^2&2+4\lambda^2\\2+4\lambda^2&1+3\lambda+5\lambda^2&7\lambda+2\lambda^2\\7\lambda+2\lambda^2&2+4\lambda^2&1+3\lambda+5\lambda^2\end{pmatrix}=A(\lambda).$$

Proposition 1.2.12

Let $A(\lambda)$ and $B(\lambda)$ be $n\times n$ circulant polynomial matrices. Then the product $A(\lambda)B(\lambda)$ is also a circulant polynomial matrix.

Proof

Let $A(\lambda)=circ\left(a_1(\lambda),\, a_2(\lambda),\, a_3(\lambda),...,\, a_n(\lambda)\right)$ and $B(\lambda)=circ\left(b_1(\lambda),\, b_2(\lambda),\, b_3(\lambda),...,\, b_n(\lambda)\right)$ be two circulant polynomial matrices.

By theorem (1.2.9), we must have both

$$A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{'}(\lambda) \text{ and } B(\lambda)=\pi_n(\lambda)B(\lambda)\pi_n^{'}(\lambda).$$

Now $A(\lambda)B(\lambda)=\left(\pi_n(\lambda)A(\lambda)\pi_n^{'}(\lambda)\right)\left(\pi_n(\lambda)B(\lambda)\pi_n^{'}(\lambda)\right)$

$$=\pi_n(\lambda)A(\lambda)\left(\pi_n^{'}(\lambda)\pi_n(\lambda)\right)B(\lambda)\pi_n^{'}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)I_n(\lambda)B(\lambda)\pi_n^{'}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)B(\lambda)\pi_n^{'}(\lambda)$$

Hence, $A(\lambda)\,B(\lambda)$ is a circulant polynomial matrix.

Result 1.2.13

If $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are circulant polynomial matrices then the following properties hold.

(i) $\left[A(\lambda)B(\lambda)\right]C(\lambda)=A(\lambda)\left[B(\lambda)C(\lambda)\right]$

(ii) $A(\lambda)\left[B(\lambda)+C(\lambda)\right]=A(\lambda)B(\lambda)+A(\lambda)C(\lambda).$

Proposition 1.2.14

Let $A(\lambda)$ be $n\times n$ circulant polynomial matrix. Then for any positive integer r, $A^r(\lambda)$ is circulant polynomial matrix.

Proof

Since $A(\lambda)$ is a circulant polynomial matrix.

By theorem (1.2.9), we have $A(\lambda)=\pi_m(\lambda)A(\lambda)\pi_m^{'}(\lambda)$. But $\pi_m(\lambda)$ is an orthogonal polynomial matrix, so

$$A^r(\lambda) = \left(\pi_m(\lambda) A(\lambda) \pi_m^{'}(\lambda)\right)^r$$

$$= \pi_m^r(\lambda) A^r(\lambda) \left(\pi_m^{'}\right)^r(\lambda)$$

$$= \pi_m(\pi) A^r(\lambda) \pi_m^{'}(\lambda)$$

Hence, $A^r(\lambda)$ is a circulant polynomial matrix.

Theorem 1.2.15

Suppose that $A(\lambda)$ and $B(\lambda)$ are $n \times n$ circulant polynomial matrices. Then their product commutes. That is, $A(\lambda)B(\lambda) = B(\lambda)A(\lambda)$.

Proof

Given that $A(\lambda)$ and $B(\lambda)$ are $n \times n$ circulant polynomial matrices. We have to prove that $A(\lambda)B(\lambda) = B(\lambda)A(\lambda)$.

If $A(\lambda) = circ\left(a_1(\lambda), a_2(\lambda), a_3(\lambda), ..., a_n(\lambda)\right)$ and $B(\lambda) = circ\left(b_1(\lambda), b_2(\lambda), b_3(\lambda), ..., b_n(\lambda)\right)$ are two circulant polynomial matrices, then from theorem (1.2.10), we find that $A(\lambda) = \sum_{i=1}^{n} a_i(\lambda) \pi_n^{i-1}(\lambda)$, $B(\lambda) = \sum_{j=1}^{n} b_j(\lambda) \pi_n^{j-1}(\lambda)$ where $\pi_n^0(\lambda) = I_n(\lambda)$.

Now $A(\lambda)B(\lambda) = \left(\sum_{i=1}^{n} a_i(\lambda) \pi_n^{i-1}(\lambda)\right)\left(\sum_{j=1}^{n} b_i(\lambda) \pi_n^{j-1}(\lambda)\right)$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_i(\lambda) \pi_n^{i-1}(\lambda)\right)\left(b_j(\lambda) \pi_n^{j-1}(\lambda)\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i(\lambda) b_j(\lambda) \pi_n^{i-1+j-1}(\lambda)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i(\lambda) b_j(\lambda) \pi_n^{i+j-2}(\lambda)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(b_j(\lambda) \pi_n^{j-1}(\lambda)\right)\left(a_i(\lambda) \pi_n^{i-1}(\lambda)\right)$$

$$= \left(\sum_{j=1}^{n} b_j(\lambda) \pi_n^{j-1}(\lambda)\right)\left(\sum_{i=1}^{n} a_i(\lambda) \pi_n^{i-1}(\lambda)\right)$$

$$= B(\lambda)A(\lambda).$$

Thus, circulant polynomial matrices commute under multiplication.

Theorem 1.2.16

If $A(\lambda)$ and $B(\lambda)$ are circulant polynomial matrices of order n, then $A(\lambda)$ and $B(\lambda)$ will commute if and only if $A(\lambda)-kI(\lambda)$ and $B(\lambda)-kI(\lambda)$ commute for every scalar k.

Proof

Given that $A(\lambda)$ and $B(\lambda)$ are two circulant polynomial matrices of order n.

Assume that the circulant polynomial matrices $A(\lambda)$ and $B(\lambda)$ commute. That is, $A(\lambda)B(\lambda)=B(\lambda)A(\lambda)$.

We have to prove that $A(\lambda)-kI(\lambda)$ and $B(\lambda)-kI(\lambda)$ commute for every scalar k.

Now $\left[A(\lambda)-kI(\lambda)\right]\left[B(\lambda)-kI(\lambda)\right]$

$$=A(\lambda)B(\lambda)-kI(\lambda)B(\lambda)-kI(\lambda)A(\lambda)+k^2\left[I(\lambda)\right]^2$$

$$=A(\lambda)B(\lambda)-k\left[A(\lambda)B(\lambda)\right]+k^2\left[I(\lambda)\right]^2$$

$$=A(\lambda)B(\lambda)-k\left[A(\lambda)B(\lambda)\right]+k^2I(\lambda) \qquad (1.1)$$

Also, $\left[B(\lambda)-kI(\lambda)\right]\left[A(\lambda)-kI(\lambda)\right]$

$$=B(\lambda)A(\lambda)-kI(\lambda)B(\lambda)-kI(\lambda)A(\lambda)+k^2\left[I(\lambda)\right]^2$$

$$=B(\lambda)A(\lambda)-k\left[B(\lambda)+A(\lambda)\right]+k^2I(\lambda)$$

$$=A(\lambda)B(\lambda)-k\left[A(\lambda)+B(\lambda)\right]+k^2I(\lambda) \qquad (1.2)$$

From (1.1) and (1.2), we get

$$\left[A(\lambda)-kI(\lambda)\right]\left[B(\lambda)-kI(\lambda)\right]=\left[B(\lambda)-kI(\lambda)\right]\left[A(\lambda)-kI(\lambda)\right]$$

That is, $A(\lambda)-kI(\lambda)$ and $B(\lambda)-kI(\lambda)$ commute.

Conversely, assume that $A(\lambda)-kI(\lambda)$ and $B(\lambda)-kI(\lambda)$ commute.

We have to prove that $A(\lambda)$ and $B(\lambda)$ commute.

$$\left[A(\lambda)-kI(\lambda)\right]\left[B(\lambda)-kI(\lambda)\right]=\left[B(\lambda)-kI(\lambda)\right]\left[A(\lambda)-kI(\lambda)\right]$$

$$A(\lambda)B(\lambda)-K\left[A(\lambda)+B(\lambda)\right]+k^2I(\lambda)=A(\lambda)B(\lambda)-k\left[A(\lambda)+B(\lambda)\right]+k^2I(\lambda)$$

$$A(\lambda)B(\lambda)=B(\lambda)A(\lambda)$$

Thus, $A(\lambda)$ and $B(\lambda)$ commute.

Theorem 1.2.17

If $A(\lambda)$ is a $n \times n$ circulant polynomial matrix, then $A^T(\lambda)$ is a circulant polynomial matrix.

Proof

Let $A(\lambda) = circ\left(a_1(\lambda), a_2(\lambda), a_3(\lambda), ..., a_n(\lambda)\right)$ be a circulant polynomial matrix.

$$A^T(\lambda) = \begin{pmatrix} a_1(\lambda) & a_2(\lambda) & a_3(\lambda) & \cdots & a_n(\lambda) \\ a_n(\lambda) & a_1(\lambda) & a_2(\lambda) & \cdots & a_{n-1}(\lambda) \\ a_{n-1}(\lambda) & a_n(\lambda) & a_1(\lambda) & \cdots & a_{n-2}(\lambda) \\ & & \vdots & & \\ a_2(\lambda) & a_3(\lambda) & a_4(\lambda) & \cdots & a_1(\lambda) \end{pmatrix}^T$$

$$= \begin{pmatrix} a_1(\lambda) & a_n(\lambda) & a_{n-1}(\lambda) & \cdots & a_2(\lambda) \\ a_2(\lambda) & a_1(\lambda) & a_n(\lambda) & \cdots & a_3(\lambda) \\ a_3(\lambda) & a_2(\lambda) & a_1(\lambda) & \cdots & a_4(\lambda) \\ \vdots & & \vdots & & \\ a_n(\lambda) & a_{n-1}(\lambda) & a_{n-2}(\lambda) & \cdots & a_1(\lambda) \end{pmatrix}$$

$$= circ\left(a_1(\lambda), a_n(\lambda), a_{n-1}(\lambda), ..., a_2(\lambda)\right)$$

Thus, $A^T(\lambda)$ is a circulant polynomial matrix.

Result 1.2.18

If $A^T(\lambda)$ and $B^T(\lambda)$ are the transpose of $A(\lambda)$ and $B(\lambda)$ respectively then

$(i)\left[A^T(\lambda)\right]^T = A(\lambda)$

$(ii)\left[A(\lambda)+B(\lambda)\right]^T = A^T(\lambda)+B^T(\lambda)$ for all $n\times n$ circulant polynomial matrices $A(\lambda)$ and $B(\lambda)$.

$(iii)\left[A(\lambda)B(\lambda)\right]^T = B^T(\lambda)A^T(\lambda)$

$(iv)\left[kA(\lambda)\right]^T = kA^T(\lambda)$.

Definition 1.2.19

The trace of a circulant polynomial matrix $A(\lambda)=\left[a_{ij}(\lambda)\right]$ is the sum of its diagonal elements.

That is, $trA(\lambda)=tr\left[a_{ij}(\lambda)\right]=\sum_{i=1}^{n} a_{ii}(\lambda)$.

Result 1.2.20

The trace of a circulant polynomial matrix has the following properties

(i) $tr\left[A(\lambda)+B(\lambda)\right]=tr\left[A(\lambda)\right]+tr\left[B(\lambda)\right]$ for all $n\times n$ circulant polynomial matrices $A(\lambda)$ and $B(\lambda)$.

(ii) $tr\left[kA(\lambda)\right]=k\,tr\left[A(\lambda)\right]$ for all $n\times n$ circulant polynomial matrices $A(\lambda)$ and all scalars k.

(iii) $tr\left[A^T(\lambda)\right]=tr\left[A(\lambda)\right]$

(iv) $tr\left[A(\lambda)B(\lambda)\right]=tr\left[B(\lambda)A(\lambda)\right]$

Definition 1.2.21

The circulant polynomial matrix obtained from any given circulant polynomial matrix $A(\lambda)$ on replacing its elements by the corresponding conjugate complex number is called the conjugate of $A(\lambda)$ and is denoted by $\overline{A(\lambda)}$.

If $A(\lambda)$ is a circulant polynomial matrix over the field of real numbers then obliviously $\overline{A(\lambda)}$ coincides with $A(\lambda)$.

Result 1.2.22

If $\overline{A(\lambda)}$ and $\overline{B(\lambda)}$ are the conjugate of $A(\lambda)$ and $B(\lambda)$ respectively then

(i) $\overline{\left(\overline{A(\lambda)}\right)} = A(\lambda)$

$(ii) \left[\overline{A(\lambda)+B(\lambda)}\right] = \left[\overline{A(\lambda)}\right] + \left[\overline{B(\lambda)}\right]$ for all $n \times n$ circulant polynomial matrices.

$(iii) \left[\overline{A(\lambda)B(\lambda)}\right] = \left[\overline{A(\lambda)}\right] + \left[\overline{B(\lambda)}\right]$

$(iv) \left[k\,\overline{A(\lambda)}\right] = k\left[\overline{A(\lambda)}\right]$, k being any complex number.

Definition 1.2.23

The transpose of the conjugate of a circulant polynomial matrix $A(\lambda)$ is called conjugate transpose of $A(\lambda)$ and is denoted by $A^{*}(\lambda)$.

That is, $\left(\overline{A^{T}(\lambda)}\right)=\left[\overline{A(\lambda)}\right]^{T}=A^{*}(\lambda)$.

Result 1.2.24

If $A^{*}(\lambda)$ and $B^{*}(\lambda)$ are the conjugate transpose of $A(\lambda)$ and $B(\lambda)$ respectively then

$(i)\left[A^{*}(\lambda)\right]^{*}=A(\lambda)$

$(ii)\ \left[A(\lambda)+B(\lambda)\right]^{*}=A^{*}(\lambda)+B^{*}(\lambda)$ for all $n\times n$ circulant polynomial matrices $A(\lambda)$ and $B(\lambda)$.

$(iii)\ \left[A(\lambda)B(\lambda)\right]^{*}=B^{*}(\lambda)A^{*}(\lambda)$

$(iv)\left[kA(\lambda)\right]^{*}=kA^{*}(\lambda)$, k being any complex number.

$(v)\left[A^{n}(\lambda)\right]^{*}=\left[A^{*}(\lambda)\right]^{n}$.

Proposition 1.2.25

Let $A(\lambda)$ be $n\times n$ circulant polynomial matrix. Then $A^{-1}(\lambda)$ is circulant polynomial matrix if $A(\lambda)$ is non-singular circulant polynomial matrix.

Proof

If $A(\lambda)$ is a non-singular circulant polynomial matrix, then by theorem (1.2.9), we have $A^{-1}(\lambda)=\left(\pi_m(\lambda)A(\lambda)\pi_m^{'}(\lambda)\right)^{-1}$

$$=\left(\pi_m^{'}\right)^{-1}(\lambda)A^{-1}(\lambda)\pi_m^{-1}(\lambda)$$

$$=\pi_m(\lambda)A^{-1}(\lambda)\pi_m^{'}(\lambda)$$

Therefore, $A^{-1}(\lambda)$ is a circulant polynomial matrix.

Result 1.2.26

The inverses of circulant polynomial matrices has the following properties

$(i)\left[A^{-1}(\lambda)\right]^{-1}=A(\lambda)$ for all invertible $n\times n$ circulant polynomial matrices $A(\lambda)$.

$(ii)\left[A(\lambda)B(\lambda)\right]^{-1}=B^{-1}(\lambda)A^{-1}(\lambda)$ for all invertible $n\times n$ circulant polynomial matrices $A(\lambda)$ and $B(\lambda)$.

$(iii)\left[A^T(\lambda)\right]^{-1}=\left[A^{-1}(\lambda)\right]^T$.

Definition 1.2.27

Let $A(\lambda)=\left[a_{ij}(\lambda)\right]$ be a circulant polynomial matrix. Then the adjoint of $A(\lambda)$ is the transpose of the cofactor polynomial matrix $C_{ij}(\lambda)$ of $A(\lambda)$. It is denoted by $adj\left[A(\lambda)\right]$.

1.2.28 Properties of the Adjoint of Circulant Polynomial Matrix

(i) If $A(\lambda)$ is a circulant polynomial matrix of order n, then

$$A(\lambda)adj\left[A(\lambda)\right]=\det\left[A(\lambda)\right]I_{n\times n}(\lambda)$$

$$=adj\left[A(\lambda)\right]A(\lambda)$$

(ii) If $A(\lambda)$ is a circulant polynomial matrix of order n, then

$$adj\left[A^T(\lambda)\right]=\left\{adj\left[A(\lambda)\right]\right\}^T$$

(iii) If $A(\lambda)$ and $B(\lambda)$ are two circulant polynomial matrices of the same order, then $adj\left[A(\lambda)B(\lambda)\right]=adj\left[A(\lambda)\right]adj\left[B(\lambda)\right]$.

(iv) $adj\left\{adj\left[A(\lambda)\right]\right\}=\left|A(\lambda)\right|^{n-2}A(\lambda)$, where $A(\lambda)$ is non- singular circulant polynomial matrix.

(v) Adjoint of diagonal polynomial matrix is also diagonal.

(vi) $adj\left[A(\lambda)B(\lambda)C(\lambda)\right]=adj\left[A(\lambda)\right]adj\left[B(\lambda)\right]adj\left[C(\lambda)\right]$

(vii) $adj\left[A(\lambda)B(\lambda)\right]^T=\left\{adjA(\lambda)\right\}^T\left\{adjB(\lambda)\right\}^T$.

Definition 1.2.29

Let $A(\lambda)=\left[a_{ij}(\lambda)\right]\in C_{m\times n}(\lambda)$ and $B(\lambda)=\left[b_{ij}(\lambda)\right]\in C_{p\times q}(\lambda)$ be two circulant polynomial matrices. Then the kronecker product (or tensor, or direct product) is that $mp\times nq$ circulant polynomial matrix defined by

$$A(\lambda)\otimes B(\lambda)=\begin{bmatrix} a_{11}(\lambda)B(\lambda) & a_{12}(\lambda)B(\lambda) & \cdots & a_{1n}(\lambda)B(\lambda) \\ a_{21}(\lambda)B(\lambda) & a_{22}(\lambda)B(\lambda) & \cdots & a_{2n}(\lambda)B(\lambda) \\ & \cdots & & \\ a_{m1}(\lambda)B(\lambda) & a_{m2}(\lambda)B(\lambda) & \cdots & a_{mn}(\lambda)B(\lambda) \end{bmatrix}.$$

1.2.30 Properties of the kronecker product of the circulant polynomial matrices:

$(i)\ \left(\alpha A(\lambda)\right)\otimes B(\lambda)=A(\lambda)\otimes\left(\alpha B(\lambda)\right)$

$=\alpha\left(A(\lambda)\otimes B(\lambda)\right);\ \alpha\ scalar$

$(ii)\ \left(A(\lambda)+B(\lambda)\right)\otimes C(\lambda)=\left(A(\lambda)\otimes C(\lambda)\right)+\left(B(\lambda)\otimes C(\lambda)\right)$

$(iii)\ A(\lambda)\otimes\left(B(\lambda)+C(\lambda)\right)=\left(A(\lambda)\otimes B(\lambda)\right)+\left(A(\lambda)\otimes C(\lambda)\right)$

$(iv)\ A(\lambda)\otimes\left(B(\lambda)\otimes C(\lambda)\right)=\left(A(\lambda)\otimes B(\lambda)\right)\otimes C(\lambda)$

$(v)\ \left(A(\lambda)\otimes B(\lambda)\right)\left(C(\lambda)\otimes D(\lambda)\right)=\left(A(\lambda)C(\lambda)\right)\otimes\left(B(\lambda)D(\lambda)\right)$

$(vi)\ \overline{A(\lambda)\otimes B(\lambda)}=\overline{A(\lambda)}\otimes\overline{B(\lambda)}$

$(vii)\ \left(A(\lambda)\otimes B(\lambda)\right)^T=A^T(\lambda)\otimes B^T(\lambda)$

$(viii)\ \left(A(\lambda)\otimes B(\lambda)\right)^*=A^*(\lambda)\otimes B^*(\lambda)$

$(ix)\ r\left(A(\lambda)\otimes B(\lambda)\right)=r\left(A(\lambda)\right).r\left(B(\lambda)\right)$

(x) If $A(\lambda)$ and $B(\lambda)$ are circulant polynomial matrices of order m and n, then $tr\left(A(\lambda)\otimes B(\lambda)\right)=tr\left(A(\lambda)\right)tr\left(B(\lambda)\right)$ **.**

(xi) If $A(\lambda)$ and $B(\lambda)$ are non-singular circulant polynomial matrices, so is $A(\lambda)\otimes B(\lambda)$ and $\left(A(\lambda)\otimes B(\lambda)\right)^{-1}=A^{-1}(\lambda)\otimes B^{-1}(\lambda)$.

$(xii)\ \det\left(A(\lambda)\otimes B(\lambda)\right)=\left(\det A(\lambda)\right)^n\left(\det B(\lambda)\right)^n$

$(xiii)$ Let $A(\lambda)$ be circulant polynomial matrix of order n and $I(\lambda)$ be the identity polynomial matrix of order n, then

$$I(\lambda)\otimes A(\lambda)=diag\left(A(\lambda),A(\lambda),...,A(\lambda)\right).$$

(xiv) There exists a permutation matrix $\pi(\lambda)$ depending only on m, n such that $B(\lambda)\otimes A(\lambda)=\pi^*(\lambda)\left(A(\lambda)\otimes B(\lambda)\right)\pi(\lambda).$

Definition 1.2.31

Eigen value of a circulant polynomial matrix $A(\lambda)$ is defined to be the zeros of the polynomial $\det\left[A(\lambda)-\mu I\right]=0.$ This polynomial is known as characteristic polynomial.

Definition 1.2.32

A non-zero vector $x\neq 0\in C_n$ is said to be a eigen vector of a complex polynomial matrix $A(\lambda)$ an associated with a eigen value μ if it satisfies $A(\lambda)x=\mu x.$

Theorem 1.2.33

$$\pi(\lambda)=F^*(\lambda)\Omega(\lambda)F(\lambda).$$

Theorem 1.2.34

If $A(\lambda)$ is a circulant polynomial matrix, it is diagonalized by $F(\lambda)$. More precisely, $A(\lambda)=F^*(\lambda)\Lambda(\lambda)F(\lambda).$

1.3 Summary of Results

In this section, a short account of the results obtained in this thesis are given.

k-Circulant and (r, s) – pair Circulant Polynomial Matrices

The concept of k-circulant and (r, s) – pair circulant polynomial matrices are introduced and some basic characterizations are derived. Many of the basic results on k-circulant and (r, s) – pair circulant matrices were extended to k-circulant and (r, s) – pair circulant polynomial matrices.

Hermitian, Normal and Conjugate Normal Circulant Polynomial Matrices

We introduced hermitian, normal and conjugate normal circulant polynomial matrices. Also, some of their basic characterizations and important results are discussed.

Block Circulant Polynomial Matrices

The concept of block circulant and circulant block polynomial matrices are developed. We give some important results concerning the diagonal polynomial matrices for block circulant, circulant block and block circulant matrices: where the blocks are circulant polynomial matrix of level

3 and type (m, n, p) is diagonalizable polynomial matrix by using the unitary polynomial matrix $F_m(\lambda) \otimes F_n(\lambda) \otimes F_p(\lambda)$.

As an applications of the circulant polynomial matrices are used to solve the travelling salesman problem.

Chapter II

k – CIRCULANT AND (r, s) – PAIR CIRCULANT POLYNOMIAL MATRICES

In this chapter we introduce the concept of k-circulant polynomial matrices[1] and (r, s) - pair circulant polynomial matrices. Some of the properties of k-circulant and (r, s)-pair circulant matrices are extended to k - circulant polynomial matrices and (r, s) - pair circulant polynomial matrices. Also, we have given characterization of k - circulant and (r, s) - pair circulant polynomial matrices.

2.1 A Study on k-circulant Polynomial Matrices

In this section some of the properties of k-circulant matrices found in [1,17,18,50] are extended to k-circulant polynomial matrices.

Definition 2.1.1

A k-circulant polynomial matrix of order n is a matrix of the form

$A(\lambda) = \text{k-circ}\left(a_1(\lambda), a_2(\lambda), \ldots, a_n(\lambda)\right)$

$$\text{That is, } \mathrm{A}(\lambda) = \begin{bmatrix} a_1(\lambda) & a_2(\lambda) & \ldots & a_n(\lambda) \\ a_{n-k+1}(\lambda) & a_{n-k+2}(\lambda) & \ldots & a_{n-k}(\lambda) \\ a_{n-2k+1}(\lambda) & a_{n-2k+2}(\lambda) & \ldots & a_{n-2k}(\lambda) \\ & \vdots & & \\ a_{k+1}(\lambda) & a_{k+2}(\lambda) & \ldots & a_k(\lambda) \end{bmatrix}.$$

Remark 2.1.2

If $0 \le k \le n$, each row of $A(\lambda)$ is the previous row moved to the right k - places or moved to the left n-k places wrap around.

If $k > n$, then a shift of k - places is the same as a shift of k mod n places. If k is negative, shifting to the right k- places will be equivalent to shifting to the left (-k)- places.

Thus, for any integers k, k' with $k' \equiv k$ (mod n) a k' circulant and a k-circulant are synonymous.

Example 2.1.3

A 3-circulant polynomial matrix of order 5 is

$$A(\lambda)=\begin{bmatrix} \lambda^3+\lambda & -\lambda^3-2\lambda^2+1 & 2\lambda^2+3 & -\lambda^2+1 & 3\lambda^2+\lambda \\ 2\lambda^2+3 & -\lambda^2+1 & 3\lambda^2+\lambda & \lambda^3+\lambda & -\lambda^3-2\lambda^2+1 \\ 3\lambda^2+\lambda & \lambda^3+\lambda & -\lambda^3-2\lambda^2+1 & 2\lambda^2+3 & -\lambda^2+1 \\ -\lambda^3-2\lambda^2+1 & 2\lambda^2+3 & -\lambda^2+1 & 3\lambda^2+\lambda & \lambda^3+\lambda \\ -\lambda^2+1 & 3\lambda^2+\lambda & \lambda^3+\lambda & -\lambda^3-2\lambda^2+1 & 2\lambda^2+3 \end{bmatrix}$$

Remark 2.1.4

A 1-circulant polynomial matrix is an ordinary circulant polynomial matrix.

A 0-circulant polynomial matrix is one in which all rows are identical.

A (-1)- circulant polynomial matrix or an (n-1)- circulant polynomial matrix has each successive row moved one place to the left.

Theorem 2.1.5

$A(\lambda)$ is a k-circulant polynomial matrix if and only if $\pi(\lambda)A(\lambda)=A(\lambda)\pi^k(\lambda)$ where $\pi(\lambda)$ is a permutation matrix (with polynomial of degree zero).

Proof

Let us assume that $A(\lambda)$ is a k-circulant polynomial matrix of order n.

That is, $A(\lambda)=$ k-circ$\left(a_1(\lambda), a_2(\lambda),..., a_n(\lambda)\right)$

$$=\begin{bmatrix} a_1(\lambda) & a_2(\lambda) & \dots & a_n(\lambda) \\ a_{n-k+1}(\lambda) & a_{n-k+2}(\lambda) & \dots & a_{n-k}(\lambda) \\ a_{n-2k+1}(\lambda) & a_{n-2k+2}(\lambda) & \dots & a_{n-2k}(\lambda) \\ & \vdots & & \\ a_{k+1}(\lambda) & a_{k+2}(\lambda) & \dots & a_k(\lambda) \end{bmatrix}$$

We have to prove that $\pi(\lambda)A(\lambda)=A(\lambda)\pi^k(\lambda)$.

Take $\sigma=\begin{pmatrix} 1 & 2 & \dots & n \\ 2 & 3 & \dots & 1 \end{pmatrix}$. Then $P_\sigma(\lambda)=\pi(\lambda)=(0,1,0\dots,0)$

$$=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

is a permutation matrix. If $A(\lambda)=(a_{ij}(\lambda))$, then

$$\pi(\lambda)A(\lambda)=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}\begin{bmatrix} a_1(\lambda) & a_2(\lambda) & \dots & a_n(\lambda) \\ a_{n-k+1}(\lambda) & a_{n-k+2}(\lambda) & \dots & a_{n-k}(\lambda) \\ a_{n-2k+1}(\lambda) & a_{n-2k+2}(\lambda) & \dots & a_{n-2k}(\lambda) \\ & \vdots & & \\ a_{k+1}(\lambda) & a_{k+2}(\lambda) & \dots & a_k(\lambda) \end{bmatrix}$$

$$=\begin{bmatrix} a_{n-k+1}(\lambda) & a_{n-k+2}(\lambda) & \dots & a_{n-k}(\lambda) \\ a_{n-2k+1}(\lambda) & a_{n-2k+2}(\lambda) & \dots & a_{n-2k}(\lambda) \\ & \vdots & & \\ a_1(\lambda) & a_2(\lambda) & \dots & a_n(\lambda) \end{bmatrix}=(a_{i+1,j}(\lambda)).$$

Take $\sigma^{-1}=\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1+k & 2+k & 3+k & \dots & k \end{pmatrix}$; then

$P_{\sigma^{-1}}(\lambda)=(P_\sigma)^{-1}(\lambda)=\pi^k(\lambda)$.

Hence, $\pi(\lambda)A(\lambda)\pi^{-k}(\lambda)=\left(a_{i+1,j+k}(\lambda)\right)$

$$=\left(a_{i,j}(\lambda)\right)$$

$$=A(\lambda)$$

Hence, $\pi(\lambda)A(\lambda)=A(\lambda)\pi^{k}(\lambda)$.

Conversely, assume that $\pi(\lambda)A(\lambda)=A(\lambda)\pi^{k}(\lambda)$.

We have to prove that $A(\lambda)$ is a k-circulant polynomial matrix.

Now $\pi(\lambda)A(\lambda)=A(\lambda)\pi^{k}(\lambda)$

$$A(\lambda)=\pi^{-1}(\lambda)A(\lambda)\pi^{k}(\lambda)$$

Therefore, $A(\lambda)$ is a k-circulant polynomial matrix.

Example 2.1.6

Let $A(\lambda)$ be a 2-circulant polynomial matrix of order 3.

That is, $A(\lambda)=\begin{bmatrix} 1+\lambda & 2+\lambda^2 & \lambda^3 \\ 2+\lambda^2 & \lambda^3 & 1+\lambda \\ \lambda^3 & 1+\lambda & 2+\lambda^2 \end{bmatrix}$

$$=A_0+A_1(\lambda)+A_2\lambda^2+A_3\lambda^3$$

Where $A_0=\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, $A_1=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $A_2=\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and

$A_3=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $\pi(\lambda)$ be a permutation matrix.

That is, $\pi(\lambda)=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Now $\pi(\lambda)A(\lambda)=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}1+\lambda&2+\lambda^2&\lambda^3\\2+\lambda^2&\lambda^3&1+\lambda\\\lambda^3&1+\lambda&2+\lambda^2\end{bmatrix}$

$$=\begin{bmatrix}2+\lambda^2&\lambda^3&1+\lambda\\\lambda^3&1+\lambda&2+\lambda^2\\1+\lambda&2+\lambda^2&\lambda^3\end{bmatrix} \qquad (2.1)$$

Now $A(\lambda)\pi^2(\lambda)=\begin{bmatrix}1+\lambda&2+\lambda^2&\lambda^3\\2+\lambda^2&\lambda^3&1+\lambda\\\lambda^3&1+\lambda&2+\lambda^2\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$

$$=\begin{bmatrix}2+\lambda^2&\lambda^3&1+\lambda\\\lambda^3&1+\lambda&2+\lambda^2\\1+\lambda&2+\lambda^2&\lambda^3\end{bmatrix} \qquad (2.2)$$

From (2.1) and (2.2), we get $\pi(\lambda)A(\lambda)=A(\lambda)\pi^2(\lambda)$.

Corollary 2.1.7

Let $A(\lambda)$ and $B(\lambda)$ be k-circulant polynomial matrices. Then $A(\lambda)\,B^*(\lambda)$ is a 1-circulant polynomial matrix.

Proof

Given that $A(\lambda)$ and $B(\lambda)$ are k-circulant polynomial matrices.

By theorem $(2.1.5)$, $A(\lambda)=\pi^*(\lambda)A(\lambda)\pi^k(\lambda)$ and

$$B(\lambda)=\pi^*(\lambda)B(\lambda)\pi^k(\lambda).$$

Now $A(\lambda)B^*(\lambda)=\left[\pi^*(\lambda)A(\lambda)\pi^k(\lambda)\right]\left[\pi^*(\lambda)B(\lambda)\pi^k(\lambda)\right]^*$

$$= \left[\pi^*(\lambda) A(\lambda) \pi^k(\lambda)\right]\left[\left(\pi^k\right)^*(\lambda) B^*(\lambda)\left(\pi^*\right)^*(\lambda)\right]$$

$$= \pi^*(\lambda) A(\lambda) \pi^k(\lambda)\left(\pi^*\right)^k(\lambda) B^*(\lambda) \pi(\lambda)$$

$$= \pi^*(\lambda) A(\lambda) B^*(\lambda) \pi(\lambda)$$

Hence, $A(\lambda)B^*(\lambda)$ is a 1-circulant polynomial matrix.

Example 2.1.8

Let $A(\lambda) = \begin{bmatrix} -3\lambda & 1-\lambda^2 & \lambda+3\lambda^2 \\ 1-\lambda^2 & \lambda+3\lambda^2 & -3\lambda \\ \lambda+3\lambda^2 & -3\lambda & 1-\lambda^2 \end{bmatrix} = A_0 + A_1\lambda + A_2\lambda^2$

where $A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} -3 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -1 & +3 \\ -1 & 3 & 0 \\ 3 & 0 & -1 \end{pmatrix}$ and

$$B(\lambda) = \begin{bmatrix} -1-\lambda^2 & -11\lambda & -13\lambda+\lambda^2 \\ -11\lambda & -13\lambda+\lambda^2 & -1-\lambda^2 \\ -13\lambda+\lambda^2 & -1-\lambda^2 & -11\lambda \end{bmatrix}$$

$$= B_0 + B_1\lambda + B_2\lambda^2$$

where $B_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0 & -11 & -13 \\ -11 & -13 & 0 \\ -13 & 0 & -11 \end{pmatrix}$, $B_2 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

be a 2-circulant polynomial matrix of order

$$A(\lambda)B^*(\lambda) = \begin{bmatrix} -3\lambda & 1-\lambda^2 & \lambda+3\lambda^2 \\ 1-\lambda^2 & \lambda+3\lambda^2 & -3\lambda \\ \lambda+3\lambda^2 & -3\lambda & 1-\lambda^2 \end{bmatrix}\begin{bmatrix} -1-\lambda^2 & -11\lambda & -13\lambda+\lambda^2 \\ -11\lambda & -13\lambda+\lambda^2 & -1-\lambda^2 \\ -13\lambda+\lambda^2 & -1-\lambda^2 & -11\lambda \end{bmatrix}$$

$$=\begin{bmatrix} \begin{matrix}(-3\lambda)(-1-\lambda^2)+(-1-\lambda^2)\\(-11\lambda)+(\lambda+3\lambda^2)(-13\lambda+\lambda^2)\end{matrix} & \begin{matrix}(-3\lambda)(-11\lambda)+(1-\lambda^2)\\(-13\lambda+\lambda^2)+(\lambda+3\lambda^2)(-1-\lambda^2)\end{matrix} & \begin{matrix}(-3\lambda)(-13\lambda+\lambda^2)+\\(1-\lambda^2)(-1-\lambda^2)+\\(\lambda+3\lambda^2)(-11\lambda)\end{matrix}\\ \begin{matrix}(1-\lambda^2)(-1-\lambda^2)+(\lambda+3\lambda^2)\\(-11\lambda)+(-3\lambda)(-13\lambda+\lambda^2)\end{matrix} & \begin{matrix}(1-\lambda^2)(-11\lambda)+(\lambda+3\lambda^2)\\(-13\lambda+\lambda^2)+(-3\lambda)(-1-\lambda^2)\end{matrix} & \begin{matrix}(1-\lambda^2)(-13\lambda+\lambda^2)+\\(\lambda+3\lambda^2)(-1-\lambda^2)+\\(-3\lambda)(-11\lambda)\end{matrix}\\ \begin{matrix}(\lambda+3\lambda^2)(-1-\lambda^2)+(-3\lambda)\\(-11\lambda)+(1-\lambda^2)(-13\lambda+\lambda^2)\end{matrix} & \begin{matrix}(\lambda+3\lambda^2)(-11\lambda)+(-3\lambda)\\(-13\lambda+\lambda^2)+(1-\lambda^2)(-1-\lambda^2)\end{matrix} & \begin{matrix}(\lambda+3\lambda^2)(-13\lambda+\lambda^2)+\\(-3\lambda)(-1-\lambda^2)+\\(1-\lambda^2)(-11\lambda)\end{matrix} \end{bmatrix}$$

$$=\begin{bmatrix} \begin{matrix}3\lambda+3\lambda^3-11\lambda+11\lambda^3-13\lambda^2\\-39\lambda^3+\lambda^3+3\lambda^4\end{matrix} & \begin{matrix}33\lambda^2-13\lambda+\lambda^2+13\lambda^3-\lambda^4\\-\lambda-3\lambda^2-\lambda^3-3\lambda^4\end{matrix} & \begin{matrix}39\lambda^2-3\lambda^3-1+\lambda^2-\lambda^2+\lambda^4\\-11\lambda^2-33\lambda^3\end{matrix}\\ \begin{matrix}-1+\lambda^2-\lambda^2+\lambda^4\\-11\lambda^2-33\lambda^3+39\lambda^2-3\lambda^3\end{matrix} & \begin{matrix}-11\lambda+11\lambda^3-13\lambda^2+\lambda^3\\-39\lambda^3+3\lambda^4+3\lambda+3\lambda^3\end{matrix} & \begin{matrix}-13\lambda+\lambda^2+13\lambda^3-\lambda^4\\-\lambda-3\lambda^2-\lambda^3-3\lambda^4+33\lambda^2\end{matrix}\\ \begin{matrix}-\lambda-3\lambda^2-\lambda^3-3\lambda^4+33\lambda^2\\-13\lambda+\lambda^2+13\lambda^3-\lambda^4\end{matrix} & \begin{matrix}-11\lambda^2-33\lambda^3+39\lambda^2-3\lambda^3\\-1+\lambda^2-\lambda^2+\lambda^4\end{matrix} & \begin{matrix}-13\lambda^2-39\lambda^3+\lambda^3+3\lambda^4\\+3\lambda+3\lambda^3-11\lambda+11\lambda^3\end{matrix} \end{bmatrix}$$

$$=\begin{bmatrix} 3\lambda^4-24\lambda^3-13\lambda^2-8\lambda & -4\lambda^4+12\lambda^3+31\lambda^2-14\lambda & \lambda^4-36\lambda^3+28\lambda^2-1\\ \lambda^4-36\lambda^3+28\lambda^2-1 & 3\lambda^4-24\lambda^3-13\lambda^2-8\lambda & -4\lambda^4+12\lambda^3+31\lambda^2-14\lambda\\ -4\lambda^4+12\lambda^3+31\lambda^2-14\lambda & \lambda^4-36\lambda^3+28\lambda^2-1 & 3\lambda^4-24\lambda^3-13\lambda^2-8\lambda \end{bmatrix}$$

Hence, $A(\lambda)B^*(\lambda)$ is a 1-circulant polynomial matrix.

Remark 2.1.9

If $A(\lambda)$ is a k-circulant polynomial matrix, then $A(\lambda)\ A^*(\lambda)$ is a 1-circulant polynomial matrix.

Theorem 2.1.10

If $A(\lambda)$ is a k-circulant polynomial matrix and $B(\lambda)$ is a h-circulant polynomial matrix then $A(\lambda)\ B(\lambda)$ is a kh-circulant polynomial matrix.

Proof

If $A(\lambda)$ is a k-circulant polynomial matrix and $B(\lambda)$ is an h-circulant polynomial matrix then, by theorem (2.1.5),

$$\pi(\lambda)A(\lambda)=A(\lambda)\pi^{k}(\lambda) \text{ and } \pi(\lambda)B(\lambda)=B(\lambda)\pi^{h}(\lambda)$$

Now $\pi(\lambda)\left[A(\lambda)B(\lambda)\right]=\left[\pi(\lambda)A(\lambda)\right]B(\lambda)$

$$=\left[A(\lambda)\pi^{k}(\lambda)\right]B(\lambda)$$

$$=A(\lambda)\pi^{k-1}(\lambda)\left[\pi(\lambda)B(\lambda)\right]$$

$$=A(\lambda)\pi^{k-1}(\lambda)\left[B(\lambda)\pi^{h}(\lambda)\right]$$

$$=A(\lambda)\pi^{k-2}(\lambda)\left[\pi(\lambda)B(\lambda)\right]\pi^{h}(\lambda)$$

$$=A(\lambda)\pi^{k-2}(\lambda)\left[B(\lambda)\pi^{h}(\lambda)\right]\pi^{h}(\lambda)$$

$$=\left[A(\lambda)\pi^{k-2}(\lambda)\right]B(\lambda)\pi^{2h}(\lambda)$$

Keep this up for h times, leading to

$$\pi(\lambda)\left[A(\lambda)B(\lambda)\right]=\left[A(\lambda)\pi^{h-h}(\lambda)\right]\left[B(\lambda)\pi^{kh}(\lambda)\right]$$

$$=A(\lambda)B(\lambda)\pi^{kh}(\lambda)$$

By theorem (2.1.5), $A(\lambda)\ B(\lambda)$ is a kh-circulant polynomial matrix.

Example 2.1.11

Let $A(\lambda)=\begin{bmatrix} -8+\lambda & 0 & 4+6\lambda^2 & 0 & \lambda \\ 0 & \lambda & -8+\lambda & 0 & 4+6\lambda^2 \\ 0 & 4+6\lambda^2 & 0 & \lambda & -8+\lambda \\ \lambda & -8+\lambda & 0 & 4+6\lambda^2 & 0 \\ 4+6\lambda^2 & 0 & \lambda & -8+\lambda & 0 \end{bmatrix}$ be a 2-circulant polynomial matrix of order 5 and

$B(\lambda)=\begin{bmatrix} 0 & 0 & 1-\lambda^2 & -\lambda+\lambda^2 & 0 \\ -\lambda+\lambda^2 & 0 & 0 & 0 & 1-\lambda^2 \\ 0 & 1-\lambda^2 & -\lambda+\lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda^2 & -\lambda+\lambda^2 \\ 1-\lambda^2 & -\lambda+\lambda^2 & 0 & 0 & 0 \end{bmatrix}$ be a 2-circulant polynomial matrix of order 5.

$$A(\lambda)B(\lambda)=\begin{bmatrix} \lambda-\lambda^3 & -6\lambda^4+\lambda^3+\lambda^2+4 & 6\lambda^4-7\lambda^3+12\lambda^2-3\lambda-8 & \lambda^3-9\lambda^2+8\lambda & 0 \\ -6\lambda^4+\lambda^3+\lambda^2+4 & 6\lambda^4-7\lambda^3+12\lambda^2-3\lambda-8 & \lambda^3-9\lambda^2+8\lambda & 0 & \lambda-\lambda^3 \\ 6\lambda^4-7\lambda^3+12\lambda^2-3\lambda-8 & \lambda^3-9\lambda^2+8\lambda & 0 & \lambda-\lambda^3 & -6\lambda^4+\lambda^3+\lambda^2+4 \\ \lambda^3-9\lambda^2+8\lambda & 0 & \lambda-\lambda^3 & -6\lambda^4+\lambda^3+\lambda^2+4 & 6\lambda^4-7\lambda^3+12\lambda^2-3\lambda-8 \\ 0 & \lambda-\lambda^3 & -6\lambda^4+\lambda^3+\lambda^2+4 & 6\lambda^4-7\lambda^3+12\lambda^2-3\lambda-8 & \lambda^3-9\lambda^2+8\lambda \end{bmatrix}$$

Hence, $A(\lambda)B(\lambda)$ is a 4-circulant polynomial matrix of order 5.

Theorem 2.1.12

Let $A(\lambda)$ be a non-singular k-circulant polynomial matrix. Then $A^{-1}(\lambda)$ is a k^{-1} circulant polynomial matrix. ($A^{-1}(\lambda)$ is a polynomial matrix obtained the inverses of coefficient matrices)

Proof

Since $A(\lambda)$ is non-singular and hence $k^{-1}(\lambda)$exists.

Now from theorem (2.1,5), $\pi(\lambda)A(\lambda)=A(\lambda)\pi^k(\lambda)$ so that

$$A^{-1}(\lambda)\pi^{-1}(\lambda)=\pi^{-k}(\lambda)A^{-1}(\lambda)$$

$$\pi(\lambda)A^{-1}(\lambda)\pi^{-1}(\lambda)\pi(\lambda)=\pi(\lambda)\pi^{-k}(\lambda)A^{-1}(\lambda)\pi(\lambda)$$

$$\pi(\lambda)A^{-1}(\lambda)=\pi^{-k+1}(\lambda)A^{-1}(\lambda)\pi(\lambda)$$

$$=\pi^{-k+1}(\lambda)A^{-1}(\lambda)\pi(\lambda)\pi^{-1}(\lambda)\pi(\lambda)\pi(\lambda)$$

$$=\pi^{-k+1}(\lambda)\left[A^{-1}(\lambda)\pi^{-1}(\lambda)\right]\pi^2(\lambda)$$

$$=\pi^{-k+1}(\lambda)\left[\pi^{-k}(\lambda)A^{-1}(\lambda)\right]\pi^2(\lambda)$$

$$=\pi^{-2k+1}(\lambda)A^{-1}(\lambda)\pi^2(\lambda)$$

Do these s times and we obtain $\pi(\lambda)A^{-1}(\lambda)=\pi^{-sk+1}(\lambda)A^{-1}(\lambda)\pi^s(\lambda)$

Put $s=k^{-1}$, we get $\pi(\lambda)A^{-1}(\lambda)=\pi^{-k^{-1}k+1}(\lambda)A^{-1}(\lambda)\pi^{k^{-1}}(\lambda)$

$$=A^{-1}(\lambda)\pi^{k^{-1}}(\lambda)$$

Therefore, $A^{-1}(\lambda)$ is a k^{-1} circulant polynomial matrix.

Example 2.1.13

Le $A(\lambda)$ be a 2-circulant polynomial matrix of order 3.

That is, $A(\lambda)=\begin{pmatrix} 3+\lambda & 2-\lambda & -1+4\lambda \\ 2-\lambda & -1+4\lambda & 3+\lambda \\ -1+4\lambda & 3+\lambda & 2-\lambda \end{pmatrix}=A_0+A_1\lambda$

Where $A_0=\begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 3 \\ -1 & 3 & 2 \end{pmatrix}$, $A_1=\begin{pmatrix} 1 & -1 & 4 \\ -1 & 4 & 1 \\ 4 & 1 & -1 \end{pmatrix}$

$$\Rightarrow A_0^{-1} = \frac{-1}{52}\begin{pmatrix} -11 & -7 & 5 \\ -7 & 5 & -11 \\ 5 & -11 & -7 \end{pmatrix},\ A_1^{-1} = \frac{-1}{70}\begin{pmatrix} -5 & 3 & -17 \\ 3 & -17 & -5 \\ -17 & -5 & 3 \end{pmatrix}$$

$$\Rightarrow A^{-1}(\lambda) = \begin{pmatrix} \frac{11}{52}+\frac{5}{70}\lambda & \frac{7}{52}-\frac{3}{70}\lambda & \frac{-5}{52}+\frac{17}{70}\lambda \\ \frac{7}{52}-\frac{3}{70}\lambda & \frac{-5}{52}+\frac{17}{70}\lambda & \frac{11}{52}+\frac{5}{70}\lambda \\ \frac{-5}{52}+\frac{17}{70}\lambda & \frac{11}{52}+\frac{5}{70}\lambda & \frac{7}{52}-\frac{3}{70}\lambda \end{pmatrix}$$

Theorem 2.1.14

$A(\lambda)$ is a k-circulant polynomial matrix if and only if $\left(A^{\dagger}\right)^{*}(\lambda)$ is a k - circulant polynomial matrix.

Proof

Let $A(\lambda)$ be a k-circulant polynomial matrix. Then by theorem (2.1.5),

$$A(\lambda) = \pi^{-1}(\lambda)A(\lambda)\pi^{k}(\lambda)$$

Since $\pi(\lambda), \pi^{-1}(\lambda), \pi^{k}(\lambda)$ are unitary polynomial matrix.

Hence, $A^{\dagger}(\lambda) = \left[\pi^{-1}(\lambda)A(\lambda)\pi^{k}(\lambda)\right]^{\dagger}$

$$= \left(\pi^{k}\right)^{\dagger}(\lambda)A^{\dagger}(\lambda)\left(\pi^{-1}\right)^{\dagger}(\lambda)$$

$$= \pi^{-k}(\lambda)A^{\dagger}(\lambda)\pi(\lambda)$$

$$\left(A^{\dagger}\right)^{*}(\lambda) = \left[\pi^{-k}(\lambda)A^{\dagger}(\lambda)\pi(\lambda)\right]^{*}$$

$$=\pi^{*}(\lambda)\left(A^{\dagger}\right)^{*}(\lambda)\left(\pi^{-k}\right)^{*}(\lambda)$$

$$=\pi^{-1}(\lambda)\left(A^{\dagger}\right)^{*}(\lambda)\pi^{k}(\lambda)$$

Therefore, $\left(A^{\dagger}\right)^{*}(\lambda)$ is a k-circulant polynomial matrix.

Conversely, let $\left(A^{\dagger}\right)^{*}$ be a k-circulant polynomial matrix.

That is, $\left(A^{\dagger}\right)^{*}(\lambda)=\pi^{-1}(\lambda)\left(A^{\dagger}\right)^{*}(\lambda)\pi^{k}(\lambda)$

$$\left(\left(A^{\dagger}\right)^{*}\right)^{\dagger}(\lambda)=\left[\pi^{-1}(\lambda)\left(A^{\dagger}\right)^{*}(\lambda)\pi^{k}(\lambda)\right]^{\dagger}$$

$$\left(\left(A^{\dagger}\right)^{\dagger}\right)^{*}(\lambda)=\left(\pi^{k}\right)^{\dagger}(\lambda)\left(\left(A^{\dagger}\right)^{*}\right)^{\dagger}(\lambda)\left(\pi^{-1}\right)^{\dagger}(\lambda)$$

$$A^{*}(\lambda)=\pi^{-k}(\lambda)\left(\left(A^{\dagger}\right)^{\dagger}\right)^{*}(\lambda)\pi(\lambda)$$

$$\left(A^{*}\right)^{*}(\lambda)=\left[\pi^{-k}(\lambda)A^{*}(\lambda)\pi(\lambda)\right]^{*}$$

$$\left(A^{*}\right)^{*}(\lambda)=\pi^{*}(\lambda)\left(A^{*}\right)^{*}(\lambda)\left(\pi^{-k}\right)^{*}(\lambda)$$

$$A(\lambda)=\pi^{-1}(\lambda)A(\lambda)\pi^{k}(\lambda)$$

Hence, $A(\lambda)$ is a k-circulant polynomial matrix.

Corollary 2.1.15

If $A(\lambda)$ is a k-circulant polynomial matrix, then $A(\lambda)A^{\dagger}(\lambda)$ is a 1-circulant polynomial matrix.

Proof

Given that $A(\lambda)$ is a k-circulant polynomial matrix.

By theorem (2.1.5), $A(\lambda)=\pi^{-1}(\lambda)A(\lambda)\pi^{k}(\lambda)$

Now $A(\lambda)A^{\dagger}(\lambda)=\left[\pi^{-1}(\lambda)A(\lambda)\pi^{k}(\lambda)\right]\left[\pi^{-1}(\lambda)A(\lambda)\pi^{k}(\lambda)\right]^{\dagger}$

$$=\pi^{-1}(\lambda)A(\lambda)\pi^{k}(\lambda)\left(\pi^{k}\right)^{\dagger}(\lambda)A^{\dagger}(\lambda)\left(\pi^{-1}\right)^{\dagger}(\lambda)$$

$$=\pi^{-1}(\lambda)A(\lambda)A^{\dagger}(\lambda)\pi(\lambda).$$

Hence, $A(\lambda)\ A^{\dagger}(\lambda)$ is a 1-circulant polynomial matrix.

Remark 2.1.16

If $A(\lambda)$ is a k-circulant polynomial matrix, then $A(\lambda)A^{*}(\lambda)$ is a 1-circulant polynomial matrix.

Theorem 2.1.17

If $A(\lambda)$ is a k-circulant polynomial matrix, then $A^{\dagger}(\lambda)=A^{*}(\lambda)\left[A(\lambda)A^{*}(\lambda)\right]^{\dagger}$.

Proof

If $A(\lambda)$ is a k-circulant polynomial matrix, then by remark (2.1.16), we have $A(\lambda)A^{*}(\lambda)$ is a 1-circulant polynomial matrix.

That is, $A(\lambda)A^{*}(\lambda)=\pi^{-1}(\lambda)A(\lambda)A^{*}(\lambda)\pi(\lambda)$

$$\left[A(\lambda)A^{*}(\lambda)\right]^{\dagger}=\left[\pi^{-1}(\lambda)A(\lambda)A^{*}(\lambda)\pi(\lambda)\right]^{\dagger}$$

$$= \pi^{\dagger}(\lambda)\left(A^{*}\right)^{\dagger}(\lambda) A^{\dagger}(\lambda)\left(\pi^{-1}\right)^{\dagger}(\lambda)$$

$$= \pi^{\dagger}(\lambda)\left(A^{*}\right)^{\dagger}(\lambda) A^{\dagger}(\lambda)\pi(\lambda)$$

$$A^{*}(\lambda)\left[A(\lambda)A^{*}(\lambda)\right]^{\dagger} = A^{*}(\lambda)\pi^{\dagger}(\lambda)\left(A^{*}\right)^{\dagger}(\lambda) A^{\dagger}(\lambda)\pi(\lambda)$$

$$= A^{*}(\lambda)\left(A^{*}\right)^{\dagger}(\lambda)\pi^{\dagger}(\lambda) A^{\dagger}(\lambda)\pi(\lambda)$$

$$= A^{\dagger}(\lambda)\pi^{\dagger}(\lambda)\pi(\lambda)$$

$$= A^{\dagger}(\lambda).$$

2.2 (r, s)- Pair Circulant Polynomial Matrices

In this section we introduce the concept of (r,s)- pair circulant polynomial matrix. Also, we give several properties, discriminance for (r,s) - pair circulant polynomial matrices found in [17,33,56].

Definition 2.2.1

A polynomial matrix $A(\lambda)$ of order n is called (r,s)- pair circulant polynomial matrix if it is of the form

$$A(\lambda)=\begin{bmatrix} a_0(\lambda) & a_1(\lambda) & a_2(\lambda) & \cdots & a_{n-2}(\lambda) & a_{n-1}(\lambda) \\ ra_{n-1}(\lambda) & a_0(\lambda)-sa_{n-1}(\lambda) & a_1(\lambda) & \cdots & a_{n-3}(\lambda) & a_{n-2}(\lambda) \\ ra_{n-2}(\lambda) & ra_{n-1}(\lambda)-sa_{n-2}(\lambda) & a_0(\lambda)-sa_{n-1}(\lambda) & \cdots & a_{n-4}(\lambda) & a_{n-3}(\lambda) \\ ra_{n-3}(\lambda) & ra_{n-2}(\lambda)-sa_{n-3}(\lambda) & ra_{n-1}(\lambda)-sa_{n-2}(\lambda) & \cdots & a_{n-5}(\lambda) & a_{n-4}(\lambda) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ ra_2(\lambda) & ra_3(\lambda)-sa_2(\lambda) & ra_4(\lambda)-sa_3(\lambda) & \cdots & a_0(\lambda)-sa_{n-1}(\lambda) & a_1(\lambda) \\ ra_1(\lambda) & ra_2(\lambda)-sa_1(\lambda) & ra_3(\lambda)-sa_2(\lambda) & \cdots & ra_{n-1}(\lambda)-sa_{n-2}(\lambda) & a_0(\lambda)-sa_{n-1}(\lambda) \end{bmatrix}$$

Which is denoted by $A(\lambda)=C_{(r,s)}\left(a_0(\lambda), a_1(\lambda),..., a_{n-1}(\lambda)\right)$.

Remark 2.2.2

(i) If $s=0$, then $A(\lambda)$ is a r-circulant polynomial matrix.

(ii) The polynomial matrix $\mathcal{B}(\lambda)=C_{(r,s)}(0,1,0,...,0)$ is called basic $(r,s)-$ pair circulant polynomial matrix.

Example 2.2.3

A 4×4 $(3,2)$-pair circulant polynomial matrix is given below.

$$A(\lambda)=\begin{pmatrix} \lambda+\lambda^2 & 1-\lambda & -3+\lambda-2\lambda^2 & 2+2\lambda+3\lambda^2 \\ 6+6\lambda+9\lambda^2 & -4-3\lambda-5\lambda^2 & 1-\lambda & -3+\lambda-2\lambda^2 \\ -9+3\lambda-6\lambda^2 & 12+4\lambda+13\lambda^2 & -4-3\lambda-5\lambda^2 & 1-\lambda \\ 3-3\lambda & -11+5\lambda-6\lambda^2 & 12+4\lambda+13\lambda^2 & -4-3\lambda-5\lambda^2 \end{pmatrix}$$

$=A_0+A_1\lambda+A_2\lambda^2$ where $A_0=C_{(3,2)}(0,1,-3,2)$, $A_1=C_{(3,2)}(1,-1,1,2)$, and $A_2=C_{(3,2)}(1,0,-2,3)$.

That is,

$$A_0=\begin{pmatrix} 0 & 1 & -3 & 2 \\ 6 & -4 & 1 & -3 \\ -9 & 12 & -4 & 1 \\ 3 & -11 & 12 & -4 \end{pmatrix}, A_1=\begin{pmatrix} 1 & -1 & 1 & 2 \\ 6 & -3 & -1 & 1 \\ 3 & 4 & -3 & -1 \\ -3 & 5 & 4 & -3 \end{pmatrix}, A_2=\begin{pmatrix} 1 & 0 & -2 & 3 \\ 9 & -5 & 0 & -2 \\ -6 & 13 & -5 & 0 \\ 0 & -6 & 13 & -5 \end{pmatrix}$$

Proposition 2.2.4

Suppose that $A(\lambda)$ and $B(\lambda)$ are (r,s)-pair circulant polynomial matrices. Then $A(\lambda)+B(\lambda)$, $A(\lambda)-B(\lambda)$ and $\alpha A(\lambda)$ are also $(r,s)-$ pair circulant polynomial matrices.

Proposition 2.2.5

A polynomial matrix $A(\lambda)$ is an (r,s)–pair circulant polynomial matrix if and only if $A(\lambda)= f_{A(\lambda)}(\mathfrak{B}(\lambda))=\sum_{i=0}^{n-1} a_i(\lambda)\mathfrak{B}^i(\lambda)$ for some polynomial $f_{A(\lambda)}(x(\lambda))=\sum_{i=0}^{n-1} a_i(\lambda)x^i(\lambda)$.

Theorem 2.2.6

A polynomial matrix $A(\lambda)\in C_{n\times n}(\lambda)$ is an (r,s) -pair circulant polynomial matrix if and only if $A(\lambda)\,\mathfrak{B}(\lambda) = \mathfrak{B}(\lambda)\,A(\lambda)$.

Proof

Assume that $A(\lambda)$ is an (r,s) -pair circulant polynomial matrix.

We have to prove that $A(\lambda)\,\mathfrak{B}(\lambda)=\mathfrak{B}(\lambda)\,A(\lambda)$.

Let $A(\lambda)=C_{(r,s)}(a_0(\lambda),a_1(\lambda),...,a_{n-1}(\lambda))$ be an (r,s)-pair circulant polynomial matrix. Then $A(\lambda)=\sum_{i=0}^{n-1} a_i(\lambda)\mathfrak{B}^i(\lambda)$.

$$\Rightarrow A(\lambda)\,\mathfrak{B}(\lambda) = \mathfrak{B}(\lambda)\,A(\lambda)$$

Conversely, assume that $A(\lambda)\,\mathfrak{B}(\lambda) = \mathfrak{B}(\lambda)\,A(\lambda)$. We have to prove that $A(\lambda)$ is an (r,s)- pair circulant polynomial matrix.

Suppose that $A(\lambda)\,\mathfrak{B}(\lambda) = \mathfrak{B}(\lambda)\,A(\lambda)$. Then

$$\left[A(\lambda)\mathfrak{B}(\lambda)\right]^T=\left[\mathfrak{B}(\lambda)A(\lambda)\right]^T$$

$$\mathcal{B}^T(\lambda)A^T(\lambda) = A^T(\lambda)\mathcal{B}^T(\lambda)$$

$$\left(\mathcal{B}^T\right)^i(\lambda)A^T(\lambda) = A^T(\lambda)\left(\mathcal{B}^T\right)^i(\lambda),\ i = 1, 2, \ldots$$

Let $e_i(\lambda)$ be the i^{th} column of $I_n(\lambda)$.

$$\Rightarrow \mathcal{B}^T(\lambda)e_i(\lambda) = e_{i+1}(\lambda) \ \ for \ \ i = 1, 2, \ldots, n-1 \ .$$

Thus, we have $\left(\mathcal{B}^T\right)^i(\lambda)e_i(\lambda) = e_{i+1}(\lambda)$ for $i = 1, 2, \ldots, n-1$.

Now $A^T(\lambda) = A^T(\lambda)I_n(\lambda)$

$$= A^T(\lambda)\left[e_1(\lambda), e_2(\lambda), \ldots, e_n(\lambda)\right]$$

$$= A^T(\lambda)\left[e_1(\lambda),\ \mathcal{B}^T(\lambda)e_1(\lambda), \ldots,\ \left(\mathcal{B}^T\right)^{n-1}(\lambda)e_1(\lambda)\right]$$

$$= \left(A^T(\lambda)e_1(\lambda),\ A^T(\lambda)\mathcal{B}^T(\lambda)e_1(\lambda), \ldots,\ A^T(\lambda)\left(\mathcal{B}^T\right)^{n-1}(\lambda)e_1(\lambda)\right)$$

$$= \left(A^T(\lambda)e_1(\lambda), \mathcal{B}^T(\lambda)A^T(\lambda)e_1(\lambda), \ldots, \left(\mathcal{B}^T\right)^{n-1}(\lambda)A^T(\lambda)e_1(\lambda)\right)$$

$$= \left(\alpha(\lambda),\ \mathcal{B}^T(\lambda)\alpha(\lambda), \ldots,\ \left(\mathcal{B}^T\right)^{n-1}(\lambda)\alpha(\lambda)\right)$$

Where $\alpha^T(\lambda)$ is the first row of $A(\lambda)$.

Let $\alpha^T(\lambda) = \left(a_0(\lambda),\ a_1(\lambda), \ldots,\ a_{n-1}(\lambda)\right)$

Thus, $\alpha(\lambda) = \sum_{i=0}^{n-1} a_i(\lambda)e_{i+1}(\lambda)$

$$A^T(\lambda) = \left(\sum_{i=0}^{n-1} a_i(\lambda)e_{i+1}(\lambda), \sum_{i=0}^{n-1} a_i(\lambda)\mathcal{B}^T(\lambda)e_{i+1}(\lambda), \ldots, \sum_{i=0}^{n-1} a_i(\lambda)\left(\mathcal{B}^T\right)^{n-1}(\lambda)e_{i+1}(\lambda)\right)$$

$$= \sum_{i=0}^{n-1} a_i \left(e_{i+1}(\lambda), \mathcal{B}^T(\lambda) e_{i+1}(\lambda), \ldots, \left(\mathcal{B}^T\right)^{n-1}(\lambda) e_{i+1}(\lambda) \right)$$

$$= \sum_{i=0}^{n-1} a_i \left(\left(\mathcal{B}^T\right)^i(\lambda) e_1(\lambda), \left(\mathcal{B}^T\right)^{i+1}(\lambda) e_1(\lambda), \ldots, \left(\mathcal{B}^T\right)^{n+i-1}(\lambda) e_1(\lambda) \right)$$

$$= \sum_{i=0}^{n-1} a_i \left(\mathcal{B}^T\right)^i(\lambda) \left(e_1(\lambda), e_2(\lambda), \ldots, e_n(\lambda) \right)$$

$$= \sum_{i=0}^{n-1} a_i(\lambda) \left(\mathcal{B}^T\right)^i(\lambda)$$

$$\Rightarrow A(\lambda) = \sum_{i=0}^{n-1} a_i(\lambda) \mathcal{B}^i(\lambda)$$

Hence, $A(\lambda)$ is an (r,s)- pair circulant polynomial matrix.

Corollary 2.2.7

If $A(\lambda)$ is an non-singular polynomial matrix, then $A(\lambda)$ is an (r,s)- pair circulant polynomial matrix if and only if $A^{-1}(\lambda)$ is an (r,s)- pair circulant polynomial matrix.

Proof

Given that $A(\lambda)$ is a non-singular polynomial matrix.

$A(\lambda)$ is an (r,s) -pair circulant polynomial matrix

$$\Leftrightarrow A(\lambda)\mathcal{B}(\lambda) = \mathcal{B}(\lambda) A(\lambda)$$

$$\Leftrightarrow A^{-1}(\lambda)\mathcal{B}(\lambda) = \mathcal{B}(\lambda) A^{-1}(\lambda)$$

$\Leftrightarrow A^{-1}(\lambda)$ is an (r,s)- pair circulant polynomial matrix.

Theorem 2.2.8

If $A(\lambda)$ and $B(\lambda)$ are (r,s)- pair circulant polynomial matrices, then $A(\lambda)B(\lambda)$ and $B(\lambda)A(\lambda)$ are also (r,s)- pair circulant polynomial matrices and $A(\lambda)B(\lambda)=B(\lambda)A(\lambda)$.

Proof

Given that $A(\lambda)$ and $B(\lambda)$ are (r,s)- pair circulant polynomial matrices.

From theorem $(2.2.6)$, we have

$$A(\lambda)\mathcal{B}(\lambda)=\mathcal{B}(\lambda)A(\lambda) \text{ and } B(\lambda)\mathcal{B}(\lambda)=\mathcal{B}(\lambda)B(\lambda).$$

Now $\left[A(\lambda)B(\lambda)\right]\mathcal{B}(\lambda)=A(\lambda)\left[B(\lambda)\mathcal{B}(\lambda)\right]$

$$=A(\lambda)\left[\mathcal{B}(\lambda)B(\lambda)\right]$$

$$=\left[A(\lambda)\mathcal{B}(\lambda)\right]B(\lambda)$$

$$=\left[\mathcal{B}(\lambda)A(\lambda)\right]B(\lambda)$$

$$=\mathcal{B}(\lambda)\left[A(\lambda)B(\lambda)\right]$$

Therefore, $A(\lambda)B(\lambda)$ is an (r,s) - pair circulant polynomial matrices.

Also, $\left[B(\lambda)A(\lambda)\right]\mathcal{B}(\lambda)=B(\lambda)\left[A(\lambda)\mathcal{B}(\lambda)\right]$

$$=B(\lambda)\left[\mathcal{B}(\lambda)A(\lambda)\right]$$

$$=\left[B(\lambda)\mathcal{B}(\lambda)\right]A(\lambda)$$

$$=\left[\mathcal{B}(\lambda)B(\lambda)\right]A(\lambda)$$

$$=\mathcal{B}(\lambda)\left[B(\lambda)A(\lambda)\right]$$

Hence, $B(\lambda)A(\lambda)$ is an (r,s)- pair circulant polynomial matrix.

From proposition $(2.2.5)$, we assume that $A(\lambda)=f\left(\mathcal{B}(\lambda)\right)$ and $B(\lambda)=g\left(\mathcal{B}(\lambda)\right)$.

$$\Rightarrow A(\lambda)B(\lambda)=B(\lambda)A(\lambda).$$

Theorem 2.2.9

Let $A(\lambda)$ be a non-singular polynomial matrix and $r\neq 0$. Then $A(\lambda)$ is an (r,s)-pair circulant polynomial matrix if and only if $A^*(\lambda)$ is an (r,s)-pair circulant polynomial matrix.

Proof

Let $A(\lambda)$ be a non-singular polynomial matrix and $r\neq 0$. Assume that $A(\lambda)$ is an $(r,s)-$pair circulant polynomial matrix.

We have to prove that $A^*(\lambda)$ is an (r,s)pair circulant polynomial matrix.

Let $A(\lambda)$ be an (r,s)-pair circulant polynomial matrix. Then

$$A(\lambda)\mathcal{B}(\lambda)=\mathcal{B}(\lambda)A(\lambda)$$

$$\left[A(\lambda)\mathcal{B}(\lambda)\right]^{*}=\left[\mathcal{B}(\lambda)A(\lambda)\right]^{*}$$

$$\mathcal{B}^{*}(\lambda)A^{*}(\lambda)=A^{*}(\lambda)\mathcal{B}^{*}(\lambda) \qquad (2.3)$$

Since $\mathcal{B}(\lambda)B^{*}(\lambda)=\left|\mathcal{B}(\lambda)\right|I_{n}(\lambda)$

$$=(-1)^{n+1}r\,I_{n}(\lambda)$$

$$\mathcal{B}^{*}(\lambda)=(-1)^{n+1}rI_{n}(\lambda)\mathcal{B}^{-1}(\lambda)$$

$$\mathcal{B}^{*}(\lambda)=(-1)^{n+1}r\,\mathcal{B}^{-1}(\lambda) \qquad (2.4)$$

sub (2.4) in (2.3), we get

$$(-1)^{n+1}r\,\mathcal{B}^{-1}(\lambda)A^{*}(\lambda)=A^{*}(\lambda)(-1)^{n+1}r\,\mathcal{B}^{-1}(\lambda)$$

$$\mathcal{B}^{-1}(\lambda)A^{*}(\lambda)=A^{*}(\lambda)\mathcal{B}^{-1}(\lambda)$$

$$A^{*}(\lambda)\mathcal{B}(\lambda)=\mathcal{B}(\lambda)A^{*}(\lambda)$$

Hence, $A^{*}(\lambda)$ is an (r,s)-pair circulant polynomial matrix.

Conversely, assume that $A^{*}(\lambda)$ is an (r,s)-pair circulant polynomial matrix.

We have to prove that $A(\lambda)$ is an (r,s)-pair circulant polynomial matrix.

Suppose that $A^{*}(\lambda)$ is an (r,s)-pair circulant polynomial matrix.

Then $$\left[A^{*}(\lambda)\mathcal{B}(\lambda)=\mathcal{B}(\lambda)A^{*}(\lambda)\right]^{*}$$

$$\mathcal{B}^{*}(\lambda)\left(A^{*}\right)^{*}(\lambda)=\left(A^{*}\right)^{*}\mathcal{B}^{*}(\lambda)$$

According to $\mathcal{B}^*(\lambda)=(-1)^{n+1} r\mathcal{B}^{-1}(\lambda)$ and we get

$$\left(A^*\right)^*(\lambda)=\left|A(\lambda)^{n-2}\right|A(\lambda) \text{ we get,}$$

$$(-1)^{n+1} r\mathcal{B}^{-1}(\lambda)\left|A(\lambda)\right|^{n-2} A(\lambda)=\left|A(\lambda)\right|^{n-2} A(\lambda)(-1)^{n+1} r\mathcal{B}^{-1}(\lambda)$$

$$\mathcal{B}^{-1}(\lambda)A(\lambda)=A(\lambda)\mathcal{B}^{-1}(\lambda)$$

$$A(\lambda)\mathcal{B}(\lambda)=\mathcal{B}(\lambda)A(\lambda)$$

Hence, $A(\lambda)$ is an (r,s)-pair circulant polynomial matrix.

Chapter III

HERMITIAN, NORMAL AND CONJUGATE NORMAL CIRCULANT POLYNOMIAL MATRICES

In this chapter we discussed the concept of Hermitian, Normal, and Conjugate normal circulant polynomial matrices. Some characterization of Hermitian, Normal and Conjugate normal circulant polynomial matrices are derived.

3.1 Hermitian Circulant Polynomial Matrix

In this section some of the properties of hermitian matrices are extended to hermitian circulant polynomial matrices. Also, we have generalized some important results of hermitian matrices found in [11,12,19,25,46] to hermitian circulant polynomial matrices.

Definition 3.1.1

A circulants polynomial matrix $A(\lambda)$ is called hermitian circulant polynomial matrix if $A(\lambda)=A^*(\lambda)$.

Example 3.1.2

Let $$A(\lambda)=\begin{pmatrix} 3+\lambda & 1+2i-4i\lambda & 1-2i+4i\lambda \\ 1-2i+4i\lambda & 3+\lambda & 1+2i-4i\lambda \\ 1+2i-4i\lambda & 1-2i+4i\lambda & 3+\lambda \end{pmatrix} \tag{3.1}$$

$$=A_0+A_1(\lambda)$$

Where $A_0=\begin{pmatrix} 3 & 1+2i & 1-2i \\ 1-2i & 3 & 1+2i \\ 1+2i & 1-2i & 3 \end{pmatrix}$, $A_1=\begin{pmatrix} 1 & -4i & 4i \\ 4i & 1 & -4i \\ -4i & 4i & 1 \end{pmatrix}$

Now $$A^*(\lambda)=\begin{pmatrix} 3+\lambda & 1+2i-4i\lambda & 1-2i+4i\lambda \\ 1-2i+4i\lambda & 3+\lambda & 1+2i-4i\lambda \\ 1+2i-4i\lambda & 1-2i+4i\lambda & 3+\lambda \end{pmatrix} \tag{3.2}$$

From (3.1) and (3.2), we get $A(\lambda)=A^*(\lambda)$

Hence, $A(\lambda)$ is a hermitian circulant polynomial matrix.

Definition 3.1.3

A circulant polynomial matrix $A(\lambda)$ is called skew hermitian circulant polynomial matrix if $A^*(\lambda) = -A(\lambda)$.

Example 3.1.4

Let

$$A(\lambda) = \begin{pmatrix} 0 & (-3+i)+(1-2i)\lambda & (3+i)+(-1-2i)\lambda \\ (3+i)+(-1-2i)\lambda & 0 & (-3+i)+(1-2i)\lambda \\ (-3+i)+(1-2i)\lambda & (3+i)+(-1-2i)\lambda & 0 \end{pmatrix} \tag{3.3}$$

$$= A_0 + A_1(\lambda)$$

where $A_0 = \begin{pmatrix} 0 & -3+i & 3+i \\ 3+i & 0 & -3+i \\ -3+i & 3+i & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 1-2i & -1-2i \\ -1-2i & 0 & 1-2i \\ 1-2i & -1-2i & 0 \end{pmatrix}$

Now

$$A^*(\lambda) = \begin{pmatrix} 0 & (3-i)+(-1+2i)\lambda & (-3-i)+(1+2i)\lambda \\ (-3-i)+(1+2i)\lambda & 0 & (3-i)+(-1+2i)\lambda \\ (3-i)+(-1+2i)\lambda & (-3-i)+(1+2i)\lambda & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & (-3+i)+(1-2i)\lambda & (3+i)+(-1-2i)\lambda \\ (3+i)+(-1-2i)\lambda & 0 & (-3+i)+(1-2i)\lambda \\ (-3+i)+(1-2i)\lambda & (3+i)+(-1-2i)\lambda & 0 \end{pmatrix} \tag{3.4}$$

From (3.3) and (3.4), we get $A^*(\lambda) = -A(\lambda)$

Hence, $A(\lambda)$ is a skew hermitian circulant polynomial matrix.

Theorem 3.1.5

If $A(\lambda)$ and $B(\lambda)$ are hermitian circulant polynomial matrices then $A(\lambda)+B(\lambda)$ is also hermitian circulant polynomial matrix.

Proof

Given that $A(\lambda)$ and $B(\lambda)$ are hermitian circulant polynomial matrices. That is, $A^*(\lambda)=A(\lambda)$ and $B^*(\lambda)=B(\lambda)$.

We have to prove that $A(\lambda)+B(\lambda)$ is a hermitian circulant polynomial matrix.

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\begin{aligned}
\left(A(\lambda)+B(\lambda)\right)^* &= \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^* \\
&= \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^* + \left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^* \\
&= \left(\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)\right)+\left(\left(\pi_n^{-1}(\lambda)\right)^* B^*(\lambda)\pi_n^*(\lambda)\right) \\
&= \left(\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)+\left(\pi_n(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\right) \\
&= \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda) \\
&= A(\lambda)+B(\lambda)
\end{aligned}$$

Thus, $A(\lambda)+B(\lambda)$ is hermitian circulant polynomial matrix.

Example 3.1.6

Let $A(\lambda)=\begin{pmatrix} 1+4\lambda & -7i+i\lambda & 7i-i\lambda \\ 7i-i\lambda & 1+4\lambda & -7i+i\lambda \\ -7i+i\lambda & 7i-i\lambda & 1+4\lambda \end{pmatrix}$ and

$B(\lambda)=\begin{pmatrix} 3+2\lambda & i-i\lambda & -i+i\lambda \\ -i+i\lambda & 3+2\lambda & i-i\lambda \\ i-i\lambda & -i+i\lambda & 3+2\lambda \end{pmatrix}$ be two circulant polynomial matrices.

Now $A(\lambda)=\pi_3(\lambda)A(\lambda)\pi_3^{-1}(\lambda)$

$$A(\lambda)+B(\lambda)=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 4+6\lambda & -6i & 6i \\ 6i & 4+6\lambda & -6i \\ -6i & 6i & 4+6\lambda \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A(\lambda)+B(\lambda)=\begin{pmatrix} 4+6\lambda & -6i & 6i \\ 6i & 4+6\lambda & -6i \\ -6i & 6i & 4+6\lambda \end{pmatrix} \qquad (3.5)$$

$$\left(A(\lambda)+B(\lambda)\right)^{*}=\begin{pmatrix} 4+6\lambda & -6i & 6i \\ 6i & 4+6\lambda & -6i \\ -6i & 6i & 4+6\lambda \end{pmatrix} \qquad (3.6)$$

From (3.5) and (3.6), we get $\left(A(\lambda)+B(\lambda)\right)^{*}=A(\lambda)+B(\lambda)$.

Hence, $A(\lambda)+B(\lambda)$ is a hermitian circulant polynomial matrix.

Theorem 3.1.7

If $A(\lambda)$ is Hermitian circulant polynomial matrix, for any scalar α, $A(\lambda)-\alpha I(\lambda)$ is a hermitian circulant polynomial matrix.

Proof

Given that $A(\lambda)$ is a hermitian circulant polynomial matrix.

That is, $A^{*}(\lambda)=A(\lambda)$

We have to prove that $A(\lambda)-\alpha I(\lambda)$ is hermitian circulant polynomial matrix.

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\left(A(\lambda)-\alpha I(\lambda)\right)^{*}=\left(A_R(\lambda)+iA_I(\lambda)-\alpha I(\lambda)\right)^{*}$$

$$=\left[\pi_n(\lambda)A_R(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)iA_I(\lambda)\pi_n^{-1}(\lambda)-\alpha\pi_n(\lambda)I(\lambda)\pi_n^{-1}(\lambda)\right]^{*}$$

$$=\left[\pi_n(\lambda)\left[A_R(\lambda)-\alpha I(\lambda)\right]\pi_n^{-1}(\lambda)+i\pi_n(\lambda)A_I(\lambda)\pi_n^{-1}(\lambda)\right]^{*}$$

$$=\left[\pi_n(\lambda)\left(A_R(\lambda)-\alpha I(\lambda)\right)\pi_n^{-1}(\lambda)\right]^{T}-i\left[\pi_n(\lambda)A_I(\lambda)\pi_n^{-1}(\lambda)\right]^{T}$$

$$=\left(\pi_n^{-1}(\lambda)\right)^{T}\left(A_R(\lambda)-\alpha I(\lambda)\right)^{T}\pi_n^{T}(\lambda)-i\left(\pi_n^{-1}(\lambda)\right)^{T}A_I^{T}(\lambda)\pi_n^{T}(\lambda)$$

$$=\pi_n(\lambda)\left(A_R^{T}(\lambda)-\alpha I^{T}(\lambda)\right)\pi_n^{-1}(\lambda)-i\pi_n(\lambda)A_I^{T}(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(A_R^{T}(\lambda)-iA_I^{T}(\lambda)\right)\pi_n^{-1}(\lambda)-\alpha\pi_n(\lambda)I^{T}(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(A_R(\lambda)+iA_I(\lambda)\right)\pi_n^{-1}(\lambda)-\alpha\pi_n(\lambda)I(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)-\alpha\pi_n(\lambda)I(\lambda)\pi_n^{-1}(\lambda)$$

$$=A(\lambda)-\alpha I(\lambda)$$

Thus, $A(\lambda)-\alpha I(\lambda)$ is hermitian circulant polynomial matrix.

Example 3.1.8

$$\text{Let } A(\lambda)=\begin{pmatrix} 2+3\lambda & i-7i\lambda & -i+7i\lambda \\ -i+7i\lambda & 2+3\lambda & i-7i\lambda \\ i-7i\lambda & -i+7i\lambda & 2+3\lambda \end{pmatrix} = A_0 + A_1\lambda$$

$$\text{where } A_0=\begin{pmatrix} 2 & i & -i \\ -i & 2 & i \\ i & -i & 2 \end{pmatrix},\ A_1=\begin{pmatrix} 3 & -7i & 7i \\ 7i & 3 & -7i \\ -7i & 7i & 3 \end{pmatrix} \text{ and } \alpha=3$$

$$\text{Now } A(\lambda)=\pi_3(\lambda)A(\lambda)\pi_3^{-1}(\lambda)$$

$$A(\lambda)-3I(\lambda)=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1+3\lambda & i-7i\lambda & -i+7i\lambda \\ -i+7i\lambda & -1+3\lambda & i-7i\lambda \\ i-7i\lambda & -i+7i\lambda & -1+3\lambda \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A(\lambda)-3I(\lambda)=\begin{pmatrix} -1+3\lambda & i-7i\lambda & -i+7i\lambda \\ -i+7i\lambda & -1+3\lambda & i-7i\lambda \\ i-7i\lambda & -i+7i\lambda & -1+3\lambda \end{pmatrix} \tag{3.7}$$

$$\left(A(\lambda)-3I(\lambda)\right)^*=\begin{pmatrix} -1+3\lambda & i-7i\lambda & -i+7i\lambda \\ -i+7i\lambda & -1+3\lambda & i-7i\lambda \\ i-7i\lambda & -i+7i\lambda & -1+3\lambda \end{pmatrix} \tag{3.8}$$

From (3.7) and (3.8), we get $\left(A(\lambda)-3I(\lambda)\right)^* = A(\lambda)-3I(\lambda)$

Hence, $A(\lambda)-3I(\lambda)$ is a hermitian circulant polynomial matrix.

Theorem 3.1.9

Any integral power of a hermitian circulant polynomial matrix is also a hermitian circulant polynomial matrix.

Proof

Let $A(\lambda)$ be a hermitian circulant polynomial matrix.

That is, $A^*(\lambda) = A(\lambda)$ $\quad (3.9)$

$$\left(A^2(\lambda)\right)^* = \left(A(\lambda)A(\lambda)\right)^*$$

$$= \left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\right]^*$$

$$= \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*$$

$$= \left(\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)\right)$$

$$= \pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)A^*(\lambda)I(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)A^*(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)\left(A^*(\lambda)\right)^2\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)\left(A(\lambda)\right)^2\pi_n^{-1}(\lambda)$$

$$= A^2(\lambda)$$

Hence, $A^2(\lambda)$ is a hermitian circulant polynomial matrix.

Assume that $A^k(\lambda)$ is a hermitian circulant polynomial matrix.

That is, $\left(A^k(\lambda)\right)^* = A^k(\lambda)$ $\quad (3.10)$

To prove that $A^{k+1}(\lambda)$ is a hermitian circulant polynomial matrix

$$\left(A^{k+1}(\lambda)\right)^* = \left(A(\lambda)A^k(\lambda)\right)^*$$

$$= \left(A^k(\lambda)\right)^* \left(A(\lambda)\right)^*$$

$$= A^k(\lambda)A(\lambda) \qquad \left(by\ (3.9)\ and\ (3.10)\right)$$

$$= A^{k+1}(\lambda)$$

Thus, any integral power of a hermitian circulant polynomial matrix is a hermitian circulant polynomial matrix.

Example 3.1.10

Let $A(\lambda) = \begin{pmatrix} 5 & -i+3i\lambda & i-3i\lambda \\ i-3i\lambda & 5 & -i+3i\lambda \\ -i+3i\lambda & i-3i\lambda & 5 \end{pmatrix} = A_0 + A_1(\lambda)$

where $A_0 = \begin{pmatrix} 5 & -i & i \\ i & 5 & -i \\ -i & i & 5 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 3i & -3i \\ -3i & 0 & 3i \\ 3i & -3i & 0 \end{pmatrix}$

Now $A(\lambda) = \pi_3(\lambda)A(\lambda)\pi_3^{-1}(\lambda)$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 5 & -i+3i\lambda & i-3i\lambda \\ i-3i\lambda & 5 & -i+3i\lambda \\ -i+3i\lambda & i-3i\lambda & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -i+3i\lambda & i-3i\lambda \\ i-3i\lambda & 5 & -i+3i\lambda \\ -i+3i\lambda & i-3i\lambda & 5 \end{pmatrix}$$

$$A^2(\lambda)=\begin{pmatrix} 27-12\lambda+18\lambda^2 & (-1-10i)+(6+30i)\lambda-9\lambda^2 & (-1+10i)+(6-30i)\lambda-9\lambda^2 \\ (-1+10i)+(6-30i)\lambda-9\lambda^2 & 27-12\lambda+18\lambda^2 & (-1-10i)+(6+30i)\lambda-9\lambda^2 \\ (-1-10i)+(6+30i)\lambda-9\lambda^2 & (-1+10i)+(6-30i)\lambda-9\lambda^2 & 27-12\lambda+18\lambda^2 \end{pmatrix}$$

Theorem 3.1.11

For any circulant polynomial matrix $A(\lambda)$, $A(\lambda)+A^*(\lambda)$ is hermitian circulant polynomial matrix.

Proof

Let $A(\lambda)$ be any circulant polynomial matrix.

We have to prove that $A(\lambda)+A^*(\lambda)$ is hermitian circulant polynomial matrix.

By using theorem$(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$

$$\left(A(\lambda)+A^*(\lambda)\right)^*=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\right]^*$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)\right]^*$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]^*$$

$$=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*+\left(\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)^*$$

$$=\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)+\left(\pi_n^{-1}(\lambda)\right)^*\left(A^*(\lambda)\right)^*\pi_n^*(\lambda)$$

$$=\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$$

$$=A^*(\lambda)+A(\lambda)$$

$$=A(\lambda)+A^*(\lambda)$$

Thus, $A(\lambda)+A^*(\lambda)$ is a hermitian circulant polynomial matrix.

Example 3.1.12

Let $A(\lambda)=\begin{pmatrix}1+2\lambda & i\lambda & 2i+\lambda\\ 2i+\lambda & 1+2\lambda & i\lambda\\ i\lambda & 2i+\lambda & 1+2\lambda\end{pmatrix}=A_0+A_1(\lambda)$

where $A_0=\begin{pmatrix}1 & 0 & 2i\\ 2i & 1 & 0\\ 0 & 2i & 1\end{pmatrix}$, $A_1=\begin{pmatrix}2 & i & 1\\ 1 & 2 & i\\ i & 1 & 2\end{pmatrix}$

Now $A(\lambda)=\pi_3(\lambda)A(\lambda)\pi_3^{-1}(\lambda)$

$$=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}1+2\lambda & i\lambda & 2i+\lambda\\ 2i+\lambda & 1+2\lambda & i\lambda\\ i\lambda & 2i+\lambda & 1+2\lambda\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

$$=\begin{pmatrix}1+2\lambda & i\lambda & 2i+\lambda\\ 2i+\lambda & 1+2\lambda & i\lambda\\ i\lambda & 2i+\lambda & 1+2\lambda\end{pmatrix}$$

$$A(\lambda)+A^*(\lambda)=\begin{pmatrix}2+4\lambda & -2i+(1+i)\lambda & 2i+(1-i)\lambda\\ 2i+(1-i)\lambda & 2+4\lambda & -2i+(1+i)\lambda\\ -2i(1+i)\lambda & 2i+(1-i)\lambda & 2+4\lambda\end{pmatrix}$$

Which is a hermitian circulant polynomial matrix.

Theorem 3.1.13

For any circulant polynomial matrix $A(\lambda)$, $A(\lambda)-A^*(\lambda)$ is skew hermitian circulant polynomial matrix.

Proof

Let $A(\lambda)$ be any circulant polynomial matrix.

We have to prove that $A(\lambda)-A^*(\lambda)$ is skew hermitian circulant polynomial matrix.

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$

Now $\left(A(\lambda)-A^*(\lambda)\right)^*=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)-\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\right]^*$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)-\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)\right]^*$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)-\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]^*$$

$$=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*-\left(\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)^*$$

$$=\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)-\left(\pi_n^{-1}(\lambda)\right)^*\left(A^*(\lambda)\right)^*\pi_n^*(\lambda)$$

$$=\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)-\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$$

$$=A^*(\lambda)-A(\lambda)$$

$$=-\left[A(\lambda)-A^*(\lambda)\right]$$

Thus, $A(\lambda)-A^*(\lambda)$ is skew hermitian circulant polynomial matrix.

Theorem 3.1.14

If $A(\lambda)$ is hermitian circulant polynomial matrix, then $iA(\lambda)$ is skew hermitian circulant polynomial matrix.

Proof

Given that $A(\lambda)$ is hermitian circulant polynomial matrix.

That is, $A^*(\lambda) = A(\lambda)$.

We have to prove that $iA(\lambda)$ is skew hermitian circulant polynomial matrix.

By using theorem$(1.2.9)$, we have $A(\lambda) = \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$

Now $(iA(\lambda))^* = \left[i\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right]^*$

$$= \left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)i^*$$

$$= -i\,\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$= -i\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$$

$$= -i\,A(\lambda)$$

Thus, $i\,A(\lambda)$ is skew hermitian circulant polynomial matrix.

Example 3.1.15

Let $A(\lambda) = \begin{pmatrix} 3+\lambda & 1+2i-4i\lambda & 1-2i+4i\lambda \\ 1-2i+4i\lambda & 3+\lambda & 1+2i-4i\lambda \\ 1+2i-4i\lambda & 1-2i+4i\lambda & 3+\lambda \end{pmatrix} = A_0 + A_1\lambda$

where $A_0 = \begin{pmatrix} 3 & 1+2i & 1-2i \\ 1-2i & 3 & 1+2i \\ 1+2i & 1-2i & 3 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & -4i & 4i \\ 4i & 1 & -4i \\ -4i & 4i & 1 \end{pmatrix}$

Now $A(\lambda) = \pi_3(\lambda)A(\lambda)\pi_3^{-1}(\lambda)$

$$=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}3+\lambda&1+2i-4i\lambda&1-2i+4i\lambda\\1-2i+4i\lambda&3+\lambda&1+2i-4i\lambda\\1+2i-4i\lambda&1-2i+4i\lambda&3+\lambda\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$=\begin{pmatrix}3+\lambda&1+2i-4i\lambda&1-2i+4i\lambda\\1-2i+4i\lambda&3+\lambda&1+2i-4i\lambda\\1+2i-4i\lambda&1-2i+4i\lambda&3+\lambda\end{pmatrix}$$

$$iA(\lambda)=\begin{pmatrix}3i+i\lambda&-2+i+4\lambda&2+i-4\lambda\\2+i-4\lambda&3i+i\lambda&-2+i+4\lambda\\-2+i+4\lambda&2+i-4\lambda&3i+i\lambda\end{pmatrix}$$

$$(iA(\lambda))^{*}=-\begin{pmatrix}3i+i\lambda&-2+i+4\lambda&2+i-4\lambda\\2+i-4\lambda&3i+i\lambda&-2+i+4\lambda\\-2+i+4\lambda&2+i-4\lambda&3i+i\lambda\end{pmatrix}$$

$$(i\,A(\lambda))^{*}=-(i\,A(\lambda))$$

Hence, $iA(\lambda)$ is skew hermitian circulant polynomial matrix.

Theorem 3.1.16

If $A(\lambda)$ is skew hermitian circulant polynomial matrix, then $iA(\lambda)$ is hermitian circulant polynomial matrix.

Proof

Given that $A(\lambda)$ is skew hermitian circulant polynomial matrix.

That is, $A^{*}(\lambda)=-A(\lambda)$.

We have to prove that $i\,A(\lambda)$ is hermitian circulant polynomial matrix.

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

Now $(iA(\lambda))^* = \left(i\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*$

$$= \left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)i^*$$

$$= -i\,\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$= -i\,\pi_n(\lambda)\left(-A(\lambda)\pi_n^{-1}(\lambda)\right)$$

$$= -i\left(-A(\lambda)\right)$$

$$= iA(\lambda)$$

Thus, $i\,A(\lambda)$ is hermitian circulant polynomial matrix.

Theorem 3.1.17

Any circulant polynomial matrix $A(\lambda)$ can be uniquely written in the form, $A(\lambda)=B(\lambda)+C(\lambda)$, where $B(\lambda)$ is hermitian circulant polynomial matrix and $C(\lambda)$ is skew hermitian circulant polynomial matrix.

Proof

Let $A(\lambda)$ be any circulant polynomial matrix which can be represented as, $A(\lambda)=\dfrac{A(\lambda)+A^*(\lambda)}{2}+\dfrac{A(\lambda)-A^*(\lambda)}{2}$.

Where $\frac{1}{2}\left(A(\lambda)+A^*(\lambda)\right)$ is hermitian circulant polynomial matrix and $\frac{1}{2}\left(A(\lambda)-A^*(\lambda)\right)$ is skew hermitian circulant polynomial matrix.

Now, to prove the uniqueness.

Let $A(\lambda)=B(\lambda)+C(\lambda)$, where $B(\lambda)$ is a hermitian circulant polynomial matrix and $C(\lambda)$ is a skew hermitian circulant polynomial matrix.

To prove that $B(\lambda)=\frac{1}{2}\left[A(\lambda)+A^{*}(\lambda)\right]$ and $C(\lambda)=\frac{1}{2}\left[A(\lambda)-A^{*}(\lambda)\right]$

$$A(\lambda)=B(\lambda)+C(\lambda) \tag{3.11}$$

$$A(\lambda)=\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)C(\lambda)\pi_n^{-1}(\lambda)$$

$$\overline{A(\lambda)^{T}}=\overline{\left[\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)C(\lambda)\pi_n^{-1}(\lambda)\right]^{T}}$$

$$A^{*}(\lambda)=\overline{\left[\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^{T}+\left(\pi_n(\lambda)C(\lambda)\pi_n^{-1}(\lambda)\right)^{T}\right]}$$

$$=\left[\overline{\left(\left(\pi_n^{-1}(\lambda)\right)^{T}B^{T}(\lambda)\pi_n^{T}(\lambda)\right)+\left(\left(\pi_n^{-1}(\lambda)\right)^{T}C^{T}(\lambda)\pi_n^{T}(\lambda)\right)}\right]$$

$$=\overline{\pi_n(\lambda)B^{T}(\lambda)\pi_n^{-1}(\lambda)}+\overline{\pi_n(\lambda)C^{T}(\lambda)\pi_n^{-1}(\lambda)}$$

$$=\overline{\pi_n(\lambda)}\;\overline{B^{T}(\lambda)}\;\overline{\pi_n^{-1}(\lambda)}+\overline{\pi_n(\lambda)}\;\overline{C^{T}(\lambda)}\;\overline{\pi_n^{-1}(\lambda)}$$

$$=\pi_n(\lambda)B^{*}(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)C^{*}(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)\left(-C(\lambda)\right)\pi_n^{-1}(\lambda)$$

$$A^{*}(\lambda)=B(\lambda)-C(\lambda) \tag{3.12}$$

From (3.11) and (3.12), we get

$$B(\lambda)=\frac{1}{2}\left[A(\lambda)+A^{*}(\lambda)\right] \text{ and } C(\lambda)=\frac{1}{2}\left[A(\lambda)-A^{*}(\lambda)\right].$$

Hence, any square polynomial matrix $A(\lambda)$ can be uniquely expressed as the sum of a hermitian circulant polynomial matrix and skew hermitian circulant polynomial matrix.

Theorem 3.1.18

If $A(\lambda)$ and $B(\lambda)$ are skew hermitian circulant polynomial matrices, then $A(\lambda)+B(\lambda)$ is also skew hermitian circulant polynomial matrix.

Proof

Given $A(\lambda)$ and $B(\lambda)$ are skew hermitian circulant polynomial matrices.

That is, $A^*(\lambda)=-A(\lambda)$ and $B^*(\lambda)=-B(\lambda)$. We have to prove that $A(\lambda)+B(\lambda)$ is a skew hermitian circulant polynomial matrix.

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

Now $\left(A(\lambda)+B(\lambda)\right)^*=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]^*$

$$=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*+\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^*$$

$$=\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)+\left(\pi_n^{-1}(\lambda)\right)^*B^*(\lambda)\pi_n^*(\lambda)$$

$$=\pi_n(\lambda)\left(-A(\lambda)\right)\pi_n^{-1}(\lambda)+\pi_n(\lambda)\left(-B(\lambda)\right)\pi_n^{-1}(\lambda)$$

$$=-\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]$$

$$=-\left[A(\lambda)+B(\lambda)\right]$$

Thus, $A(\lambda)+B(\lambda)$ is skew hermitian circulant polynomial matrix.

Theorem 3.1.19

If $A(\lambda)$ and $B(\lambda)$ are hermitian circulant polynomial matrices then $A(\lambda)B(\lambda)+B(\lambda)A(\lambda)$ is a hermitian circulant polynomial matrix.

Proof

Given that $A(\lambda)$ and $B(\lambda)$ are hermitian circulant polynomial matrices. That is, $A^*(\lambda)=A(\lambda)$ and $B^*(\lambda)=B(\lambda)$.

To prove that $A(\lambda)B(\lambda)+B(\lambda)A(\lambda)$ is a hermitian circulant polynomial matrix.

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\left[A(\lambda)B(\lambda)+B(\lambda)A(\lambda)\right]^*=\begin{pmatrix}\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right]\left[\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]+\\ \left[\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right]\end{pmatrix}^*$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]^*+\left[\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\right]^*$$

$$=\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^*\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*+\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^*$$

$$=\left(\left(\pi_n^{-1}(\lambda)\right)^*B^*(\lambda)\pi_n^*(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)\right)+\left(\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^*B^*(\lambda)\pi_n^*(\lambda)\right)$$

$$= \left(\pi_n(\lambda) B^*(\lambda) \pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda) A^*(\lambda) \pi_n^{-1}(\lambda)\right) + \left(\pi_n(\lambda) A^*(\lambda) \pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda) B^*(\lambda) \pi_n^{-1}(\lambda)\right)$$

$$= \left(\pi_n(\lambda) B(\lambda) \pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda) A(\lambda) \pi_n^{-1}(\lambda)\right) + \left(\pi_n(\lambda) A(\lambda) \pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda) B(\lambda) \pi_n^{-1}(\lambda)\right)$$

$$= B(\lambda)A(\lambda) + A(\lambda)B(\lambda)$$

$$= A(\lambda)B(\lambda) + B(\lambda)A(\lambda)$$

Thus, $A(\lambda)B(\lambda)+B(\lambda)A(\lambda)$ is a hermitian circulant polynomial matrix.

Theorem 3.1.20

If $A(\lambda)$ and $B(\lambda)$ are hermitian circulant polynomial matrices then $A(\lambda)B(\lambda)-B(\lambda)A(\lambda)$ is a skew hermitian circulant polynomial matrix.

Proof

Given that $A(\lambda)$ and $B(\lambda)$ are hermitian circulant polynomial matrices. That is, $A^{*}(\lambda)=A(\lambda)$ and $B^{*}(\lambda)=B(\lambda)$.

To prove that $A(\lambda)B(\lambda)-B(\lambda)A(\lambda)$ is a skew hermitian circulant polynomial matrices.

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\left[A(\lambda)B(\lambda)-B(\lambda)A(\lambda)\right]^{*}=\begin{bmatrix}\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\\ -\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\end{bmatrix}^{*}$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]^{*}$$
$$-\left[\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\right]^{*}$$

$$=\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^{*}\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{*}$$
$$-\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{*}\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^{*}$$

$$=\left(\left(\pi_n^{-1}(\lambda)\right)^{*}B^{*}(\lambda)\pi_n^{*}(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^{*}A^{*}(\lambda)\pi_n^{*}(\lambda)\right)$$
$$-\left(\left(\pi_n^{-1}(\lambda)\right)^{*}A^{*}(\lambda)\pi_n^{*}(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^{*}B^{*}(\lambda)\pi_n^{*}(\lambda)\right)$$

$$=\left(\pi_n(\lambda)B^{*}(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\right)$$
$$-\left(\pi_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)B^{*}(\lambda)\pi_n^{-1}(\lambda)\right)$$

$$=\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)$$
$$-\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)$$

$$=B(\lambda)A(\lambda)-A(\lambda)B(\lambda)$$

$$=-\left[A(\lambda)B(\lambda)-B(\lambda)A(\lambda)\right]$$

Thus, $A(\lambda)B(\lambda)-B(\lambda)A(\lambda)$ is a skew hermitian circulant polynomial matrix.

Theorem 3.1.21

Let $A(\lambda)$ be any circulant polynomial matrix. Then

$(i)\, A(\lambda)+A^*(\lambda)$ is a hermitian circulant polynomial matrix.

$(ii)\, A(\lambda)-A^*(\lambda)$ is a skew hermitian circulant polynomial matrix.

Proof

(i) Let $C(\lambda)=A(\lambda)+A^*(\lambda)$.

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$C^*(\lambda)=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)^*$$

$$=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*+\left(\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)^*$$

$$=\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)+\left(\pi_n^{-1}(\lambda)\right)^*\left(A^*(\lambda)\right)^*\pi_n^*(\lambda)$$

$$=\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$$

$$=A^*(\lambda)+A(\lambda)$$

$$=A(\lambda)+A^*(\lambda)$$

That is, $\left[A(\lambda)+A^*(\lambda)\right]^*=A(\lambda)+A^*(\lambda)$.

Thus, $A(\lambda)+A^*(\lambda)$ is a hermitian circulant polynomial matrix.

(ii) Let $C(\lambda)=A(\lambda)-A^*(\lambda)$

$$C^*(\lambda)=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)-\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)^*$$

$$=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^* - \left(\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)^*$$

$$=\left(\pi_n^{-1}(\lambda)\right)^* A(\lambda)^* \pi_n^*(\lambda) - \left(\pi_n^{-1}(\lambda)\right)^* \left(A^*(\lambda)\right)^* \pi_n^*(\lambda)$$

$$=\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) - \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$$

$$=A^*(\lambda) - A(\lambda)$$

$$=-\left(A(\lambda) - A^*(\lambda)\right)$$

That is, $\left(A(\lambda) - A^*(\lambda)\right)^* = -\left(A(\lambda) - A^*(\lambda)\right)$

Thus, $A(\lambda) - A^*(\lambda)$ is a skew hermitian circulant polynomial matrix.

Theorem 3.1.22

Conjugate of a hermitian circulant polynomial matrix is a hermitian circulant polynomial matrix.

Proof

Let $A(\lambda)$ be a hermitian circulant polynomial matrix.

That is, $A^*(\lambda) = A(\lambda)$.

By using theorem $(1.2.9)$, we have $A(\lambda) = \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\Rightarrow \overline{A(\lambda)} = \overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}$$

$$\Rightarrow \left(\overline{A(\lambda)}\right)^* = \left(\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)^*$$

$$=\left[\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)^*}\right]$$

$$=\left(\overline{\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)}\right)$$

$$=\left(\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$=\overline{A(\lambda)}$$

Thus, $\overline{A(\lambda)}$ is a hermitian circulant polynomial matrix.

Theorem 3.1.23

Conjugate of a skew hermitian circulant polynomial matrix is skew hermitian circulant polynomial matrix.

Proof

Let $A(\lambda)$ be a skew hermitian circulant polynomial matrix.

That is, $A^*(\lambda)=-A(\lambda)$.

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\Rightarrow \overline{A(\lambda)}=\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}$$

$$\Rightarrow \left(\overline{A(\lambda)}\right)^* = \left(\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)^*$$

$$=\left(\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)^*$$

$$=\left(\overline{\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)}\right)$$

$$=\left(\overline{\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$=\left(\overline{\pi_n(\lambda)\left(-A(\lambda)\pi_n^{-1}(\lambda)\right)}\right)$$

$$=-\left(\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$=-\overline{A(\lambda)}$$

Thus, $\overline{A(\lambda)}$ is a skew hermitian circulant polynomial matrix.

3.2 Normal Circulant Polynomial matrices

In this section some of the properties of normal matrices are extended to normal circulant polynomial matrices. Some important results of normal matrices found in [15,38,40,41,47] are generalized to normal circulants polynomial matrices.

Definition 3.2.1

A circulant polynomial matrix $A(\lambda)$ is called normal circulant polynomial matrix if $A(\lambda)A^*(\lambda)=A^*(\lambda)A(\lambda)$.

Example 3.2.2

Let $A(\lambda)=\begin{pmatrix}1+\lambda+i & 1+2i\lambda\\ 1+2i\lambda & 1+\lambda+i\end{pmatrix}=A_0+A_1\lambda$ where the coefficient matrix of $A(\lambda)$ are $A_0=\begin{pmatrix}1+i & 1\\ 1 & 1+i\end{pmatrix}$, $A_1=\begin{pmatrix}1 & 2i\\ 2i & 1\end{pmatrix}$

$$A(\lambda)A^*(\lambda)=\begin{pmatrix}3+2\lambda+5\lambda^2 & 2+6\lambda\\ 2+6\lambda & 3+2\lambda+5\lambda^2\end{pmatrix}=A^*(\lambda)A(\lambda)$$

Hence, $A(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.3

If $A(\lambda)$ is a normal circulant polynomial matrix and α is a complex number, then

(i) $A(\lambda)+\alpha I_n(\lambda)$ is a normal circulant polynomial matrix.

(ii) $A(\lambda)-\alpha I_n(\lambda)$ is a normal circulant polynomial matrix.

Proof

Given that $A(\lambda)$ is a normal circulant polynomial matrix. We have

$$A(\lambda)A^*(\lambda)=A^*(\lambda)A(\lambda)$$

By using (3), we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

Proof of (i)

Now $\left[A(\lambda)+\alpha I_n(\lambda)\right]\left[A(\lambda)+\alpha I_n(\lambda)\right]^*$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]^*$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*+\left(\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right)^*\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)+\left(\pi_n^{-1}(\lambda)\right)^*I_n^*(\lambda)\pi_n^*(\lambda)\overline{\alpha}\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\overline{\alpha}\right]$$

$$=\left(\pi_n(\lambda)\right)^2A(\lambda)A^*(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2+\alpha\left(\pi_n(\lambda)\right)^2I_n(\lambda)A^*(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2+$$
$$\left(\pi_n(\lambda)\right)^2A(\lambda)I_n(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2\overline{\alpha}+\alpha\left(\pi_n(\lambda)\right)^2\left(I_n(\lambda)\right)^2\left(\pi_n^{-1}(\lambda)\right)^2\overline{\alpha}$$

$$=\pi_n(\lambda)\left[\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)$$
$$+\alpha\pi_n(\lambda)\left[\pi_n(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)$$
$$+\pi_n(\lambda)\left[\pi_n(\lambda)A(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)\overline{\alpha}$$
$$+\alpha\pi_n(\lambda)\left[\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)\overline{\alpha}$$

$$=\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)+\alpha\pi_n(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$
$$+\pi_n(\lambda)A(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\overline{\alpha}+\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\overline{\alpha}$$
$$=A(\lambda)A^*(\lambda)+\alpha I_n(\lambda)A^*(\lambda)+A(\lambda)I_n(\lambda)\overline{\alpha}+\alpha I_n(\lambda)\overline{\alpha}$$

$$=A^*(\lambda)A(\lambda)+\alpha I_n(\lambda)A^*(\lambda)+\overline{\alpha}I_n(\lambda)A(\lambda)+\alpha\overline{\alpha}I_n(\lambda)$$

$$=A^*(\lambda)A(\lambda)+\alpha I_n(\lambda)A^*(\lambda)+\overline{\alpha}I_n(\lambda)A(\lambda)+\alpha\overline{\alpha}I_n(\lambda)$$

$$=A^*(\lambda)\left[A(\lambda)+\alpha I_n(\lambda)\right]+\overline{\alpha}I_n(\lambda)\left[A(\lambda)+\alpha I_n(\lambda)\right]$$

$$=\left[A^*(\lambda)+\overline{\alpha}I_n(\lambda)\right]\left[A(\lambda)+\alpha I_n(\lambda)\right]$$

$$=\left[\left(A(\lambda)\right)^{*}+\left(\alpha I_{n}(\lambda)\right)^{*}\right]\left[A(\lambda)+I_{n}(\lambda)\alpha\right]$$

$$=\left[A(\lambda)+\alpha I_{n}(\lambda)\right]^{*}\left[A(\lambda)+\alpha I_{n}(\lambda)\right]$$

Thus, $A(\lambda)+\alpha I_{n}(\lambda)$ is a normal circulant polynomial matrix.

Proof of (ii)

Now $\left[A(\lambda)-\alpha I_{n}(\lambda)\right]\left[A(\lambda)-\alpha I_{n}(\lambda)\right]^{*}$

$$=\left[\pi_{n}(\lambda)A(\lambda)\pi_{n}^{-1}(\lambda)-\alpha\pi_{n}(\lambda)I_{n}(\lambda)\pi_{n}^{-1}(\lambda)\right]$$
$$\left[\pi_{n}(\lambda)A(\lambda)\pi_{n}^{-1}(\lambda)-\alpha\pi_{n}(\lambda)I_{n}(\lambda)\pi_{n}^{-1}(\lambda)\right]^{*}$$

$$=\left[\pi_{n}(\lambda)A(\lambda)\pi_{n}^{-1}(\lambda)-\alpha\pi_{n}(\lambda)I_{n}(\lambda)\pi_{n}^{-1}(\lambda)\right]$$
$$\left[\left(\pi_{n}(\lambda)A(\lambda)\pi_{n}^{-1}(\lambda)\right)^{*}-\left(\alpha\pi_{n}(\lambda)I_{n}(\lambda)\pi_{n}^{-1}(\lambda)\right)^{*}\right]$$

$$=\left[\pi_{n}(\lambda)A(\lambda)\pi_{n}^{-1}(\lambda)-\alpha\pi_{n}(\lambda)I_{n}(\lambda)\pi_{n}^{-1}(\lambda)\right]$$
$$\left[\left(\pi_{n}^{-1}(\lambda)\right)^{*}A^{*}(\lambda)\pi_{n}^{*}(\lambda)-\left(\pi_{n}^{-1}(\lambda)\right)^{*}I_{n}^{*}(\lambda)\pi_{n}^{*}(\lambda)\overline{\alpha}\right]$$

$$=\left[\pi_{n}(\lambda)A(\lambda)\pi_{n}^{-1}(\lambda)-\alpha\pi_{n}(\lambda)I_{n}(\lambda)\pi_{n}^{-1}(\lambda)\right]$$
$$\left[\pi_{n}(\lambda)A^{*}(\lambda)\pi_{n}^{-1}(\lambda)-\pi_{n}(\lambda)I_{n}(\lambda)\pi_{n}^{-1}(\lambda)\overline{\alpha}\right]$$

$$=\left(\pi_{n}(\lambda)\right)^{2}A(\lambda)A^{*}(\lambda)\left(\pi_{n}^{-1}(\lambda)\right)^{2}-\alpha\left(\pi_{n}(\lambda)\right)^{2}I_{n}(\lambda)A^{*}(\lambda)\left(\pi_{n}^{-1}(\lambda)\right)^{2}$$
$$-\overline{\alpha}\left(\pi_{n}(\lambda)\right)^{2}A(\lambda)I_{n}(\lambda)\left(\pi_{n}^{-1}(\lambda)\right)^{2}+\alpha\overline{\alpha}\left(\pi_{n}(\lambda)\right)^{2}\left(I_{n}(\lambda)\right)^{2}\left(\pi_{n}^{-1}(\lambda)\right)^{2}$$

$$
\begin{aligned}
&= \pi_n(\lambda)\left[\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda) \\
&\qquad -\alpha\pi_n(\lambda)\left[\pi_n(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda) \\
&\qquad -\overline{\alpha}\pi_n(\lambda)\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda) \\
&\qquad +\alpha\overline{\alpha}\pi_n(\lambda)\left[\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)
\end{aligned}
$$

$$
\begin{aligned}
&= \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)-\alpha\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) \\
&\qquad -\overline{\alpha}\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\alpha\overline{\alpha}\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)
\end{aligned}
$$

$$= A(\lambda)A^*(\lambda)-\alpha A^*(\lambda)-\overline{\alpha}A(\lambda)+\alpha\overline{\alpha}I_n(\lambda)$$

$$= A^*(\lambda)A(\lambda)-\alpha A^*(\lambda)-\overline{\alpha}A(\lambda)+\alpha\overline{\alpha}I_n(\lambda)$$

$$= A^*(\lambda)A(\lambda)-\alpha A^*(\lambda)-\overline{\alpha}A(\lambda)+\alpha\overline{\alpha}I_n(\lambda)$$

$$= A^*(\lambda)\left[A(\lambda)-\overline{\alpha}I_n(\lambda)\right]-\overline{\alpha}I_n(\lambda)\left[A(\lambda)-\alpha I_n(\lambda)\right]$$

$$= \left[A^*(\lambda)-\overline{\alpha}I_n(\lambda)\right]\left[A(\lambda)-\alpha I_n(\lambda)\right]$$

$$= \left[\left(A(\lambda)\right)^*-\left(I_n(\lambda)\alpha\right)^*\right]\left[A(\lambda)-\alpha I_n(\lambda)\right]$$

$$= \left[A(\lambda)-I_n(\lambda)\alpha\right]^*\left[A(\lambda)-\alpha I_n(\lambda)\right]$$

$$= \left[A(\lambda)-\alpha I_n(\lambda)\right]^*\left[A(\lambda)-\alpha I_n(\lambda)\right]$$

Hence, $A(\lambda)-\alpha I_n(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.4

Let $A(\lambda)$ and $B(\lambda)$ be normal circulant polynomial matrices and that $A(\lambda)B^*(\lambda)=B^*(\lambda)A(\lambda)$ and $A^*(\lambda)B(\lambda)=B(\lambda)A^*(\lambda)$. Then $A(\lambda)+B(\lambda)$ is a normal circulant polynomial matrix.

Proof

Let $A(\lambda)$ and $B(\lambda)$ be normal circulant polynomial matrices and that $A(\lambda)B^{*}(\lambda)=B^{*}(\lambda)A(\lambda)$ and $A^{*}(\lambda)B(\lambda)=B(\lambda)A^{*}(\lambda)$.

By using (3), we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\left[A(\lambda)+B(\lambda)\right]\left[A(\lambda)+B(\lambda)\right]^{*}$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]^{*}$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{*}+\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^{*}\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\pi_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B^{*}(\lambda)\pi_n^{-1}(\lambda)\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n^{-1}(\lambda)\right)^{*}A^{*}(\lambda)\pi_n^{*}(\lambda)+\left(\pi_n^{-1}(\lambda)\right)^{*}B^{*}(\lambda)\pi_n^{*}(\lambda)\right]$$

$$=\left(\pi_n(\lambda)\right)^{2}A(\lambda)A^{*}(\lambda)\left(\pi_n^{-1}(\lambda)\right)^{2}+\left(\pi_n(\lambda)\right)^{2}B(\lambda)A^{*}(\lambda)\left(\pi_n^{-1}(\lambda)\right)^{2}+$$
$$+\left(\pi_n(\lambda)\right)^{2}A(\lambda)B^{*}(\lambda)\left(\pi_n^{-1}(\lambda)\right)^{2}+\left(\pi_n(\lambda)\right)^{2}B(\lambda)B^{*}(\lambda)\left(\pi_n^{-1}(\lambda)\right)^{2}$$

$$
\begin{aligned}
&= \pi_n(\lambda)\left[\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda) \\
&\qquad\qquad + \pi_n(\lambda)\left[\pi_n(\lambda)B(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right] \\
&\qquad\qquad + \pi_n(\lambda)\left[\pi_n(\lambda)A(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda) \\
&\qquad\qquad + \pi_n(\lambda)\left[\pi_n(\lambda)B(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda) \\
&= \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) + \pi_n(\lambda)B(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) \\
&\qquad + \pi_n(\lambda)A(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda) + \pi_n(\lambda)B(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda) \\
&= A(\lambda)A^*(\lambda) + B(\lambda)A^*(\lambda) + A(\lambda)B^*(\lambda) + B(\lambda)B^*(\lambda) \\
&= A^*(\lambda)A(\lambda) + B(\lambda)A^*(\lambda) + A(\lambda)B^*(\lambda) + B^*(\lambda)B(\lambda) \\
&= A^*(\lambda)\left[A(\lambda) + B(\lambda)\right] + B^*(\lambda)\left[A(\lambda) + B(\lambda)\right] \\
&= \left[A^*(\lambda) + B^*(\lambda)\right]\left[A(\lambda) + B(\lambda)\right] \\
&= \left[A(\lambda) + B(\lambda)\right]^*\left[A(\lambda) + B(\lambda)\right]
\end{aligned}
$$

Hence, $A(\lambda)+B(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.5

Let $A(\lambda)$ and $B(\lambda)$ be normal circulant polynomial matrices and that $A(\lambda)B^*(\lambda)=B^*(\lambda)A(\lambda)$ and $A^*(\lambda)B(\lambda)=B(\lambda)A^*(\lambda)$. Then $A(\lambda)B(\lambda)$ is a normal circulant polynomial matrix.

Proof

Let $A(\lambda)$ and $B(\lambda)$ be normal circulant polynomial matrices and that $A(\lambda)B^*(\lambda)=B^*(\lambda)A(\lambda)$ and $A^*(\lambda)B(\lambda)=B(\lambda)A^*(\lambda)$.

By using (3), we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\left[A(\lambda)B(\lambda)\right]\left[A(\lambda)B(\lambda)\right]^*$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]^*$$

$$=\left[\pi_n(\lambda)A(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^*+\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n^{-1}(\lambda)\right)^*B^*(\lambda)\pi_n^*(\lambda)\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)\right]$$

$$=\pi_n(\lambda)A(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)B(\lambda)B^*(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$=A(\lambda)B(\lambda)B^*(\lambda)A^*(\lambda)$$

$$=A(\lambda)B^*(\lambda)B(\lambda)A^*(\lambda)$$

$$=B^*(\lambda)A(\lambda)B(\lambda)A^*(\lambda)$$

$$=B^*(\lambda)A(\lambda)A^*(\lambda)B(\lambda)$$

$$=B^*(\lambda)A^*(\lambda)A(\lambda)B(\lambda)$$

$$=\left(A(\lambda)B(\lambda)\right)^*\left(A(\lambda)B(\lambda)\right)$$

Hence, $A(\lambda)B(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.6

If $A(\lambda)$ and $B(\lambda)$ are normal circulant polynomial matrices, then so is $A(\lambda)\otimes B(\lambda)$.

Proof

Given that $A(\lambda)$ and $B(\lambda)$ are normal circulant polynomial matrices.

We have to prove that $A(\lambda)\otimes B(\lambda)$ is a circulant polynomial matrix

$$\left(A(\lambda)\otimes B(\lambda)\right)\left(A(\lambda)\otimes B(\lambda)\right)^{*}=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\otimes\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\otimes\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]^{*}$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\otimes\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{*}\otimes\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^{*}\right]$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\otimes\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$\left[\left(\left(\pi_n^{-1}(\lambda)\right)^{*}A^{*}(\lambda)\pi_n^{*}(\lambda)\right)\otimes\left(\left(\pi_n^{-1}(\lambda)\right)^{*}B^{*}(\lambda)\pi_n^{*}(\lambda)\right)\right]$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\otimes\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$\left[\left(\pi_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\right)\otimes\left(\pi_n(\lambda)B^{*}(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\right)\right]\otimes$$
$$\left[\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)B^{*}(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)\right]\otimes$$
$$\left[\left(\pi_n(\lambda)B(\lambda)I_n(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]\otimes$$
$$\left[\left(\pi_n(\lambda)B(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$

$$=\left(A(\lambda)A^*(\lambda)\right)\otimes\left(B(\lambda)B^*(\lambda)\right)$$

$$=\left(A^*(\lambda)A(\lambda)\right)\otimes\left(B^*(\lambda)B(\lambda)\right)$$

$$=\left(A^*(\lambda)\otimes B^*(\lambda)\right)\left(A(\lambda)\otimes B(\lambda)\right)$$

$$=\left(A(\lambda)\otimes B(\lambda)\right)^*\left(A(\lambda)\otimes B(\lambda)\right)$$

Hence, $A(\lambda)\otimes B(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.7

Transpose of a normal circulant polynomial matrix is a normal circulant polynomial matrix.

Proof

Let $A(\lambda)$ be a normal circulant polynomial matrix.

That is, $A(\lambda)A^*(\lambda)=A^*(\lambda)A(\lambda)$.

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

Now $A^T(\lambda)\left(A^T\right)^*(\lambda)=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^T\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^T\right]^*$

$$=\left(\left(\pi_n^{-1}\right)^T A^T(\lambda)\pi_n^T(\lambda)\right)\left(\left(\pi_n^{-1}\right)^T A^T(\lambda)\pi_n^T(\lambda)\right)^*$$

$$=\left(\pi_n^{-1}\right)^T(\lambda)A^T(\lambda)\pi_n^T(\lambda)\left(\pi_n^T\right)^*(\lambda)\left(A^T\right)^*(\lambda)\left(\left(\pi_n^{-1}\right)^T\right)^*(\lambda)$$

$$=\pi_n(\lambda)A^T(\lambda)\pi_n^T(\lambda)\pi_n(\lambda)\left(A^T\right)^*(\lambda)\pi_n^*(\lambda)$$

$$=\pi_n(\lambda)A^T(\lambda)I_n(\lambda)\left(A^T\right)^*(\lambda)\pi_n^*(\lambda)$$

$$=\pi_n(\lambda)A^T(\lambda)\left(A^T\right)^*(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A^T(\lambda)\left(A^*\right)^T(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left[A^*(\lambda)A(\lambda)\right]^T\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left[A(\lambda)A^*(\lambda)\right]^T\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(A^*\right)^T(\lambda)A^T(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(A^T\right)^*(\lambda)A^T(\lambda)\pi_n^{-1}(\lambda)$$

$$=\left(A^T\right)^*(\lambda)A^T(\lambda)$$

Thus, transpose of a normal circulant polynomial matrix is a normal circulant polynomial matrix.

Example 3.2.8

Let $A(\lambda)=\begin{pmatrix} 0 & -2+i & 3i\lambda \\ 3i\lambda & 0 & -2+i \\ -2+i & 3i\lambda & 0 \end{pmatrix}=A_0+A_1\lambda$

where $A_0 = \begin{pmatrix} 0 & -2+i & 0 \\ 0 & 0 & -2+i \\ -2+i & 0 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 0 & 3i \\ 3i & 0 & 0 \\ 0 & 3i & 0 \end{pmatrix}$

Now $A(\lambda) = \pi_3(\lambda) A(\lambda) \pi_3^{-1}(\lambda)$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2+i & 3i\lambda \\ 3i\lambda & 0 & -2+i \\ -2+i & 3i\lambda & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2+i & 3i\lambda \\ 3i\lambda & 0 & -2+i \\ -2+i & 3i\lambda & 0 \end{pmatrix}$$

$$A^T(\lambda)\left(A^T(\lambda)\right)^* = \begin{pmatrix} 5+9\lambda^2 & (3+6i)\lambda & (3-6i)\lambda \\ (3-6i)\lambda & 5+9\lambda^2 & (3+6i)\lambda \\ (3+6i)\lambda & (3-6i)\lambda & 5+9\lambda^2 \end{pmatrix} \qquad (3.15)$$

$$\left(A^T(\lambda)\right)^* A^T(\lambda) = \begin{pmatrix} 5+9\lambda^2 & (3+6i)\lambda & (3-6i)\lambda \\ (3-6i)\lambda & 5+9\lambda^2 & (3+6i)\lambda \\ (3+6i)\lambda & (3-6i)\lambda & 5+9\lambda^2 \end{pmatrix} \qquad (3.16)$$

From (3.15) and (3.16), we get

$$A^T(\lambda)\left(A^T(\lambda)\right)^* = \left(A^T(\lambda)\right)^* A^T(\lambda)$$

Hence, $A^T(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.9

Conjugate of a normal circulant polynomial matrix is a normal circulants polynomial matrix.

Proof

Let $A(\lambda)$ be a normal circulant polynomial matrix.

That is, $A(\lambda)A^*(\lambda)=A^*(\lambda)A(\lambda)$

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

Now $\left(\overline{A(\lambda)}\right)\left(\overline{A(\lambda)}\right)^* = \left(\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)\left(\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)^*$

$$=\left(\overline{\pi_n(\lambda)}\ \overline{A(\lambda)}\ \overline{\pi_n^{-1}(\lambda)}\right)\left(\overline{\pi_n(\lambda)}\,\overline{A(\lambda)}\ \overline{\pi_n^{-1}(\lambda)}\right)^*$$

$$=\left(\pi_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)\overline{A(\lambda)}\ \pi_n^{-1}(\lambda)\right)^*$$

$$=\left(\pi_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right)\left(\pi_n^{-1}(\lambda)\right)^*\left(\overline{A(\lambda)}\right)^*\left(\pi_n(\lambda)\right)^*$$

$$=\pi_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\pi_n(\lambda)\left(\overline{A(\lambda)}\right)^*\pi_n^*(\lambda)$$

$$=\pi_n(\lambda)\overline{A(\lambda)}I_n(\lambda)\left(\overline{A(\lambda)}\right)^*\pi_n^*(\lambda)$$

$$=\pi_n(\lambda)\overline{A(\lambda)}\left(\overline{A(\lambda)}\right)^*\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\overline{A(\lambda)}\ \left(\overline{A^*(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(\overline{A(\lambda)A^*(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(\overline{A^*(\lambda)A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\overline{A^*(\lambda)}\ \overline{A(\lambda)}\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(\overline{A(\lambda)}\right)^*\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\left(\overline{A(\lambda)}\right)^*\left(\overline{A(\lambda)}\right)$$

Thus, conjugate of a normal circulant polynomial matrix is a normal circulants polynomial matrix.

Example 3.2.10

Let $A(\lambda)=\begin{pmatrix} 3i & i+(2+i)\lambda \\ i+(2+i)\lambda & 3i \end{pmatrix}=A_0+A_1\lambda$

where $A_0=\begin{pmatrix} 3i & i \\ i & 3i \end{pmatrix}$, $A_1=\begin{pmatrix} 0 & 2+i \\ i & 0 \end{pmatrix}$

Now $A(\lambda)=\pi_2(\lambda)A(\lambda)\pi_2^{-1}(\lambda)$

$$=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3i & i+(2+i)\lambda \\ i+(2+i)\lambda & 3i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 3i & i+(2+i)\lambda \\ i+(2+i)\lambda & 3i \end{pmatrix}$$

$$\left(\overline{A(\lambda)}\right)\left(\overline{A(\lambda)}\right)^*=\begin{pmatrix} 10+2\lambda+5\lambda^2 & 6+6\lambda \\ 6+6\lambda & 10+2\lambda+5\lambda^2 \end{pmatrix} \tag{3.17}$$

$$\left(\overline{A(\lambda)}\right)^*\left(\overline{A(\lambda)}\right)=\begin{pmatrix} 10+2\lambda+5\lambda^2 & 6+6\lambda \\ 6+6\lambda & 10+2\lambda+5\lambda^2 \end{pmatrix} \tag{3.18}$$

From (3.17) and (3.18), we get

$$\left(\overline{A(\lambda)}\right)\left(\overline{A(\lambda)}\right)^*=\left(\overline{A(\lambda)}\right)^*\left(\overline{A(\lambda)}\right).$$

Hence, Conjugate of a normal circulant polynomial matrix is normal circulant polynomial matrix.

Theorem 3.2.11

Conjugate transpose of a normal circulant polynomial matrix is a normal circulant polynomial matrix.

Proof

Let $A(\lambda)$ be a normal circulant polynomial matrix.

That is, $A(\lambda)A^*(\lambda) = A^*(\lambda)A(\lambda)$

By using theorem $(1.2.9)$, we have $A(\lambda) = \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

Now $A(\lambda)A^*(\lambda) = \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*$

$$= \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)\right)$$

$$= \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)A(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

Taking Conjugate transpose on both side

$$\left(A(\lambda)A^*(\lambda)\right)^* = \left(\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)^*$$

$$\left(A^*(\lambda)\right)^*\left(A^*(\lambda)\right) = \left(\pi_n^{-1}(\lambda)\right)^*\left(A^*(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)$$

$$A(\lambda)A^*(\lambda) = \pi_n(\lambda)\left[A(\lambda)A^*(\lambda)\right]\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)A^*(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$$

$$= A^*(\lambda)A(\lambda)$$

Thus, conjugate transpose of a normal circulant polynomial matrix is a normal circulant polynomial matrix.

Example 3.2.12

Let $A(\lambda)=\begin{pmatrix} 5+2\lambda & -i\lambda \\ -i\lambda & 5+2\lambda \end{pmatrix} = A_0 + A_1\lambda$

where $A_0 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$, $A_1 = \begin{pmatrix} 2 & -i \\ -i & 2 \end{pmatrix}$

Now $A(\lambda)=\pi_2(\lambda)A(\lambda)\pi_2^{-1}(\lambda)$

$$=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5+2\lambda & -i\lambda \\ -i\lambda & 5+2\lambda \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 5+2\lambda & -i\lambda \\ -i\lambda & 5+2\lambda \end{pmatrix}$$

$$A^*(\lambda)\left(A^*(\lambda)\right)^* = A^*(\lambda)A(\lambda)$$

$$=\begin{pmatrix} 25+20\lambda+5\lambda^2 & 0 \\ 0 & 25+20\lambda+5\lambda^2 \end{pmatrix} \tag{3.19}$$

$$\left(A^*(\lambda)\right)^* A^*(\lambda) = A(\lambda)A^*(\lambda)$$

$$=\begin{pmatrix} 25+20\lambda+5\lambda^2 & 0 \\ 0 & 25+20\lambda+5\lambda^2 \end{pmatrix} \tag{3.20}$$

From (3.19) and (3.20), we get

$$\left(A^*(\lambda)\right)\left(A^*(\lambda)\right)^* = \left(A^*(\lambda)\right)^*\left(A^*(\lambda)\right)$$

Hence, $A^*(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.13

If $A(\lambda)$ is invertible normal circulant polynomial matrix, then $A^{-1}(\lambda)$ is a normal circulant polynomial matrix.

Proof

Given that $A(\lambda)$ is invertible normal circulant polynomial matrix.

For a normal circulant polynomial matrix $A(\lambda)$,

$$A(\lambda)A^*(\lambda) = A^*(\lambda)A(\lambda)$$

By using theorem $(1.2.9)$, we have $A(\lambda) = \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\left(A^{-1}(\lambda)\right)\left(A^{-1}(\lambda)\right)^* = \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{-1}\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\right]^*$$

$$= \left[\left(\pi_n^{-1}(\lambda)\right)^{-1} A^{-1}(\lambda)\pi_n^{-1}(\lambda)\right]\left(\left(\pi_n^{-1}(\lambda)\right)^{-1} A^{-1}(\lambda)\pi_n^{-1}(\lambda)\right)^*$$

$$= \left(\pi_n(\lambda)A^{-1}(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A^{-1}(\lambda)\pi_n^{-1}(\lambda)\right)^*$$

$$= \left(\pi_n(\lambda)A^{-1}(\lambda)\pi_n^{-1}(\lambda)\right)\left[\left(\pi_n^{-1}(\lambda)\right)^*\left(A^{-1}(\lambda)\right)^*\pi_n^*(\lambda)\right]$$

$$= \pi_n(\lambda)A^{-1}(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)\left(A^{-1}(\lambda)\right)^*\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)A^{-1}(\lambda)I_n(\lambda)\left(A^{-1}(\lambda)\right)^*\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)A^{-1}(\lambda)\left(A^*(\lambda)\right)^*\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)\left[A(\lambda)A^*(\lambda)\right]^{-1}\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)\left(A^*(\lambda)\right)^{-1}A^{-1}(\lambda)\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)\left(A^{-1}(\lambda)\right)^{*}A^{-1}(\lambda)\pi_n^{-1}(\lambda)$$

$$= \left(A^{-1}(\lambda)\right)^{*}A^{-1}(\lambda)$$

Thus, $A^{-1}(\lambda)$ is a normal circulant polynomial matrix.

Example 3.2.14

Let $A(\lambda) = \begin{pmatrix} i\lambda & -1 \\ -1 & i\lambda \end{pmatrix} = A_0 + A_1(\lambda)$

Where $A_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

Now $A(\lambda) = \pi_2(\lambda)A(\lambda)\pi_2^{-1}(\lambda)$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i\lambda & -1 \\ -1 & i\lambda \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i\lambda & -1 \\ -1 & i\lambda \end{pmatrix}$$

$$A^{-1}(\lambda)\left(A^{-1}(\lambda)\right)^{*} = \begin{pmatrix} 1+\lambda^2 & 0 \\ 0 & 1+\lambda^2 \end{pmatrix} \quad (3.21)$$

$$\left(A^{-1}(\lambda)\right)^{*}A^{-1}(\lambda) = \begin{pmatrix} 1+\lambda^2 & 0 \\ 0 & 1+\lambda^2 \end{pmatrix} \quad (3.22)$$

Form (3.21) and (3.22), we get

$$A^{-1}(\lambda)\left(A^{-1}(\lambda)\right)^{*} = \left(A^{-1}(\lambda)\right)^{*}A^{-1}(\lambda)$$

Hence, $A^{-1}(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.15

If $A(\lambda)$ is a normal circulant polynomial matrix and α is a real number, then $\alpha A(\lambda)$ is a normal circulant polynomial matrix.

Proof

Given that $A(\lambda)$ is a normal circulant polynomial matrix.

We have to prove that $\alpha A(\lambda)$ is a normal circulant polynomial matrix.

For a normal circulant polynomial matrix

$$A(\lambda), A(\lambda)A^*(\lambda) = A^*(\lambda)A(\lambda)$$

By using theorem $(1.2.9)$, we have $A(\lambda) = \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\begin{aligned}
(\alpha A(\lambda))(\alpha A(\lambda))^* &= \left[\alpha\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\right]\left[\alpha\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\right]^* \\
&= \left[\alpha\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right]\left(\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)\alpha^*\right) \\
&= \alpha\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\alpha \\
&= \alpha\pi_n(\lambda)A(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\alpha \\
&= \alpha\pi_n(\lambda)\left[A(\lambda)A^*(\lambda)\right]\pi_n^{-1}(\lambda)\alpha \\
&= \alpha\pi_n(\lambda)A^*(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\alpha \\
&= \pi_n(\lambda)(\alpha A(\lambda))^*(\alpha A(\lambda))\pi_n^{-1}(\lambda) \\
&= (\alpha A(\lambda))^*(\alpha A(\lambda))
\end{aligned}$$

Thus, $\alpha A(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.16

If $A(\lambda)$ is a normal circulant polynomial matrix, then $iA(\lambda)$ is a normal circulant polynomial matrix.

Proof

Given that $A(\lambda)$ is a normal circulant polynomial matrix.

We have to prove that $iA(\lambda)$ is a normal circulant polynomial matrix.

For a normal circulant polynomial matrix $A(\lambda)$,

$$A(\lambda)A^*(\lambda) = A^*(\lambda)A(\lambda)$$

By using theorem $(1.2.9)$, we have $A(\lambda) = \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\begin{aligned}
(iA(\lambda))(iA(\lambda))^* &= \left(i\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(i\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^* \\
&= \left(i\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)i^*\right) \\
&= \left(i\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)(-i)\right) \\
&= i\pi_n(\lambda)A(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)(-i) \\
&= i\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)(-i) \\
&= i\pi_n(\lambda)A^*(\lambda)A(\lambda)\pi_n^{-1}(\lambda)(-i) \\
&= \pi_n(\lambda)\left(-iA^*(\lambda)\right)\left(iA(\lambda)\right)\pi_n^{-1}(\lambda) \\
&= \pi_n(\lambda)\left(-iA^*(\lambda)\right)I_n(\lambda)\left(iA(\lambda)\right)\pi_n^{-1}(\lambda) \\
&= \left[\pi_n(\lambda)\left(-iA^*(\lambda)\right)\pi_n^{-1}(\lambda)\right]\left[\pi_n(\lambda)\left(iA(\lambda)\right)\pi_n^{-1}(\lambda)\right]
\end{aligned}$$

$$=\left(-iA^{*}(\lambda)\right)\left(iA(\lambda)\right)$$

$$=\left(i^{*}A^{*}(\lambda)\right)\left(iA(\lambda)\right)$$

$$=\left(A(\lambda)i\right)^{*}\left(iA(\lambda)\right)$$

$$=\left(iA(\lambda)\right)^{*}\left(iA(\lambda)\right)$$

Hence, $iA(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.17

Every hermitian circulant polynomial is normal circulant polynomial matrix.

Proof

For a hermitian circulant polynomial matrix $A(\lambda)$, $A^{*}(\lambda)=A(\lambda)$

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$A(\lambda)A^{*}(\lambda)=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{*}$$

$$=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^{*}A^{*}(\lambda)\pi_n^{*}(\lambda)\right)$$

$$=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\right)$$

$$=\pi_n(\lambda)A(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(A(\lambda)\right)^{2}\pi_n^{-1}(\lambda)$$

$$A(\lambda)A^*(\lambda)=\left(A(\lambda)\right)^2 \tag{3.23}$$

Similarly, we can prove that $A^*(\lambda)A(\lambda)=\left(A(\lambda)\right)^2$ (3.24)

From (3.23) and (3.24) , we get $A(\lambda)A^*(\lambda)=A^*(\lambda)A(\lambda)$.

Thus, $A(\lambda)$ is a normal circulant polynomial matrix.

Example 3.2.18

Let $A(\lambda)=\begin{pmatrix} 1 & 2i+i\lambda & -2i-i\lambda \\ -2i-i\lambda & 1 & 2i+i\lambda \\ 2i+i\lambda & -2i-i\lambda & 1 \end{pmatrix} = A_0+A_1(\lambda)$

Where $A_0=\begin{pmatrix} 1 & 2i & -2i \\ -2i & 1 & 2i \\ 2i & -2i & 1 \end{pmatrix}$, $A_1=\begin{pmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{pmatrix}$

Now $A(\lambda)=\pi_3(\lambda)A(\lambda)\pi_3^{-1}(\lambda)$

$$=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2i+i\lambda & -2i-i\lambda \\ -2i-i\lambda & 1 & 2i+i\lambda \\ 2i+i\lambda & -2i-i\lambda & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A(\lambda)A^*(\lambda)=\begin{pmatrix} 9+8\lambda+2\lambda^2 & (-4+4i)+(-4+2i)\lambda-\lambda^2 & (-4-4i)+(-4-2i)\lambda-\lambda^2 \\ (-4-4i)+(-4-2i)\lambda-\lambda^2 & 9+8\lambda+2\lambda^2 & (-4+4i)+(-4+2i)\lambda-\lambda^2 \\ (-4+4i)+(-4+2i)\lambda-\lambda^2 & (-4-4i)+(-4-2i)\lambda-\lambda^2 & 9+8\lambda+2\lambda^2 \end{pmatrix} \tag{3.25}$$

$$A^*(\lambda)A(\lambda)=\begin{pmatrix} 9+8\lambda+2\lambda^2 & (-4+4i)+(-4+2i)\lambda-\lambda^2 & (-4-4i)+(-4-2i)\lambda-\lambda^2 \\ (-4-4i)+(-4-2i)\lambda-\lambda^2 & 9+8\lambda+2\lambda^2 & (-4+4i)+(-4+2i)\lambda-\lambda^2 \\ (-4+4i)+(-4+2i)\lambda-\lambda^2 & (-4-4i)+(-4-2i)\lambda-\lambda^2 & 9+8\lambda+2\lambda^2 \end{pmatrix} \tag{3.26}$$

From (3.25) and (3.26), we get $A(\lambda)A^{*}(\lambda)=A^{*}(\lambda)A(\lambda)$.

Hence, $A(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.19

Every skew-hermitian circulant polynomial matrix is a normal circulant polynomial matrix.

Proof

For a skew-hermitian circulant polynomial matrix $A(\lambda)$,

$$A^{*}(\lambda)=-A(\lambda)$$

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\begin{aligned}
A(\lambda)A^{*}(\lambda)&=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{*}\\
&=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^{*}A^{*}(\lambda)\pi_n^{*}(\lambda)\right)\\
&=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\right)\\
&=\pi_n(\lambda)A(\lambda)I_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\\
&=\pi_n(\lambda)A(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\\
&=\pi_n(\lambda)A(\lambda)\left(-A(\lambda)\right)\pi_n^{-1}(\lambda)\\
&=\pi_n(\lambda)-\left(A(\lambda)\right)^{2}\pi_n^{-1}(\lambda)\\
&=-\left(A(\lambda)\right)^{2} \qquad (3.27)
\end{aligned}$$

Similarly, we can prove that $A^*(\lambda)A(\lambda) = -(A(\lambda))^2$ (3.28)

From (3.27) and (3.28),we get $A(\lambda)A^*(\lambda) = A^*(\lambda)A(\lambda)$.

Thus, every skew-hermitian circulant polynomial matrix is a normal circulant polynomial matrix.

Example 3.2.20

Let $A(\lambda) = \begin{pmatrix} 0 & 1+i+i\lambda & -1+i+i\lambda \\ -1+i+i\lambda & 0 & 1+i+i\lambda \\ 1+i+i\lambda & -1+i+i\lambda & 0 \end{pmatrix} = A_0 + A_1\lambda$

where $A_0 = \begin{pmatrix} 0 & 1+i & -1+i \\ -1+i & 0 & 1+i \\ 1+i & -1+i & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{pmatrix}$

Now $A(\lambda) = \pi_3(\lambda)A(\lambda)\pi_3^{-1}(\lambda)$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1+i+i\lambda & -1+i+i\lambda \\ -1+i+i\lambda & 0 & 1+i+i\lambda \\ 1+i+i\lambda & -1+i+i\lambda & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1+i+i\lambda & -1+i+i\lambda \\ -1+i+i\lambda & 0 & 1+i+i\lambda \\ 1+i+i\lambda & -1+i+i\lambda & 0 \end{pmatrix}$$

$$A(\lambda)A^*(\lambda) = \begin{pmatrix} 4+4\lambda+2\lambda^2 & 2i+(2+2i)\lambda+\lambda^2 & -2i+(2-2i)\lambda+\lambda^2 \\ -2i+(2-2i)\lambda+\lambda^2 & 4+4\lambda+2\lambda^2 & 2i+(2+2i)\lambda+\lambda^2 \\ 2i+(2+2i)\lambda+\lambda^2 & -2i+(2-2i)\lambda+\lambda^2 & 4+4\lambda+2\lambda^2 \end{pmatrix} \tag{3.29}$$

$$A^*(\lambda)A(\lambda)=\begin{pmatrix} 4+4\lambda+2\lambda^2 & 2i+(2+2i)\lambda+\lambda^2 & -2i+(2-2i)\lambda+\lambda^2 \\ -2i+(2-2i)\lambda+\lambda^2 & 4+4\lambda+2\lambda^2 & 2i+(2+2i)\lambda+\lambda^2 \\ 2i+(2+2i)\lambda+\lambda^2 & -2i+(2-2i)\lambda+\lambda^2 & 4+4\lambda+2\lambda^2 \end{pmatrix} \quad (3.30)$$

From (3.29) and (3.30), we get $A(\lambda)A^*(\lambda)=A^*(\lambda)A(\lambda)$.

Hence, $A(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.21

Every real symmetric circulant polynomial matrix is a normal circulant polynomial matrix.

Proof

For a real symmetric circulant polynomial matrix $A(\lambda)$, we have

$$A(\lambda)=A^T(\lambda)=A^*(\lambda) \quad (3.31)$$

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

Consider $A(\lambda)A^T(\lambda)=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^T$

$$=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^T(\lambda)\pi_n^{-1}(\lambda)$$

$$=\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)^T\right)A^T(\lambda)\pi_n^T(\lambda)\right)$$

$$=\pi_n(\lambda)A(\lambda)I_n(\lambda)A^T(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)A^T(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)A(\lambda)\pi_n^{-1}(\lambda) \qquad (by(3.31))$$

$$=\pi_n(\lambda)\left(A(\lambda)\right)^2\pi_n^{-1}(\lambda)$$

$$A(\lambda)A^T(\lambda)=(A(\lambda))^2$$

$$A(\lambda)A^*(\lambda)=(A(\lambda))^2 \qquad (3.32)$$

Similarly, we can prove that $A^*(\lambda)A(\lambda)=(A(\lambda))^2$ $\qquad (3.33)$

Form (3.32) and (3.33), we get $A(\lambda)A^*(\lambda)=A^*(\lambda)A(\lambda)$.

Thus, every real symmetric circulant polynomial matrix is a normal circulant polynomial matrix.

Example 3.2.22

Let $A(\lambda)=\begin{pmatrix} 4+2\lambda & 1+3\lambda \\ 1+3\lambda & 4+2\lambda \end{pmatrix}$ be a real symmetric circulant polynomial matrix.

$$=A_0+A_1\lambda \quad \text{where } A_0=\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix},\ A_1=\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

Now $A(\lambda)=\pi_2(\lambda)A(\lambda)\pi_2^{-1}(\lambda)$

$$=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4+2\lambda & 1+3\lambda \\ 1+3\lambda & 4+2\lambda \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} 4+2\lambda & 1+3\lambda \\ 1+3\lambda & 4+2\lambda \end{pmatrix}$$

$$A(\lambda)A^*(\lambda)=\begin{pmatrix} 17+22\lambda+13\lambda^2 & 8+28\lambda+12\lambda^2 \\ 8+28\lambda+12\lambda^2 & 17+22\lambda+13\lambda^2 \end{pmatrix} \qquad (3.34)$$

$$A^*(\lambda)A(\lambda)=\begin{pmatrix} 17+22\lambda+13\lambda^2 & 8+28\lambda+12\lambda^2 \\ 8+28\lambda+12\lambda^2 & 17+22\lambda+13\lambda^2 \end{pmatrix} \qquad (3.35)$$

From (3.34) and (3.35), we get $A(\lambda)A^*(\lambda)=A^*(\lambda)A(\lambda)$.

Hence, $A(\lambda)$ is a normal circulant polynomial matrix.

Theorem 3.2.23

Every real skew symmetric circulant polynomial matrix is a normal circulant polynomial matrix.

Proof

For a real skew-symmetric circulant polynomial matrix $A(\lambda)$,

we have $A^T(\lambda) = A^*(\lambda) = -A(\lambda)$. $\quad (3.36)$

Consider $A(\lambda)A^T(\lambda) = \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^T$

$$= \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^T A^T(\lambda)\pi_n^T(\lambda)\right)$$

$$= \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^T(\lambda)\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)A(\lambda)I_n(\lambda)A^T(\lambda)\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)A(\lambda)A^T(\lambda)\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)A(\lambda)\left(-A(\lambda)\right)\pi_n^{-1}(\lambda) \qquad \text{by } (3.36)$$

$$= \pi_n(\lambda)\left(-A(\lambda)^2\right)\pi_n^{-1}(\lambda)$$

$$A(\lambda)A^T(\lambda) = -\left(A(\lambda)\right)^2$$

Thus, $A(\lambda)A^*(\lambda) = -\left(A(\lambda)\right)^2$ $\quad (3.37)$

Similarly, we can prove that $A^*(\lambda)A(\lambda) = -\left(A(\lambda)\right)^2$. $\quad (3.38)$

From (3.37) and (3.38), we get $A(\lambda)A^*(\lambda) = A^*(\lambda)A(\lambda)$.

Thus, every real skew symmetric circulant polynomial matrix is a normal circulant polynomial matrix.

Example 3.2.24

Let $A(\lambda)=\begin{pmatrix} 0 & 3+2\lambda \\ 3+2\lambda & 0 \end{pmatrix}$ be a real skew symmetric circulant polynomial matrix.

$$= A_0 + A_1\lambda \text{ where } A_0 = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

Now $A(\lambda)=\pi_2(\lambda)A(\lambda)\pi_2^{-1}(\lambda)$

$$=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 3+2\lambda \\ 3+2\lambda & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3+2\lambda \\ 3+2\lambda & 0 \end{pmatrix}$$

$$A(\lambda)A^*(\lambda)=\begin{pmatrix} 9+12\lambda+4\lambda^2 & 0 \\ 0 & 9+12\lambda+4\lambda^2 \end{pmatrix} \tag{3.39}$$

$$A^*(\lambda)A(\lambda)=\begin{pmatrix} 9+12\lambda+4\lambda^2 & 0 \\ 0 & 9+12\lambda+4\lambda^2 \end{pmatrix} \tag{3.40}$$

From (3.39) and (3.40), we get $A(\lambda)A^*(\lambda)=A^*(\lambda)A(\lambda)$.

Hence, $A(\lambda)$ is a normal circulant polynomial matrix.

3.3 Conjugate Normal Circulant Polynomial Matrices

In this section some important results and characterization of conjugate normal matrix is found in [15,16,19,45] generalized to conjugate normal circulant polynomial matrix.

Definition 3.3.1

A circulant polynomial matrix $A(\lambda)\in C_{n\times n}(\lambda)$ is said to be conjugate normal circulant polynomial matrix if $A(\lambda)A^{*}(\lambda)=\overline{A^{*}(\lambda)A(\lambda)}$.

That is, $A(\lambda)A^{*}(\lambda)=A^{T}(\lambda)\overline{A(\lambda)}$ (or) $A^{*}(\lambda)A(\lambda)=\overline{A(\lambda)}A^{T}(\lambda)$.

Example 3.3.2

Let $A(\lambda)=\begin{pmatrix}(1+i)-3i\lambda & i+(6-i)\lambda\\ i+(6-i)\lambda & (1+i)-3i\lambda\end{pmatrix}$ be a circulant polynomial matrix.

$$A(\lambda)A^{*}(\lambda)=\begin{pmatrix}3-8\lambda+46\lambda^{2} & 2+4\lambda+6\lambda^{2}\\ 2+4\lambda+6\lambda^{2} & 3-8\lambda+46\lambda^{2}\end{pmatrix} \qquad (3.41)$$

$$\overline{A^{*}(\lambda)A(\lambda)}=\begin{pmatrix}3-8\lambda+46\lambda^{2} & 2+4\lambda+6\lambda^{2}\\ 2+4\lambda+6\lambda^{2} & 3-8\lambda+46\lambda^{2}\end{pmatrix}$$

$$A^{*}(\lambda)A(\lambda)=\begin{pmatrix}3-8\lambda+46\lambda^{2} & 2+4\lambda+6\lambda^{2}\\ 2+4\lambda+6\lambda^{2} & 3-8\lambda+46\lambda^{2}\end{pmatrix} \qquad (3.42)$$

From (3.41) and (3.42), we get $A^{*}(\lambda)A(\lambda)=\overline{A^{*}(\lambda)A(\lambda)}$.

Hence, $A(\lambda)$ is a conjugate normal circulant polynomial matrix.

Remark 3.3.3

For any polynomial matrix $A(\lambda)\in C_{n\times n}(\lambda)$ we can write $A(\lambda)=S(\lambda)+K(\lambda)$, Such that $S(\lambda)$ is symmetric and $K(\lambda)$ is skew-symmetric. This decomposition for polynomial matrix is $A(\lambda)$ uniquely determined by $S(\lambda)=\frac{1}{2}\left[A(\lambda)+A^{*}(\lambda)\right]$ and $K(\lambda)=\frac{1}{2}\left[A(\lambda)-A^{*}(\lambda)\right]$.

We introduce the polynomial matrices $A_L(\lambda)=\overline{A(\lambda)}A(\lambda)$ and $A_K(\lambda)=A(\lambda)\overline{A(\lambda)}=\overline{A_L(\lambda)}$.

Theorem 3.3.4

If $A(\lambda)$ is a conjugate normal circulant polynomial matrix then $A_L(\lambda)$ and $A_R(\lambda)$ are normal circulant polynomial matrix.

Proof

Given that $A(\lambda)$ is a conjugate normal circulant polynomial matrix.

That is, $A(\lambda)A^{*}(\lambda)=\overline{A^{*}(\lambda)A(\lambda)}$.

That is, $A(\lambda)A^{*}(\lambda)=A^{T}(\lambda)\overline{A(\lambda)}$ (or) $A^{*}(\lambda)A(\lambda)=\overline{A(\lambda)}A^{T}(\lambda)$.

Using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

Now $A_R(\lambda)A_R^{*}(\lambda)=\left(A(\lambda)\overline{A(\lambda)}\right)\left(A(\lambda)\overline{A(\lambda)}\right)^{*}$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\overline{\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}\right]\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\overline{\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}\right]^{*}$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\overline{\pi_n(\lambda)}\left(\overline{A(\lambda)}\right)\left(\overline{\pi_n^{-1}(\lambda)}\right)\right]\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\overline{\pi_n(\lambda)}\left(\overline{A(\lambda)}\right)\right)\overline{\pi_n^{-1}(\lambda)}\right]^{*}$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right]$$

$$\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right]^*$$

$$=\left[\pi_n(\lambda)A(\lambda)I_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right]\left[\pi_n(\lambda)A(\lambda)I_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right]^*$$

$$=\left[\pi_n(\lambda)A(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right]\left[\pi_n(\lambda)A(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right]^*$$

$$=\left[\pi_n(\lambda)A(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right]\left[\left(\pi_n^{-1}(\lambda)\right)^*\left(\overline{A(\lambda)}\right)^*A^*(\lambda)\pi_n^*(\lambda)\right]$$

$$=\pi_n(\lambda)A(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\pi_n(\lambda)\left(\overline{A(\lambda)}\right)^*A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)I_n(\lambda)A^T(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}A^T(\lambda)\right)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)A^*(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(A(\lambda)A^*(\lambda)\right)^2\pi_n^{-1}(\lambda)$$

$$A_R(\lambda)A_R^*(\lambda)=\left(A(\lambda)A^*(\lambda)\right)^2 \tag{3.43}$$

Now $A_R^*(\lambda)A_R(\lambda)=\left(A(\lambda)\overline{A(\lambda)}\right)^*\left(A(\lambda)\overline{A(\lambda)}\right)$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\overline{\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}\right]^*$$

$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\overline{\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}\right]$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\overline{\pi_n(\lambda)}\ \overline{A(\lambda)}\ \overline{\pi_n^{-1}(\lambda)}\right)\right]^*$$

$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\overline{\pi_n(\lambda)}\left(\overline{A(\lambda)}\right)\left(\overline{\pi_n^{-1}(\lambda)}\right)\right)\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right]^*$$

$$\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)I_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right]^*\left[\pi_n(\lambda)A(\lambda)I_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda)\right]^*\left[\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]$$

$$=\left(\pi_n^{-1}(\lambda)\right)^*\left(\overline{A(\lambda)}\right)^*A^*(\lambda)\pi_n^*(\lambda)\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A^T(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A^T(\lambda)A^*(\lambda)I_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A^T(\lambda)\left(A^*(\lambda)A(\lambda)\right)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(A^T(\lambda)\overline{A(\lambda)}\right)\left(A^T(\lambda)\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(A(\lambda)A^*(\lambda)\right)\left(A(\lambda)A^*(\lambda)\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(A(\lambda)A^*(\lambda)\right)^2\pi_n^{-1}(\lambda)$$

$$A_R^*(\lambda)A_R(\lambda)=\left(A(\lambda)A^*(\lambda)\right)^2 \qquad (3.44)$$

From (3.43) and (3.44), we get $A_R(\lambda)A_R^*(\lambda)=A_R^*(\lambda)A_R(\lambda)$.

Hence, $A_R(\lambda)$ is normal circulant polynomial matrix.

Therefore, $A_L(\lambda)=\overline{A_R(\lambda)}$ is normal circulant polynomial matrix as well.

Remark 3.3.5

To state the next theorem, we associate with each polynomial matrix $A(\lambda)\in C_{n\times n}(\lambda)$ the polynomial matrix $\hat{A}(\lambda)=\begin{pmatrix} O(\lambda) & A(\lambda) \\ A(\lambda) & O(\lambda) \end{pmatrix}$.

Theorem 3.3.6

A polynomial matrix $A(\lambda)\in C_{n\times n}(\lambda)$ is conjugate normal circulant polynomial matrix if and only if $\hat{A}(\lambda)$ is normal circulant polynomial matrix.

Proof

Let $\hat{A}(\lambda)=\begin{pmatrix} O(\lambda) & A(\lambda) \\ \overline{A(\lambda)} & O(\lambda) \end{pmatrix}$.

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\left(\hat{A}(\lambda)\right)^{*}=\begin{pmatrix} O(\lambda) & A(\lambda) \\ \overline{A(\lambda)} & O(\lambda) \end{pmatrix}^{*}$$

$$=\begin{bmatrix} \pi_n(\lambda)O(\lambda)\pi_n^{-1}(\lambda) & \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda) \\ \pi_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda) & \pi_n(\lambda)O(\lambda)\pi_n^{-1}(\lambda) \end{bmatrix}^{*}$$

$$=\begin{bmatrix} O(\lambda) & \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda) \\ \pi_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda) & O(\lambda) \end{bmatrix}^{*}$$

$$=\begin{bmatrix} O(\lambda) & \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{*} \\ \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{*} & O(\lambda) \end{bmatrix}$$

$$=\begin{bmatrix} O(\lambda) & \left(\pi_n^{-1}(\lambda)\right)^* A^T(\lambda)\pi_n^T(\lambda) \\ \left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda) & O(\lambda) \end{bmatrix}$$

$$=\begin{bmatrix} O(\lambda) & \pi_n(\lambda)A^T(\lambda)\pi_n^{-1}(\lambda) \\ \pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) & O(\lambda) \end{bmatrix}$$

$$\hat{A}(\lambda)\left(\hat{A}(\lambda)\right)^* = \begin{bmatrix} O(\lambda) & \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda) \\ \pi_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda) & O(\lambda) \end{bmatrix} \begin{bmatrix} O(\lambda) & \pi_n(\lambda)A^T(\lambda)\pi_n^{-1}(\lambda) \\ \pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) & O(\lambda) \end{bmatrix}$$

$$=\begin{bmatrix} \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) & O(\lambda) \\ O(\lambda) & \pi_n(\lambda)\overline{A(\lambda)}A^T(\lambda)\pi_n^{-1}(\lambda) \end{bmatrix}$$

$$=\begin{bmatrix} A(\lambda)A^*(\lambda) & O(\lambda) \\ O(\lambda) & \overline{A(\lambda)}A^T(\lambda) \end{bmatrix}$$

$$\hat{A}(\lambda)\left(\hat{A}(\lambda)\right)^* = A(\lambda)A^*(\lambda) \oplus \overline{A(\lambda)}A^T(\lambda) \tag{3.45}$$

Now

$$\left(\hat{A}(\lambda)\right)^* \hat{A}(\lambda) = \begin{bmatrix} O(\lambda) & \pi_n(\lambda)A^T(\lambda)\pi_n^{-1}(\lambda) \\ \pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) & O(\lambda) \end{bmatrix} \begin{bmatrix} O(\lambda) & \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda) \\ \pi_n(\lambda)\overline{A(\lambda)}\pi_n^{-1}(\lambda) & O(\lambda) \end{bmatrix}$$

$$=\begin{bmatrix} \pi_n(\lambda)A^T(\lambda)\,\overline{A(\lambda)}\pi_n^{-1}(\lambda) & O(\lambda) \\ O(\lambda) & \pi_n(\lambda)A^*(\lambda)A(\lambda)\pi_n^{-1}(\lambda) \end{bmatrix}$$

$$=\begin{bmatrix} A^T(\lambda)\overline{A(\lambda)} & O(\lambda) \\ O(\lambda) & A^*(\lambda)A(\lambda) \end{bmatrix}$$

$$\left(\hat{A}(\lambda)\right)^*\hat{A}(\lambda)=A^T(\lambda)\overline{A(\lambda)}\oplus A^*(\lambda)A(\lambda) \qquad (3.46)$$

From (3.45) and (3.46), we get $\left(\hat{A}(\lambda)\right)\left(\hat{A}(\lambda)\right)^*=\left(\hat{A}(\lambda)\right)^*\left(\hat{A}(\lambda)\right)$.

Hence, $\hat{A}(\lambda)$ is normal circulant polynomial matrix.

Theorem 3.3.7

If $A(\lambda)\in C_{n\times n}(\lambda)$ is conjugate normal circulant polynomial matrix, then $im\left(A(\lambda)\right)=im\left(A^T(\lambda)\right)$ and $\ker\left(A(\lambda)\right)=\ker\left(A^T(\lambda)\right)$.

Proof

For any circulant polynomial matrix $A(\lambda)$, we have $im\left(A(\lambda)A^*(\lambda)\right)=im\left(A^T(\lambda)\right)$ and $\ker\left(A(\lambda)A^*(\lambda)\right)=\ker\left(A^*(\lambda)\right)$. So $imA(\lambda)=im\left(A(\lambda)A^*(\lambda)\right)$.

By using theorem$(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$=im\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\right]$$

$$=im\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)\right)\right]$$

$$=im\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\right]$$

$$= im\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$

$$= im\left[\left(\pi_n(\lambda)A(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$

$$= im\left[\pi_n(\lambda)\left(A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$

$$= im\left[\pi_n(\lambda)\left(A^T(\lambda)\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]$$

$$= im\left[\pi_n(\lambda)\left(A^T(\lambda)\left(A^T(\lambda)\right)^*\right)\pi_n^{-1}(\lambda)\right]$$

$$= im\left[\pi_n(\lambda)A^T(\lambda)\pi_n^{-1}(\lambda)\right]$$

$$= im\left[A^T(\lambda)\right]$$

Hence, $im\left(A(\lambda)\right) = im\left(A^T(\lambda)\right)$.

Now $\ker\left(A^*(\lambda)\right) = \ker\left(A(\lambda)A^*(\lambda)\right)$

$$= \ker\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\right]$$

$$= \ker\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)\right)\right]$$

$$= \ker\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]$$

$$= \ker\left[\pi_n(\lambda)A(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]$$

$$= \ker\left[\pi_n(\lambda)\left(A(\lambda)A^*(\lambda)\right)\pi_n^{-1}(\lambda)\right]$$

$$= \ker\left[\pi_n(\lambda)\left(A^T(\lambda)\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]$$

$$\ker\left(A^*(\lambda)\right) = \ker\left(\overline{A(\lambda)}\right)$$

$$\ker\left(\left(A^{*}(\lambda)\right)^{*}\right)=\left(\ker\left(\overline{A(\lambda)}\right)^{*}\right)$$

$$\ker\left(A(\lambda)\right)=\ker\left(A^{T}(\lambda)\right)$$

Hence, $\ker\left(A(\lambda)\right)=\ker\left(A^{T}(\lambda)\right)$.

Theorem 3.3.8

Let $A(\lambda)\in C_{n\times n}(\lambda)$ be a circulant polynomial matrix. Then the following statements are equivalent.

(i) $A(\lambda)$ is a conjugate normal circulant polynomial matrix.

(ii) $\overline{A(\lambda)}$ is a conjugate normal circulant polynomial matrix.

(iii) $A^{T}(\lambda)$ is a conjugate normal circulant polynomial matrix.

(iv) $A^{*}(\lambda)$ is a conjugate normal circulant polynomial matrix.

(v) $A^{-1}(\lambda)$ is a conjugate normal circulant polynomial matrix, if $A^{-1}(\lambda)$ exists.

(vi) $A^{\dagger}(\lambda)$ is a conjugate normal circulant polynomial matrix.

(vii) $\alpha A(\lambda)$ is a conjugate normal circulant polynomial matrix, where λ is a real number.

Proof

Let $A(\lambda)$ be a circulant polynomial matrix. By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

Proof of $(i)\Leftrightarrow(ii)$

$A(\lambda)$ is a conjugate normal circulant polynomial matrix

$$\Leftrightarrow A(\lambda)A^*(\lambda)=\overline{A^*(\lambda)A(\lambda)}$$

$$\Leftrightarrow\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*=\overline{\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}$$

$$\Leftrightarrow\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)=\left(\overline{\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}\right)$$

$$\Leftrightarrow\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\overline{\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}$$

$$\Leftrightarrow\pi_n(\lambda)A(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\overline{\pi_n(\lambda)A^*(\lambda)I_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}$$

$$\Leftrightarrow\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\left(\overline{\pi_n(\lambda)A^*(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\overline{\pi_n(\lambda)}\left(\overline{A^*(\lambda)}\right)\left(\overline{A(\lambda)}\right)\overline{\left(\pi_n^{-1}(\lambda)\right)}$$

$$\Leftrightarrow\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow\left(\overline{\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)}\right)=\left(\overline{\pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow\overline{\pi_n(\lambda)}\left(\overline{A(\lambda)}\right)\left(\overline{A^*(\lambda)}\right)\left(\overline{\pi_n^{-1}(\lambda)}\right)=\left(\overline{\pi_n(\lambda)}\right)\left(\overline{A^T(\lambda)}\right)\overline{\left(\overline{A(\lambda)}\right)}\overline{\left(\pi_n^{-1}(\lambda)\right)}$$

$$\Leftrightarrow\pi_n(\lambda)\left(\overline{A(\lambda)}\right)\left(A^T(\lambda)\right)\pi_n^{-1}(\lambda)=\pi_n(\lambda)A^*(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow\left(\overline{A(\lambda)}\right)\left(A^T(\lambda)\right)=A^*(\lambda)A(\lambda)$$

$\Leftrightarrow\left(\overline{A(\lambda)}\right)$ is a conjugate normal circulant polynomial matrix.

Proof of $(i) \Leftrightarrow (iii)$

$A(\lambda)$ is a conjugate normal circulant polynomial matrix

$$\Leftrightarrow A(\lambda)A^{*}(\lambda) = \overline{A^{*}(\lambda)A(\lambda)}$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{*} = \overline{\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{*}\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n^{-1}(\lambda)\right)^{*}A^{*}(\lambda)\pi_n^{*}(\lambda) = \left(\overline{\left(\pi_n^{-1}(\lambda)\right)^{*}A^{*}(\lambda)\pi_n^{*}(\lambda)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda) = \left(\overline{\pi_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)I_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda) = \left(\overline{\pi_n(\lambda)A^{*}(\lambda)I_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda) = \left(\overline{\pi_n(\lambda)A^{*}(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda) = \overline{\pi_n(\lambda)}\,\overline{A^{*}(\lambda)}\left(\overline{A(\lambda)}\right)\left(\overline{\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda) = \pi_n(\lambda)A^{T}(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\right)^{T} = \left(\pi_n(\lambda)A^{T}(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right)^{T}$$

$$\Leftrightarrow \left(\pi_n^{-1}(\lambda)\right)^{T}\left(A^{*}(\lambda)\right)^{T}A^{T}(\lambda)\pi_n^{T}(\lambda) = \left(\pi_n^{-1}(\lambda)\right)^{T}\left(\overline{A(\lambda)}\right)^{T}\left(A^{T}(\lambda)\right)^{T}\pi_n^{T}(\lambda)$$

$$\Leftrightarrow \pi_n(\lambda)\left(\overline{A(\lambda)}\right)A^{T}(\lambda)\pi_n^{-1}(\lambda) = \pi_n(\lambda)A^{*}(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow \left(\overline{A(\lambda)}\right)A^{T}(\lambda) = A^{*}(\lambda)A(\lambda)$$

$\Leftrightarrow A^{T}(\lambda)$ is a conjugate normal circulant polynomial matrix.

Proof of $(i) \Leftrightarrow (vi)$

$A(\lambda)$ is a conjugate normal circulant polynomial matrix

$$\Leftrightarrow A(\lambda)A^*(\lambda) = \overline{A^*(\lambda)A(\lambda)}$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*$$
$$= \overline{\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)\right) =$$
$$\left(\overline{\left(\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}\right)$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)$$
$$= \left(\overline{\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) = \left(\overline{\pi_n(\lambda)A^*(\lambda)I_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) = \left(\overline{\pi_n(\lambda)A^*(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) = \left(\overline{\pi_n(\lambda)}\right)\left(\overline{A^*(\lambda)}\right)\left(\overline{A(\lambda)}\right)\left(\overline{\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) = \pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)^* = \left(\pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right)^*$$

$$\Leftrightarrow \left(\pi_n^{-1}(\lambda)\right)^*\left(A^*(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda) = \left(\pi_n^{-1}(\lambda)\right)^*\left(\overline{A(\lambda)}\right)^*\left(A^T(\lambda)\right)^*\pi_n^*(\lambda)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) = \pi_n(\lambda)\left(\overline{A^*(\lambda)}\right)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)^{\dagger}=\left(\pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right)^{\dagger}$$

$$\Leftrightarrow \left(\pi_n^{-1}(\lambda)\right)^{\dagger}\left(A^*(\lambda)\right)^{\dagger}A^{\dagger}(\lambda)\pi_n^{\dagger}(\lambda)=\left(\pi_n^{-1}(\lambda)\right)^{\dagger}\left(\overline{A(\lambda)}\right)^{\dagger}\left(A^T(\lambda)\right)^{\dagger}\pi_n^{\dagger}(\lambda)$$

$$\Leftrightarrow \pi_n(\lambda)\left(A^{\dagger}(\lambda)\right)^{*}A^{\dagger}(\lambda)\pi_n^{-1}(\lambda)=\pi_n(\lambda)\left(\overline{A^{\dagger}(\lambda)}\right)\left(A^{\dagger}(\lambda)\right)^{\dagger}\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow \left(A^{\dagger}(\lambda)\right)^{*}\left(A^{\dagger}(\lambda)=\right)\left(\overline{A^{\dagger}(\lambda)}\right)\left(A^{\dagger}(\lambda)\right)^{\dagger}$$

$\Leftrightarrow A^{\dagger}(\lambda)$ is a conjugate normal circulant polynomial matrix.

Proof of $(i)\Leftrightarrow(vii)$

$A(\lambda)$ is a conjugate normal circulant polynomial matrix

$$\Leftrightarrow A(\lambda)A^*(\lambda)=\overline{A^*(\lambda)A(\lambda)}$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{*}$$
$$=\overline{\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^{*}\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n^{-1}(\lambda)\right)^{*}\left(A^*(\lambda)\pi_n^{*}(\lambda)\right)$$
$$=\left(\overline{\left(\pi_n^{-1}(\lambda)\right)^{*}A^*(\lambda)\pi_n^{*}(\lambda)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$
$$=\overline{\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\overline{\pi_n(\lambda)A^*(\lambda)I_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) = \left(\overline{\pi_n(\lambda)A^*(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) = \left(\overline{\pi_n(\lambda)}\right)\left(\overline{A^*(\lambda)}\right)\left(\overline{A(\lambda)}\right)\overline{\left(\pi_n^{-1}(\lambda)\right)}$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) = \pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow \alpha^2\left(\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right) = \alpha^2\left(\pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right)$$

$$\Leftrightarrow \pi_n(\lambda)\left(\alpha A(\lambda)\right)\left(\alpha A^*(\lambda)\right)\pi_n^{-1}(\lambda) = \pi_n(\lambda)\left(\alpha A^T(\lambda)\right)\left(\overline{\alpha A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow \pi_n(\lambda)\left(\alpha A(\lambda)\right)\left(\alpha A(\lambda)*\right)\pi_n^{-1}(\lambda) = \pi_n(\lambda)\left(\alpha A(\lambda)\right)^T\left(\overline{\alpha A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow \left(\alpha A(\lambda)\right)\left(\alpha A(\lambda)\right)^* = \left(\alpha A(\lambda)\right)^T\left(\overline{\alpha A(\lambda)}\right)$$

$\Leftrightarrow \alpha A(\lambda)$ is a conjugate normal circulant polynomial matrix.

Proof of $(i) \Leftrightarrow (v)$

$A(\lambda)$ is a conjugate normal circulant polynomial matrix

$$\Leftrightarrow A(\lambda)A^*(\lambda) = \overline{A^*(\lambda)A(\lambda)}$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^* = \overline{\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n^{-1}(\lambda)\right)^*\left(A^*(\lambda)\pi_n^*(\lambda)\right) = \left(\overline{\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda) = \overline{\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\left(\overline{\pi_n(\lambda)A^*(\lambda)I_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\left(\overline{\pi_n(\lambda)A^*(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\overline{\pi_n(\lambda)}\left(\overline{A^*(\lambda)}\right)\left(\overline{A(\lambda)}\right)\left(\overline{\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow \pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)^{-1}=\left(\pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right)^{-1}$$

$$\Leftrightarrow \left(\pi_n^{-1}(\lambda)\right)^{-1}\left(A^*(\lambda)\right)^{-1}A^{-1}(\lambda)\pi_n^{-1}(\lambda)$$
$$=\left(\pi_n^{-1}(\lambda)\right)^{-1}\left(\overline{A(\lambda)}\right)^{-1}\left(A^T(\lambda)\right)^{-1}\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow \pi_n(\lambda)\left(A^{-1}(\lambda)\right)^*A^{-1}(\lambda)\pi_n^{-1}(\lambda)=\pi_n(\lambda)\left(\overline{A^{-1}(\lambda)}\right)\left(A^{-1}(\lambda)\right)^T\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow \left(A^{-1}(\lambda)\right)^*A^{-1}(\lambda)=\left(\overline{A^{-1}(\lambda)}\right)\left(A^{-1}(\lambda)\right)^T$$

$\Leftrightarrow A^{-1}(\lambda)$ is a conjugate normal circulant polynomial matrix if $A^{-1}(\lambda)$ exists.

Proof of $(i)\Leftrightarrow(iv)$

$A(\lambda)$ is a conjugate normal circulant polynomial matrix.

$$\Leftrightarrow A(\lambda)A^*(\lambda)=\overline{A^*(\lambda)A(\lambda)}$$
$$\Leftrightarrow \left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*$$
$$=\overline{\left(\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}$$

$$\Leftrightarrow\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\left(\pi_n^{-1}(\lambda)\right)^{*}A^{*}(\lambda)\pi_n^{*}(\lambda)\right)$$
$$=\overline{\left(\left(\left(\pi_n^{-1}(\lambda)\right)^{*}A^{*}(\lambda)\pi_n^{*}(\lambda)\right)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}$$

$$\Leftrightarrow\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\right)$$
$$=\overline{\left(\left(\pi_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\right)}$$

$$\Leftrightarrow\left(\pi_n(\lambda)A(\lambda)I_n(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)\right)=\overline{\left(\left(\pi_n(\lambda)A^{*}(\lambda)I_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\right)}$$

$$\Leftrightarrow\pi_n(\lambda)A(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)=\overline{\left(\left(\pi_n(\lambda)A^{*}(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\right)}$$

$$\Leftrightarrow\pi_n(\lambda)A(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)=\left(\overline{\pi_n(\lambda)}\right)\left(\overline{A^{*}(\lambda)}\right)\left(\overline{A(\lambda)}\right)\left(\overline{\pi_n^{-1}(\lambda)}\right)$$

$$\Leftrightarrow\pi_n(\lambda)A(\lambda)A^{*}(\lambda)\pi_n^{-1}(\lambda)=\pi_n(\lambda)A^{T}(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$\Leftrightarrow A(\lambda)A^{*}(\lambda)=A^{T}(\lambda)\left(\overline{A(\lambda)}\right)$$

$\Leftrightarrow A^{*}(\lambda)$ is a conjugate normal circulant polynomial matrix.

Theorem 3.3.9

Let $A(\lambda)\in C_{n\times n}(\lambda)$ be a circulant polynomial matrix.

(i) If $A(\lambda)$ is a conjugate normal circulant polynomial matrix, then $iA(\lambda)$ is a conjugate normal circulant polynomial matrix.

(ii) If $A(\lambda)$ is a conjugate normal circulant polynomial matrix, then $-iA(\lambda)$ is a conjugate normal circulant polynomial matrix.

Proof

Given that $A(\lambda)$ is a conjugate normal circulant polynomial matrix.

That is, $A(\lambda)A^*(\lambda)=\overline{A^*(\lambda)A(\lambda)}$.

By using theorem $(1.2.9)$, we have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

Now $A(\lambda)A^*(\lambda)=\overline{A^*(\lambda)A(\lambda)}$

$$\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*$$
$$=\overline{\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}$$

$$\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)$$
$$=\left(\overline{\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)}\right)$$

$$\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$
$$=\left(\overline{\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\pi_n(\lambda)A(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\left(\overline{\pi_n(\lambda)A^*(\lambda)I_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\left(\overline{\pi_n(\lambda)A^*(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)$$

$$\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\overline{\pi_n(\lambda)}\left(\overline{A^*(\lambda)}\right)\left(\overline{A(\lambda)}\right)\left(\overline{\pi_n^{-1}(\lambda)}\right)$$

$$\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=\pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$i^2\,\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)=i^2\,\pi_n(\lambda)A^T(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$\pi_n(\lambda)\left(iA(\lambda)\right)\left(iA^*(\lambda)\right)\pi_n^{-1}(\lambda)=\pi_n(\lambda)\left(iA^T(\lambda)\right)\left(i\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$\pi_n(\lambda)(iA(\lambda))\left(-(iA(\lambda))^*\right)\pi_n^{-1}(\lambda)=\pi_n(\lambda)(iA(\lambda))^T\left(-\left(\overline{iA(\lambda)}\right)\right)\pi_n^{-1}(\lambda)$$

$$\pi_n(\lambda)(iA(\lambda))\left((iA(\lambda))\right)^*\pi_n^{-1}(\lambda)=\pi_n(\lambda)(iA(\lambda))^T\left(\left(\overline{iA(\lambda)}\right)\right)\pi_n^{-1}(\lambda)$$

$$(iA(\lambda))\left((iA(\lambda))\right)^*=(iA(\lambda))^T\left(\overline{iA(\lambda)}\right)$$

Hence, $iA(\lambda)$ is a conjugate normal circulant polynomial matrix.

Similarly, we can prove that $-iA(\lambda)$ is a conjugate normal circulant polynomial matrix.

Theorem 3.3.10

If $A(\lambda)\in C_{n\times n}(\lambda)$ is a conjugate normal circulant polynomial matrix, then $A(\lambda)\left(\overline{A(\lambda)}\right)$ and $\left(\overline{A(\lambda)}\right)A(\lambda)$ are normal circulant polynomial matrices.

Proof

Given that $A(\lambda)$ is a conjugate normal circulant polynomial matrix.

That is, $A(\lambda)A^*(\lambda)=\overline{A^*(\lambda)A(\lambda)}$

That is, $A(\lambda)A^*(\lambda)=A^T(\lambda)\left(\overline{A(\lambda)}\right)$

Now

$$\left(A(\lambda)\left(\overline{A(\lambda)}\right)\right)\left(A(\lambda)\left(\overline{A(\lambda)}\right)\right)^*=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)\right]^*$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\overline{\pi_n(\lambda)}\right)\left(\overline{A(\lambda)}\right)\left(\overline{\pi_n^{-1}(\lambda)}\right)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\overline{\pi_n(\lambda)}\right)\left(\overline{A(\lambda)}\right)\left(\overline{\pi_n^{-1}(\lambda)}\right)\right]^*$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)\right)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]^*$$

$$=\left[\pi_n(\lambda)A(\lambda)I_n(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)I_n(\lambda)\right)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]^*$$

$$=\left[\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]\left[\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]^*$$

$$=\left[\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]\left[\left(\pi_n^{-1}(\lambda)\right)^*\left(\overline{A(\lambda)}\right)^*A^*(\lambda)\pi_n^*(\lambda)\right]$$

$$=\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\pi_n(\lambda)\left(\overline{A^*(\lambda)}\right)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)I_n(\lambda)A^T(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)\left[\overline{A(\lambda)}A^T(\lambda)\right]A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A(\lambda)A^*(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(A(\lambda)A^*(\lambda)\right)^2\pi_n^{-1}(\lambda)$$

$$\left(A(\lambda)\left(\overline{A(\lambda)}\right)\right)\left(A(\lambda)\left(\overline{A(\lambda)}\right)\right)^*=\left(A(\lambda)\left(\overline{A(\lambda)}\right)\right)^2 \qquad (3.47)$$

Now

$$\left(A(\lambda)\left(\overline{A(\lambda)}\right)\right)^{*}\left(A(\lambda)\left(\overline{A(\lambda)}\right)\right)=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)\right]^{*}$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\overline{\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)}\right)\right]$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\overline{\pi_n(\lambda)}\right)\left(\overline{A(\lambda)}\right)\left(\overline{\pi_n^{-1}(\lambda)}\right)\right]^{*}$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\overline{\pi_n(\lambda)}\left(\overline{A(\lambda)}\right)\left(\overline{\pi_n^{-1}(\lambda)}\right)\right]$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)\right)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]^{*}$$
$$\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)I_n(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]^{*}$$
$$\left[\pi_n(\lambda)A(\lambda)I_n(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]^{*}\left[\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]$$

$$=\left[\left(\pi_n^{-1}(\lambda)\right)^{*}\left(\overline{A(\lambda)}\right)^{*}A^{*}(\lambda)\pi_n^{*}(\lambda)\right]\left[\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)\right]$$

$$=\pi_n(\lambda)\left(\overline{A^{*}(\lambda)}\right)A^{*}(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A^{T}(\lambda)A^{*}(\lambda)I_n(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A^{T}(\lambda)A^{*}(\lambda)A(\lambda)\left(\overline{A(\lambda)}\right)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)A^{T}(\lambda)A^{*}(\lambda)\left(\overline{A(\lambda)}\right)A(\lambda)\pi_n^{-1}(\lambda)$$

$$=\pi_n(\lambda)\left(A^{T}(\lambda)\overline{A(\lambda)}\right)A^{*}(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)\left(A(\lambda)A^*(\lambda)\right)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$

$$= \pi_n(\lambda)\left(A(\lambda)A^*(\lambda)\right)^2 \pi_n^{-1}(\lambda)$$

$$= \left(A(\lambda)A^*(\lambda)\right)^2$$

$$\left(A(\lambda)\overline{A(\lambda)}\right)^* A(\lambda)\overline{A(\lambda)} = \left(A(\lambda)A^*(\lambda)\right)^2 \qquad (3.48)$$

From (3.47) and (3.48), we get

$$\left(A(\lambda)\overline{A(\lambda)}\right)\left(A(\lambda)\overline{A(\lambda)}\right)^* = \left(A(\lambda)\overline{A(\lambda)}\right)^* \left(A(\lambda)\overline{A(\lambda)}\right)$$

Hence, $A(\lambda)\left(\overline{A(\lambda)}\right)$ is a normal circulant polynomial matrix.

Similarly, we can prove that $\left(\overline{A(\lambda)}\right)A(\lambda)$ is a normal circulant polynomial matrix.

Chapter IV

BLOCK CIRCULNT POLYNOMIAL MATRICES

This chapter is devoted to the concept of block circulant and circulant block polynomial matrices. Some characterization of block circulant and circulant block polynomial matrices and all block circulant matrix with circulant polynomial matrices as its blocks are diagonalizable polynomial matrices by using the unitary polynomial matrix.

4.1 Block Circulant Polynomial Matrices

In this section we define block circulant polynomial matrices and we extend some of the properties of block circulant matrices found in [3,9,42,52] to block circulant polynomial matrices.

Definition 4.1.1

A block circulant polynomial matrix is a polynomial matrix in the following form

$$\text{b circ}\left(A_1(\lambda), A_2(\lambda), \ldots, A_m(\lambda)\right) = \begin{pmatrix} A_1(\lambda) & A_2(\lambda) & \ldots & A_m(\lambda) \\ A_m(\lambda) & A_1(\lambda) & \ldots & A_{m-1}(\lambda) \\ & & \vdots & \\ A_2(\lambda) & A_3(\lambda) & \ldots & A_1(\lambda) \end{pmatrix}.$$

We denote the set of all block circulant polynomial matrices of order $m \times n$ as $\mathscr{BC}_{\mathrm{m,n}}(\lambda)$.

Example 4.1.2

The polynomial matrix $\begin{bmatrix} 1-\lambda^2 & \lambda^3 & 2+\lambda^2 & -11\lambda \\ \lambda+3\lambda^2 & 1+\lambda & 4+6\lambda^2 & -8+\lambda \\ 2+\lambda^2 & -11\lambda & 1-\lambda^2 & \lambda^3 \\ 4+6\lambda^2 & -8+\lambda & \lambda+3\lambda^2 & 1+\lambda \end{bmatrix}$ is a block circulant polynomial matrix.

Theorem 4.1.3

$A(\lambda) \in \mathscr{BC}_{\mathrm{m,n}}(\lambda)$ if and only if $A(\lambda)$ commutes with the unitary polynomial matrix $\pi_m(\lambda) \otimes I_n(\lambda)$:

$$A(\lambda)\left(\pi_m(\lambda) \otimes I_n(\lambda)\right) = \left(\pi_m(\lambda) \otimes I_n(\lambda)\right) A(\lambda)$$

Proof

Assume that $A(\lambda)$ is a block circulant polynomial matrix.

That is, b circ $\left(A_1(\lambda), A_2(\lambda), \ldots, A_m(\lambda)\right) = \begin{pmatrix} A_1(\lambda) & A_2(\lambda) & \ldots & A_m(\lambda) \\ A_m(\lambda) & A_1(\lambda) & \ldots & A_{m-1}(\lambda) \\ & & \vdots & \\ A_2(\lambda) & A_3(\lambda) & \ldots & A_1(\lambda) \end{pmatrix}$.

We have to prove that $A(\lambda)\left(\pi_m(\lambda) \otimes I_n(\lambda)\right) = \left(\pi_m(\lambda) \otimes I_n(\lambda)\right) A(\lambda)$.

Now the polynomial matrix $\pi_m(\lambda) \otimes I_n(\lambda) \in \mathscr{BC}_{\mathrm{m,n}}(\lambda)$ is given by

$$\pi_m(\lambda) \otimes I_n(\lambda) = \begin{bmatrix} O_n(\lambda) & I_n(\lambda) & O_n(\lambda) & \ldots & O_n(\lambda) \\ O_n(\lambda) & O_n(\lambda) & I_n(\lambda) & \ldots & O_n(\lambda) \\ & & \vdots & & \\ O_n(\lambda) & O_n(\lambda) & O_n(\lambda) & \ldots & I_n(\lambda) \\ I_n(\lambda) & O_n(\lambda) & O_n(\lambda) & \ldots & O_n(\lambda) \end{bmatrix}$$

$$A(\lambda)\left(\pi_m(\lambda) \otimes I_n(\lambda)\right) = \begin{pmatrix} A_m(\lambda) & A_1(\lambda) & A_2(\lambda) & \ldots & A_{m-1}(\lambda) \\ A_{m-1}(\lambda) & A_m(\lambda) & A_1(\lambda) & \ldots & A_{m-2}(\lambda) \\ & & \vdots & & \\ A_1(\lambda) & A_2(\lambda) & A_3(\lambda) & \ldots & A_m(\lambda) \end{pmatrix} \quad (4.1)$$

$$\left(\pi_m(\lambda) \otimes I_n(\lambda)\right) A(\lambda) = \begin{pmatrix} A_m(\lambda) & A_1(\lambda) & A_2(\lambda) & \ldots & A_{m-1}(\lambda) \\ A_{m-1}(\lambda) & A_m(\lambda) & A_1(\lambda) & \ldots & A_{m-2}(\lambda) \\ & & \vdots & & \\ A_1(\lambda) & A_2(\lambda) & A_3(\lambda) & \ldots & A_m(\lambda) \end{pmatrix} \quad (4.2)$$

From (4.1) and (4.2), we get

$$A(\lambda)\left(\pi_m(\lambda) \otimes I_n(\lambda)\right) = \left(\pi_m(\lambda) \otimes I_n(\lambda)\right) A(\lambda).$$

Conversely, assume that

$A(\lambda)\left(\pi_m(\lambda)\otimes I_n(\lambda)\right)=\left(\pi_m(\lambda)\otimes I_n(\lambda)\right)A(\lambda)$. We have to prove that

$A(\lambda)$ is a block circulant polynomial matrix.

$$I_m(\lambda)\otimes A_1(\lambda)=\begin{bmatrix} A_1(\lambda) & 0 & \dots & 0 \\ 0 & A_1(\lambda) & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & A_1(\lambda) \end{bmatrix}$$

$$\pi_m(\lambda)\otimes A_2(\lambda)=\begin{bmatrix} 0 & A_2(\lambda) & 0 & \dots & 0 \\ 0 & 0 & A_2(\lambda) & \dots & 0 \\ & & \vdots & & \\ A_2(\lambda) & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$\pi_m^2(\lambda)\otimes A_3(\lambda)=\begin{bmatrix} 0 & 0 & A_3(\lambda) & 0 & \dots & 0 \\ 0 & 0 & 0 & A_3(\lambda) & \dots & 0 \\ & & \vdots & & & \\ 0 & A_3(\lambda) & 0 & 0 & \dots & 0 \end{bmatrix} \quad etc$$

$$\left(I_m(\lambda)\otimes A_1(\lambda)\right)+\left(\pi_m(\lambda)\otimes A_2(\lambda)\right)+\dots+\left(\pi_m^{m-1}(\lambda)\otimes A_m(\lambda)\right)$$

$$= \text{b circ }\left(A_1(\lambda),A_2(\lambda),\dots,A_m(\lambda)\right).$$

Hence, $A(\lambda)$ is a block circulant polynomial matrix.

Theorem 4.1.4

$$\text{b circ }\left(A_1(\lambda),A_2(\lambda),\dots,A_n(\lambda)\right)=\sum_{k=0}^{m-1}\left[\pi_m^k(\lambda)\otimes A_{K+1}(\lambda)\right].$$

Proof

Given that $A(\lambda)=bcirc\left(A_1(\lambda), A_2(\lambda),..., A_n(\lambda)\right)$ is a block circulant polynomial matrix.

That is, $$A(\lambda)=\begin{bmatrix} A_1(\lambda) & A_2(\lambda) & \cdots & A_n(\lambda) \\ A_n(\lambda) & A_1(\lambda) & \cdots & A_{n-1}(\lambda) \\ \vdots & \vdots & & \vdots \\ A_2(\lambda) & A_3(\lambda) & \cdots & A_1(\lambda) \end{bmatrix}$$

Now $$\mathrm{I_m}(\lambda)\otimes A_1(\lambda)=\begin{bmatrix} A_1(\lambda) & 0 & \cdots & 0 \\ 0 & A_1(\lambda) & \cdots & 0 \\ 0 & 0 & \cdots & A_1(\lambda) \end{bmatrix}$$

$$\pi_{\mathrm{m}}(\lambda)\otimes A_2(\lambda)=\begin{bmatrix} 0 & A_2(\lambda) & 0 & \cdots & 0 \\ 0 & 0 & A_2(\lambda) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ A_2(\lambda) & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\pi_m^2(\lambda)\otimes A_3(\lambda)=\begin{bmatrix} 0 & 0 & A_3(\lambda) & 0 & \cdots & 0 \\ 0 & 0 & 0 & A_3(\lambda) & \cdots & 0 \\ & & \vdots & & & \\ 0 & A_3(\lambda) & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Since the pre direct product of any $n\times n$ polynomial matrix by $\pi_m(\lambda)$ shifts the columns of that matrix one place to the right. Therefore, we find that

$$\pi_m^{m-1}\otimes A_n(\lambda)=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & A_n(\lambda) \\ A_n(\lambda) & 0 & 0 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & A_n(\lambda) & 0 \end{bmatrix}$$

Hence, $bcirc\left(A_1(\lambda), A_2(\lambda), \ldots, A_n(\lambda)\right) = \sum_{k=0}^{m-1}\left[\pi_m^k(\lambda) \otimes A_{K+1}(\lambda)\right]$.

Remark 4.1.5

Block circulant polynomial matrix of the same type do not necessarily commute.

Example 4.1.6

$$\begin{pmatrix} A(\lambda) & O(\lambda) \\ O(\lambda) & A(\lambda) \end{pmatrix}\begin{pmatrix} B(\lambda) & O(\lambda) \\ O(\lambda) & B(\lambda) \end{pmatrix} = \begin{pmatrix} A(\lambda)B(\lambda) & O(\lambda) \\ O(\lambda) & A(\lambda)B(\lambda) \end{pmatrix}$$

$$\begin{pmatrix} B(\lambda) & O(\lambda) \\ O(\lambda) & B(\lambda) \end{pmatrix}\begin{pmatrix} A(\lambda) & O(\lambda) \\ O(\lambda) & A(\lambda) \end{pmatrix} = \begin{pmatrix} B(\lambda)A(\lambda) & O(\lambda) \\ O(\lambda) & B(\lambda)A(\lambda) \end{pmatrix}$$

Theorem 4.1.7

Let $A(\lambda) = bcirc\left(A_1(\lambda), A_1(\lambda), \ldots, A_m(\lambda)\right)$,

$B(\lambda) = bcirc\left(B_1(\lambda), B_2(\lambda), \ldots, B_m(\lambda)\right) \in \mathscr{BC}_{\mathrm{m,n}}(\lambda)$.

Then, if the $A_j(\lambda)$'s commutes with the $B_K(\lambda)$'s, $A(\lambda)$ and $B(\lambda)$ commute.

Proof

By theorem (4.1.4), we have

$$A(\lambda) = \sum_{j=0}^{m-1}\pi^j(\lambda) \otimes A_{j+1}(\lambda), \quad B(\lambda) = \sum_{j=0}^{m-1}\pi^K(\lambda) \otimes B_{K+1}(\lambda)$$

$$A(\lambda)B(\lambda) = \left[\sum_{j=0}^{m-1}\left(\pi^j(\lambda) \otimes A_{j+1}(\lambda)\right)\right]\left[\sum_{k=0}^{m-1}\left(\pi^k(\lambda) \otimes B_{k+1}(\lambda)\right)\right]$$

$$= \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left[\pi^{j+k}(\lambda) \otimes A_{j+1}(\lambda)B_{K+1}(\lambda)\right]$$

$$=\sum_{k=0}^{m-1}\sum_{j=0}^{m-1}\left[\pi^{k+j}(\lambda)\otimes B_{K+1}(\lambda)A_{j+1}(\lambda)\right]$$

$$=\left[\sum_{k=0}^{m-1}\left(\pi^{k}(\lambda)\otimes B_{k+1}(\lambda)\right)\right]\left[\sum_{j=0}^{m-1}\left(\pi^{j}(\lambda)\otimes A_{j+1}(\lambda)\right)\right]$$

$$=B(\lambda)A(\lambda)$$

Theorem 4.1.8

$A(\lambda)\in \mathscr{BC}_{\mathrm{m,n}}(\lambda)$ if and only if it is of the form $A(\lambda)=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*$

diag$\left[M_1(\lambda),M_2(\lambda),...,M_n(\lambda)\right]\left[F_M(\lambda)\otimes F_n(\lambda)\right]$

where the $M_K(\lambda)$'s are arbitrary polynomial square matrices of order n.

Proof

Assume that $A(\lambda)$ is a block circulant polynomial matrix.

From theorem (4.1.4), we have

$$A(\lambda)=b\,circ\left(A_1(\lambda),A_2(\lambda),...,A_n(\lambda)\right)=\sum_{k=0}^{m-1}\left(\pi_m^k(\lambda)\otimes A_{K+1}(\lambda)\right)$$

for some $A_K(\lambda)$.

Now

$$\pi_m^k(\lambda)\otimes A_{k+1}(\lambda)=\left[F_m^*(\lambda)\,\Omega^k(\lambda)F_m(\lambda)\right]\otimes\left[F_n^*(\lambda)\left(F_n(\lambda)A_{K+1}(\lambda)F_n^*(\lambda)\right)F_n(\lambda)\right]$$

Let $B_K(\lambda)=F_n(\lambda)A_{K+1}(\lambda)F_n^*(\lambda)$

$$\pi_m^k(\lambda)\otimes A_{k+1}(\lambda)=\left[F_m^*(\lambda)\overset{k}{\Omega}(\lambda)\ F_m(\lambda)\right]\otimes\left[F_n^*(\lambda)B_K(\lambda)F_n(\lambda)\right]$$

$$=\left(F_m^*(\lambda)\otimes F_n^*(\lambda)\right)\left(\Omega^k(\lambda)\otimes B_k(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$$

$$\sum_{k=0}^{m-1}\left[\pi_m^k(\lambda)\otimes A_{k+1}(\lambda)\right]=\sum_{k=0}^{m-1}\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\left(\Omega^k(\lambda)\otimes B_k(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$$

$$A(\lambda)=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\sum_{k=0}^{m-1}\left(\Omega^k(\lambda)\otimes B_k(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$$

$$=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\ \mathrm{diag}\left(M_1(\lambda),M_2(\lambda),\ldots,M_n(\lambda)\right)\left(F_m(\lambda)\otimes F_m(\lambda)\right)$$

Where $\begin{pmatrix} M_1(\lambda) \\ M_2(\lambda) \\ \vdots \\ M_m(\lambda) \end{pmatrix}=\left(m^{\frac{1}{2}}F_m^*(\lambda)\otimes I_n(\lambda)\right)\begin{pmatrix} B_0(\lambda) \\ B_1(\lambda) \\ \vdots \\ B_{m-1}(\lambda) \end{pmatrix}$ $\qquad (4.3)$

Thus,

$$A(\lambda)=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*diag\left(M_1(\lambda),M_2(\lambda),\ldots,M_n(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$$

From (4.3), $\begin{pmatrix} B_0(\lambda) \\ B_1(\lambda) \\ \vdots \\ B_{m-1}(\lambda) \end{pmatrix}=m^{-\frac{1}{2}}\left(F_m(\lambda)\otimes I_n(\lambda)\right)\begin{pmatrix} M_1(\lambda) \\ M_2(\lambda) \\ \vdots \\ M_m(\lambda) \end{pmatrix}$

Since $A_{k+1}(\lambda)=F_n^*(\lambda)B_k(\lambda)F_n(\lambda)$

$M_k(\lambda)$ arbitrary $\Leftrightarrow B_k(\lambda)$ are arbitrary.

$\Leftrightarrow A_k(\lambda)$ are arbitrary.

Hence, $A(\lambda) \in \mathscr{BC}_{\mathrm{m,n}}(\lambda)$.

4.2 Circulant Block Polynomial Matrices

In this section we have given a characterization of circulant block polynomial matrices analogous to that of the results found in [4,10,49].

Definition: 4.2.1

Let $A(\lambda)$ be of type (m,n):

$$A(\lambda)=\begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & \cdots & A_{1m}(\lambda) \\ & \vdots & & \\ A_{m1}(\lambda) & A_{m2}(\lambda) & \cdots & A_{mm}(\lambda) \end{pmatrix}$$

$m \times m$ blocks, each block of order n is circulant block polynomial matrices if each block $A_{ij}(\lambda)$ is a circulant polynomial matrix. Which is denoted by $\mathscr{CB}_{\mathrm{m,n}}(\lambda)$.

Theorem 4.2.2

$A(\lambda) \in \mathscr{CB}_{\mathrm{m,n}}(\lambda)$ if and only if $A(\lambda)$ commutes with $I_m(\lambda) \otimes \pi_n(\lambda)$. That is, $A(\lambda)\left(I_m(\lambda) \otimes \pi_n(\lambda)\right) = \left(I_m(\lambda) \otimes \pi_n(\lambda)\right) A(\lambda)$.

Proof

Assume that $A(\lambda)$ is a circulant block polynomial matrices.

We have to prove that $A(\lambda)\left(I_m(\lambda) \otimes \pi_n(\lambda)\right) = \left(I_m(\lambda) \otimes \pi_n(\lambda)\right) A(\lambda)$.

We have $I_m(\lambda) \otimes \pi_n(\lambda) \in \mathscr{CB}_{\mathrm{m,n}}(\lambda)$, and is given by

$$I_m(\lambda)\otimes\pi_n(\lambda)=\begin{pmatrix}\pi_n(\lambda) & 0 & \cdots & 0\\ 0 & \pi_n(\lambda) & \cdots & 0\\ & \vdots & & \\ 0 & 0 & \dots & \pi_n(\lambda)\end{pmatrix}.$$

By block multiplication, $A(\lambda)\left(I_m(\lambda)\otimes\pi_n(\lambda)\right)=\left(A_{jk}(\lambda)\pi_n(\lambda)\right)_{j,k=1,2,\dots,m}$

Similarly, $\left(I_m(\lambda)\otimes\pi_n(\lambda)\right)A(\lambda)=\left(\pi_n(\lambda)A_{jk}(\lambda)\right)_{j,k=1,2,,\dots,m}$.

Since $A(\lambda)$ is a circulant block polynomial matrices.

Therefore, $A_{jk}(\lambda)\pi_n(\lambda)=\pi_n(\lambda)A_{jk}(\lambda),\ j,k=1,2,\dots,m.$

Conversely, assume that $A(\lambda)\left(\pi_m(\lambda)\otimes I_n(\lambda)\right)=\left(\pi_m(\lambda)\otimes I_n(\lambda)\right)A(\lambda).$

We have to prove that $A(\lambda)$ is a circulant block polynomial matrices.

By (2.1.9), equality holds if and only if each block $A_{jk}(\lambda)$ is a circulant polynomial matrices.

Therefore, $A(\lambda)$ is a circulant block polynomial matrices.

Theorem 4.2.3

$A(\lambda)\in\mathscr{CB}_{\mathrm{m,n}}(\lambda)$ if and only if it is of the form $A(\lambda)=\sum_{k=0}^{m-1}\left(A_{k+1}(\lambda)\otimes\pi_n^k(\lambda)\right)$ where $A_{k+1}(\lambda)$ are arbitrary polynomial square matrices of order n.

Proof

By (1.2.10), $A(\lambda)=\left(A_{jk}(\lambda)\right)\in\mathscr{CB}_{\mathrm{m,n}}(\lambda)$ if and only if

$$A_{jk}(\lambda)=a_{jk1}(\lambda)I_n(\lambda)+a_{jk2}(\lambda)\pi_n(\lambda)+\ldots+a_{jkn}(\lambda)\pi_n^{n-1}(\lambda)$$

Now set $(a_{jk1}(\lambda))=A_1(\lambda),\ldots,(a_{jkn}(\lambda))=A_n(\lambda)$. Then

$$\begin{aligned}
&A_1(\lambda)\otimes I_n(\lambda)=(a_{jk1}(\lambda)),\ j,k=1,2,\ldots,m\\
&\qquad\vdots\\
&A_n(\lambda)\otimes \pi_n^{n-1}(\lambda)=(a_{jkn}(\lambda)\pi_n^{n-1}(\lambda)),\ j,k=1,2,\ldots,m
\end{aligned}$$

$$\begin{aligned}
(A_1(\lambda)\otimes I_n(\lambda))+\ldots+(A_n(\lambda)\otimes \pi_n^{n-1}(\lambda))&=(a_{jk1}(\lambda)I_n(\lambda))\\
&\quad+\ldots+(a_{jkn}(\lambda)\pi_n^{n-1}(\lambda)),\ j,k=1,2,\ldots,m
\end{aligned}$$

Therefore, $\sum_{k=0}^{m-1}(A_{k+1}(\lambda)\otimes \pi_n^{k}(\lambda))=A(\lambda)$.

Theorem 4.2.4

If $A(\lambda)\in\mathscr{CB}_{\mathrm{m,n}}(\lambda)$, then

$$A(\lambda)=(F_m(\lambda)\otimes F_n(\lambda))^{*}(\odot_{ij}(\lambda))(F_m(\lambda)\otimes F_n(\lambda))$$

where the $\odot_{jk}(\lambda),\ j,k=1,2,\ldots,m$ are arbitrary diagonal polynomial matrices of order n.

Proof

Let $A(\lambda)\in\mathscr{CB}_{\mathrm{m,n}}(\lambda)$. Then by (1.2.34) for certain diagonal polynomial matrices $\Lambda_{jk}(\lambda)$ of order n, $A_{jk}(\lambda)=F_n^{*}(\lambda)\Lambda_{jk}(\lambda)F_n(\lambda)$

$$A(\lambda)=(F_n^{*}(\lambda)\Lambda_{jk}(\lambda)F_n(\lambda))$$

$$
=\begin{pmatrix} F_n^*(\lambda) & 0 & \cdots & 0 \\ 0 & F_n^*(\lambda) & \cdots & 0 \\ 0 & 0 & \cdots & F_n^*(\lambda) \end{pmatrix} \begin{pmatrix} \Lambda_{11}(\lambda) & \Lambda_{12}(\lambda) & \cdots & \Lambda_{1m}(\lambda) \\ & \cdots & & \\ \Lambda_{m1}(\lambda) & \Lambda_{m2}(\lambda) & & \Lambda_{mn}(\lambda) \end{pmatrix}
$$

$$
\begin{pmatrix} F_n(\lambda) & 0 & \cdots & 0 \\ 0 & F_n(\lambda) & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & F_n(\lambda) \end{pmatrix}
$$

$$
A(\lambda)=\left(I_m(\lambda)\otimes F_n(\lambda)\right)^* \Lambda_{jk}(\lambda)\left(I_m(\lambda)\otimes F_n(\lambda)\right)
$$

Hence, any $A(\lambda)\in\mathscr{CB}_{\mathrm{m,n}}(\lambda)$ is unitarily similar to a polynomial matrix with diagonal blocks.

Now

$$
A(\lambda)=\left[\left(F_m^*(\lambda)\otimes I_n(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)\right]^* \Lambda_{jk}(\lambda)
$$

$$
\left[\left(F_m^*(\lambda)\otimes I_n(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)\right]
$$

Since $\left(I_m(\lambda)\otimes F_n(\lambda)\right)=\left(\left(F_m^*(\lambda)\otimes I_n(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)\right)$

$$
\Leftrightarrow\left[\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\left(F_m^*(\lambda)\otimes I_n(\lambda)\right)^*\right]\Lambda_{jk}(\lambda)\left[\left(F_m^*(\lambda)\otimes I_n(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)\right]
$$

$$
\Leftrightarrow\left(F_m^*(\lambda)\otimes F_n^*(\lambda)\right)\left(\left(F_m^{**}(\lambda)\otimes I_n^*(\lambda)\right)\right)\Lambda_{jk}(\lambda)\left(F_m^*(\lambda)\otimes I_n^*(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)
$$

$$
\Leftrightarrow\left(F_m(\lambda)\otimes F_n(\lambda)\right)^* F_m(\lambda)\otimes I_n(\lambda)\Lambda_{jk}(\lambda)\left(F_m(\lambda)\otimes I_n(\lambda)\right)^*\left(F_m(\lambda)\otimes F_n(\lambda)\right)
$$

Since $F_m(\lambda)\otimes I_n(\lambda)$ and $\Lambda_{jk}(\lambda)$ consist of diagonal blocks polynomial matrices and diagonal block polynomial matrices are closed with respect to matrix addition and multiplication.

Therefore, $\left(F_m(\lambda)\otimes I_n(\lambda)\right)\Lambda_{jk}(\lambda)\left(F_m(\lambda)\otimes I_n(\lambda)\right)^* = \left(\odot_{jk}(\lambda)\right)$

where $\odot_{jk}(\lambda)$ are diagonal block polynomial matrix.

Thus, $A(\lambda)=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\odot_{ij}(\lambda)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$.

4.3 Block Circulant Matrices: Where the Blocks are Circulant Polynomial Matrices

In this section, we have given a characterization of block circulant matrix with circulant polynomial matrices as its blocks analogous to that of the results found in [9,10,44,49,52].

Definition 4.3.1

Let $A(\lambda)$ be of type (m, n). $A(\lambda)$ is said to be a block circulant matrix with circulant polynomial matrices as its blocks if it is circulant block wise and each block is a circulant polynomial matrix and is denoted by $\mathscr{CPBC}_{m,\mathrm{n}}(\lambda)$.

Example 4.3.2

$$A(\lambda)=\begin{bmatrix} 3+\lambda & -\lambda & 12-\lambda & -1+\lambda & 1+7\lambda & -10+\lambda \\ -\lambda & 3+\lambda & -1+\lambda & 12-\lambda & -10+\lambda & 1+7\lambda \\ 1+7\lambda & -10+\lambda & 3+\lambda & -\lambda & 12-\lambda & -1+\lambda \\ -10+\lambda & 1+7\lambda & -\lambda & 3+\lambda & -1+\lambda & 12-\lambda \\ 12-\lambda & -1+\lambda & 1+7\lambda & -10+\lambda & 3+\lambda & -\lambda \\ -1+\lambda & 12-\lambda & -10+\lambda & 1+7\lambda & -\lambda & 3+\lambda \end{bmatrix}$$

is in $\mathscr{CPBC}_{m,\mathrm{n}}(\lambda)$.

Remark 4.3.3

A polynomial matrix in $\mathscr{CPBC}_{m,\mathrm{n}}(\lambda)$ is not necessarily a circulant polynomial matrix.

Lemma 4.3.4

$F_m(\lambda)$ and $F_n(\lambda)$ satisfies the following equalities.

(i) $\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]=\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]$

(ii) $\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]$.

Proof

$$(i)\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[\left(F_m^*(\lambda)\Omega_m(\lambda)F_m(\lambda)\right)\otimes\left(F_n^*(\lambda)I_n(\lambda)F_n(\lambda)\right)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[F_m^*(\lambda)\otimes F_n^*(\lambda)\right]\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$(ii)\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*$$

$$=\left[\left(F_m^*(\lambda)\Omega_m(\lambda)F_m(\lambda)\right)\otimes\left(F_n^*(\lambda)I_n(\lambda)F_n(\lambda)\right)\right]\left[F_m^*(\lambda)\otimes F_n^*(\lambda)\right]$$

$$=\left(F_m^*(\lambda)\otimes F_n^*(\lambda)\right)\left(\Omega_m(\lambda)\otimes I_n(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*$$

$$=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\left(\Omega_m(\lambda)\otimes I_n(\lambda)\right)$$

Lemma 4.3.5

If $A(\lambda)=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\Lambda(\lambda)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$ where $\Lambda(\lambda)$ is diagonal polynomial matrix, then $A(\lambda)\in\mathscr{CPBC}_{m,n}(\lambda)$ or equivalently, that $A(\lambda)$ commutes with both $\pi_m(\lambda)\otimes I_n(\lambda)$ and $I_m(\lambda)\otimes\pi_n(\lambda)$.

Proof

Now $A(\lambda)\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\Lambda(\lambda)\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\Lambda(\lambda)\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]\Lambda(\lambda)\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\Lambda(\lambda)\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]A(\lambda).$$

Also, $A(\lambda)\left[I_m(\lambda)\otimes\pi_n(\lambda)\right]$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\Lambda(\lambda)\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[I_m(\lambda)\otimes\pi_n(\lambda)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\Lambda(\lambda)\left[I_m(\lambda)\otimes\Omega_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\left[I_m(\lambda)\otimes\Omega_n(\lambda)\right]\Lambda(\lambda)\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[I_m(\lambda)\otimes\pi_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\Lambda(\lambda)\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[I_m(\lambda)\otimes\pi_n(\lambda)\right]A(\lambda).$$

Theorem 4.3.6

All polynomial matrices in $\mathscr{CPBC}_{m,\mathrm{n}}(\lambda)$ are simultaneously diagonalizable polynomial matrices by the unitary polynomial matrix $F_m(\lambda)\otimes F_n(\lambda)$, and they commute. If the eigen values of the circulant block polynomial matrices are given $\Lambda_{k+1}(\lambda), k=0,1,\ldots,m-1,$ then the diagonal polynomial matrix of the eigen values of the $\mathscr{CPBC}(\lambda)$ polynomial matrix is given by $\sum_{k=0}^{m-1}\Omega_m^k(\lambda)\otimes\Lambda_{k+1}(\lambda)$. Conversely, any matrix of the form

$A(\lambda)=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\Lambda(\lambda)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$ where $\Lambda(\lambda)$ is diagonal polynomial matrix is in $\mathscr{CPBC}_{m,\mathrm{n}}(\lambda)$.

Proof

Assume that $A(\lambda)$ is a block circulant polynomial matrix. From $(4.1.14)$, $A(\lambda)$ can be written as $A(\lambda)=\sum_{k=0}^{m-1}\pi_m^k(\lambda)\otimes A_{k+1}(\lambda)$ where the blocks are $A_1(\lambda), A_2(\lambda),\ldots,A_m(\lambda)$. The $A_{k+1}(\lambda)$ are circulant polynomial matrices if and only if $A_{k+1}(\lambda)=F_n^*(\lambda)\Lambda_{k+1}(\lambda)F_n(\lambda)$, where $F_n(\lambda)$ is the Fourier polynomial matrix of order n and $\Lambda_{k+1}(\lambda)$ is a diagonal polynomial matrix of order n.

From $(1.2.33)$, we have $\pi_m^k(\lambda)=F_m^*(\lambda)\Omega_m^k(\lambda)F(\lambda)$ where $\Omega_m(\lambda)$ is the $\Omega(\lambda)$ polynomial matrix of order m, $\Omega(\lambda)=\mathrm{diag}\left(1,\omega(\lambda),\omega^k(\lambda),\ldots,\omega^{m-1}(\lambda)\right)$ where $\omega(\lambda)=e^{\frac{2\pi i}{m}(\lambda)}$.

Hence, $A(\lambda)=\sum_{k=0}^{m-1}\left(F_m^*(\lambda)\Omega_m^k(\lambda)F_m(\lambda)\right)\otimes\left(F_n^*(\lambda)\Lambda_{k+1}(\lambda)F_n(\lambda)\right)$

$$=\sum_{k=0}^{m-1}\left(F_m^*(\lambda)\otimes F_n^*(\lambda)\right)\left(\Omega_m^k(\lambda)\otimes\Lambda_{k+1}(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$$

$$=\sum_{k=0}^{m-1}\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\left(\Omega_m^k(\lambda)\otimes\Lambda_{k+1}(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$$

$$A(\lambda)=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\Lambda(\lambda)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$$

Conversely, assume that

$$A(\lambda)=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^{*}\Lambda(\lambda)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$$

where $\Lambda(\lambda)$ is diagonal polynomial matrix.

We have to prove that $A(\lambda)\in \mathscr{BCCB}_{\mathrm{m,n}}(\lambda)$. It is enough to prove that $A(\lambda)$ commutes with both $\pi_m(\lambda)\otimes I_n(\lambda)$ and $I_m(\lambda)\otimes\pi_n(\lambda)$.

From lemma $(4.3.5)$, we have

$$A(\lambda)\left(\pi_m(\lambda)\otimes I_n(\lambda)\right)=\left(\pi_m(\lambda)\otimes I_n(\lambda)\right)A(\lambda) \text{ and}$$

$$A(\lambda)\left(I_m(\lambda)\otimes\pi_n(\lambda)\right)=\left(I_m(\lambda)\otimes\pi_n(\lambda)\right)A(\lambda)$$

Hence, $A(\lambda)$ is a block circulant matrix with circulant polynomial matrices as its blocks.

Lemma 4.3.7

Let j, k be nonnegative integers. Let $A_m(\lambda),B_m(\lambda)$ be of order m and n. Then $\left[A_m(\lambda)\otimes I_n(\lambda)\right]^k\left[I_m(\lambda)\otimes B_n(\lambda)\right]^j=A_m^k(\lambda)\otimes B_n^j(\lambda)$.

Proof

$$\left(A_m(\lambda)\otimes I_n(\lambda)\right)\left(A_m(\lambda)\otimes I_n(\lambda)\right)=A_m(\lambda)A_m(\lambda)\otimes I_n(\lambda)I_n(\lambda)$$

$$=A_m^2(\lambda)\otimes I_n^2(\lambda)$$

$$\left(A_m(\lambda)\otimes I_n(\lambda)\right)^2=A_m^2(\lambda)\otimes I_n(\lambda)$$

By induction, $\left(A_m(\lambda)\otimes I_n(\lambda)\right)^k=A_m^k(\lambda)\otimes I_n(\lambda)$

Similarly, $\left(I_m(\lambda)\otimes B_n(\lambda)\right)^j = I_m(\lambda)\otimes B_n^j(\lambda)$

Now

$$\left[A_m(\lambda)\otimes I_n(\lambda)\right]^k\left[I_m(\lambda)\otimes B_n(\lambda)\right]^j = \left[A_m^k(\lambda)\otimes I_n(\lambda)\right]\left[I_m(\lambda)\otimes B_n^j(\lambda)\right]$$

$$= \left[A_m^k(\lambda)I_m(\lambda)\otimes I_n(\lambda)B_n^j(\lambda)\right]$$

$$= A_m^k(\lambda)\otimes B_n^j(\lambda).$$

Theorem 4.3.8

Let $A(\lambda)\in \mathscr{CPBC}_{m,\mathrm{n}}(\lambda)$. Then $A(\lambda)$ is a polynomial (of two variables) in $\pi_m(\lambda)\otimes I_n(\lambda)$ and $\pi_m(\lambda)\otimes \pi_n(\lambda)$.

Proof

Since $A(\lambda)$ is a block circulant polynomial matrix.

Therefore, by (1.10), we have $A(\lambda)=\sum_{k=0}^{m-1}\left[\pi_m^k(\lambda)\otimes A_{k+1}(\lambda)\right]$. Where the blocks $A_{k+1}(\lambda)$ are themselves circulant polynomial matrices. Then

$$A_{k+1}(\lambda)=\sum_{j=0}^{n-1} a_{k+1,j+i}(\lambda)\pi_n^j(\lambda)$$

Now $A(\lambda)=\sum_{k=0}^{m-1}\left[\pi_m^k(\lambda)\otimes\left(\sum_{j=0}^{n-1} a_{k+1,j+1}(\lambda)\pi_n^j(\lambda)\right)\right]$

$$=\sum_{k=0}^{m-1}\sum_{j=0}^{n-1}\left[\pi_m^k(\lambda)\otimes a_{k+1,j+1}(\lambda)\pi_n^j(\lambda)\right]$$

$$= \sum_{k,j=0}^{m-1,n-1} a_{k+1,j+1}(\lambda)\left[\pi_m^k(\lambda)\otimes\pi_n^j(\lambda)\right]$$

$$= \sum_{k,j=0}^{m-1,n-1} a_{k+1,j+1}(\lambda)\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]^k\left[I_m(\lambda)\otimes\pi_n(\lambda)\right]^j$$

[by lemma (4.3.7)]

This is a polynomial in $\pi_n(\lambda)\otimes I_n(\lambda)$ and $I_m(\lambda)\otimes\pi_n(\lambda)$.

Remark 4.3.9

(i) A circulant polynomial matrix of level 1 is an ordinary circulant polynomial matrix.

(ii) A circulant polynomial matrix of level 2 is in $\mathcal{CPBC}_{m,\mathrm{n}}(\lambda)$.

(iii) A circulant polynomial matrix of level 3 is a block circulant polynomial whose block polynomials are level 2 circulant polynomial matrices.

Theorem 4.3.10

A circulant polynomial matrix of level 3 and type (m, n, p) is diagonalizable polynomial matrix by the unitary polynomial matrix $F_m(\lambda)\otimes F_n(\lambda)\otimes F_p(\lambda)$.

Proof

Let $A(\lambda)$ be a level 3 circulant polynomial matrix of type (m, n, p).

From $(4.1.14)$, we have $A(\lambda)=\sum_{k=0}^{m-1}\pi_m^k(\lambda)\otimes A_{k+1}(\lambda)$ $\qquad(4.4)$

Where each $A_{k+1}(\lambda)$ is a level 2 circulant polynomial matrix of type (n, p).

Thus, $A_{k+1}(\lambda)=\sum_{j=0}^{n-1}\pi_n^j(\lambda)\otimes A_{k+1,j+1}(\lambda)$ $\qquad(4.5)$

Where each $A_{k+1,j+1}(\lambda)$ is a circulant polynomial matrix of level 1 and of order p.

Thus, from $(1.2.10)$, $A_{k+1,j+1}(\lambda)=\sum_{r=0}^{p-1}a_{k+1,j+1,r+1}(\lambda)\pi_p^r(\lambda)$. $\qquad(4.6)$

Combining (4.4),(4.5) and (4.6) we have

$$A(\lambda)=\left[\sum_{k=0}^{m-1}\pi_m^k(\lambda)\otimes\left[\sum_{j=0}^{n-1}\pi_n^j(\lambda)\otimes\left[\sum_{r=0}^{p-1}a_{k+1,j+1,r+1}(\lambda)\pi_p^r(\lambda)\right]\right]\right] \qquad (4.7)$$

$$=\left[\sum_{k=0}^{m-1}\pi_m^k(\lambda)\otimes\left[\sum_{j=0}^{n-1}\sum_{r=0}^{p-1}a_{k+1,j+1,r+1}(\lambda)\pi_n^j(\lambda)\otimes\pi_p^r(\lambda)\right]\right]$$

$$=\sum_{k,j,r=0}^{m-1,n-1,p-1}a_{k+1,j+1,r+1}(\lambda)\left(\pi_m^k(\lambda)\otimes\pi_n^j(\lambda)\otimes\pi_p^r(\lambda)\right)$$

Since,

$$\pi_m^k(\lambda)=F_m^*(\lambda)\Omega_m^k(\lambda)F_m(\lambda),\ \pi_m^j(\lambda)=F_n^*(\lambda)\Omega_n^k(\lambda)F_n(\lambda)\ and\ \pi_p^r(\lambda)=F_p^*(\lambda)\Omega_p^k(\lambda)F_p(\lambda)$$

Therefore,

$$A(\lambda)=\sum_{k,j,r=0}^{m-1,n-1,p-1} a_{k+1,j+1,r+1}(\lambda)\left[F_m^*(\lambda)\Omega_m^k(\lambda)F_m(\lambda)\otimes F_n^*(\lambda)\Omega_n^k(\lambda)F_n(\lambda)\otimes F_p^*(\lambda)\Omega_p^k(\lambda)F_p(\lambda)\right]$$

$$=\sum_{k,j,r=0}^{m-1,n-1,p-1} a_{k+1,j+1,r+1}(\lambda)\left(F_m^*(\lambda)\otimes F_n^*(\lambda)\otimes F_p^*(\lambda)\right)\left(\Omega_m^k(\lambda)\otimes\Omega_n^k(\lambda)\otimes\Omega_p^k(\lambda)\right)$$
$$\left(F_m(\lambda)\otimes F_n(\lambda)\otimes F_p(\lambda)\right)$$

$$=\sum_{k,j,r=0}^{m-1,n-1,p-1} a_{k+1,j+1,r+1}(\lambda)\left(F_m(\lambda)\otimes F_n(\lambda)\otimes F_p(\lambda)\right)^*\left(\Omega_m^k(\lambda)\otimes\Omega_n^k(\lambda)\otimes\Omega_p^k(\lambda)\right)$$
$$\left(F_m(\lambda)\otimes F_n(\lambda)\otimes F_p(\lambda)\right)$$

$$=\left(F_m(\lambda)\otimes F_n(\lambda)\otimes F_p(\lambda)\right)^*\sum_{k,j,r=0}^{m-1,n-1,p-1} a_{k+1,j+1,r+1}(\lambda)\left(\Omega_m^k(\lambda)\otimes\Omega_n^k(\lambda)\otimes\Omega_p^k(\lambda)\right)$$
$$\left(F_m(\lambda)\otimes F_n(\lambda)\otimes F_p(\lambda)\right)$$

Thus, a circulant polynomial matrix of level 3 and type (m, n, p) is diagonalizable polynomial matrix.

Lemma 4.3.11

$$A_m(\lambda)\otimes B_n(\lambda)=\left[A_m(\lambda)\otimes I_n(\lambda)\right]\left[I_m(\lambda)\otimes B_n(\lambda)\right]$$

Proof

$$\left[A_m(\lambda)\otimes I_n(\lambda)\right]\left[I_m(\lambda)\otimes B_n(\lambda)\right]=\left(A_m(\lambda)I_m(\lambda)\otimes I_n(\lambda)B_n(\lambda)\right)$$
$$=A_m(\lambda)\otimes B_n(\lambda).$$

Lemma 4.3.12

$$A_m(\lambda)\otimes B_n(\lambda)\otimes C_p(\lambda)$$
$$=\left(A_m(\lambda)\otimes I_{np}(\lambda)\right)\left(I_m(\lambda)\otimes B_n(\lambda)\otimes I_P(\lambda)\right)\left(I_{mn}(\lambda)\otimes C_P(\lambda)\right)$$

Proof

$$A_m(\lambda)\otimes B_n(\lambda)\otimes C_p(\lambda)=\left[A_m(\lambda)\otimes B_n(\lambda)\right]\otimes C_p(\lambda)$$

$$=\left[\left(A_m(\lambda)\otimes B_n(\lambda)\right)\otimes I_p(\lambda)\right]\left[I_{mn}(\lambda)\otimes C_p(\lambda)\right]$$

$$=\left(A_m(\lambda)\otimes\left[B_n(\lambda)\otimes I_p(\lambda)\right]\right)\left(I_{mn}(\lambda)\otimes C_p(\lambda)\right)$$

$$=\left(A_m(\lambda)\otimes I_{np}(\lambda)\right)\left[I_m(\lambda)\left(B_n(\lambda)\otimes I_p(\lambda)\right)\right]\left(I_{mn}(\lambda)\otimes C_p(\lambda)\right)$$

$$\left[by\, lemma\left(4.3.11\right)\right]$$

Lemma 4.3.13

For nonnegative integers k, j, r

$$A_m^k(\lambda)\otimes B_n^j(\lambda)\otimes C_p^r(\lambda)=\left(A_m(\lambda)\otimes I_{np}(\lambda)\right)^k\left[I_m(\lambda)\otimes B_n(\lambda)\otimes I_P(\lambda)\right]^j \left[I_{mn}(\lambda)\otimes C_p(\lambda)\right]^r$$

Proof

By the lemma $\left(4.3.12\right)$

$$A_m^k(\lambda)\otimes B_n^j(\lambda)\otimes C_p^r(\lambda)=\left(A_m^k(\lambda)\otimes I_{np}(\lambda)\right)\left(I_m(\lambda)\otimes B_n^j(\lambda)\otimes I_P(\lambda)\right)\left(I_{mn}(\lambda)\otimes C_P^r(\lambda)\right)$$

We have $\left[A_m^k(\lambda)\otimes I_{np}(\lambda)\right]=\left[A_m(\lambda)\otimes I_{np}(\lambda)\right]^k$

$$\left[I_m(\lambda)\otimes B_n^j(\lambda)\otimes I_p(\lambda)\right]=\left[I_m(\lambda)\otimes B_n(\lambda)\otimes I_p(\lambda)\right]^j$$

$$\left[I_{mn}(\lambda)\otimes C_p^r(\lambda)\right]=\left[I_{mn}(\lambda)\otimes C_p(\lambda)\right]^r$$

Hence, $A_m^k(\lambda)\otimes B_n^j(\lambda)\otimes C_p^r(\lambda)=\left[A_m(\lambda)\otimes I_{np}(\lambda)\right]^k$

$$\left[I_m(\lambda)\otimes B_n(\lambda)\otimes I_p(\lambda)\right]^j\left[I_{mn}(\lambda)\otimes C_p(\lambda)\right]^r$$

Theorem 4.3.14

Let $A(\lambda)$ be of type (m,n,p) and be a circulant polynomial matrix of level 3. Then

$$A(\lambda)=\sum_{k,j,r=0}^{m-1,n-1,p-1} a_{k+1,j+1,r+1}(\lambda)\left[\pi_m(\lambda)\otimes I_{np}(\lambda)\right]^k\left[I_m(\lambda)\otimes\pi_n(\lambda)\otimes I_p(\lambda)\right]^j\left[I_{m,m}(\lambda)\otimes\pi_p(\lambda)\right]^r$$

Proof

Since $A(\lambda)$ is of type (m,n,p) and is a circulants polynomial matrix of level 3. Therefore, from (4.7) it can be written as

$$A(\lambda)=\sum_{k=0}^{m-1}\left[\pi_m^k(\lambda)\otimes\left[\sum_{j=0}^{n-1}\pi_n^j(\lambda)\otimes\left[\sum_{r=0}^{p-1}a_{k+1,j+1,r+1}(\lambda)\pi_p^r(\lambda)\right]\right]\right]$$

$$=\sum_{k=0}^{m-1}\left[\pi_m^k(\lambda)\otimes\left[\sum_{j=0}^{n-1}\sum_{r=0}^{p-1}a_{k+1,j+1,r+1}(\lambda)\pi_n^j(\lambda)\otimes\pi_p^r(\lambda)\right]\right]$$

$$=\sum_{k,j,r=0}^{m-1,n-1,p-1} a_{k+1,j+1,r+1}(\lambda)\pi_m^k(\lambda)\otimes\pi_n^j(\lambda)\otimes\pi_p^r(\lambda)$$

Hence, by lemma $(4.3.13)$ and (4.7), the above can be written as

$$A(\lambda)=\sum_{k,j,r=0}^{m-1,n-1,p-1} a_{k+1,j+1,r+1}(\lambda)\left(\pi_m(\lambda)\otimes I_{np}(\lambda)\right)^k\left(I_m(\lambda)\otimes\pi_n(\lambda)\otimes I_p(\lambda)\right)^j$$
$$\left(I_{mn}(\lambda)\otimes\pi_p(\lambda)\right)^r.$$

Chapter V

APPLICATIONS OF CIRCULANT POLYNOMIAL MATRICES IN FINDING SOLUTIONS TO TRAVELLING SALESMAN PROBLEM

In this chapter we use circulant polynomial matrices to find the solution for travelling salesman problem.

5.1 Basics of graph theory

Many real world situations can conveniently be described by means of a diagram consisting of a set of vertices together with edges joining certain pairs of those vertices. The vertices could represent communication centres, with edges representing communication links. A mathematical abstraction of situations of this type gives rise to the concept of a graph [6]. Graphs can be used to model situations that occur within certain kind of problems. These problems then can be studied with the aid of graphs [20].

Graph theory is a delightful playground for the exploration of proof techniques in discrete mathematics, and its results have applications in many areas of the computing, social and natural sciences [14]. The world of theoretical physics discovered graph theory for its own purposes. In the study of statistical mechanics, the points stand for molecules and two adjacent points indicate nearest neighbour interaction such as magnetic attraction on repulsion. The study of Markov chains in probability theory involves directed graphs in which events are represented by vertices and a directed edge form one vertex to other indicates a positive probability of direct succession of these two events. The representation of directed graphs also arises in numerical analysis involving matrix inversion and the calculation of eigen values. The rapidly growing fields of linear programming and operations research also have made use of a graph theoretic approach by the study of flows in networks. The psychologists use graphs in which people are represented by vertices and interpersonal relations by edges.

A graph G is an ordered triple $G=(V(G),E(G),\psi_G)$ such that $V(G)$, the set of all vertices of G, $E(G)$ the set of all edges of G and ψ_G the incidence function of G which maps each edge of G to an unordered pairs of vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e)=uv$, then e is said to join u and v; the vertices u and v are called the ends of e.

An edge with identical ends is called a loop and an edge with distinct ends a link.

Two or more edges having the same end vertices are said to be parallel edges.

The end vertices of an edge are said to be incident with the edge and conversely the edge is incident on the vertices.

The end vertices of an edge are said to be adjacent vertices. If two or more edges are incident on the same vertex then they are said to be adjacent edges.

A graph without loops and parallel edges is called a simple graph.

A graph is said to be null graph it is edge set is empty. That is, a graph which is a null graph contains only a vertex and no edges.

A graph is finite if both its vertex set edge set are finite otherwise, it is called infinite.

A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph.

The degree $d_G(v)$ of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees respectively, of vertices of G.

A graph is said to be regular if all its vertices are of the same degree.

A walk in G is a finite non-null sequence $W = v_0e_1v_1e_2v_2...e_kv_k$, whose terms are alternatively vertices and edges, such that, for $1 \le i \le k$, the ends of e_i are v_{i-1} and v_i. The vertices v_0 and v_k are called the initial and terminus of W, respectively, and $v_1, v_2,..., v_{k-1}$ its internal vertices.

If the initial and terminal vertices of a walk W coincide then the walk is called a closed walk. If the initial and terminal vertices of a walk distinct then it is called an open walk. If the edges $e_1, e_2,..., e_k$ of a walk W are distinct, then W is called a trail. If the vertices $v_0, v_1, v_2,..., v_k$ of a walk W are distinct, then W is called a path.

A closed walk in which each is of degree two is called a cycle or circuit.

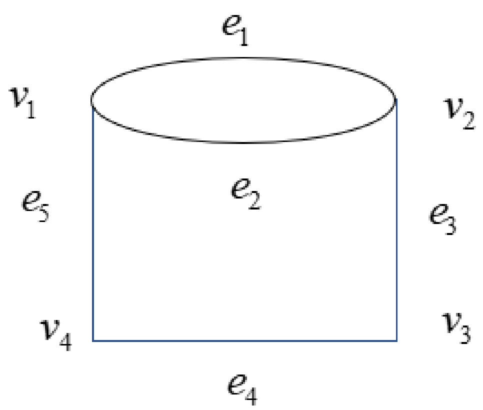
A graph G is said to be connected if there is atleast one path joining every pair of vertices. If a graph is not connected then it is called disconnected.

A graph H is a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. A spanning subgraph of G is a subgraph H with $V(H) = V(G)$.

Let G be a graph with vertices $v_1, v_2,..., v_\nu$ edges $e_1, e_2,..., e_\varepsilon$.Then the $\nu \times \varepsilon$ matrix $M(G) = [m_{ij}]$ is called the incidence matrix of G. Where

$$m_{ij} = \begin{cases} 1, & \text{if the vertex } v_i \text{ is incident with the edge } e_j \\ 0, & \text{otherwise} \end{cases}.$$

Consider the graph G,

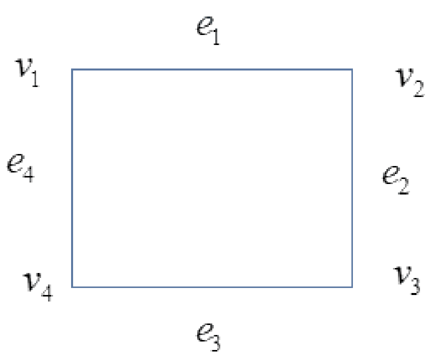


The incident matrix $M(G)$ is $\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$.

Let G be a simple graph with vertices $v_1, v_2, \ldots, v_\nu$. Then the $\nu \times \nu$ matrix $A(G) = [a_{ij}]$ is called the adjacency matrix of G where

$$a_{ij} = \begin{cases} 1, & if\ u_i\ is\ adjacent\ to\ v_j \\ 0, & \qquad\qquad otherwise \end{cases}$$

Consider the graph G,



The adjacency matrix A(G) is $\begin{array}{c|cccc|} & v_1 & v_2 & v_3 & v_4 \\ v_1 & 0 & 1 & 0 & 1 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \end{array}$.

A circulant graph $G = G(n,s)$ is a graph on the set of n vertices $V(G) = \{v_1,...,v_n\}$ with an edge incident with v_i and v_j whenever $|i - j| \in s$; The set S is said to be the symbol of G. In particular, $k =\neq s$ is the degree of a circulant graph $G(n;s)$.

5.2 Decomposition of graphs into circuits

Circulant graphs form a class of highly symmetric mathematical (graphical) structures. Catalan introduced circulant matrices in 1846 [8,36-39] and properties of circulant graphs have been investigated by many authors [2,23,34,43,54]. If a graph G is circulant, then its adjacency matrix A(G) is circulant. Circulant graphs are often used as models of communication networks, being a popular class of fault-tolerant network topologies, which include rings and complete graphs. In particular, the interest in circulant graphs mainly arises in the design and implementation of distributed computed networks, communication networks, parallel processing architecture, and in VLSI-design.

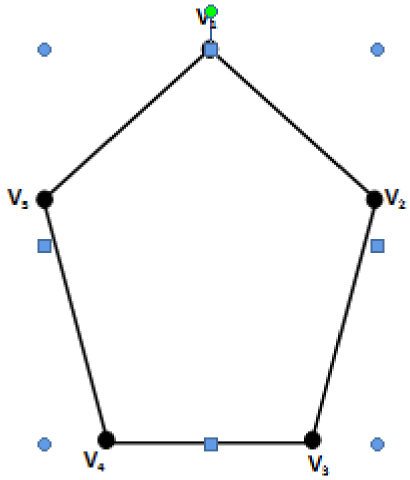
A path that contains every vertex of G is called a Hamilton path of G. A Hamilton cycle of G is a cycle that contains every vertex of G. A graph is Hamiltonian if it contains a Hamilton cycle.

Hamiltonian circuits are helpful in solving travelling salesman problem. From circulant polynomial matrix we can find different Hamiltonian circuits of a complete graph. From these different Hamiltonian circuits, we can find the weight of each circuits. The minimum weight circuit will give the optimal solution to the travelling salesman problem.

A circulant graph is a graph which has a circulant adjacency matrix. For a given positive integer k, let $n_1, n_2, ..., n_k$ be a sequence of integers where $0 < n_1 < n_2 < ... < n_k < \frac{(p+1)}{2}$. Then the circulant graph $C_P(n_1, n_2, ..., n_k)$ is the graph on P vertices $v_1, v_2, ..., v_p$ with vertex v_i adjacent to each vertex $V_{i \pm n_{j(\mathrm{mod}\, p)}}$. The values n_i are called jump sizes.

The graph of Figure 5.1 is a circulant graph having jump-1 and figure 5.2 is also a circulant graph having jump-2 for the complete graph K_5.

The circulant matrix of the circulant graph of figure 5.1 and figure 5.2 are shown in figure 5.3 and 5.4 respectively.



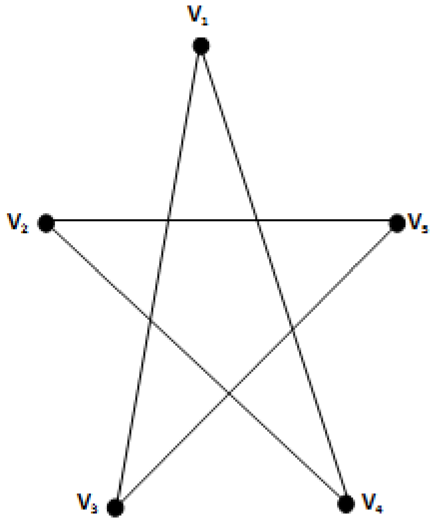


Figure-5.1 $C_5(1)$ Figure-5.1 $C_5(2)$

$$\begin{array}{c|c} & \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Figure -5.3 Circulant matrix of $C_5(1)$

$$\begin{array}{c|c} & \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Figure -5.4 Circulant matrix of $C_5(2)$

Example

Consider a complete graph of seven vertices, which is shown below.

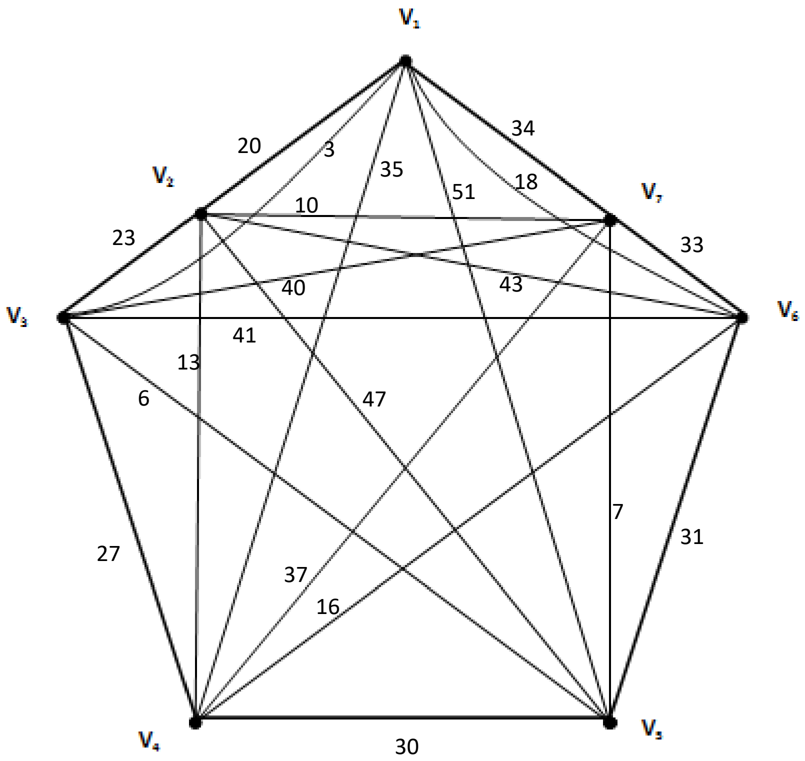


Figure 5.5

From this complete graph we can get a circulant polynomial matrix.

$$A(\lambda)=\begin{bmatrix} 0 & 1 & \lambda & \lambda^2 & \lambda^2 & \lambda & 1 \\ 1 & 0 & 1 & \lambda & \lambda^2 & \lambda^2 & \lambda \\ \lambda & 1 & 0 & 1 & \lambda & \lambda^2 & \lambda^2 \\ \lambda^2 & \lambda & 1 & 0 & 1 & \lambda & \lambda^2 \\ \lambda^2 & \lambda^2 & \lambda & 1 & 0 & 1 & \lambda \\ \lambda & \lambda^2 & \lambda^2 & \lambda & 1 & 0 & 1 \\ 1 & \lambda & \lambda^2 & \lambda^2 & \lambda & 1 & 0 \end{bmatrix}$$

Now we decompose the complete graph (figure 5.5) by using the above circulant polynomial matrix with respect to the degree of the parameter λ.

Case (i)

We collect the coefficient of λ^0 and consider the remaining terms (entries) as zero. We get the circulant matrix which is named as A_0. That is,

$$A_0 = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & V_7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

The corresponding Hamiltonian circuit is

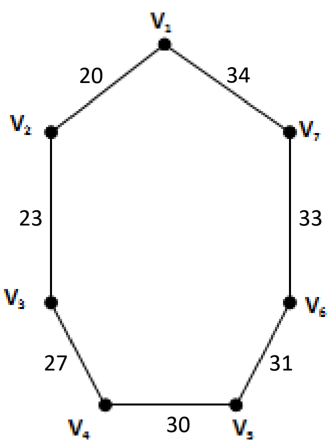


Figure 5.6 $C_7(1)$

The weight of $C_7(1)$ is 198.

Case (ii)

We collect the coefficient of λ and consider the remaining terms(entries) as zero. We get the circulant matrix which is named as A_1.

That is,
$$A_1 = \begin{array}{c c} & \begin{array}{ccccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & V_7 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The corresponding Hamiltonian circuit is

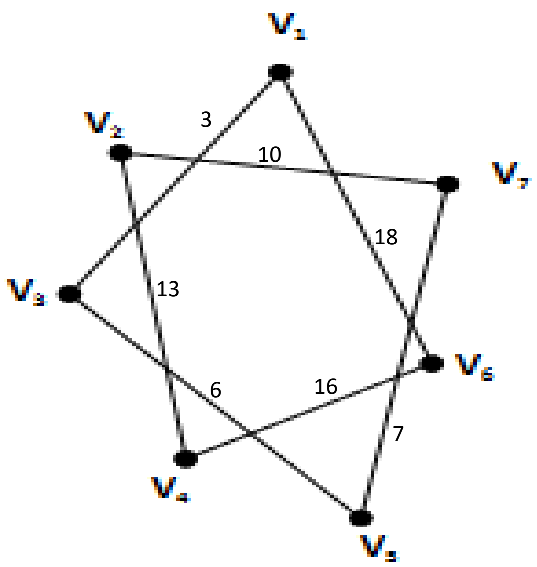


Figure 5.7 $C_7(2)$

Theweight of $C_7(2)$ is 73.

Case (iii)

We collect the coefficient of λ^2 and consider the remaining terms(entries) as zero. We get the circulant matrix which is named as A_2.

That is,
$$A_2 = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & V_7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The corresponding Hamiltonian circuit is

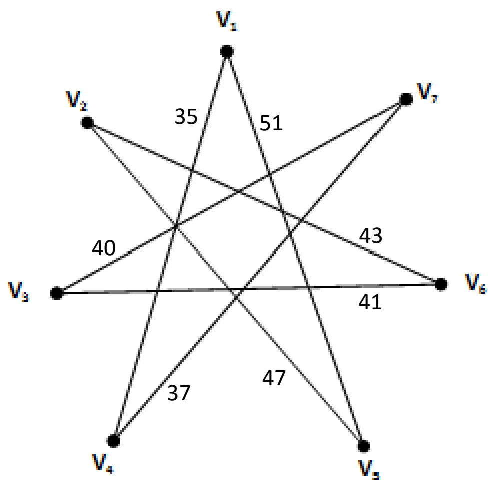


Figure 5.8 $C_7(3)$

The weight of $C_7(3)$ is 294.

From all the three cases we can conclude that $C_7(2)$ has the minimum weight. Therefore, $C_7(2)$ is the optimum solution for the travelling salesman problem.

Conclusion

CONCLUSION

In this dissertation we have introduced the notion of circulant polynomial matrices. Some properties of k-circulant and (r,s)-pair circulant polynomial matrices are analyzed. We have defined hermitian, normal and conjugate normal circulant polynomial matrices. Also, some important results and characterizations are discussed.

We have introduced the concept of block circulant polynomial matrices. Circulant block polynomial matrices and block circulant matrices: where the blocks are circulant polynomial matrices are obtained. Also, some results relating to block circulant and circulant block polynomial matrices are found. Finally, we have studied an application of circulant polynomial matrix in travelling salesman problems. Further this can be extended to orthogonal, unitary and any other type of matrices.

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Publications

List of Publications

List of publications of Mr. R. Muthamilselvam during his Ph.D., Degree course, Ramanujan Research Centre, P.G and Research Department of Mathematics, Government Arts College (Autonomous), Kumbakonam under the guidance of Dr. G. Ramesh.

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Stochastic Modeling and Applications
Vol25 No. 1 (January-June, 2021)
ISSN: 0972-3641

Received: 02nd April 2021 *Revised: 10th May 2021* *Selected: 07th July 2021*

ON BLOCK CIRCULANT POLYNOMIAL MATRICES

R. MUTHAMILSELVAM AND G. RAMESH

Abstract. The characterization of block circulant polynomial matrices are derived as a generalization of the block circulant matrices.

1. Introduction

Let $(a_1(\alpha), a_2(\alpha), ...a_n(\alpha))$ be an ordered n-tuple of polynomial complex numbers and let them generate the circulant polynomial matrix [2] [3] [5] [7] of order n:

$$A(\alpha) = \begin{pmatrix} a_1(\alpha) & a_2(\alpha) & \dots & a_n(\alpha) \\ a_n(\alpha) & a_1(\alpha) & \dots & a_2(\alpha) \\ \dots & \dots & \dots & \dots \\ a_2(\alpha) & a_3(\alpha) & \dots & a_1(\alpha) \end{pmatrix} \tag{1.1}$$

We shall often denote this circulant polynomial matrix as

$$A(\alpha) = Circ(a_1(\alpha), a_2(\alpha), ..., a_n(\alpha)) \tag{1.2}$$

It is well known that all circulant polynomial matrices of order n are simultaneously diagonalizable by the polynomial matrix $F(\alpha)$ associated with the finite Fourier transform.

Specifically, let

$$\omega(\alpha) = exp(\frac{2\pi i}{n}(\alpha)), i = \sqrt{-1} \tag{1.3}$$

and set

$$F^*(\alpha) = n^{(\frac{-1}{2})} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega(\alpha) & \omega^2(\alpha) & \dots & \omega^{n-1}(\alpha) \\ 1 & \omega^2(\alpha) & \omega^4(\alpha) & \dots & \omega^{2(n-1)}(\alpha) \\ \dots & \dots & \dots & \dots & \\ 1 & \omega^{(n-1)}(\alpha) & \omega^{(n-2)}(\alpha) & \dots & \omega(\alpha) \end{pmatrix} \tag{1.4}$$

The Fourier polynomial matrix $F(\alpha)$ depends only on n. This matrix is also symmetric polynomial and unitary polynomial $F(\alpha)F^*(\alpha) = F^*(\alpha)F(\alpha) = I(\alpha)$ and we have

$$A(\alpha) = F^*(\alpha) \wedge (\alpha) F(\alpha) \tag{1.5}$$

2000 *Mathematics Subject Classification.* 15A09, 15A15,15A57.

Key words and phrases. Circulant polynomial matrices, Block Circulant polynomial matrices.

* This research is supported by G. Ramesh.

where $\wedge(\alpha) = diag(\alpha_1, \alpha_2, \ldots \alpha_n)$

The symbol * designates the conjugate transpose.

From the spectral mapping theorem, we may represent $A(\alpha)$ in the form

$$A(\alpha) = a_1(\alpha) + a_2(\alpha)\pi(\alpha) + a_3(\alpha)\pi^2(\alpha) + \cdots + a_n(\alpha)\pi^{n-1}(\alpha) \tag{1.6}$$

where $\pi(\alpha)$ is the permutation matrix $circ(0, 1, 0, 0, \ldots)$.

Also, let $A(\alpha)$ be an $n \times n$ polynomial matrix. Then $A(\alpha)$ is a circulant polynomial matrix if and only if

$$A(\alpha)\pi(\alpha) = \pi(\alpha)A(\alpha) \tag{1.7}$$

The matrix $\pi(\alpha) = circ(0, 1, 0, \ldots, 0)$

This paper is devoted to the study of block circulant polynomial matrices.

2. Block Circulant Polynomial Matrices

In this section we define block circulant polynomial matrices and we extend some of the properties of block circulant matrices found in [1], [4], [6], [8], [9] to block circulant polynomial matrices.

Definition 2.1. A block circulant polynomial matrix is a polynomial matrix in the following form

$$\text{b circ}(A_1(\alpha), A_2(\alpha), \ldots, A_n(\alpha)) = \begin{pmatrix} A_1(\alpha) & A_2(\alpha) & \ldots & A_m(\alpha) \\ A_m(\alpha) & A_1(\alpha) & \ldots & A_{m-1}(\alpha) \\ \ldots & \ldots & \ldots & \ldots \\ A_2(\alpha) & A_3(\alpha) & \ldots & A_1(\alpha) \end{pmatrix}$$

We denote the set of all block circulant polynomial matrices of order $m \times n$ as $\mathbb{BC}_{m,n}(\alpha)$.

Example 2.2. The polynomial matrix

$$\begin{pmatrix} 1-\alpha^2 & \alpha^3 & 2+\alpha^2 & -11\alpha \\ \alpha+3\alpha^2 & 1+\alpha & 4+6\alpha^2 & -8+\alpha \\ 2+\alpha^2 & -11\alpha & 1-\alpha^2 & \alpha^3 \\ 4+6\alpha^2 & -8+\alpha & \alpha+3\alpha^2 & 1+\alpha \end{pmatrix}$$

is a block circulant polynomial matrix.

Theorem 2.3. *$A(\alpha) \in \mathbb{BC}_{m,n}(\alpha)$ iff $A(\alpha)$ commutes with the unitary polynomial matrix*
$\pi_m(\alpha) \otimes I_n(\alpha) : A(\alpha)(\pi_m(\alpha) \otimes I_n(\alpha)) = (\pi_m(\alpha) \otimes I_n(\alpha))A(\alpha)$

Proof. Assume that $A(\alpha)$ is a block circulant polynomial matrix. That is □

$$\text{b circ}(A_1(\alpha), A_2(\alpha), \ldots A_n(\alpha)) = \begin{pmatrix} A_1(\alpha) & A_2(\alpha) & \ldots & A_m(\alpha) \\ A_m(\alpha) & A_1(\alpha) & \ldots & A_{m-1}(\alpha) \\ \ldots & \ldots & \ldots & \ldots \\ A_2(\alpha) & A_3(\alpha) & \ldots & A_1(\alpha) \end{pmatrix}$$

We have to prove that $A(\alpha)(\pi_m(\alpha) \otimes I_n(\alpha)) = (\pi_m(\alpha) \otimes I_n(\alpha))A(\alpha)$.
Now the polynomial matrix $\pi_m(\alpha) \otimes I_n(\alpha) \in \mathbb{BC}_{m,n}(\alpha)$ is given by

$$\pi_m(\alpha) \otimes I_n(\alpha) = \begin{pmatrix} O_n(\alpha) & I_n(\alpha) & O_n(\alpha) & \dots & O_n(\alpha) \\ O_n(\alpha) & O_n(\alpha) & I_n(\alpha) & \dots & O_n(\alpha) \\ \dots & \dots & \dots & \dots & \dots \\ O_n(\alpha) & O_n(\alpha) & O_n(\alpha) & \dots & I_n(\alpha) \\ I_n(\alpha) & O_n(\alpha) & O_n(\alpha) & \dots & O_n(\alpha) \end{pmatrix}$$

$$A(\alpha)(\pi_m(\alpha) \otimes I_n(\alpha)) = \begin{pmatrix} A_m(\alpha) & A_1(\alpha) & A_2(\alpha) & \dots & A_{m-1}(\alpha) \\ A_{m-1}(\alpha) & A_m(\alpha) & A_1(\alpha) & \dots & A_{m-2}(\alpha) \\ \dots & \dots & \dots & \dots & \dots \\ A_1(\alpha) & A_2(\alpha) & A_3(\alpha) & \dots & A_m(\alpha) \end{pmatrix} \tag{2.1}$$

$$(\pi_m(\alpha) \otimes I_n(\alpha))A(\alpha) = \begin{pmatrix} A_m(\alpha) & A_1(\alpha) & A_2(\alpha) & \dots & A_{m-1}(\alpha) \\ A_{m-1}(\alpha) & A_m(\alpha) & A_1(\alpha) & \dots & A_{m-2}(\alpha) \\ \dots & \dots & \dots & \dots & \dots \\ A_1(\alpha) & A_2(\alpha) & A_3(\alpha) & \dots & A_m(\alpha) \end{pmatrix} \tag{2.2}$$

From (8) and (9), we get $A(\alpha)(\pi_m(\alpha) \otimes I_n(\alpha)) = (\pi_m(\alpha) \otimes I_n(\alpha))A(\alpha)$.

Conversely, assume that $A(\alpha)(\pi_m(\alpha) \otimes I_n(\alpha)) = (\pi_m(\alpha) \otimes I_n(\alpha))A(\alpha)$.

We have to prove that $A(\alpha)$ is a block circulant polynomial matrix.

$$I_m(\alpha) \otimes A_1(\alpha) = \begin{pmatrix} A_1(\alpha) & 0 & \dots & 0 \\ 0 & A_1(\alpha) & \dots & 0 \\ \dots & \dots & \dots & \\ 0 & 0 & \dots & A_1(\alpha) \end{pmatrix}$$

$$\pi_m(\alpha) \otimes A_2(\alpha) = \begin{pmatrix} 0 & A_2(\alpha) & 0 & \dots & 0 \\ 0 & 0 & A_2(\alpha) & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ A_2(\alpha) & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\pi_m^2(\alpha) \otimes A_3(\alpha) = \begin{pmatrix} 0 & 0 & A_3(\alpha) & 0 & \dots & 0 \\ 0 & 0 & 0 & A_3(\alpha) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & A_3(\alpha) & 0 & 0 & \dots & 0 \end{pmatrix}$$

etc

$(I_m(\alpha) \otimes A_1(\alpha) + (\pi_m(\alpha) \otimes A_2(\alpha)) + \dots + (\pi_m^{m-1}(\alpha) \otimes A_m(\alpha)) = \text{b circ}(A_1(\alpha), A_2(\alpha), \dots, A_m(\alpha).$ $\blacksquare$

Hence, $A(\alpha)$ is a block circulant polynomial matrix.

Theorem 2.4. *b circ*$(A_1(\alpha), A_2(\alpha), \dots, A_n(\alpha)) = \sum_{k=0}^{m-1} [\pi_m^k(\alpha) \otimes A_{K+1}(\alpha)]$.

Proof. Given that $A(\alpha) = \text{b circ}(A_1(\alpha), A_2(\alpha), \dots, A_n(\alpha))$ is a block circulant polynomial matrix.

That is,

$$A(\alpha) = \begin{pmatrix} A_1(\alpha) & A_2(\alpha) & \dots & A_n(\alpha) \\ A_n(\alpha) & A_1(\alpha) & \dots & A_{n-1}(\alpha) \\ \dots & \dots & \dots & \dots \\ A_2(\alpha) & A_3(\alpha) & \dots & A_1(\alpha) \end{pmatrix}$$

Now

$$I_m(\alpha) \otimes A_1(\alpha) = \begin{pmatrix} A_1(\alpha) & 0 & \dots & 0 \\ 0 & A_1(\alpha) & \dots & 0 \\ \dots & \dots & \dots & \\ 0 & 0 & \dots & A_1(\alpha) \end{pmatrix}$$

$$\pi_m(\alpha) \otimes A_2(\alpha) = \begin{pmatrix} 0 & A_2(\alpha) & 0 & \dots & 0 \\ 0 & 0 & A_2(\alpha) & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ A_2(\alpha) & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\pi_m^{m-1}(\alpha) \otimes A_3(\alpha) = \begin{pmatrix} 0 & 0 & A_3(\alpha) & 0 & \dots & 0 \\ 0 & 0 & 0 & A_3(\alpha) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & A_3(\alpha) & 0 & 0 & \dots & 0 \end{pmatrix}$$

Since the pre direct of any $n \times n$ polynomial matrix by $\pi_m(\alpha)$ shifts the columns of the matrix one place to the right. Therefore, we find that

$$\pi_{m-1}^2(\alpha) \otimes A_n(\alpha) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & A_n(\alpha) \\ A_n(\alpha) & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_n(\alpha) & 0 \end{pmatrix}$$

$$\text{b circ}(A_1(\alpha), A_2(\alpha), \dots, A_n(\alpha) = \sum_{k=0}^{m-1} [\pi_m^k(\alpha) \otimes A_{K+1}(\alpha)]. \qquad \square$$

Remark 2.5. Block circulant polynomial matrix of the same type do not necessarily commute.

Example 2.6.

$$\begin{pmatrix} A(\alpha) & O(\alpha) \\ O(\alpha) & A(\alpha) \end{pmatrix} \begin{pmatrix} B(\alpha) & O(\alpha) \\ O(\alpha) & B(\alpha) \end{pmatrix} = \begin{pmatrix} A(\alpha)B(\alpha) & O(\alpha) \\ O(\alpha) & A(\alpha)B(\alpha) \end{pmatrix}$$

$$\begin{pmatrix} B(\alpha) & O(\alpha) \\ O(\alpha) & B(\alpha) \end{pmatrix} \begin{pmatrix} A(\alpha) & O(\alpha) \\ O(\alpha) & A(\alpha) \end{pmatrix} = \begin{pmatrix} B(\alpha)A(\alpha) & O(\alpha) \\ O(\alpha) & B(\alpha)A(\alpha) \end{pmatrix}$$

Theorem 2.7. *Let $A(\alpha) = b\ circ(A_1(\alpha), A_2(\alpha), \dots, A_m(\alpha))$,
$B(\alpha) =\ b\ circ(B_1(\alpha), B_2(\alpha), \dots, B_m(\alpha)) \in \mathbb{BC}_{m\times n}(\alpha)$.*

Then, if the $A_j(\alpha)'s$ commutes with the $B_K(\alpha)'s$, $A(\alpha)$ and $B(\alpha)$ commute.

Proof. By theorem (2.4), we have

$$A(\alpha) = \sum_{j=0}^{m-1} [\pi^j(\alpha) \otimes A_{j+1}(\alpha)],\ B(\alpha) = \sum_{k=0}^{m-1} [\pi^k(\alpha) \otimes B_{k+1}(\alpha)]$$

$$
\begin{aligned}
A(\alpha)B(\alpha) &= [\sum_{j=0}^{m-1}[\pi^j(\alpha)\otimes A_{j+1}(\alpha)]][\sum_{k=0}^{m-1}[\pi^k(\alpha)\otimes B_{k+1}(\alpha)]] \\
&= \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}[\pi^{j+k}(\alpha)\otimes A_{j+1}(\alpha)B_{k+1}(\alpha)] \\
&= \sum_{k=0}^{m-1}\sum_{j=0}^{m-1}[\pi^{k+j}(\alpha)\otimes B_{k+1}(\alpha)A_{j+1}(\alpha)] \\
&= [\sum_{k=0}^{m-1}[\pi^k(\alpha)\otimes B_{k+1}(\alpha)]][\sum_{j=0}^{m-1}[\pi^j(\alpha)\otimes A_{j+1}(\alpha)]] \\
&= B(\alpha)A(\alpha)
\end{aligned}
$$
□

Theorem 2.8. *$A(\alpha)\in\mathbb{BC}_{m\times n}(\alpha)$ if and only if it is of the form*
$A(\alpha)=[F_m(\alpha)\otimes F_n(\alpha)^*]diag[M_1(\alpha),M_2(\alpha),\dots,M_n(\alpha)][F_m(\alpha)\otimes F_n(\alpha)]$
where the $M_k(\alpha)$ are arbitrary polynomial square matrices of order n.

Proof. Assume that $A(\alpha)$ is a block circulant polynomial matrix.
From theorem (2.4), we have

$A(\alpha)=\text{b circ}(A_1(\alpha),A_2(\alpha),\dots,A_m(\alpha))=\sum_{k=0}^{m-1}[\pi_m^k(\alpha)\otimes A_{k+1}(\alpha)]$ for some $A_k(\alpha)$.
Now $\pi_m^k(\alpha)\otimes A_{k+1}(\alpha)=[F_m^*(\alpha)\Omega^k(\alpha)F_m(\alpha)]\otimes[F_n^*(\alpha)(F_n(\alpha)A_{k+1}(\alpha)F_n^*(\alpha)F_n(\alpha))]$
Let $B_K(\alpha)=(F_n(\alpha)A_{k+1}(\alpha)F_n^*(\alpha)$
$\pi_m^k(\alpha)\otimes A_{k+1}(\alpha)=[F_m^*(\alpha)\Omega^k(\alpha)F_m(\alpha)]\otimes[F_n^*(\alpha)B_K(\alpha)F_n(\alpha))]$
$=(F_m^*(\alpha)\otimes F_n^*(\alpha))(\Omega^k(\alpha)\otimes B_K(\alpha))(F_m(\alpha)\otimes F_n(\alpha))$
$\sum_{k=0}^{m-1}\pi_m^k(\alpha)\otimes A_{k+1}(\alpha)=\sum_{k=0}^{m-1}(F_m(\alpha)\otimes F_n(\alpha))^*(\Omega^k(\alpha)\otimes B_K(\alpha))(F_m(\alpha)\otimes F_n(\alpha))$
$A(\alpha)=(F_m(\alpha)\otimes F_n(\alpha))^*\sum_{k=0}^{m-1}(\Omega^k(\alpha)\otimes B_K(\alpha))(F_m(\alpha)\otimes F_n(\alpha))$
$=(F_m(\alpha)\otimes F_n(\alpha))^*diag(M_1(\alpha),M_2(\alpha),\dots,M_n(\alpha))(F_m(\alpha)\otimes F_n(\alpha))$
where

$$
\begin{pmatrix} M_1(\alpha)\\ M_2(\alpha)\\ \dots\\ M_m(\alpha)\end{pmatrix}=(m^{\frac{1}{2}}F_m^*(\alpha)\otimes I_m(\alpha))\begin{pmatrix} B_0(\alpha)\\ B_1(\alpha)\\ \dots\\ B_{(m-1)}(\alpha)\end{pmatrix} \tag{2.3}
$$

Thus, $A(\alpha)=(F_m(\alpha)\otimes F_n(\alpha))^*diag(M_1(\alpha),M_2(\alpha),\dots,M_n(\alpha))(F_m(\alpha)\otimes F_n(\alpha))$
From (10),

$$
\begin{pmatrix} B_0(\alpha)\\ B_1(\alpha)\\ \dots\\ B_{(m-1)}(\alpha)\end{pmatrix}=(m^{\frac{-1}{2}}F_m^*(\alpha)\otimes I_m(\alpha))\begin{pmatrix} M_1(\alpha)\\ M_2(\alpha)\\ \dots\\ M_m(\alpha)\end{pmatrix}
$$

Since $A_{(k+1)}(\alpha)=F_n^*(\alpha)B_k(\alpha)F_n(\alpha)$
$M_k(\alpha)$ arbitrary $\Leftrightarrow B_k(\alpha)$ are arbitrary.
$\Leftrightarrow A_k(\alpha)$ are arbitrary.
Hence, $A(\alpha)\in\mathbb{BC}_{(m,n)}(\alpha)$. □

3. Conclusion

some of the characterization of block circulant polynomial matrices are discussed here. Further we can study the circulant block polynomial matrices.

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Stochastic Modeling and Applications
Vol25 No. 1 (January-June, 2021)
ISSN: 0972-3641

Received: 18th April 2021 *Revised: 18th May 2021* *Selected: 19th July 2021*

A STUDY ON (ρ,ζ)-CIRCULANT POLYNOMIAL MATRICES

R. MUTHAMILSELVAM AND G. RAMESH

ABSTRACT. (ρ,ζ)- Circulant polynomial matrices are defined. Its additive properties are investigated and characterizations are also given.

1. Introduction

Let $(a_1(\alpha), a_2(\alpha), ...a_n(\alpha))$ be an ordered n-tuple of polynomials with coefficients in the field of complex numbers and let them generate the circulant polynomial matrix [1][3] [4] of order n:

$$A(\alpha) = \begin{pmatrix} a_1(\alpha) & a_2(\alpha) & ... & a_n(\alpha) \\ a_n(\alpha) & a_1(\alpha) & ... & a_2(\alpha) \\ ... & ... & ... & ... \\ a_2(\alpha) & a_3(\alpha) & ... & a_1(\alpha) \end{pmatrix} \tag{1.1}$$

We shall often denote this circulant polynomial matrix as

$$A(\alpha) = Circ(a_1(\alpha), a_2(\alpha), ..., a_n(\alpha)) \tag{1.2}$$

In this paper, we define the (ρ,ζ)-circulant polynomial matrix and also, we examine some fundamental properties.

We found a characterization of (ρ,ζ)-circulant polynomial matrix. Let $I_n(\alpha)$ be the unit n x n polynomial matrix.

Let $A(\alpha) \in C_{n\times n}(\alpha)$, then $A^T(\alpha), A^*(\alpha)$ and $|A(\alpha)|$ be its transpose, adjoint and the determinant respectively.

2. (ρ,ζ)-Circulant Polynomial Matrices

Here we define (ρ,ζ)-circulant polynomial matrix. Also, we generalize some properties of (ρ,ζ)-circulant matrices found in [2], [5], [6], [7].

Definition 2.1. If a polynomial matrix is of the form,

$$A(\alpha) =$$

2000 *Mathematics Subject Classification.* 15A09, 15A15,15A57.
Key words and phrases. Circulant polynomial matrices, (ρ,ζ)-Circulant polynomial matrices.
* This research is supported by G. Ramesh.

$$\begin{pmatrix} a_0(\alpha) & a_1(\alpha) & a_2(\alpha) & ... & a_{n-2}(\alpha) & a_{n-1}(\alpha) \\ \rho a_{n-1}(\alpha) & a_0(\alpha)-\zeta a_{n-1}(\alpha) & a_1(\alpha) & ... & a_{n-3}(\alpha) & a_{n-2}(\alpha) \\ \rho a_{n-2}(\alpha) & \rho a_{n-1}(\alpha)-\zeta a_{n-2}(\alpha) & a_0(\alpha)-\zeta a_{n-1}(\alpha) & ... & a_{n-4}(\alpha) & a_{n-3}(\alpha) \\ \rho a_{n-3}(\alpha) & \rho a_{n-2}(\alpha)-\zeta a_{n-3}(\alpha) & a_{n-1}(\alpha)-\zeta a_{n-2}(\alpha) & ... & a_{n-5}(\alpha) & a_{n-4}(\alpha) \\ ... & ... & ... & ... & ... & ... \\ \rho a_2(\alpha) & \rho a_3(\alpha)-\zeta a_2(\alpha) & \rho a_4(\alpha)-\zeta a_3(\alpha) & ... & a_0(\alpha)-\zeta a_{n-1}(\alpha) & a_1(\alpha) \\ \rho a_1(\alpha) & \rho a_2(\alpha)-\zeta a_1(\alpha) & \rho a_3(\alpha)-\zeta a_2(\alpha) & ... & a_{n-1}(\alpha)-\zeta a_{n-2}(\alpha) & a_0(\alpha)-\zeta a_{n-1}(\alpha) \end{pmatrix}$$

it is known as a (ρ,ζ)-circulant polynomial matrix. which is denoted by $A(\alpha) = C_{(\rho,\zeta)}(a_0(\alpha), a_1(\alpha), ..., a_{n-1}(\alpha))$.

Remark 2.2. (i) If $\zeta = 0$, then $A(\alpha)$ is a ρ-circulant polynomial matrix.

(ii)The polynomial matrix $b(\alpha) = C_{(\rho,\zeta)}(0, 1, 0, ...0)$ is referred to as fundamental (ρ,ζ) circulant matrix.

Example 2.3. A 4X4 (3,2)-circulant polynomial matrix is given below.

$$A(\alpha) = \begin{pmatrix} \alpha+\alpha^2 & 1-\alpha & -3+\alpha-2\alpha^2 & 2+2\alpha+3\alpha^2 \\ 6+6\alpha+9\alpha^2 & -4-3\alpha-5\alpha^2 & 1-\alpha & -3+\alpha-2\alpha^2 \\ -9+3\alpha-6\alpha^2 & 12+4\alpha+13\alpha^2 & -4-3\alpha-5\alpha^2 & 1-\alpha \\ 3-3\alpha & 11-5\alpha-6\alpha^2 & 12+4\alpha-13\alpha^2 & -4-3\alpha+5\alpha^2 \end{pmatrix}$$

$= A_0 + A_1\alpha + A_2\alpha^2$ where $A_0 = C_{(3,2)}(0, 1, -3, 2)$, $A_1 = C_{(3,2)}(1, -1, 1, 2)$ and $A_2 = C_{(3,2)}(1, 0, -2, 3)$.
that is

$$A_0 = \begin{pmatrix} 0 & 1 & -3 & 2 \\ 6 & -4 & 1 & -3 \\ -9 & 12 & -4 & 1 \\ 3 & -11 & 12 & -4 \end{pmatrix} A_1 = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 6 & -3 & -1 & 1 \\ 3 & 4 & -3 & -1 \\ -3 & 5 & 4 & -3 \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0 & -2 & 3 \\ 9 & -5 & 0 & -2 \\ -6 & 13 & -5 & 0 \\ 0 & -6 & 13 & -5 \end{pmatrix}$$

Proposition 2.4. *If $A(\alpha), B(\alpha)$ are (ρ,ζ)-circulant polynomial matrices, then $A(\alpha) + B(\alpha)$, $A(\alpha) - B(\alpha)$, $\alpha A(\alpha)$ where α is a scalar, are also (ρ,ζ)-circulant polynomial matrices.*

Proposition 2.5. *A polynomial matrix $A(\alpha)$ is a (ρ,ζ)-circulant polynomial matrix if and only if $A(\alpha) = f_{A(\alpha)}(b(\alpha)) = \left(\sum_{i=0}^{n-1} a_i(\alpha)b^i(\alpha)\right)$.*

Theorem 2.6. *A matrix with polynomial coefficients $A(\alpha) \in C^{(n\times n)}(\alpha)$ is a (ρ,ζ)-circulant polynomial matrix if and only if $A(\alpha)b(\alpha) = b(\alpha)A(\alpha)$.*

Proof. Assume $A(\alpha)$ is a (ρ,ζ)-circulant polynomial matrix.
We must demonstrate our worth $A(\alpha)b(\alpha) = b(\alpha)A(\alpha)$
Let $A(\alpha)b(\alpha) = C_{(\rho,\zeta)}(a_0(\alpha), a_1(\alpha), \ldots a_{n-1}(\alpha))$ be a (ρ,ζ)-circulant polynomial matrix. Then $A(\alpha) = \left(\sum_{i=0}^{n-1} a_i(\alpha)b^i(\alpha)\right)$.
$\Rightarrow (\alpha)b(\alpha) = b(\alpha)A(\alpha)$.

Conversely, assume that $A(\alpha)b(\alpha) = b(\alpha)A(\alpha)$. Let us prove $A(\alpha)$ is a (ρ,ζ)-circulant polynomial matrix.

If $A(\alpha)b(\alpha) = b(\alpha)A(\alpha)$, then

$$b^T(\alpha)A^T(\alpha) = A^T(\alpha)b^T(\alpha)$$
$$(b^T)^i(\alpha)A^T(\alpha) = A^T(\alpha)(b^T)^i(\alpha), \text{ i = 1,2,...}$$

If $e_i(\alpha)$ is the i^{th} column of $I_n(\alpha)$, then

$$b^T(\alpha)e_i(\alpha) = e_{i+1}(\alpha) \text{ for } i = 1, 2, ...n-1.$$

Thus, we have $(b^T)^i(\alpha)e_i(\alpha) = e_{i+1}(\alpha)$ for $i = 1, 2, ...n-1$.

$$\begin{aligned}
\text{Now } A^T(\alpha) &= A^T(\alpha)I_n(\alpha)\\
&= A^T(\alpha)[e_1(\alpha), e_2\alpha, \ldots, e_n(\alpha)]\\
&= A^T(\alpha)[e_1(\alpha), b^T(\alpha)e_1\alpha, \ldots, (b^T)^{n-1}(\alpha)e_1(\alpha)]\\
&= [A^T(\alpha)e_1(\alpha), A^T(\alpha)b^T(\alpha)e_1\alpha, \ldots, A^T(\alpha)(b^T)^{n-1}(\alpha)e_1(\alpha)]\\
&= [A^T(\alpha)e_1(\alpha), b^T(\alpha)A^T(\alpha)e_1\alpha, \ldots, (b^T)^{n-1}(\alpha)A^T(\alpha)e_1(\alpha)]\\
&= [\beta(\alpha), (b^T)(\alpha)\beta(\lambda), \ldots, (b^T)^{n-1}(\alpha)\beta(\lambda)]
\end{aligned}$$

where $\lambda^T(\alpha)$ is the first row of $A(\alpha)$.

Let $\lambda^T(\alpha) = (a_0(\alpha), a_1(\alpha, \ldots, a_{n-1}(\alpha)))$

$$\text{Thus } \lambda(\alpha) = \left(\sum_{i=0}^{n-1} a_i(\alpha)e_{i+1}(\alpha)\right)$$

$$\begin{aligned}
A^T(\alpha) &= \left(\sum_{i=0}^{n-1} a_i(\alpha)e_{i+1}(\alpha), \sum_{i=0}^{n-1} a_i(\alpha)b^T(\alpha)e_{i+1}(\alpha), \cdots \sum_{i=0}^{n-1} a_i(\alpha)(b^T)^{n-1}(\alpha)e_{i+1}(\alpha)\right)\\
&= \sum_{i=0}^{n-1} a_i\left(e_{i+1}(\alpha), b^T(\alpha)e_{i+1}(\alpha), \ldots, (b^T)^{n-1}(\alpha)e_{i+1}(\alpha)\right)\\
&= \sum_{i=0}^{n-1} a_i\left((b^T)^i(\alpha)e_1(\alpha), (b^T)^{i+1}(\alpha)e_1(\alpha), \ldots, (b^T)^{n+i-1}(\alpha)e_1(\alpha)\right)\\
&= \sum_{i=0}^{n-1} a_i(b^T)^i(\alpha)(e_1(\alpha), e_2(\alpha), \ldots, e_n(\alpha))\\
&= \sum_{i=0}^{n-1} a_i(\alpha)(b^T)^i(\alpha)
\end{aligned}$$

$$\Rightarrow A(\alpha) = \sum_{i=0}^{n-1} a_i(\alpha)b(\alpha)$$

Hence $A(\alpha)$ is a (ρ, ζ)-circulant polynomial matrix. □

Corollary 2.7. $|A(\alpha)| \neq 0$ *is a* (ρ, ζ)*-circulant polynomial matrix if and only if* $A^{-1}(\alpha)$ *is a* (ρ, ζ)*-circulant polynomial matrix.*

Proof. Given that $|A(\alpha)| \neq 0$ is a (ρ, ζ)-circulant polynomial matrix.
$\Longleftrightarrow A(\alpha)b(\alpha) = b(\alpha)A(\alpha)$
$\Longleftrightarrow A^{-1}(\alpha)b(\alpha) = b(\alpha)A^{-1}(\alpha)$
$\Longleftrightarrow A^{-1}(\alpha)$ is a (ρ, ζ)-circulant polynomial matrix. □

Theorem 2.8. *If* $A(\alpha), B(\alpha)$ *are* (ρ, ζ)*-circulant polynomial matrices, then* $A(\alpha)B(\alpha)$ *and* $B(\alpha)A(\alpha)$ *are* (ρ, ζ)*-circulant polynomial matrices and* $A(\alpha)B(\alpha) = B(\alpha)A(\alpha)$.

Proof. Given that $A(\alpha), B(\alpha)$ are (ρ, ζ)-circulant polynomial matrices.
From theorem (2.6), we have $A(\alpha)b(\alpha) = b(\alpha)A(\alpha)$ and $B(\alpha)b(\alpha) = b(\alpha)B(\alpha)$.

$$\begin{aligned}
\text{Now } [A(\alpha)B(\alpha)]b(\alpha) &= A(\alpha)[B(\alpha)b(\alpha)]\\
&= A(\alpha)[b(\alpha)B(\alpha)]\\
&= [A(\alpha)b(\alpha)]B(\alpha)\\
&= b(\alpha)[A(\alpha)B(\alpha)]
\end{aligned}$$

Thus $A(\alpha)B(\alpha)$ is a (ρ, ζ)-circulant polynomial matrix.

Also, $[B(\alpha)A(\alpha)]b(\alpha) = B(\alpha)[A(\alpha)b(\alpha)]$

$$= B(\alpha)[b(\alpha)A(\alpha)]$$
$$= [B(\alpha)b(\alpha)]A(\alpha)]$$
$$= [b(\alpha)B(\alpha)]A(\alpha)]$$
$$= b(\alpha)[B(\alpha)A(\alpha)]$$

Hence $B(\alpha)A(\alpha)$ is a (ρ, ζ)-circulant polynomial matrix.

We can deduce from proposition that (2.5), we assume that $A(\alpha) = f(b(\alpha))$ and $B(\alpha) = g(b(\alpha))$.

$$\Rightarrow A(\alpha)B(\alpha) = B(\alpha)A(\alpha) \qquad \square$$

3. Conclusion

some of the characterization of (ρ, ζ)-circulant polynomial matrices are discussed here. In the same way, the other properties can be extended.

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Block Circulant Matrices: Where the Blocks are Circulant Polynomial Matrices

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Abstract: *The characterization of block* circulant matrix with circulant polynomial matrices as its blocks ar*e derived as a generalization of the block circulant matrices with circulant block matrices.*
Keywords: *Circulant polynomial matrix, Block circulant polynomial matrix, Circulant block polynomial matrix.*

AMS Classification: *15A09, 15A15, 15A57.*

I. Introduction

Let $\left(a_1(\lambda), a_2(\lambda), ..., a_n(\lambda)\right)$ be an ordered n-tuple of polynomial with complex coefficients , and let them generate the circulant polynomial matrix of order n $[5]$:

$$A(\lambda)=\begin{pmatrix} a_1(\lambda) & a_2(\lambda) & ... & a_n(\lambda) \\ a_n(\lambda) & a_1(\lambda) & ... & a_2(\lambda) \\ & & \vdots & \\ a_2(\lambda) & a_3(\lambda) & ... & a_1(\lambda) \end{pmatrix} \quad (1.1)$$

We shall often denote this circulant polynomial matrix as

$$A(\lambda)=circ\left(a_1(\lambda), a_2(\lambda), ..., a_n(\lambda)\right) \quad (1.2)$$

It is well known that all circulant polynomial matrices of order n are simultaneously diagonalizable by the polynomial matrix $F(\lambda)$ associated with the finite Fourier transforms.

Specifically, let $\omega(\lambda)=\exp\left(\frac{2\pi i}{n}(\lambda)\right), i=\sqrt{-1}$ $\quad (1.3)$

and set
$$F^*(\lambda)=n^{\frac{-1}{2}}\begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & \omega(\lambda) & \omega^2(\lambda) & & \omega^{n-1}(\lambda) \\ 1 & \omega^2(\lambda) & \omega^4(\lambda) & & \omega^{2(n-1)}(\lambda) \\ & & \vdots & & \\ 1 & \omega^{n-1}(\lambda) & \omega^{n-2}(\lambda) & ... & \omega(\lambda) \end{pmatrix} \quad (1.4)$$

The Fourier polynomial matrix $\mathrm{F}(\lambda)$ depends only on n. This matrix is also symmetric polynomial and unitary polynomial $\left(F(\lambda)F^*(\lambda)=F^*(\lambda)F(\lambda)=I(\lambda)\right)$ and we have

$$A(\lambda)=F^*(\lambda)\Lambda(\lambda)F(\lambda) \quad (1.5)$$

Where

$$\Lambda(\lambda)=diag(\lambda_1, \lambda_2, ..., \lambda_n) \quad (1.6)$$

The symbol * designates the conjugate transpose.

From the spectral mapping theorem, we may represent $A(\lambda)$ in the form

$$A(\lambda)=a_1(\lambda)+a_2(\lambda)\pi(\lambda)+a_3(\lambda)\pi^2(\lambda)+...+a_n(\lambda)\pi^{n-1}(\lambda) \quad (1.7)$$ Where

$\pi(\lambda)$ is the permutation matrix $circ(0,1,0,0,...)$.

Also, let $A(\lambda)$ be an $n\times n$ polynomial matrix. Then $A(\lambda)$ is a circulant polynomial matrix if and only if $A(\lambda)\pi(\lambda)=\pi(\lambda)A(\lambda)$ (1.8)

The matrix $\pi(\lambda)=circ(0,1,0,...,0)$

From the diagonalization of the circulant polynomial matrix $\pi(\lambda)$, we have

$$\pi(\lambda)=F^*(\lambda)\Omega(\lambda)F(\lambda) \quad (1.9)$$

Let $(A_1(\lambda),A_2(\lambda),...,A_n(\lambda))$ be a collection of square polynomial matrices, each of order n. By a block circulant polynomial matrix $[2,3,7,8]$ of type (m,n) $(and\ of\ order\ mn)$ is meant an $mn\times mn$ polynomial matrix of the form

$$bcirc(A_1(\lambda),A_2(\lambda),...,A_m(\lambda))=\begin{pmatrix} A_1(\lambda) & A_2(\lambda) & ... & A_m(\lambda) \\ A_m(\lambda) & A_1(\lambda) & ... & A_{m-1}(\lambda) \\ & & \vdots & \\ A_2(\lambda) & A_3(\lambda) & ... & A_1(\lambda) \end{pmatrix}$$

We denote the set of all block circulant polynomial matrices of order mn as $\mathscr{BC}_{m,n}(\lambda)$.

A representation of block circulant polynomial matrices can be developed as

$$bcirc(A_1(\lambda),A_2(\lambda),...,A_n(\lambda))=\sum_{k=0}^{m-1}\left(\pi_m^k(\lambda)\otimes A_{k+1}(\lambda)\right) \quad (1.10)$$

Also, let $A(\lambda)$ be of type (m,n):

$$A(\lambda)=\begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & \cdots & A_{1m}(\lambda) \\ & & \vdots & \\ A_{m1}(\lambda) & A_{m2}(\lambda) & \cdots & A_{mm}(\lambda) \end{pmatrix}$$

$(m\times m\ \ block\ matrix\ each\ block\ is\ of\ oreder\ n)$ is a circulant block polynomial matrix $[7,5]$ if each block $A_{ij}(\lambda)$ is a circulant polynomial matrix.

We denote the set of all circulant block polynomial matrices of order n as $\mathscr{CB}_{m,n}(\lambda)$.

This paper is devoted to the study of block circulant matrix with circulant polynomial matrices as its blocks.

II Block Circulant Matrices: Where the Blocks are Circulant Polynomial Matrices

In this section, we have given a characterization of block circulant matrix with circulant polynomial matrices as its blocks analogous to that of the results found in $[1,4,6,9,10]$.

Definition: 2.1

Let $A(\lambda)$ be of type (m,n). $A(\lambda)$ is said to be a block circulant matrix with circulant polynomial matrices as its blocks if it is circulant block wise and each block is a circulant polynomial matrix and is denoted by $\mathcal{CPBC}_{m,\mathrm{n}}(\lambda)$.

Example: 2.2

$$A(\lambda)=\begin{bmatrix} 3+\lambda & -\lambda & 12-\lambda & -1+\lambda & 1+7\lambda & -10+\lambda \\ -\lambda & 3+\lambda & -1+\lambda & 12-\lambda & -10+\lambda & 1+7\lambda \\ 1+7\lambda & -10+\lambda & 3+\lambda & -\lambda & 12-\lambda & -1+\lambda \\ -10+\lambda & 1+7\lambda & -\lambda & 3+\lambda & -1+\lambda & 12-\lambda \\ 12-\lambda & -1+\lambda & 1+7\lambda & -10+\lambda & 3+\lambda & -\lambda \\ -1+\lambda & 12-\lambda & -10+\lambda & 1+7\lambda & -\lambda & 3+\lambda \end{bmatrix}$$

is in $\mathcal{CPBC}_{m,\mathrm{n}}(\lambda)$.

Remark: 2.3

A polynomial matrix in $\mathcal{CPBC}_{m,\mathrm{n}}(\lambda)$ is not necessarily a circulant polynomial matrix.

Lemma: 2.4

$F_m(\lambda)$ and $F_n(\lambda)$ satisfies the following equalities.

(i) $\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]=\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]$

(ii) $\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]$

Proof:

$$(i)\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[\left(F_m^*(\lambda)\Omega_m(\lambda)F_m(\lambda)\right)\otimes\left(F_n^*(\lambda)I_n(\lambda)F_n(\lambda)\right)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[F_m^*(\lambda)\otimes F_n^*(\lambda)\right]\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$(ii)\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*$$

$$=\left[\left(F_m^*(\lambda)\Omega_m(\lambda)F_m(\lambda)\right)\otimes\left(F_n^*(\lambda)I_n(\lambda)F_n(\lambda)\right)\right]\left[F_m^*(\lambda)\otimes F_n^*(\lambda)\right]$$

$$=\left(F_m^*(\lambda)\otimes F_n^*(\lambda)\right)\left(\Omega_m(\lambda)\otimes I_n(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*$$

$$=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\left(\Omega_m(\lambda)\otimes I_n(\lambda)\right)$$

Lemma: 2.5

If $A(\lambda)=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\Lambda(\lambda)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$ where $\Lambda(\lambda)$ is diagonal polynomial matrix, then $A(\lambda)\in\mathcal{CPBC}_{m,n}(\lambda)$ or equivalently, that $A(\lambda)$ commutes with both $\pi_m(\lambda)\otimes I_n(\lambda)$ and $I_m(\lambda)\otimes\pi_n(\lambda)$.

Proof:

Now $A(\lambda)\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\Lambda(\lambda)\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\Lambda(\lambda)\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\left[\Omega_m(\lambda)\otimes I_n(\lambda)\right]\Lambda(\lambda)\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\Lambda(\lambda)\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]A(\lambda).$$

Also, $A(\lambda)\left[I_m(\lambda)\otimes\pi_n(\lambda)\right]$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\Lambda(\lambda)\left[F_m(\lambda)\otimes F_n(\lambda)\right]\left[I_m(\lambda)\otimes\pi_n(\lambda)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\Lambda(\lambda)\left[I_m(\lambda)\otimes\Omega_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\left[I_m(\lambda)\otimes\Omega_n(\lambda)\right]\Lambda(\lambda)\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[I_m(\lambda)\otimes\pi_n(\lambda)\right]\left[F_m(\lambda)\otimes F_n(\lambda)\right]^*\Lambda(\lambda)\left[F_m(\lambda)\otimes F_n(\lambda)\right]$$

$$=\left[I_m(\lambda)\otimes\pi_n(\lambda)\right]A(\lambda).$$

Theorem: 2.6

All polynomial matrices in $\mathcal{CPBC}_{m,n}(\lambda)$ are simultaneously diagonalizable polynomial matrices by the unitary polynomial matrix $F_m(\lambda)\otimes F_n(\lambda)$, and they commute. If the eigen values of the circulant block polynomial matrices are given $\Lambda_{k+1}(\lambda), k = 0,1,\dots,m-1,$ then the diagonal polynomial matrix of the eigen values of the $\mathscr{CPBC}(\lambda)$ polynomial matrix is given by $\sum_{k=0}^{m-1}\Omega_m^k(\lambda)\otimes\Lambda_{k+1}(\lambda)$. Conversely, any matrix of the form $A(\lambda)=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\Lambda(\lambda)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$ where $\Lambda(\lambda)$ is diagonal polynomial matrix is in $\mathscr{CPBC}_{m,n}(\lambda)$.

Proof:

Assume that $A(\lambda)$ is a block circulant polynomial matrix.

From (1.10), $A(\lambda)$ can be written as $A(\lambda)=\sum_{k=0}^{m-1}\pi_m^k(\lambda)\otimes A_{k+1}(\lambda)$ where the blocks are $A_1(\lambda), A_2(\lambda),\dots,A_m(\lambda)$. The $A_{k+1}(\lambda)$ are circulant polynomial matrices if and only if $A_{k+1}(\lambda)=F_n^*(\lambda)\Lambda_{k+1}(\lambda)F_n(\lambda)$, where $F_n(\lambda)$ is the Fourier polynomial matrix of order n and $\Lambda_{k+1}(\lambda)$ is a diagonal polynomial matrix of order n.

From (1.9), we have $\pi_m^k(\lambda)=F_m^*(\lambda)\Omega_m^k(\lambda)F(\lambda)$ where $\Omega_m(\lambda)$ is the $\Omega(\lambda)$ polynomial matrix of order m, $\Omega(\lambda)=\operatorname{diag}\left(1,\omega(\lambda),\omega^k(\lambda),\dots,\omega^{m-1}(\lambda)\right)$ where $\omega(\lambda)=e^{\frac{2\pi i}{m}(\lambda)}$

Hence, $$A(\lambda)=\sum_{k=0}^{m-1}\left(F_m^*(\lambda)\Omega_m^k(\lambda)F_m(\lambda)\right)\otimes\left(F_n^*(\lambda)\Lambda_{k+1}(\lambda)F_n(\lambda)\right)$$

$$=\sum_{k=0}^{m-1}\left(F_m^*(\lambda)\otimes F_n^*(\lambda)\right)\left(\Omega_m^k(\lambda)\otimes\Lambda_{k+1}(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$$

$$=\sum_{k=0}^{m-1}\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\left(\Omega_m^k(\lambda)\otimes\Lambda_{k+1}(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$$

$$A(\lambda)=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\Lambda(\lambda)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$$

Conversely, assume that $A(\lambda)=\left(F_m(\lambda)\otimes F_n(\lambda)\right)^*\Lambda(\lambda)\left(F_m(\lambda)\otimes F_n(\lambda)\right)$ where $\Lambda(\lambda)$ is diagonal polynomial matrix.

We have to prove that $A(\lambda)\in\mathscr{BCCB}_{m,n}(\lambda)$. It is enough to prove that $A(\lambda)$ commutes with both $\pi_m(\lambda)\otimes I_n(\lambda)$ and $I_m(\lambda)\otimes\pi_n(\lambda)$.

From lemma (2.5), we have $A(\lambda)(\pi_m(\lambda)\otimes I_n(\lambda)) = (\pi_m(\lambda)\otimes I_n(\lambda))A(\lambda)$ and

$$A(\lambda)(I_m(\lambda)\otimes\pi_n(\lambda)) = (I_m(\lambda)\otimes\pi_n(\lambda))A(\lambda)$$

Hence, $A(\lambda)$ is a block circulant matrix with circulant polynomial matrices as its blocks.

Lemma: 2.7

Let j, k be nonnegative integers. Let $A_m(\lambda), B_m(\lambda)$ be of order m and n. Then

$\left[A_m(\lambda)\otimes I_n(\lambda)\right]^k\left[I_m(\lambda)\otimes B_n(\lambda)\right]^j = A_m^k(\lambda)\otimes B_n^j(\lambda)$.

Proof:

$$(A_m(\lambda)\otimes I_n(\lambda))(A_m(\lambda)\otimes I_n(\lambda)) = A_m(\lambda)A_m(\lambda)\otimes I_n(\lambda)I_n(\lambda)$$

$$= A_m^2(\lambda)\otimes I_n^2(\lambda)$$

$$(A_m(\lambda)\otimes I_n(\lambda))^2 = A_m^2(\lambda)\otimes I_n(\lambda)$$

By induction, $(A_m(\lambda)\otimes I_n(\lambda))^k = A_m^k(\lambda)\otimes I_n(\lambda)$

Similarly, $(I_m(\lambda)\otimes B_n(\lambda))^j = I_m(\lambda)\otimes B_n^j(\lambda)$

Now $\left[A_m(\lambda)\otimes I_n(\lambda)\right]^k\left[I_m(\lambda)\otimes B_n(\lambda)\right]^j = \left[A_m^k(\lambda)\otimes I_n(\lambda)\right]\left[I_m(\lambda)\otimes B_n^j(\lambda)\right]$

$$= \left[A_m^k(\lambda)I_m(\lambda)\otimes I_n(\lambda)B_n^j(\lambda)\right]$$

$$= A_m^k(\lambda)\otimes B_n^j(\lambda).$$

Theorem: 2.8

Let $A(\lambda)\in \mathscr{CPBC}_{m,n}(\lambda)$. Then $A(\lambda)$ is a polynomial (of two variables) in $\pi_m(\lambda)\otimes I_n(\lambda)$ and $\pi_m(\lambda)\otimes\pi_n(\lambda)$.

Proof:

Since $A(\lambda)$ is a block circulant polynomial matrix.

Therefore, by (1.10), we have $A(\lambda) = \sum_{k=0}^{m-1}\left[\pi_m^k(\lambda)\otimes A_{k+1}(\lambda)\right]$

Where the blocks $A_{k+1}(\lambda)$ are themselves circulant polynomial matrices. Then

$$A_{k+1}(\lambda)=\sum_{j=0}^{n-1} a_{k+1,j+i}(\lambda)\pi_n^j(\lambda)$$

$$\text{Now } A(\lambda)=\sum_{k=0}^{m-1}\left[\pi_m^k(\lambda)\otimes\left(\sum_{j=0}^{n-1} a_{k+1,j+1}(\lambda)\pi_n^j(\lambda)\right)\right]$$

$$=\sum_{k=0}^{m-1}\sum_{j=0}^{n-1}\left[\pi_m^k(\lambda)\otimes a_{k+1,j+1}(\lambda)\pi_n^j(\lambda)\right]$$

$$=\sum_{k,j=0}^{m-1,n-1} a_{k+1,j+1}(\lambda)\left[\pi_m^k(\lambda)\otimes\pi_n^j(\lambda)\right]$$

$$=\sum_{k,j=0}^{m-1,n-1} a_{k+1,j+1}(\lambda)\left[\pi_m(\lambda)\otimes I_n(\lambda)\right]^k\left[I_m(\lambda)\otimes\pi_n(\lambda)\right]^j$$

[by lemma (2.7)]

This is a polynomial in $\pi_n(\lambda)\otimes I_n(\lambda)$ and $I_m(\lambda)\otimes\pi_n(\lambda)$.

Remark: 2.9

(i) A circulant polynomial matrix of level 1 is an ordinary circulant polynomial matrix.

(ii) A circulant polynomial matrix of level 2 is in $\mathcal{CPBC}_{m,n}(\lambda)$.

(iii) A circulant polynomial matrix of level 3 is a block circulant polynomial whose block Polynomials are level 2 circulant polynomial matrices.

Theorem: 2.10

A circulant polynomial matrix of level 3 and type (m,n,p) is diagonalizable polynomial matrix by the unitary polynomial matrix $F_m(\lambda)\otimes F_n(\lambda)\otimes F_p(\lambda)$.

Proof :

Let $A(\lambda)$ be a level 3 circulant polynomial matrix of type (m, n, p).

From (1.10), we have $A(\lambda)=\sum_{k=0}^{m-1}\pi_m^k(\lambda)\otimes A_{k+1}(\lambda)$ $\quad(2.1)$

Where each $A_{k+1}(\lambda)$ is a level 2 circulant polynomial matrix of type (n,p).

Thus, $A_{k+1}(\lambda)=\sum_{j=0}^{n-1}\pi_n^j(\lambda)\otimes A_{k+1,j+1}(\lambda)$ $\quad(2.2)$

Where each $A_{k+1,j+1}(\lambda)$ is a circulant polynomial matrix of level 1 and of order p.

Thus, from (1.7)

$$A_{k+1,j+1}(\lambda)=\sum_{r=0}^{p-1} a_{k+1,j+1,r+1}(\lambda)\pi_p^r(\lambda) \quad (2.3)$$

Combining (2.1),(2.2) and (2.3) we have

$$A(\lambda)=\left[\sum_{k=0}^{m-1}\pi_m^k(\lambda)\otimes\left[\sum_{j=0}^{n-1}\pi_n^j(\lambda)\otimes\left[\sum_{r=0}^{p-1} a_{k+1,j+1,r+1}(\lambda)\pi_p^r(\lambda)\right]\right]\right] \quad (2.4)$$

$$=\left[\sum_{k=0}^{m-1}\pi_m^k(\lambda)\otimes\left[\sum_{j=0}^{n-1}\sum_{r=0}^{p-1}a_{k+1,j+1,r+1}(\lambda)\pi_n^j(\lambda)\otimes\pi_p^r(\lambda)\right]\right]$$

$$=\sum_{k,j,r=0}^{m-1,n-1,p-1}a_{k+1,j+1,r+1}(\lambda)\left(\pi_m^k(\lambda)\otimes\pi_n^j(\lambda)\otimes\pi_p^r(\lambda)\right)$$

Since,

$\pi_m^k(\lambda)=F_m^*(\lambda)\Omega_m^k(\lambda)F_m(\lambda),\ \pi_m^j(\lambda)=F_n^*(\lambda)\Omega_n^k(\lambda)F_n(\lambda)$ *and* $\pi_p^r(\lambda)=F_p^*(\lambda)\Omega_p^k(\lambda)F_p(\lambda)$

There fore,

$$A(\lambda)=\sum_{k,j,r=0}^{m-1,n-1,p-1}a_{k+1,j+1,r+1}(\lambda)\left[F_m^*(\lambda)\Omega_m^k(\lambda)F_m(\lambda)\otimes F_n^*(\lambda)\Omega_n^k(\lambda)F_n(\lambda)\otimes F_p^*(\lambda)\Omega_p^k(\lambda)F_p(\lambda)\right]$$

$$=\sum_{k,j,r=0}^{m-1,n-1,p-1}a_{k+1,j+1,r+1}(\lambda)\left(F_m^*(\lambda)\otimes F_n(\lambda)\otimes F_p^*(\lambda)\right)\left(\Omega_m^k(\lambda)\otimes\Omega_n^k(\lambda)\otimes\Omega_p^k(\lambda)\right)\left(F_m^*(\lambda)\otimes F_n^*(\lambda)\otimes F_p(\lambda)\right)$$

$$=\sum_{k,j,r=0}^{m-1,n-1,p-1}a_{k+1,j+1,r+1}(\lambda)\left(F_m(\lambda)\otimes F_n(\lambda)\otimes F_p(\lambda)\right)^*\left(\Omega_m^k(\lambda)\otimes\Omega_n^k(\lambda)\otimes\Omega_p^k(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\otimes F_p(\lambda)\right)$$

$$=\left(F_m(\lambda)\otimes F_n(\lambda)\otimes F_p(\lambda)\right)^*\sum_{k,j,r=0}^{m-1,n-1,p-1}a_{k+1,j+1,r+1}(\lambda)\left(\Omega_m^k(\lambda)\otimes\Omega_n^k(\lambda)\otimes\Omega_p^k(\lambda)\right)\left(F_m(\lambda)\otimes F_n(\lambda)\otimes F_p(\lambda)\right)$$

Thus, a circulant polynomial matrix of level 3 and type (m, n,p) is diagonalizable polynomial matrix.

Lemma: 2.11

$$A_m(\lambda)\otimes B_n(\lambda)=\left[A_m(\lambda)\otimes I_n(\lambda)\right]\left[I_m(\lambda)\otimes B_n(\lambda)\right]$$

Proof:

$$\left[A_m(\lambda)\otimes I_n(\lambda)\right]\left[I_m(\lambda)\otimes B_n(\lambda)\right]=\left(A_m(\lambda)I_m(\lambda)\otimes I_n(\lambda)B_n(\lambda)\right)$$
$$=A_m(\lambda)\otimes B_n(\lambda)$$

Lemma: 2.12

$$A_m(\lambda)\otimes B_n(\lambda)\otimes C_p(\lambda)$$
$$=\left(A_m(\lambda)\otimes I_{np}(\lambda)\right)\left(I_m(\lambda)\otimes B_n(\lambda)\otimes I_P(\lambda)\right)\left(I_{mn}(\lambda)\otimes C_P(\lambda)\right)$$

Proof:

$$A_m(\lambda)\otimes B_n(\lambda)\otimes C_p(\lambda)=\left[A_m(\lambda)\otimes B_n(\lambda)\right]\otimes C_p(\lambda)$$
$$=\left[\left(A_m(\lambda)\otimes B_n(\lambda)\right)\otimes I_p(\lambda)\right]\left[I_{mn}(\lambda)\otimes C_p(\lambda)\right]$$
$$=\left(A_m(\lambda)\otimes\left[B_n(\lambda)\otimes I_p(\lambda)\right]\right)\left(I_{mn}(\lambda)\otimes C_p(\lambda)\right)$$
$$=\left(A_m(\lambda)\otimes I_{np}(\lambda)\right)\left[I_m(\lambda)\left(B_n(\lambda)\otimes I_p(\lambda)\right)\right]\left(I_{mn}(\lambda)\otimes C_p(\lambda)\right)$$
$$\left[by\,lemma(2.11)\right]$$

Lemma: 2.13

For nonnegative integers k, j, r

$$A_m^k(\lambda)\otimes B_n^j(\lambda)\otimes C_p^r(\lambda)=\left(A_m(\lambda)\otimes I_{np}(\lambda)\right)^k\left[I_m(\lambda)\otimes B_n(\lambda)\otimes I_P(\lambda)\right]^j\left[I_{mn}(\lambda)\otimes C_p(\lambda)\right]^r$$

Proof:

By the lemma (2.12)

$$A_m^k(\lambda)\otimes B_n^j(\lambda)\otimes C_p^r(\lambda)=\left(A_m^k(\lambda)\otimes I_{np}(\lambda)\right)\left(I_m(\lambda)\otimes B_n^j(\lambda)\otimes I_P(\lambda)\right)\left(I_{mn}(\lambda)\otimes C_p^r(\lambda)\right)$$

We have $\left[A_m^k(\lambda)\otimes I_{np}(\lambda)\right]=\left[A_m(\lambda)\otimes I_{np}(\lambda)\right]^k$

$$\left[I_m(\lambda)\otimes B_n^j(\lambda)\otimes I_p(\lambda)\right]=\left[I_m(\lambda)\otimes B_n(\lambda)\otimes I_p(\lambda)\right]^j$$

$$\left[I_{mn}(\lambda)\otimes C_p^r(\lambda)\right]=\left[I_{mn}(\lambda)\otimes C_p(\lambda)\right]^r$$

Hence, $A_m^k(\lambda)\otimes B_n^j(\lambda)\otimes C_p^r(\lambda)=\left[A_m(\lambda)\otimes I_{np}(\lambda)\right]^k\left[I_m(\lambda)\otimes B_n(\lambda)\otimes I_p(\lambda)\right]^j\left[I_{mn}(\lambda)\otimes C_p(\lambda)\right]^r$

Theorem: 2.14

Let $A(\lambda)$ be of type (m,n,p) and be a circulant polynomial matrix of level 3. Then

$$A(\lambda)=\sum_{k,j,r=0}^{m-1,n-1,p-1}a_{k+1,j+1,r+1}(\lambda)\left[\pi_m(\lambda)\otimes I_{np}(\lambda)\right]^k\left[I_m(\lambda)\otimes\pi_n(\lambda)\otimes I_p(\lambda)\right]^j\left[I_{m,m}(\lambda)\otimes\pi_p(\lambda)\right]^r$$

Proof:

Since $A(\lambda)$ is of type (m,n,p) and is a circulants polynomial matrix of level 3. Therefore, from (2.4) it can be written as

$$A(\lambda)=\sum_{k=0}^{m-1}\left[\pi_m^k(\lambda)\otimes\left[\sum_{j=0}^{n-1}\pi_n^j(\lambda)\otimes\left[\sum_{r=0}^{p-1}a_{k+1,j+1,r+1}(\lambda)\pi_p^r(\lambda)\right]\right]\right]$$

$$=\sum_{k=0}^{m-1}\left[\pi_m^k(\lambda)\otimes\left[\sum_{j=0}^{n-1}\sum_{r=0}^{p-1}a_{k+1,j+1,r+1}(\lambda)\pi_n^j(\lambda)\otimes\pi_p^r(\lambda)\right]\right]$$

$$=\sum_{k,j,r=0}^{m-1,n-1,p-1}a_{k+1,j+1,r+1}(\lambda)\pi_m^k(\lambda)\otimes\pi_n^j(\lambda)\otimes\pi_p^r(\lambda)$$

Hence, by (2.14) & lemma (2.13), the above can be written as

$$A(\lambda)=\sum_{k,j,r-0}^{m-1,n-1,p-1}a_{k+1,j+1,r+1}(\lambda)\left(\pi_m(\lambda)\otimes I_{np}(\lambda)\right)^k\left(I_m(\lambda)\otimes\pi_n(\lambda)\otimes I_p(\lambda)\right)^j\left(I_{mn}(\lambda)\otimes\pi_p(\lambda)\right)^r$$

Conclusion:

Some of the characterizations of block circulant matrix with circulant polynomial matrices as its blocks are discussed here.

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A STUDY ON k- CIRCULANT POLYNOMIAL MATRICES

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Abstract: Some of the properties of k-circulant matrices are extended to k-circuant polynmial matrices.

Keywords: *k-circuant matrix, k-circulant Polynomail matrix*
AMS Classification: *15A09, 15A15, 15A57.*

I. Introduction

A circulant matrix is one in which a basic row of numbers is repeated again and again [2,5,6] but with a shift in position circulant matrices have many connections to problems in physics, to image processing, to probability and statistics, to numerical analysis, to number theory, to geometry. The built –in periodicity means that circulants tie in with Fourier analysis and group theory.

In this paper we have derived characterizations for k-circulant polynomial matrices and proved some results.

II. k-Circulant Polynomial Matrices:

In this section some of the properties of k-circulant matrices found in [1,3,4,7] are extended to k-circulant polynomial matrices.

Definition: 2.1

A k-circulant polynomial matrix of order n is a matrix of the form

$$A(\lambda) = \text{k-circ}\left(a_1(\lambda), a_2(\lambda), ..., a_n(\lambda)\right)$$

$$\text{That is, } \mathrm{A}(\lambda) = \begin{bmatrix} a_1(\lambda) & a_2(\lambda) & ... & a_n(\lambda) \\ a_{n-k+1}(\lambda) & a_{n-k+2}(\lambda) & ... & a_{n-k}(\lambda) \\ a_{n-2k+1}(\lambda) & a_{n-2k+2}(\lambda) & ... & a_{n-2k}(\lambda) \\ & \mathrm{M} & & \\ a_{k+1}(\lambda) & a_{k+2}(\lambda) & ... & a_k(\lambda) \end{bmatrix}$$

Remark: 2.2

If $0 \le k \le n$, Each row of $A(\lambda)$ is the previous row moved to the right k places or moved to the left n-k places wraparound.

If $k > n$, then a shift of k places is the same as a shift of k mod n places. If k is negative, shifting to the right k places will be equivalent to shifting to the left (-k) places.

Thus, for any integers k, k' with $k' \equiv k$ (mod n) a k' circulant and a k-circulant are synonymous.

Example: 2.3

A-3-circulant polynomial of order 5 is

$$A(\lambda) = \begin{bmatrix} \lambda^3+\lambda & -\lambda^3-2\lambda^2+1 & 2\lambda^2+3 & -\lambda^2+1 & 3\lambda^2+\lambda \\ 2\lambda^2+3 & -\lambda^2+1 & 3\lambda^2+\lambda & \lambda^3+\lambda & -\lambda^3-2\lambda^2+1 \\ 3\lambda^2+\lambda & \lambda^3+\lambda & -\lambda^3-2\lambda^2+1 & 2\lambda^2+3 & -\lambda^2+1 \\ -\lambda^3-2\lambda^2+1 & 2\lambda^2+3 & -\lambda^2+1 & 3\lambda^2+\lambda & \lambda^3+\lambda \\ -\lambda^2+1 & 3\lambda^2+\lambda & \lambda^3+\lambda & -\lambda^3-2\lambda^2+1 & 2\lambda^2+3 \end{bmatrix}$$

Remark: 2.4

A 1-circulant polynomial matrix is an ordinary circulant polynomial matrix.

A 0-circulant polynomial matrix is one in which all rows are identical.

A (-1)- circulant polynomial matrix or an (n-1)- circulant polynomial matrix has each successive row moved one place to the left.

Theorem: 2.5

$A(\lambda)$ is a k-circulant polynomial matrix if and only if $\pi(\lambda)A(\lambda)=A(\lambda)\pi^{k}(\lambda)$ where $\pi(\lambda)$ is a permutation matrix (with polynomial of degree zero).

Proof:

Let us assume that $A(\lambda)$ is a k-circulant polynomial matrix of order n.

That is, $A(\lambda)=$ k-circ$\left(a_1(\lambda),a_2(\lambda),...,a_n(\lambda)\right)$

$$=\begin{bmatrix} a_1(\lambda) & a_2(\lambda) & ... & a_n(\lambda) \\ a_{n-k+1}(\lambda) & a_{n-k+2}(\lambda) & ... & a_{n-k}(\lambda) \\ a_{n-2k+1}(\lambda) & a_{n-2k+2}(\lambda) & ... & a_{n-2k}(\lambda) \\ & \vdots & & \\ a_{k+1}(\lambda) & a_{k+2}(\lambda) & ... & a_k(\lambda) \end{bmatrix}$$

We have to prove that $\pi(\lambda)A(\lambda)=A(\lambda)\pi^{k}(\lambda)$.

Take $\sigma=\begin{pmatrix} 1 & 2 & ... & n \\ 2 & 3 & ... & 1 \end{pmatrix}$. Then $P_{\sigma}(\lambda)=\pi(\lambda)=(0,1,0...,0)$

$$=\begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ & & ... & & \\ 1 & 0 & 0 & ... & 0 \end{bmatrix}$$

is a permutation matrix.

If $A(\lambda)=\left(a_{ij}(\lambda)\right)$, then

$$\pi(\lambda)A(\lambda)=\begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ & & ... & & \\ 1 & 0 & 0 & ... & 0 \end{bmatrix}\begin{bmatrix} a_1(\lambda) & a_2(\lambda) & ... & a_n(\lambda) \\ a_{n-k+1}(\lambda) & a_{n-k+2}(\lambda) & ... & a_{n-k}(\lambda) \\ a_{n-2k+1}(\lambda) & a_{n-2k+2}(\lambda) & ... & a_{n-2k}(\lambda) \\ & \vdots & & \\ a_{k+1}(\lambda) & a_{k+2}(\lambda) & ... & a_k(\lambda) \end{bmatrix}$$

$$=\begin{bmatrix} a_{n-k+1}(\lambda) & a_{n-k+2}(\lambda) & ... & a_{n-k}(\lambda) \\ a_{n-2k+1}(\lambda) & a_{n-2k+2}(\lambda) & ... & a_{n-2k}(\lambda) \\ & \vdots & & \\ a_1(\lambda) & a_1(\lambda) & ... & a_n(\lambda) \end{bmatrix}=\left(a_{i+1,j}(\lambda)\right).$$

Take $\sigma^{-1}=\begin{pmatrix} 1 & 2 & 3 & ... & n \\ 1+k & 2+k & 3+k & ... & k \end{pmatrix}$; then $P_{\sigma^{-1}}(\lambda)=(P_{\sigma})^{-1}(\lambda)=\pi^{k}(\lambda)$.

Hence, $\pi(\lambda)A(\lambda)\pi^{-k}(\lambda)=\left(a_{i+1,j+k}(\lambda)\right)$

$$=\left(a_{i,j}(\lambda)\right)$$

$$=A(\lambda)$$

Hence, $\pi(\lambda)A(\lambda)=A(\lambda)\pi^{k}(\lambda)$.

Conversely, assume that $\pi(\lambda)A(\lambda)=A(\lambda)\pi^k(\lambda)$.

We have to prove that $A(\lambda)$ is a k-circulant polynomial matrix.

Now $\pi(\lambda)A(\lambda)=A(\lambda)\pi^k(\lambda)$

$$A(\lambda)=\pi^{-1}(\lambda)A(\lambda)\pi^k$$

Therefore, $A(\lambda)$ is a k-circulant polynomial matrix.

Example: 2.6

Let $A(\lambda)$ be a 2-circulant polynomial matrix of order 3.

That is, $A(\lambda)=\begin{bmatrix} 1+\lambda & 2+\lambda^2 & \lambda^3 \\ 2+\lambda^2 & \lambda^3 & 1+\lambda \\ \lambda^3 & 1+\lambda & 2+\lambda^2 \end{bmatrix}$

$$=A_0+A_1(\lambda)+A_2\lambda^2+A_3\lambda^3$$

Where $A_0=\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}, A_1=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, A_2=\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $A_3=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

and $\pi(\lambda)$ be a permutation matrix. That is, $\pi(\lambda)=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Now $\pi(\lambda)A(\lambda)=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1+\lambda & 2+\lambda^2 & \lambda^3 \\ 2+\lambda^2 & \lambda^3 & 1+\lambda \\ \lambda^3 & 1+\lambda & 2+\lambda^2 \end{bmatrix}$

$$=\begin{bmatrix} 2+\lambda^2 & \lambda^3 & 1+\lambda \\ \lambda^3 & 1+\lambda & 2+\lambda^2 \\ 1+\lambda & 2+\lambda^2 & \lambda^3 \end{bmatrix} \qquad (1)$$

Now $A(\lambda)\pi^2(\lambda)=\begin{bmatrix} 1+\lambda & 2+\lambda^2 & \lambda^3 \\ 2+\lambda^2 & \lambda^3 & 1+\lambda \\ \lambda^3 & 1+\lambda & 2+\lambda^2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$=\begin{bmatrix} 2+\lambda^2 & \lambda^3 & 1+\lambda \\ \lambda^3 & 1+\lambda & 2+\lambda^2 \\ 1+\lambda & 2+\lambda^2 & \lambda^3 \end{bmatrix} \qquad (2)$$

From (1) and (2), we get $\pi(\lambda)A(\lambda)=A(\lambda)\pi^2(\lambda)$.

Corollary: 2.7

Let $A(\lambda)$ and $B(\lambda)$ be k-circulant polynomial matrices .Then $A(\lambda)B^*(\lambda)$ is a 1-circulant polynomial matrix $A(\lambda)$.

Proof:

Given that $A(\lambda)$ and $B(\lambda)$ are k-circulant polynomial matrices.

By theorem (2.5), $A(\lambda)=\pi^*(\lambda)A(\lambda)\pi^k(\lambda)$ and $B(\lambda)=\pi^*(\lambda)B(\lambda)\pi^k(\lambda)$.

Now $A(\lambda)B^*(\lambda)=\left[\pi^*(\lambda)A(\lambda)\pi^k(\lambda)\right]\left[\pi^*(\lambda)B(\lambda)\pi^k(\lambda)\right]^*$

$$=\left[\pi^*(\lambda)A(\lambda)\pi^k(\lambda)\right]\left[\left(\pi^k\right)^*(\lambda)B^*(\lambda)\left(\pi^*\right)^*(\lambda)\right]$$

$$=\pi^*(\lambda)A(\lambda)\pi^k(\lambda)\left(\pi^*\right)^k(\lambda)B^*(\lambda)\pi(\lambda)$$

$$=\pi^*(\lambda)A(\lambda)B^*(\lambda)\pi(\lambda)$$

Hence, $A(\lambda)B^*(\lambda)$ is a 1-circulant polynomial matrix.

Example: 2.8

Let $A(\lambda)=\begin{bmatrix} -3\lambda & 1-\lambda^2 & \lambda+3\lambda^2 \\ 1-\lambda^2 & \lambda+3\lambda^2 & -3\lambda \\ \lambda+3\lambda^2 & -3\lambda & 1-\lambda^2 \end{bmatrix}=A_0+A_1\lambda+A_2\lambda^2$

where $A_0=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_1=\begin{pmatrix} -3 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 0 \end{pmatrix}, A_2=\begin{pmatrix} 0 & 1 & -3 \\ -1 & 3 & 0 \\ 3 & 0 & -1 \end{pmatrix}$ and

$$B(\lambda)=\begin{bmatrix} -1-\lambda^2 & -11\lambda & -13\lambda+\lambda^2 \\ -11\lambda & -13\lambda+\lambda^2 & -1-\lambda^2 \\ -13\lambda+\lambda^2 & -1-\lambda^2 & -11\lambda \end{bmatrix}=B_0+B_1\lambda+B_2\lambda^2$$

Where $B_0=\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$, $B_1=\begin{pmatrix} 0 & -11 & -13 \\ -11 & -13 & 0 \\ -13 & 0 & -11 \end{pmatrix}, B_2=\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

be a 2-circulant polynomial matrix of order 3.

$$A(\lambda)B^*(\lambda)=\begin{bmatrix} -3\lambda & 1-\lambda^2 & \lambda+3\lambda^2 \\ 1-\lambda^2 & \lambda+3\lambda^2 & -3\lambda \\ \lambda+3\lambda^2 & -3\lambda & 1-\lambda^2 \end{bmatrix}\begin{bmatrix} -1-\lambda^2 & -11\lambda & -13\lambda+\lambda^2 \\ -11\lambda & -13\lambda+\lambda^2 & -1-\lambda^2 \\ -13\lambda+\lambda^2 & -1-\lambda^2 & -11\lambda \end{bmatrix}$$

$$=\begin{bmatrix} \begin{matrix}(-3\lambda)(-1-\lambda^2)+(-1-\lambda^2) \\ (-11\lambda)+(\lambda+3\lambda^2)(-13\lambda+\lambda^2)\end{matrix} & \begin{matrix}(-3\lambda)(-11\lambda)+(1-\lambda^2) \\ (-13\lambda+\lambda^2)+(\lambda+3\lambda^2)(-1-\lambda^2)\end{matrix} & \begin{matrix}(-3\lambda)(-13\lambda+\lambda^2)+ \\ (1-\lambda^2)(-1-\lambda^2)+ \\ (\lambda+3\lambda^2)(-11\lambda)\end{matrix} \\ \begin{matrix}(1-\lambda^2)(-1-\lambda^2)+(\lambda+3\lambda^2) \\ (-11\lambda)+(-3\lambda)(-13\lambda+\lambda^2)\end{matrix} & \begin{matrix}(1-\lambda^2)(-11\lambda)+(\lambda+3\lambda^2) \\ (-13\lambda+\lambda^2)+(-3\lambda)(-1-\lambda^2)\end{matrix} & \begin{matrix}(1-\lambda^2)(-13\lambda+\lambda^2)+ \\ (\lambda+3\lambda^2)(-1-\lambda^2)+ \\ (-3\lambda)(-11\lambda)\end{matrix} \\ \begin{matrix}(\lambda+3\lambda^2)(-1-\lambda^2)+(-3\lambda) \\ (-11\lambda)+(1-\lambda^2)(-13\lambda+\lambda^2)\end{matrix} & \begin{matrix}(\lambda+3\lambda^2)(-11\lambda)+(-3\lambda) \\ (-13\lambda+\lambda^2)+(1-\lambda^2)(-1-\lambda^2)\end{matrix} & \begin{matrix}(\lambda+3\lambda^2)(-13\lambda+\lambda^2)+ \\ (-3\lambda)(-1-\lambda^2)+ \\ (1-\lambda^2)(-11\lambda)\end{matrix} \end{bmatrix}$$

$$=\begin{bmatrix} 3\lambda+3\lambda^3-11\lambda+11\lambda^3-13\lambda^2 & 33\lambda^2-13\lambda+\lambda^2+13\lambda^3-\lambda^4 & 39\lambda^2-3\lambda^3-1+\lambda^2-\lambda^2+\lambda^4 \\ -39\lambda^3+\lambda^3+3\lambda^4 & -\lambda-3\lambda^2-\lambda^3+3\lambda^4 & -11\lambda^2-33\lambda^3 \\ -1+\lambda^2-\lambda^2+\lambda^4 & -11\lambda+11\lambda^3-13\lambda^2+\lambda^3 & -13\lambda+\lambda^2+13\lambda^3-\lambda^4 \\ -11\lambda^2-33\lambda^3+39\lambda^2-3\lambda^3 & -39\lambda^3+3\lambda^4+3\lambda+3\lambda^3 & -\lambda-3\lambda^2-\lambda^3-3\lambda^4+33\lambda^2 \\ -\lambda-3\lambda^2-\lambda^3-3\lambda^4 & -11\lambda^2-33\lambda^3+39\lambda^2-3\lambda^3 & -13\lambda^2-39\lambda^3+\lambda^3+3\lambda^4 \\ -13\lambda+\lambda^2+13\lambda^3-\lambda^4 & -1+\lambda^2-\lambda^2+\lambda^4 & +3\lambda+3\lambda^3-11\lambda+11\lambda^3 \end{bmatrix}$$

$$=\begin{bmatrix} 3\lambda^4-24\lambda^3-13\lambda^2-8\lambda & -4\lambda^4+12\lambda^3+31\lambda^2-14\lambda & \lambda^4-36\lambda^3+28\lambda^2-1 \\ \lambda^4-36\lambda^3+28\lambda^2-1 & 3\lambda^4-24\lambda^3-13\lambda^2-8\lambda & -4\lambda^4+12\lambda^3+31\lambda^2-14\lambda \\ -4\lambda^4+12\lambda^3+31\lambda^2-14\lambda & \lambda^4-36\lambda^3+28\lambda^2-1 & 3\lambda^4-24\lambda^3-13\lambda^2-8\lambda \end{bmatrix}$$

Hence, $A(\lambda)B^*(\lambda)$ is a 1-circulant polynomial matrix.

Remark: 2.9

If $A(\lambda)$ is a k-circulant polynomial matrix, then $A(\lambda)\ A^*(\lambda)$ is a 1-circulant polynomial matrix.

Theorem: 2.10

If $A(\lambda)$ is a k-circulant polynomial matrix and $B(\lambda)$ is a h-circulant polynomial matrix then $A(\lambda)\ B(\lambda)$ is a kh-circulant polynomial matrix.

Proof:

If $A(\lambda)$ is a k-circulant polynomial matrix and $B(\lambda)$ is an h-circulant polynomial matrix then, by theorem (2.5),

$$\pi(\lambda)A(\lambda)=A(\lambda)\pi^k(\lambda) \text{ and } \pi(\lambda)B(\lambda)=B(\lambda)\pi^h(\lambda)$$

$$\begin{aligned} \text{Now } \pi(\lambda)\left[A(\lambda)B(\lambda)\right] &= \left[\pi(\lambda)A(\lambda)\right]B(\lambda) \\ &= \left[A(\lambda)\pi^k(\lambda)\right]B(\lambda) \\ &= A(\lambda)\pi^{k-1}(\lambda)\left[\pi(\lambda)B(\lambda)\right] \\ &= A(\lambda)\pi^{k-1}(\lambda)\left[B(\lambda)\pi^h(\lambda)\right] \\ &= A(\lambda)\pi^{k-2}(\lambda)\left[\pi(\lambda)B(\lambda)\right]\pi^h(\lambda) \\ &= A(\lambda)\pi^{k-2}(\lambda)\left[B(\lambda)\pi^h(\lambda)\right]\pi^h(\lambda) \\ &= \left[A(\lambda)\pi^{k-2}(\lambda)\right]B(\lambda)\pi^{2h}(\lambda) \end{aligned}$$

Keep this up for h times, leading to $\pi(\lambda)\left[A(\lambda)B(\lambda)\right]=\left[A(\lambda)\pi^{h-h}(\lambda)\right]\left[B(\lambda)\pi^{kh}(\lambda)\right]$

$$=A(\lambda)B(\lambda)\pi^{kh}(\lambda)$$

By theorem (2.5), $A(\lambda)\ B(\lambda)$ is a kh-circulant polynomial matrix.

Example: 2.11

$$\text{Let } A(\lambda)=\begin{bmatrix} -8+\lambda & 0 & 4+6\lambda^2 & 0 & \lambda \\ 0 & \lambda & -8+\lambda & 0 & 4+6\lambda^2 \\ 0 & 4+6\lambda^2 & 0 & \lambda & -8+\lambda \\ \lambda & -8+\lambda & 0 & 4+6\lambda^2 & 0 \\ 4+6\lambda^2 & 0 & \lambda & -8+\lambda & 0 \end{bmatrix}$$

be a 2-circulant polynomial matrix of order 5 and

$$B(\lambda)=\begin{bmatrix} 0 & 0 & 1-\lambda^2 & -\lambda+\lambda^2 & 0 \\ -\lambda+\lambda^2 & 0 & 0 & 0 & 1-\lambda^2 \\ 0 & 1-\lambda^2 & -\lambda+\lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda^2 & -\lambda+\lambda^2 \\ 1-\lambda^2 & -\lambda+\lambda^2 & 0 & 0 & 0 \end{bmatrix}$$

be a 2-circulant polynomial matrix of order 5

$$A(\lambda)B(\lambda)=\begin{bmatrix} \lambda-\lambda^3 & -6\lambda^4+\lambda^3+\lambda^2+4 & 6\lambda^4-7\lambda^3+12\lambda^2-3\lambda-8 & \lambda^3-9\lambda^2+8\lambda & 0 \\ -6\lambda^4+\lambda^3+\lambda^2+4 & 6\lambda^4-7\lambda^3+12\lambda^2-3\lambda-8 & \lambda^3-9\lambda^2+8\lambda & 0 & \lambda-\lambda^3 \\ 6\lambda^4-7\lambda^3+12\lambda^2-3\lambda-8 & \lambda^3-9\lambda^2+8\lambda & 0 & \lambda-\lambda^3 & -6\lambda^4+\lambda^3+\lambda^2+4 \\ \lambda^3-9\lambda^2+8\lambda & 0 & \lambda-\lambda^3 & -6\lambda^4+\lambda^3+\lambda^2+4 & 6\lambda^4-7\lambda^3+12\lambda^2-3\lambda-8 \\ 0 & \lambda-\lambda^3 & -6\lambda^4+\lambda^3+\lambda^2+4 & 6\lambda^4-7\lambda^3+12\lambda^2-3\lambda-8 & \lambda^3-9\lambda^2+8\lambda \end{bmatrix}$$

Hence, $A(\lambda)B(\lambda)$ is a 4-circulant polynomial matrix of order 5.

Theorem: 2.12

Let $A(\lambda)$ be a non-singular k-circulant polynomial matrix. Then $A^{-1}(\lambda)$ is a k^{-1} circulant polynomial matrix. ($A^{-1}(\lambda)$ is a polynomial matrix obtained the inverses of coefficient matrices)

Proof:

Since $A(\lambda)$ is non-singular and hence $k^{-1}(\lambda)$ exists.

Now from theorem (2.5), $\pi(\lambda)A(\lambda)=A(\lambda)\pi^k(\lambda)$ so that $A^{-1}(\lambda)\pi^{-1}(\lambda)=\pi^{-k}(\lambda)A^{-1}(\lambda)$

$$\pi(\lambda)A^{-1}(\lambda)\pi^{-1}(\lambda)\pi(\lambda)=\pi(\lambda)\pi^{-k}(\lambda)A^{-1}(\lambda)\pi(\lambda)$$

$$\pi(\lambda)A^{-1}(\lambda)=\pi^{-k+1}(\lambda)A^{-1}(\lambda)\pi(\lambda)$$

$$=\pi^{-k+1}(\lambda)A^{-1}(\lambda)\pi(\lambda)\pi^{-1}(\lambda)\pi(\lambda)\pi(\lambda)$$

$$=\pi^{-k+1}(\lambda)\left[A^{-1}(\lambda)\pi^{-1}(\lambda)\right]\pi^2(\lambda)$$

$$=\pi^{-k+1}(\lambda)\left[\pi^{-k}(\lambda)A^{-1}(\lambda)\right]\pi^2(\lambda)$$

$$=\pi^{-2k+1}(\lambda)A^{-1}(\lambda)\pi^2(\lambda)$$

Do these s times and we obtain $\pi(\lambda)A^{-1}(\lambda)=\pi^{-sk+1}(\lambda)A^{-1}(\lambda)\pi^s(\lambda)$

Put $s=k^{-1}$, we get $\pi(\lambda)A^{-1}(\lambda)=\pi^{-k^{-1}k+1}(\lambda)A^{-1}(\lambda)\pi^{k^{-1}}(\lambda)$

$$=A^{-1}(\lambda)\pi^{k^{-1}}(\lambda)$$

Therefore, $A^{-1}(\lambda)$ is a k^{-1} circulant polynomial matrix.

Example: 2.13

Le $A(\lambda)$ be a 2-circulant polynomial matrix of order 3.

That is, $A(\lambda)=\begin{pmatrix} 3+\lambda & 2-\lambda & -1+4\lambda \\ 2-\lambda & -1+4\lambda & 3+\lambda \\ -1+4\lambda & 3+\lambda & 2-\lambda \end{pmatrix}=A_0+A_1\lambda$

Where $A_0=\begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 3 \\ -1 & 3 & 2 \end{pmatrix}, A_1=\begin{pmatrix} 1 & -1 & -4 \\ -1 & 4 & 1 \\ 4 & 1 & -1 \end{pmatrix}$

$$\Rightarrow A_0^{-1} = \frac{-1}{52}\begin{pmatrix} -11 & -7 & 5 \\ -7 & 5 & -11 \\ 5 & -11 & -7 \end{pmatrix}, A_1^{-1} = \frac{-1}{70}\begin{pmatrix} -5 & 3 & -17 \\ 3 & -17 & -5 \\ -17 & -5 & 3 \end{pmatrix}$$

$$\Rightarrow A^{-1}(\lambda) = \begin{pmatrix} \frac{-11}{52}+\frac{5}{70}\lambda & \frac{7}{52}-\frac{3}{70}\lambda & \frac{-5}{52}+\frac{17}{70}\lambda \\ \frac{7}{52}-\frac{3}{70}\lambda & \frac{-5}{52}+\frac{17}{70}\lambda & \frac{11}{52}+\frac{5}{70}\lambda \\ \frac{-5}{52}+\frac{17}{70}\lambda & \frac{11}{52}+\frac{5}{70}\lambda & \frac{7}{52}-\frac{3}{70}\lambda \end{pmatrix}$$

Theorem: 2.14

$A(\lambda)$ is a k-circulant polynomial matrix if and only if $\left(A^{\dagger}\right)^{*}(\lambda)$ is a k^{-1} circulant polynomial matrix.

Proof:

Let $A(\lambda)$ be a k-circulant polynomial matrix. Then by theorem (2.5),

$$A(\lambda) = \pi^{-1}(\lambda)A(\lambda)\pi^{k}(\lambda)$$

Since $\pi(\lambda), \pi^{-1}(\lambda), \pi^{k}(\lambda)$ are unitary polynomial matrix.

Hence $A^{\dagger}(\lambda) = \left[\pi^{-1}(\lambda)A(\lambda)\pi^{k}(\lambda)\right]^{\dagger}$

$$= \left(\pi^{k}\right)^{\dagger}(\lambda)A^{\dagger}(\lambda)\left(\pi^{-1}\right)^{\dagger}(\lambda)$$

$$= \pi^{-k}(\lambda)A^{\dagger}(\lambda)\pi(\lambda)$$

$$\left(A^{\dagger}\right)^{*}(\lambda) = \left[\pi^{-k}(\lambda)A^{\dagger}(\lambda)\pi(\lambda)\right]^{*}$$

$$= \pi^{*}(\lambda)\left(A^{\dagger}\right)^{*}(\lambda)\left(\pi^{-k}\right)^{*}(\lambda)$$

$$= \pi^{-1}(\lambda)\left(A^{\dagger}\right)^{*}(\lambda)\pi^{k}(\lambda)$$

Therefore, $\left(A^{\dagger}\right)^{*}(\lambda)$ is a k-circulant polynomial matrix.

Conversely, let $\left(A^{\dagger}\right)^{*}$ be a k-circulant polynomial matrix.

That is, $\left(A^{\dagger}\right)^{*}(\lambda) = \pi^{-1}(\lambda)\left(A^{\dagger}\right)^{*}(\lambda)\pi^{k}(\lambda)$

$$\left(\left(A^{\dagger}\right)^{*}\right)^{\dagger}(\lambda) = \left[\pi^{-1}(\lambda)\left(A^{\dagger}\right)^{*}(\lambda)\pi^{k}(\lambda)\right]^{\dagger}$$

$$\left(\left(A^{\dagger}\right)^{\dagger}\right)^{*}(\lambda) = \left(\pi^{k}\right)^{\dagger}(\lambda)\left(\left(A^{\dagger}\right)^{*}\right)^{\dagger}(\lambda)\left(\pi^{-1}\right)^{\dagger}(\lambda)$$

$$A^{*}(\lambda) = \pi^{-k}(\lambda)\left(\left(A^{\dagger}\right)^{\dagger}\right)^{*}(\lambda)\pi(\lambda)$$

$$\left(A^{*}\right)^{*}(\lambda) = \left[\pi^{-k}(\lambda)A^{*}(\lambda)\pi(\lambda)\right]^{*}$$

$$\left(A^{*}\right)^{*}(\lambda) = \pi^{*}(\lambda)\left(A^{*}\right)^{*}(\lambda)\left(\pi^{-k}\right)^{*}(\lambda)$$

$$A(\lambda) = \pi^{-1}(\lambda)A(\lambda)\pi^{k}(\lambda)$$

Hence, $A(\lambda)$ is a k-circulant polynomial matrix.

Corollary: 2.15

If $A(\lambda)$ is a k-circulant polynomial matrix, then $A(\lambda)A^{\dagger}(\lambda)$ is a 1-circulant polynomial matrix.

Proof:

Given that $A(\lambda)$ is a k-circulant polynomial matrix. By theorem (2.5),

$$A(\lambda) = \pi^{-1}(\lambda) A(\lambda) \pi^k(\lambda)$$

Now $A(\lambda) A^{\dagger}(\lambda) = \left[\pi^{-1}(\lambda) A(\lambda) \pi^k(\lambda)\right]\left[\pi^{-1}(\lambda) A(\lambda) \pi^k(\lambda)\right]^{\dagger}$

$$= \pi^{-1}(\lambda) A(\lambda) \pi^k(\lambda) \left(\pi^k\right)^{\dagger}(\lambda) A^{\dagger}(\lambda) \left(\pi^{-1}\right)^{\dagger}(\lambda)$$

$$= \pi^{-1}(\lambda) A(\lambda) A^{\dagger}(\lambda) \pi(\lambda).$$

Hence, $A(\lambda)\ A^{\dagger}(\lambda)$ is a 1-circulant polynomial matrix.

Remark: 2.16

If $A(\lambda)$ is a k-circulant polynomial matrix, then $A(\lambda)\ A^{*}(\lambda)$ is a 1-circulant polynomial matrix.

Theorem: 2.17

If $A(\lambda)$ is a k-circulant polynomial matrix, then $A^{\dagger}(\lambda) = A^{*}(\lambda)\left[A(\lambda) A^{*}(\lambda)\right]^{\dagger}$

Proof:

If $A(\lambda)$ is a k-circulant polynomial matrix, then by remark (2.5), we have $A(\lambda)\ A^{*}(\lambda)$ is a 1-circulant polynomial matrix.

That is, $A(\lambda) A^{*}(\lambda) = \pi^{-1}(\lambda) A(\lambda) A^{*}(\lambda) \pi(\lambda)$

$$\left[A(\lambda) A^{*}(\lambda)\right]^{\dagger} = \left[\pi^{-1}(\lambda) A(\lambda) A^{*}(\lambda) \pi(\lambda)\right]^{\dagger}$$

$$= \pi^{\dagger}(\lambda) \left(A^{*}\right)^{\dagger}(\lambda) A^{\dagger}(\lambda) \left(\pi^{-1}\right)^{\dagger}(\lambda)$$

$$= \pi^{\dagger}(\lambda) \left(A^{*}\right)^{\dagger}(\lambda) A^{\dagger}(\lambda) \pi(\lambda)$$

$$A^{*}(\lambda)\left[A(\lambda) A^{*}(\lambda)\right]^{\dagger} = A^{*}(\lambda) \pi^{\dagger}(\lambda) \left(A^{*}\right)^{\dagger}(\lambda) A^{\dagger}(\lambda) \pi(\lambda)$$

$$= A^{*}(\lambda) \left(A^{*}\right)^{\dagger}(\lambda) \pi^{\dagger}(\lambda) A^{\dagger}(\lambda) \pi(\lambda)$$

$$= A^{\dagger}(\lambda) \pi^{\dagger}(\lambda) \pi(\lambda)$$

$$= A^{\dagger}(\lambda).$$

Conclusion

We have extended some properties of k-circulant matrices to k-circulant polynomial matrices. All other properties can also be extended in a similar way.

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ISSN: 1110-8703
Pages: 820 – 828
Volume: 5 Issue 2

Sums And Products Of Normal Circulant Polynomial Matrices

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Abstract: We Have Introduced Normal Circulant Polynomial Matrices As A Generalization Of Normal Polynomial Matrices. Sums, Products And Direct Product Of Normal Circulant Polynomial Matrices Are Investigated.

Keywords: Circulant Polynomial Matrices, Normal Circulant Polynomial Matrices.
AMS Classification: 15A09, 15A15, 15A57.

1. Introduction

Let $(a_1(\lambda), a_2(\lambda), \dots, a_n(\lambda))$ Be An Ordered N-Tuple Of Polynomials With Coefficients In The Field Of Complex Numbers, And Let Them Generate The Circulant Polynomial Matrix [1,3,4,8] Of Order N:

$$A(\lambda) = \begin{pmatrix} a_1(\lambda) & a_2(\lambda) & \dots & a_n(\lambda) \\ a_n(\lambda) & a_1(\lambda) & \dots & a_2(\lambda) \\ & & \vdots & \\ a_2(\lambda) & a_3(\lambda) & \dots & a_1(\lambda) \end{pmatrix} \qquad (1)$$

We Shall Often Denote This Circulant Polynomial Matrix As

$$A(\lambda) = circ(a_1(\lambda), a_2(\lambda), \dots, a_n(\lambda)) \qquad (2)$$

Let $A(\lambda)$ Be An $n \times n$ Polynomial Matrix And $\pi(\lambda) = circ(0,1,0,\dots,0)$. $A(\lambda)$ Is A Circulant Polynomial Matrix If And Only If

$$A(\lambda)\pi(\lambda) = \pi(\lambda)A(\lambda) \qquad (3)$$

In This Paper, We Defined A New Type Of Polynomial Matrix Called Normal Circulant Polynomial Matrix, Which Is A Generalization Of The Normalpolynomial Matrix. For This Class Of Matrices, We Investigate Thesums, Products And Direct Products.

Before Proceeding, We Introduce Some Notation Needed Throughout This Paper. Let $C_{n\times n}(\lambda)$ Denote The Set Of All $n \times n$ Polynomial Matrices Over The Complex Field C And $I_n(\lambda)$ Denote The

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Identity Polynomial Matrix Of Order N. For A Matrix $A(\lambda) \in C_{n\times n}(\lambda)$, We Denote The Transpose, The Adjoint And The Determinant Of $A(\lambda)$ By $A^T(\lambda)$, $A^*(\lambda)$ And $|A(\lambda)|$, Respectively.

2 Normal Circulant Polynomial Matrices

In This Section Some Of The Properties Of Normal Polynomial Matrices Are Extended To Normal Circulant Polynomial Matrices. Some Results Of Normal Matrices Found In $[2,5,6,7]$ Are Generalized To Normal Circulant Polynomial Matrices.

Definition 2.1

A Circulant Polynomial Matrix $A(\lambda)$ Is Called Normal Circulant Polynomial Matrix If $A(\lambda)A^*(\lambda) = A^*(\lambda)A(\lambda)$.

Example 2.2

Let $A(\lambda) = \begin{pmatrix} 1+\lambda+i & 1+2i\lambda \\ 1+2i\lambda & 1+\lambda+i \end{pmatrix} = A_0 + A_1\lambda$ Where The Coefficient Matrix Of $A(\lambda)$ Are $A_0 = \begin{pmatrix} 1+i & 1 \\ 1 & 1+i \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix}$

$$A(\lambda)A^*(\lambda) = \begin{pmatrix} 3+2\lambda+5\lambda^2 & 2+6\lambda \\ 2+6\lambda & 3+2\lambda+5\lambda^2 \end{pmatrix} = A^*(\lambda)A(\lambda)$$

Hence, $A(\lambda)$ Is A Normal Circulant Polynomial Matrix.

Theorem 2.3

If $A(\lambda)$ Is A Normal Circulant Polynomial Matrix And α Is A Complex Number, Then

(i) $A(\lambda)+\alpha I_n(\lambda)$ Is A Normal Circulant Polynomial Matrix.

(ii) $A(\lambda)-\alpha I_n(\lambda)$ Is A Normal Circulant Polynomial Matrix.

Proof

Given That $A(\lambda)$ Is A Normal Circulant Polynomial Matrix. We Have

$$A(\lambda)A^*(\lambda) = A^*(\lambda)A(\lambda)$$

By Using (3), We Have $A(\lambda) = \pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

Proof Of (i)

Now $\left[A(\lambda)+\alpha I_n(\lambda)\right]\left[A(\lambda)+\alpha I_n(\lambda)\right]^*$

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$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]^*$$
$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*+\left(\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right)^*\right]$$
$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)+\left(\pi_n^{-1}(\lambda)\right)^*I_n^*(\lambda)\pi_n^*(\lambda)\overline{\alpha}\right]$$
$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\overline{\alpha}\right]$$
$$=\left(\pi_n(\lambda)\right)^2A(\lambda)A^*(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2+\alpha\left(\pi_n(\lambda)\right)^2I_n(\lambda)A^*(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2+$$
$$\left(\pi_n(\lambda)\right)^2A(\lambda)I_n(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2\overline{\alpha}+\alpha\left(\pi_n(\lambda)\right)^2\left(I_n(\lambda)\right)^2\left(\pi_n^{-1}(\lambda)\right)^2\overline{\alpha}$$
$$=\pi_n(\lambda)\left[\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)$$
$$+\alpha\pi_n(\lambda)\left[\pi_n(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)$$
$$+\pi_n(\lambda)\left[\pi_n(\lambda)A(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)\overline{\alpha}$$
$$+\alpha\pi_n(\lambda)\left[\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)\overline{\alpha}$$
$$=\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)+\alpha\pi_n(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$
$$+\pi_n(\lambda)A(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\overline{\alpha}+\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\overline{\alpha}$$
$$=A(\lambda)A^*(\lambda)+\alpha I_n(\lambda)A^*(\lambda)+A(\lambda)I_n(\lambda)\overline{\alpha}+\alpha I_n(\lambda)\overline{\alpha}$$
$$=A^*(\lambda)A(\lambda)+\alpha I_n(\lambda)A^*(\lambda)+\overline{\alpha}I_n(\lambda)A(\lambda)+\alpha\overline{\alpha}I_n(\lambda)$$
$$=A^*(\lambda)A(\lambda)+\alpha I_n(\lambda)A^*(\lambda)+\overline{\alpha}I_n(\lambda)A(\lambda)+\alpha\overline{\alpha}I_n(\lambda)$$
$$=A^*(\lambda)\left[A(\lambda)+\alpha I_n(\lambda)\right]+\overline{\alpha}I_n(\lambda)\left[A(\lambda)+\alpha I_n(\lambda)\right]$$
$$=\left[A^*(\lambda)+\overline{\alpha}I_n(\lambda)\right]\left[A(\lambda)+\alpha I_n(\lambda)\right]$$
$$=\left[\left(A(\lambda)\right)^*+\left(\alpha I_n(\lambda)\right)^*\right]\left[A(\lambda)+I_n(\lambda)\alpha\right]$$
$$=\left[A(\lambda)+\alpha I_n(\lambda)\right]^*\left[A(\lambda)+\alpha I_n(\lambda)\right]$$

Thus, $A(\lambda)+\alpha I_n(\lambda)$ Is A Normal Circulant Polynomial Matrix.

Proof Of (ii)

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Now $\left[A(\lambda)-\alpha I_n(\lambda)\right]\left[A(\lambda)-\alpha I_n(\lambda)\right]^*$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)-\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)-\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]^*$$
$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)-\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*-\left(\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right)^*\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)-\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)-\left(\pi_n^{-1}(\lambda)\right)^*I_n^*(\lambda)\pi_n^*(\lambda)\overline{\alpha}\right]$$

$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)-\alpha\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)-\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\overline{\alpha}\right]$$
$$=\left(\pi_n(\lambda)\right)^2A(\lambda)A^*(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2-\alpha\left(\pi_n(\lambda)\right)^2I_n(\lambda)A^*(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2$$
$$-\overline{\alpha}\left(\pi_n(\lambda)\right)^2A(\lambda)I_n(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2+\alpha\overline{\alpha}\left(\pi_n(\lambda)\right)^2\left(I_n(\lambda)\right)^2\left(\pi_n^{-1}(\lambda)\right)^2$$
$$=\pi_n(\lambda)\left[\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)$$
$$-\alpha\pi_n(\lambda)\left[\pi_n(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)$$
$$-\overline{\alpha}\pi_n(\lambda)\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)$$
$$+\alpha\overline{\alpha}\pi_n(\lambda)\left[\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)$$
$$=\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)-\alpha\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$
$$-\overline{\alpha}\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\alpha\overline{\alpha}\pi_n(\lambda)I_n(\lambda)\pi_n^{-1}(\lambda)$$
$$=A(\lambda)A^*(\lambda)-\alpha A^*(\lambda)-\overline{\alpha}A(\lambda)+\alpha\overline{\alpha}I_n(\lambda)$$
$$=A^*(\lambda)A(\lambda)-\alpha A^*(\lambda)-\overline{\alpha}A(\lambda)+\alpha\overline{\alpha}I_n(\lambda)$$
$$=A^*(\lambda)A(\lambda)-\alpha A^*(\lambda)-\overline{\alpha}A(\lambda)+\alpha\overline{\alpha}I_n(\lambda)$$
$$=A^*(\lambda)\left[A(\lambda)-\overline{\alpha}I_n(\lambda)\right]-\overline{\alpha}I_n(\lambda)\left[A(\lambda)-\alpha I_n(\lambda)\right]$$
$$=\left[A^*(\lambda)-\overline{\alpha}I_n(\lambda)\right]\left[A(\lambda)-\alpha I_n(\lambda)\right]$$
$$=\left[\left(A(\lambda)\right)^*-\left(I_n(\lambda)\alpha\right)^*\right]\left[A(\lambda)-\alpha I_n(\lambda)\right]$$

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$$=\left[A(\lambda)-I_n(\lambda)\alpha\right]^*\left[A(\lambda)-\alpha I_n(\lambda)\right]$$
$$=\left[A(\lambda)-\alpha I_n(\lambda)\right]^*\left[A(\lambda)-\alpha I_n(\lambda)\right]$$

Hence, $A(\lambda)-\alpha I_n(\lambda)$ Is A Normal Circulant Polynomial Matrix.

Theorem 2.4

Let $A(\lambda)$ And $B(\lambda)$ Be Normal Circulant Polynomial Matrices And That $A(\lambda)B^*(\lambda)=B^*(\lambda)A(\lambda)$ And $A^*(\lambda)B(\lambda)=B(\lambda)A^*(\lambda)$. Then $A(\lambda)+B(\lambda)$ Is A Normal Circulant Polynomial Matrix.

Proof

Let $A(\lambda)$ And $B(\lambda)$ Be Normal Circulant Polynomial Matrices And That $A(\lambda)B^*(\lambda)=B^*(\lambda)A(\lambda)$ And $A^*(\lambda)B(\lambda)=B(\lambda)A^*(\lambda)$.

By Using (3), We Have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\left[A(\lambda)+B(\lambda)\right]\left[A(\lambda)+B(\lambda)\right]^*$$
$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]^*$$
$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*+\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^*\right]$$
$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]\left[\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$=\left[\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]\left[\left(\pi_n^{-1}(\lambda)\right)^*A^*(\lambda)\pi_n^*(\lambda)+\left(\pi_n^{-1}(\lambda)\right)^*B^*(\lambda)\pi_n^*(\lambda)\right]$$
$$=\left(\pi_n(\lambda)\right)^2A(\lambda)A^*(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2+\left(\pi_n(\lambda)\right)^2B(\lambda)A^*(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2+$$
$$+\left(\pi_n(\lambda)\right)^2A(\lambda)B^*(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2+\left(\pi_n(\lambda)\right)^2B(\lambda)B^*(\lambda)\left(\pi_n^{-1}(\lambda)\right)^2$$

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$$=\pi_n(\lambda)\left[\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)$$
$$+\pi_n(\lambda)\left[\pi_n(\lambda)B(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$+\pi_n(\lambda)\left[\pi_n(\lambda)A(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)$$
$$+\pi_n(\lambda)\left[\pi_n(\lambda)B(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\right]\pi_n^{-1}(\lambda)$$
$$=\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$
$$+\pi_n(\lambda)A(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)+\pi_n(\lambda)B(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)$$
$$=A(\lambda)A^*(\lambda)+B(\lambda)A^*(\lambda)+A(\lambda)B^*(\lambda)+B(\lambda)B^*(\lambda)$$
$$=A^*(\lambda)A(\lambda)+B(\lambda)A^*(\lambda)+A(\lambda)B^*(\lambda)+B^*(\lambda)B(\lambda)$$
$$=A^*(\lambda)\left[A(\lambda)+B(\lambda)\right]+B^*(\lambda)\left[A(\lambda)+B(\lambda)\right]$$
$$=\left[A^*(\lambda)+B^*(\lambda)\right]\left[A(\lambda)+B(\lambda)\right]$$
$$=\left[A(\lambda)+B(\lambda)\right]^*\left[A(\lambda)+B(\lambda)\right]$$

Hence, $A(\lambda)+B(\lambda)$ Is A Normal Circulant Polynomial Matrix.

Theorem 2.5

Let $A(\lambda)$ And $B(\lambda)$ Be Normal Circulant Polynomial Matrices And That $A(\lambda)B^*(\lambda)=B^*(\lambda)A(\lambda)$ And $A^*(\lambda)B(\lambda)=B(\lambda)A^*(\lambda)$. Then $A(\lambda)B(\lambda)$ Is A Normal Circulant Polynomial Matrix.

Proof

Let $A(\lambda)$ And $B(\lambda)$ Be Normal Circulant Polynomial Matrices And That $A(\lambda)B^*(\lambda)=B^*(\lambda)A(\lambda)$ And $A^*(\lambda)B(\lambda)=B(\lambda)A^*(\lambda)$.

By Using (3), We Have $A(\lambda)=\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)$.

$$\left[A(\lambda)B(\lambda)\right]\left[A(\lambda)B(\lambda)\right]^*$$
$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]^*$$
$$=\left[\pi_n(\lambda)A(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^*+\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\right]$$

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$$=\left[\pi_n(\lambda)A(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right]$$
$$\left[\left(\pi_n^{-1}(\lambda)\right)^* B^*(\lambda)\pi_n^*(\lambda)\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)\right]$$
$$=\pi_n(\lambda)A(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$
$$=\pi_n(\lambda)A(\lambda)B(\lambda)B^*(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)$$
$$=A(\lambda)B(\lambda)B^*(\lambda)A^*(\lambda)$$
$$=A(\lambda)B^*(\lambda)B(\lambda)A^*(\lambda)$$
$$=B^*(\lambda)A(\lambda)B(\lambda)A^*(\lambda)$$
$$=B^*(\lambda)A(\lambda)A^*(\lambda)B(\lambda)$$
$$=B^*(\lambda)A^*(\lambda)A(\lambda)B(\lambda)$$
$$=\left(A(\lambda)B(\lambda)\right)^*\left(A(\lambda)B(\lambda)\right)$$

Hence, $A(\lambda)B(\lambda)$ Is A Normal Circulant Polynomial Matrix.

Theorem 2.6

If $A(\lambda)$ And $B(\lambda)$ Are Normal Circulant Polynomial Matrices, Then So Is $A(\lambda)\otimes B(\lambda)$.

Proof

Given That $A(\lambda)$ And $B(\lambda)$ Are Normal Circulant Polynomial Matrices.

We Have To Prove That $A(\lambda)\otimes B(\lambda)$ Is A Circulant Polynomial Matrix

$$\left(A(\lambda)\otimes B(\lambda)\right)\left(A(\lambda)\otimes B(\lambda)\right)^*=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\otimes\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\otimes\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]^*$$
$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\otimes\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)^*\otimes\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)^*\right]$$

$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\otimes\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$\left[\left(\left(\pi_n^{-1}(\lambda)\right)^* A^*(\lambda)\pi_n^*(\lambda)\right)\otimes\left(\left(\pi_n^{-1}(\lambda)\right)^* B^*(\lambda)\pi_n^*(\lambda)\right)\right]$$
$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\otimes\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$\left[\left(\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)\otimes\left(\pi_n(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$=\left[\left(\pi_n(\lambda)A(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)\right]\otimes$$
$$\left[\left(\pi_n(\lambda)B(\lambda)\pi_n^{-1}(\lambda)\right)\left(\pi_n(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$=\left[\left(\pi_n(\lambda)A(\lambda)I_n(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right)\right]\otimes$$
$$\left[\left(\pi_n(\lambda)B(\lambda)I_n(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$=\left[\pi_n(\lambda)A(\lambda)A^*(\lambda)\pi_n^{-1}(\lambda)\right]\otimes$$
$$\left[\left(\pi_n(\lambda)B(\lambda)B^*(\lambda)\pi_n^{-1}(\lambda)\right)\right]$$
$$=\left(A(\lambda)A^*(\lambda)\right)\otimes\left(B(\lambda)B^*(\lambda)\right)$$
$$=\left(A^*(\lambda)A(\lambda)\right)\otimes\left(B^*(\lambda)B(\lambda)\right)$$
$$=\left(A^*(\lambda)\otimes B^*(\lambda)\right)\left(A(\lambda)\otimes B(\lambda)\right)$$
$$=\left(A(\lambda)\otimes B(\lambda)\right)^*\left(A(\lambda)\otimes B(\lambda)\right)$$

Hence, $A(\lambda)\otimes B(\lambda)$ Is A Normal Circulant Polynomial Matrix.

Conclusion

Some Of The Properties Of Normal Circulant Polynomial Matrices Are Discussed Here. All Other Properties Can Also Be Extended In A Similar Way.

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