PORTRAYAL OF TOPOGRAPHIES OF BEGUILING NUMBER PATTERNS AND RESTRICTED CATEGORIES OF DIOPHANTINE EQUATIONS

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By

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Under the Guidance of

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Since 1970

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CERTIFICATE

This is to certify that this thesis entitled **"PORTRAYAL OF TOPOGRAPHIES OF BEGUILING NUMBER PATTERNS AND RESTRICTED CATEGORIES OF DIOPHANTINE EQUATIONS"** in partial fulfillment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY** in **MATHEMATICS** is a record of original research work carried out by **Ms. SANDHYA P**, (Ref.No.25216/Ph.D.K1/Mathematics/PT(3-5)/Prov.Regn./January 2019) during her period of study from 2019 to 2022 under the Part-Time Programme at Urumu Dhanalakshmi College affiliated to Bharathidasan University, Tiruchirappalli, Tamil Nadu, India under my supervision and guidance and the thesis has not been submitted for the award of any other degree / Associationship / Fellowship or similar title of this or any other University.

Place : Tiruchirappalli **(Dr. V. Pandichelvi)**

Date : **Research Supervisor**

DECLARATION

I do hereby declare that the thesis entitled **"PORTRAYAL OF TOPOGRAPHIES OF BEGUILING NUMBER PATTERNS AND RESTRICTED CATEGORIES OF DIOPHANTINE EQUATIONS"** has been originally carried out by me during the period of my study from 2019 – 2022 in the department of Mathematics, Urumu Dhanalakshmi College affiliated to Bharathidasan University, Tiruchirappalli, Tamil Nadu under the guidance of **Dr. V. Pandichelvi, M. Sc., M. Phil., Ph. D.,** Research Supervisor, Assistant Professor, PG and Research Department of Mathematics, Urumu Dhanalakshmi College, Tiruchirappalli and this work has not been submitted elsewhere for any other Degree or Diploma or any other University.

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This is to certify that this thesis entitled **"PORTRAYAL OF TOPOGRAPHIES OF BEGUILING NUMBER PATTERNS AND RESTRICTED CATEGORIES OF DIOPHANTINE EQUATIONS"** of **Ms. SANDHYA P**, Part-time Ph.D. Research Scholar, **(Ref.No.25216/Ph.D.K1/Mathematics/PT(3-5)/Prov.Regn./January 2019)**, Department of Mathematics, Urumu Dhanalakshmi College, Tiruchirappalli – 19, has been examined. No sentence, equation, table, paragraph or section has been copied verbatim from previous work unless it is placed under quotation marks and duly referenced. The work presented is original and own work of the author. The thesis has been checked using **Ouriginal Plagiarism Checker** (copy of originality report attached) and found within limits as per the Ph. D. Regulations of the Bharathidasan University.

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CONTENTS

Chapter - I

Introduction

CHAPTER – I

Introduction

State of the art of the research topic:

The exploration of natural numbers and integers is the emphasis of the pure Mathematics discipline known as Number Theory. "The Queen of Mathematics" refers to Number Theory's position as the discipline's linchpin. The study of numbers [6, 27, 30, 32, 35, 38, 43, 52] provides a framework for identifying patterns and establishing the veracity of those patterns through the use of Number Theory. Number Theory is a mixture of both experimental and theoretical aspects. Issues are raised and viable solutions are proposed in the experimental component of the course. It is the objective of the theoretical part of the research to provide an argument that satisfies all of the issues raised [66, 67, 110, 111, 113, 129, 132, 136, 150, 151, 152].

Diophantus of Alexandria, the creator of Arithmetica and one of the most prominent later Greek Mathematicians, deserves particular mention. Diophantine equations, the most important of the many problems in this book, have been considered the most significant [8, 68]. These are equations in which the answers must be integers [3, 4, 10, 13, 18, 31, 34, 42, 48, 50, 60, 64, 81, 94, 98, 100]. As an illustration, Diophantus requested for two numbers, one of which was a square and the other a cube, such that the total of their squares was also a square itself. In contemporary symbols, he hunted numbers $x, y,$ and z such that $(x^2)^2 + (y^3)^2 = z^2$. Generating real numbers that meet this condition is straightforward (e.g., $x = \sqrt{2}, y = 1,$ and $z = \sqrt{5}$), but the constraint that solutions be integers brands the task more complex. Work of Diophantus

was incredibly influential on subsequent Mathematics [105, 107, 112, 120, 121, 128, 131, 133, 147, 155].

Due to the wide range of Diophantine equations, there exists a plethora of them [71, 74, 80, 82, 99, 154]. Diophantine equations have no uniform mechanism for determining whether a solution exists or finding all of them if they do. One of the most well-known and renowned Diophantine equations is the Fermat equation $x^d + y^d - z^d = 0$ [126, 142, 143]. $d = 2$ yields an infinite number of integer solutions, but $d = 3$ yields no positive integer solutions.

The equation $y^2 = x^3 + k$ for $k \in \mathcal{Z}$ is alluded to as Mordell's equation due to Mordell's deep passion in it [56, 137]. Mordell [109] established in 1920 that the equation $y^2 = x^3 + k$ has an infinite number of integral solutions for any $k \in \mathcal{Z}$. Michael A. Bennett and Amir Ghadermarzi [11] cast-off the traditional link between Mordell and cubic Thue equations to solve the Diophantine problem $y^2 = x^3 + k$ for all non-zero integers k with $|k| \leq 10^7$.

Since prehistoric days, mastering the equations with variables as exponents has glinted the inquisitiveness of countless Mathematicians [1,7,17, 19, 20-25, 44, 55, 63, 65, 90, 91, 97]. J. L. Brenner and Lorraine L. Foster [14] explored several Diophantine exponential equations and derived conclusions. Maohua Le, Reese Scott, and Robert Styer [93] discussed many unresolved problems and related studies involving positive integer solutions to the ternary exponential Diophantine equation [106, 127, 130, 134, 135, 138-141, 144, 145, 146].

A Diophantine l-tuple is a set of l unique positive integers that has the feature that the product of any two of its distinct members plus 1 is a square. Fermat discovered

the world's first Diophantine quadruple in the integers $(1, 3, 8, 120)$. Diophantus discovered the first example of a rational Diophantine quadruple: $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$ [9, 16, 40, 41, 45, 47, 49, 53, 54, 57]. Certain of the most prominent Mathematicians of the history, such as Diophantus, Fermat, and Euler, as well as some contemporary Mathematicians, such as Fields Medalist Alan Baker, have made significant contributions to issues involving Diophantine l-tuples [58, 59, 61, 62, 102, 115, 118, 119, 123, 149], yet many of these problems remain unsolved.

A Number pattern is a form of arithmetic pattern that is often seen. Number patterns are a series of numbers that are sorted in a certain way according to a set of rules. Mathematics is extremely significant when it can assist you in making predictions, and Number patterns are all about making predictions. In Mathematics, dealing with Number patterns leads straight to the notion of functions, which is a structured representation of the interactions between multiple variables. It is also crucial to be able to recognize patterns in numbers while problem-solving [76, 104].

Until recently, the importance of Fibonacci's work in Mathematics was generally unnoticed. Modern Mathematicians are familiar with his work mostly because of the Fibonacci sequence he devised. The first recursive number series is made up of the numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, and 55, where each number is the sum of its two previous counterparts. Robert Simson, a Mathematician in 1753 at Glasgow University, discovered that when numbers became larger, the ratio between them neared the golden number, which is 1.6180 ... or $(1 + \sqrt{5})/2$. As early as the 19th century, Edouard Lucas coined the term "Fibonacci sequence" and scientists began to discover such

sequences across the natural world, such as the spirals of sunflower heads, pine cones and the male bee [33, 39, 46, 69, 70, 78, 79, 88, 96, 103, 122, 148].

As a result of the investigation of Pell's equation $x^2 - dy^2 = (-1)^n$, where d is a positive non-square integer, Pell numbers were called after the English Mathematician John Pell (1611–1685). On the other hand, Pell–Lucas numbers are called after him and Lucas, despite the fact that neither of them was involved with them in any way. Pell and Pell–Lucas numbers are Mathematical twins, much as Fibonacci and Lucas numbers are; they are both widespread and have a number of characteristics in common with one another [15, 26, 28, 29, 37, 51, 72, 73, 75, 85, 89, 95, 101, 114, 116, 117, 125, 153].

Congruence techniques are a valuable tool for calculating the number of solutions to a Diophantine equation. When applied to the simplest Diophantine equation, $ax + by = c$, where $a, b,$ and c are nonzero integers, these approaches demonstrate that the equation has either no solutions or infinitely many solutions, depending on whether the greatest common divisor (GCD) of a and b divides c: if it does not, there are no solutions; if it does, there are infinitely many solutions, which constitute a one-parameter family of solutions [77, 86].

In the 20th century, there was a surge in the field of Number Theory. In addition to classical and analytic Number Theory, researchers are currently exploring specific subdisciplines such as algebraic Number Theory, geometric Number Theory, and combinatorial Number Theory [2, 5]. Concepts got increasingly abstract, while the methodologies used to implement them because it was more sophisticated.

Undoubtedly, Fermat's greatest ambitions have been surpassed by the scope of the topic [108].

In the mid-twentieth century, Number Theory was regarded as the purest area of study in Mathematics, with no concrete applications in the real world. The emergence of digital communication and digital computers emphasized that Number Theory may bring surprising solutions to real-world issues [83, 87]. Factoring big numbers, finding primes, testing hypotheses, and resolving numerical problems previously believed impossible have all been made possible by breakthroughs in computer technology during the last several decades.

Furthermore, practical issues involving splicing of telephone lines have been resolved by using techniques of basic Number Theory [92]. Many more fascinating applications may be found in the book Number Theory and the Periodicity of Matter [12], which has a large number of additional examples. Finally, new and interesting applications of Number Theory include cryptography [84], coding theory, chemistry [36, 124] and random number generation among other things. These areas are developing at a breakneck pace as a result of the widespread use of computers, and their significance is growing all the time.

Since the theory of numbers has existed since the dawn of Mathematics, it is both timeless and up-to-the-minute. Because of its apparent (sometimes deceptive) simplicity and seductive beauty, it retains its interest. Because it has such a long and glorious history, Number Theory has rightfully been referred to as "The Queen of Mathematics," in the words of Gauss.

Objective and scope of research work:

The novel variations of patterns of special numbers, as well as fascinating relationships among them by using congruences and divisibility, are being investigated. Aside from that, processes for obtaining infinitely large number of non-zero integer solutions in Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers to some quadratic Diophantine equations are discussed.

Also, it is proved that there exists finite number of integer solutions or no solutions to some Mordell type Diophantine equations and exponential Diophantine equations that include Prime numbers and natural numbers are being explored as well. Furthermore, the application of linear Diophantine equations with certain restrictions for finding molecular formulae of chemical substances are investigated.

Results and discussion:

This doctoral thesis has nine Chapters. **Chapter I** delivers an overview of the history and literature required to analyze the different types of problems and their integral solutions in Chapters II through IX, which include number patterns and their related properties.

In **Chapter II**, new sequences and their characteristics are explored in two sections.

Section 2.1

Manifestation of Two Tremendous Sequences Cheldhiya and Cheldhiya Companion Sequences

Section 2.2

Invention of Four Novel Sequences and their Properties

Section 2.1 deals with the general solution to the Pell equation $x^2 - dy^2 = \pm 1$ for some particular positive values of d and are developed as Cheldhiya and Cheldhiya Companion sequences. Based on these sequences some interesting results are provided.

In **Section 2.2**, four disparate sequences and their recurrence relations named as Pan-San, Pan-San Buddy, Pan-San Comrade and Pan-San Mate sequences are established by utilizing the generalized solutions (x, y) to the universal equation called as Pell equation for two non-zero square-free integers $d = k^2 + 2, d = k^2 - 2$ where $k \in \mathcal{N} - \{1\}$. Also, the general formulae and few theorems are proved involving such sequences for distinct values of d and can analyze the corresponding results.

Chapter III comprehends certain patterns of Diophantine Triples, incorporating some of the sequences that were developed in the previous chapter.

Section 3.1

The Patterns of Diophantine Triples Engross Cheldhiya Companion Sequence with Inspiring Properties.

Section 3.2

Demonstration of Two Disparate Structures of Integer Triples Concerning Pan-San and Pan-San Comrade Numbers.

In **Section 3.1**, the following patterns of Diophantine triples

$\{x_{2m},\ x_{2m+2},\ x_{2m} + 2x_{2m+1} + x_{2m+2}\}$,

$\{x_{2m+2},\ x_{2m} + 2x_{2m+1} + x_{2m+2},\ 3x_{2m} + (2 + 2P_{k+1})x_{2m+1} + 2x_{2m+2}\}$,

$\{x_{2m} + 2x_{2m+1} + x_{2m+2}, 3x_{2m} + (2 + 2P_{k+1})x_{2m+1} + 2x_{2m+2}, 6x_{2m} +$

$(10 + 2P_{k+1})x_{2m+1} + x_{2m+2}\}$,

$\{3x_{2m} + (2 + 2P_{k+1})x_{2m+1} + 2x_{2m+2},\quad 6x_{2m} + (10 + 2P_{k+1})x_{2m+1} + 7x_{2m+2}, 13x_{2m} +$

$(26 + 4P_{k+1})x_{2m+1} + 21x_{2m+2}\}$ …

comprising the Cheldhiya companion sequence with the property $D\left(-(k^2+1)\right)$ is studied.

Also, a pattern of Diophantine triples

$\{x_{2m-1},\ x_{2m+1}, x_{2m-1} + 2x_{2m} + x_{2m+1}\}$,

$\{x_{2m+1},\ x_{2m-1} + 2x_{2m} + x_{2m+1},\ x_{2m-1} + 4x_{2m} + 4x_{2m+1}\}$,

$\{x_{2m-1} + 2x_{2m} + x_{2m+1},\ x_{2m-1} + 4x_{2m} + 4x_{2m+1},\ 4x_{2m-1} + 12x_{2m} + 9x_{2m+1}\}$,

$\{x_{2m-1} + 4x_{2m} + 4x_{2m+1},\ 4x_{2m-1} + 12x_{2m} + 9x_{2m+1},\ 9x_{2m-1} + 30x_{2m} + 25x_{2m+1}\}$,…

involving Cheldhiya companion sequence with the property $D(k^2+1)$ is obtained.

In **Section 3.2**, two different patterns of triples

$\{C_{n-,k},\ C_{n+1,k}, 2k^2C_{n,k}\}$, $\{C_{n+1,k}, 2k^2C_{n,k}, 3C_{n+1,k} + 2C_{n,k}(k^2-1)\}$,

$\{2k^2C_{n,k}, 3C_{n+1,k} + 2C_{n,k}(k^2-1), 2C_{n-1,k} + 7C_{n+1,k} + 4C_{n,k}(k^2-2)\}$,

$\{3C_{n+1,k} + 2C_{n,k}(k^2-1), 2C_{n-1,k} + 7C_{n+1,k} + 4C_{n,k}(k^2-2), 6C_{n-1,k} +$

$22C_{n+1,k} + 6C_{n,k}(k^2-4)\}$, etc

and

$\{R_{n-,k},\ R_{n+1,k}, 2k^2R_{n,k}\}$, $\{R_{n+1,k}, 2k^2R_{n,k}, 3R_{n+1,k} + 2R_{n,k}(k^2+1)\}$,

$\{2k^2R_{n,k}, 3R_{n+1,k} + 2R_{n,k}(k^2+1), 2R_{n-1,k} + 7R_{n+1,k} + 4R_{n,k}(k^2+2)\}$,

$\{3R_{n+1,k} + 2R_{n,k}(k^2+1), 2R_{n-1,k} + 7R_{n+1,k} + 4R_{n,k}(k^2+2),$

$6R_{n-1,k} + 22R_{n+1,k} + 6R_{n,k}(k^2+4)\}$, etc

entailing Pan-San and Pan-San Comrade sequences respectively whereas the multiplication of two basics raised by k^2 is a perfect square where $k \in \mathcal{N} - \{1\}$ are engendered.

Chapter IV enlightens the artwork of some integer quadruples and quintuples with exclusive properties.

Section 4.1

Fabrication of Gorgeous Integer Quadruple.

Section 4.2

Incomparable Integer Quintuple in Arithmetic Progression with Prominent Condition.

In **Section 4.1**, Three alternative processes are used to examine the quadruple (a, b, c, d) in order to confirm that the total of any three of them is a cubical integer. These approaches are outlined below.

(i) $a = 8(6m^3 - 9mn^2)^3 + 6(18m^2n - 3n^3)^2(6m^3 - 9mn^2) + 5(18m^2n - 3n^3)^3$

$b = (6m^3 - 9mn^2)^3 + 6(6m^3 - 9mn^2)^2(18m^2n - 3n^3)$

$+12(6m^3 - 9mn^2) \times (18m^2n - 3n^3)^2 - 2(18m^2n - 3n^3)^3$

$c = -8(6m^3 - 9mn^2)^3 - 6(6m^3 - 9mn^2)(18m^2n - 3n^3)^2 + 5(18m^2n - 3n^3)^3$

$d = -(6m^3 - 9mn^2)^3 + 6(6m^3 - 9mn^2)^2(18m^2n - 3n^3)$

$-12(6m^3 - 9mn^2) \times (18m^2n - 3n^3)^2 - 2(18m^2n - 3n^3)^3$

(ii) $a = 8(2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 + 6(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times$

$(8m^3 + 6m^2n - 12mn^2 - n^3)^2 + 5(8m^3 + 6m^2n - 12mn^2 - n^3)^3$

$b = (2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 + 6(2m^3 - 12m^2n - 3mn^2 + 2n^3)^2 \times$

$(8m^3 + 6m^2n - 12mn^2 - n^3) + 12(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times$

$(8m^3 + 6m^2n - 12mn^2 - n^3)^2 - 2(8m^3 + 6m^2n - 12mn^2 - n^3)^3$

$c = -8(2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 - 6(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times$

$(8m^3 + 6m^2n - 12mn^2 - n^3)^2 + 5(8m^3 + 6m^2n - 12mn^2 - n^3)^3$

$$d = -(2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 + 6(2m^3 - 12m^2n - 3mn^2 + 2n^3)^2 \times (8m^3 + 6m^2n - 12mn^2 - n^3) - 12(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times (8m^3 + 6m^2n - 12mn^2 - n^3)^2 - 2(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

(iii) $$a = 1728m^9 + 3888m^7n^2 + 1080m^6n^3 + 3240m^5n^4 + 1620m^4n^5 + 1188m^3n^6 + 810m^2n^7 + 162mn^8 + 135n^9$$

$$b = 216m^9 + 1296m^8n + 2916m^7n^2 + 1512m^6n^3 + 4050m^5n^4 + 324m^4n^5 + 1971m^3n^6 - 162m^2n^7 + 324mn^8 - 54n^9$$

$$c = -1728m^9 - 3888m^7n^2 + 1080m^6n^3 - 3240m^5n^4 + 1620m^4n^5 - 1188m^3n^6 + 810m^2n^7 - 162mn^8 + 135n^9$$

$$d = -216m^9 + 1296m^8n - 2916m^7n^2 + 1512m^6n^3 - 4050m^5n^4 + 324m^4n^5 - 1971m^3n^6 - 162m^2n^7 - 324mn^8 - 54n^9$$

In **Section 4.2**, an elegant integer quintuple(p, q, r, s, t) in three different ways where the components make ensure in arithmetic progression with the conjecture that the sum of any three consecutive elements designates a perfect square is recognized.

(i) $(p, q, r, s, t) = \{48(U^2 + V^2)^2 - 384UV(V^2 - U^2), 48(U^2 + V^2)^2 - 192UV(V^2 - U^2), 48(U^2 + V^2)^2, 48(U^2 + V^2)^2 + 192UV(V^2 - U^2), 48(U^2 + V^2)^2 + 384UV(V^2 - U^2)\}$

(ii) $(p, q, r, s, t) = \{30000(U^2 + V^2)^2 + 9600(6U^4 + 6V^4 + 7U^3V - 7UV^3 - 36U^2V^2), 30000(U^2 + V^2)^2 + 4800(6U^4 + 6V^4 + 7U^3V - 7UV^3 - 36U^2V^2), 30000(U^2 + V^2)^2, 30000(U^2 + V^2)^2 - 4800(6U^4 + 6V^4 + 7U^3V - 7UV^3 - 36U^2V^2), 30000(U^2 + V^2)^2 - 9600(6U^4 + 6V^4 + 7U^3V - 7UV^3 - 36U^2V^2)\}$

(iii) $(p,q,r,s,t) = \{12(m^2+n^2)^2 - 96mn(n^2-m^2), 12(m^2+n^2)^2 - 48mn(n^2-m^2), 12(m^2+n^2)^2, 12(m^2+n^2)^2 + 48mn(n^2-m^2), 2(m^2+n^2)^2 + 96mn(n^2-m^2)\}$

Chapter V establishes the art of sums, congruence relations and divisibility properties of Pell and Pell Lucas Numbers

Section 5.1

Sums and Congruences of Pell and Pell-Lucas Numbers

Section 5.2

Divisibility Properties of Pell and Pell-Lucas Numbers

In **Section 5.1**, numerous innovative identities about Pell and Pell-Lucas numbers empower to deliver certain congruence relations for the above stated numbers are reflected in the following theorems.

Theorem 5.2

If $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$, then

$$Q_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i Q_{mi+k}$$

and $$P_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i P_{mi+k}$$

Corollary 5.2.1

$$Q_{2mn+k} \equiv (-1)^{(m+1)n} Q_k \ (mod\ Q_m) \qquad (5.6)$$

and $$P_{2mn+k} \equiv (-1)^{(m+1)n} P_k \ (mod\ Q_m) \qquad (5.7)$$

for every $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$.

Theorem 5.3

If $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$, then

$$Q_{2mn+k} = (-1)^{mn}\left\{\sum_{i=o}^{\lfloor n/2 \rfloor}\binom{n}{2i} 8^i\, {P_m}^{2i} Q_{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2 \rfloor}\binom{n}{2i+1}(-1)^{mi} 8^i\, {P_m}^{2i+1} P_{2mi+m+k}\right\} \text{ and}$$

$$P_{2mn+k} = (-1)^{mn}\left\{\sum_{i=o}^{\lfloor n/2 \rfloor}\binom{n}{2i} 8^i\, {P_m}^{2i} P_{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2 \rfloor}\binom{n}{2i+1}(-1)^{mi} 8^i\, {P_m}^{2i+1} Q_{2mi+m+k}\right\}$$

Corollary 5.3.1

If $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$, then

$$Q_{2mn+k} \equiv (-1)^{mn} Q_k \ (mod\ P_m) \qquad (5.8)$$

and $$P_{2mn+k} \equiv (-1)^{mn} P_k \ (mod\ P_m) \qquad (5.9)$$

In **Section 5.2**, divisibility properties of Pell and Pell-Lucas numbers are revealed by means of the derived congruence relations in Section 5.1.

Theorem 5.4

The necessary and sufficient conditions for $Q_m | Q_n$ are

i. $m|n$ and

ii. $\frac{n}{m}$ is an odd integer

for all $m, n \in N$ and $m \geq 2$.

Theorem 5.5

Let $m, n \in N$ and $m \geq 2$. Then $Q_m | P_n$ if and only if $m|n$ and $\frac{n}{m}$ is an even integer.

Theorem 5.6:

For all $m, n \in N$ and $m \geq 3$, $P_m | P_n$ if and only if $m|n$.

Chapter VI deals with several Quadratic Diophantine equation with solutions as familiar Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers.

Section 6.1

Assessment of Solutions in Pell and Pell – Lucas Numbers to Disparate Polynomial Equations of Degree Two

Section 6.2

Conception of Positive Integer Solutions Relating Jacobsthal and Jacobsthal – Lucas Numbers to Restricted Number of Quadratic Equations with Double Variables

In **Section 6.1**, the solutions in Pell and Pell-Lucas numbers for the following explicit polynomial equations of degree two in two variables are derived.

(i) $x^2 - 2xy - y^2 = \pm k$ when $k = 1, 8$

(ii) $x^2 - 6xy + y^2 = \pm l$ when $l = 4, 32$

(iii) $x^2 - 2xy - y^2 \pm x = 0$

(iv) $x^2 - 2xy - y^2 \pm y = 0$

(v) $x^2 - 2xy - y^2 \pm 8x = 0$

(vi) $x^2 - 6xy + y^2 \pm 4x = 0$ and

(vii) $x^2 - 6xy + y^2 \pm 32x = 0$

In **Section 6.2**, sequences of non-negative integer solutions encircling Jacobsthal and Jacobsthal-Lucas numbers for restricted number of quadratic equations with double variables by utilizing the appropriate erections connecting these two numbers and the concepts of divisibility are investigated.

(i) $X^2 - XY - 2Y^2 = \pm C$

(ii) $X^2 - 5XY \pm 4Y^2 = \pm C$

(iii) $X^2 - XY - 2Y^2 \pm CX = 0$

(iv) $X^2 - XY - 2Y^2 \pm CY = 0$

(v) $X^2 - 5XY - 4Y^2 \pm CX = 0$

(vi) $X^2 - 5XY - 4Y^2 \pm CY = 0$

(vii) $X^2 - XY - 2Y^2 = \pm 9C$

(viii) $X^2 - XY - 2Y^2 \pm 9CX = 0$

(ix) $X^2 - XY - 2Y^2 \pm 9CY = 0$

(x) $X^2 - 5XY + 4Y^2 = \pm 9C$

(xi) $X^2 - 5XY + 4Y^2 \pm 9CX = 0$ and

(xii) $X^2 - 5XY + 4Y^2 \pm 9CY = 0$

where C is a constant denoting some powers of the number 2.

Chapter VII discovers solutions to certain Mordell Type Diophantine Equations.

Section 7.1

Methodology of Proving No Solutions to Three Categories of Mordell Diophantine Equations $B^2 = A^3 + K, K = U^3 - V^2, U^3 - 2V^2\, 2V^2 + U^3, U, V \in \mathcal{Z}$.

Section 7.2

Attesting finite number of integer solutions or no integer solutions to four Mordell Kinds Equations $Y^2 = X^3 + C, C = \pm 9,36, -16$.

In **Section 7.1**, an unsurpassed Diophantine equation $B^2 = A^3 + K$ for three distinct values of K is studied and it is exposed that no integer solution arises by using some classical congruence relations and Legendre symbols.

In **Section 7.2**, four groups of Mordell equations $Y^2 = X^3 + C$, $C = \pm 9, -16,36$ are preferred and exposed that two of the equations

$Y^2 = X^3 - 9$, $Y^2 = X^3 - 16$ among them have no integer solutions and the lingering two equations $Y^2 = X^3 + 9, Y^2 = X^3 + 36$ have partial integer solutions by mostly focused on the ideas of properties of congruences.

Chapter VIII deals with Exponential Diophantine Equations in three sections 8.1, 8.2 and 8.3.

Section 8.1

Exploration of solutions for an Exponential Diophantine Equation $p^x + (p+1)^y = z^2$

Section 8.2

Tactics of achieving non-negative integer solutions to an Exponential Equation with Base as Natural Numbers $n^x + (n+1)^y = z^2$.

Section 8.3

Investigation of Solutions to an Exponential Diophantine Equation ${p_1}^x + {p_2}^y + {p_3}^z = M^2$

Section 8.1 lists the infinite numbers of integer solutions of the equation $p^x + (p+1)^y = z^2$ where p is a prime number by using the basic concept of Mathematics and the theory of divisibility.

In **Section 8.2**, an exclusive exponential Diophantine equation $n^x + (n+1)^y = z^2$ where $n \in \mathcal{N}$, the set of all-natural numbers are examined for all choices of two exponents x and y such that their sum $x + y = 1,2,3,4$ in order to discover integral solutions by using inspiring fundamental concepts of Mathematics.

In **Section 8.3**, the Diophantine equation $p_1{}^x + p_2{}^y + p_3{}^z = M^2$ is established and results are analyzed for the prime triplets where p is of the form $4n + 1$ or $4n + 3$ and the powers of primes are either 1 or 2.

In **Chapter IX**, the application of linear Diophantine equation in chemistry is displayed.

Section 9.1

Usage of Linear Diophantine Equation in the Resolution of Molecular Formulae for Various Chemical Substances.

Section 9.1 examines the application of the linear Diophantine equation with certain constraints in the determination of the chemical molecular formulas for three distinct compounds and its effectiveness.

Chapter – II

Exploration of New Sequences and their Characteristics

CHAPTER – II

Exploration of New Sequences and their Characteristics

This chapter consists of two sections 2.1 and 2.2

In Section 2.1, two peculiar sequences named as Cheldhiya sequence and Cheldhiya Companion sequence are discovered. General formula for Cheldhiya sequence is enumerated by using the special property called as normalization of the matrix. Also, few theorems involving these sequences are elucidated.

In Section 2.2, four novel sequences named as Pan-San, Pan-San Buddy, Pan-San Comrade and Pan-San Mate sequences are discovered by employing the general solutions (x, y) to the worldwide equation called as Pell equation for two discrete non-zero square-free integers $d = k^2 + 2$, $d = k^2 - 2$ where $k \in \mathcal{N} - \{1\}$. Also, the recurrence relations, the general formulae for all sequences and some theorems interrelated to all these sequences are invented by exploiting basic perceptions of matrices.

2.1 Manifestation of Two Tremendous Sequences Cheldhiya and Cheldhiya Companion Sequences

The Pell Equation is a quadratic Diophantine equation of the form $x^2 - dy^2 = 1$ where d is a positive square-free integer. The equation $x^2 - dy^2 = 1$ has infinitely many solutions whereas the negative Pell equation $x^2 - dy^2 = -1$ does not always have a solution. In this section, the process of developing new sequences named as Cheldhiya and Cheldhiya Companion sequences by using sequence of solutions to the equation $x^2 - dy^2 = \pm 1$ for certain d and few theorems based on these sequences are explained as follows.

The y values of the equation $x^2 - dy^2 = \pm 1$ for certain non-zero square-free integer d can be generally sequenced as $0, 1, 2k, 4k^2 + 1, 8k^3 + 4k$, etc called as **Cheldhiya sequence**. The n[th] term of this sequence is generalized by the recurrence relation

$$y_n = 2ky_{n-1} + y_{n-2}, k = 1,2,3, \dots, n \geq 1.$$

with initial values $y_0 = 0, y_1 = 1$.

The x values of the equation $x^2 - \mathrm{d}y^2 = \pm 1$ for certain non-zero square-free integer d can be generally sequenced as $1, k, 2k^2 + 1, 4k^3 + 3k$, etc called as **Cheldhiya Companion Sequence.** The n[th] term of this sequence is generalized by the recurrence relation

$$x_n = 2kx_{n-1} + x_{n-2}, k = 1,2,3, \dots, n \geq 1$$

with initial values $x_0 = 1, x_1 = k$.

Here k and d can be related as $d = k^2 + 1$ where k indicates the order of the sequence while n indicates the number of terms in the k^{th} order sequence.

Define the Cheldhiya sequence matrix as

$$\mathbb{y} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}$$

Now,

$$\mathbb{y}\begin{pmatrix} y_1 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2k \\ 1 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}$$

Also,

$$\mathbb{y}\begin{pmatrix} y_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2k \\ 1 \end{pmatrix} = \begin{pmatrix} 4k^2+1 \\ 2k \end{pmatrix} = \begin{pmatrix} y_3 \\ y_2 \end{pmatrix}$$

In general,

$$\mathbb{y}\begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ y_n \end{pmatrix}$$

Theorem 2.1

If $\mathbb{y} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}$ is a Cheldhiya sequence matrix, then

(i) $\mathbb{y}^n = \begin{pmatrix} y_{n+1} & y_n \\ y_n & y_{n-1} \end{pmatrix}$ for all $n \in \mathcal{Z}^+$.

(ii) $\mathbb{y}^n \begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} y_{k+n} \\ y_{k+n-1} \end{pmatrix}$ for all $n, k \in \mathcal{Z}^+$.

Proof:

This theorem can be proved by using the principle of mathematical induction on n.

(i) Since $y_0 = 0, y_1 = 1, y_2 = 2k$, the exclusive Cheldhiya sequence matrix is interpreted by

$$\mathbb{y} = \begin{pmatrix} y_2 & y_1 \\ y_1 & y_0 \end{pmatrix}$$

Therefore, the theorem is valid for $n = 1$.

Assume that the result is true for $n = k$.

That is

$$\mathbb{y}^k = \begin{pmatrix} y_{k+1} & y_k \\ y_k & y_{k-1} \end{pmatrix}$$

Now,

$$\mathbb{y}^{k+1} = \mathbb{y}^k \mathbb{y} = \begin{pmatrix} 2ky_{k+1} + y_k & y_{k+1} \\ 2ky_k + y_{k-1} & y_k \end{pmatrix} = \begin{pmatrix} y_{k+2} & y_{k+1} \\ y_{k+1} & y_k \end{pmatrix}$$

Thus, the theorem is valid for $n = k + 1$.

Hence, the conclusion of the theorem is perceived by $\mathbb{y}^n = \begin{pmatrix} y_{n+1} & y_n \\ y_n & y_{n-1} \end{pmatrix}$

(ii) Since,

$$\mathbb{y}\begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} y_{k+1} \\ y_k \end{pmatrix}$$

the theorem is valid for $n = 1$.

Assume that the theorem is true for $n = t$.

That is,

$$\mathbb{y}^t \begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} y_{k+t} \\ y_{k+t-1} \end{pmatrix}$$

Now, consider

$$\mathbb{y}^{t+1} \begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \mathbb{y}.\mathbb{y}^t \begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix}$$

$$= \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_{t+1} & y_t \\ y_t & y_{t-1} \end{pmatrix}\begin{pmatrix} y_{k+1} \\ y_k \end{pmatrix} = \begin{pmatrix} y_{k+t+1} \\ y_{k+t} \end{pmatrix}$$

Hence,

$$\mathbb{y}^n \begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} y_{k+n} \\ y_{k+n-1} \end{pmatrix}$$

Theorem 2.2 Generalization of Cheldhiya sequence

If $\mathbb{y} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}$ is a Cheldhiya sequence matrix, then the nth term of the Cheldhiya Sequence is generalized by

$$y_n = \frac{1}{2\sqrt{k^2+1}}\left[\left(k+\sqrt{k^2+1}\right)^n - \left(k-\sqrt{k^2+1}\right)^n\right] \text{ where } n = 0,1,2,3, \ldots.$$

Proof:

Given

$$\mathbb{y} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}$$

The two eigenvalues of the above matrix can be derived from the characteristic equation

$$|\mathbb{y} - \lambda I| = 0$$

as $\lambda_1 = k + \sqrt{k^2+1}$ and $\lambda_2 = k - \sqrt{k^2+1}$.

The eigenvectors of $\mathbb{y}$ are given by

$$(\mathbb{y} - \lambda I)V = 0$$

which implies that

$$\begin{pmatrix} 2k-\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \qquad (2.1)$$

Case 1: If $\lambda_1 = k + \sqrt{k^2+1}$, then one of the eigen vector of $\mathbb{y}$ is performed from (2.1) by

$$V_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$$

Case 2: If $\lambda_2 = k - \sqrt{k^2+1}$, then the other eigen vector of $\mathbb{y}$ is computed from (2.1) by

$$V_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

If the Diagonal matrix of $\mathbb{y}$ is given by

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \text{ then } D^n = \begin{pmatrix} {\lambda_1}^n & 0 \\ 0 & {\lambda_2}^n \end{pmatrix}$$

Let the Normalized eigenvector matrix be

$$N = \begin{pmatrix} \frac{\lambda_1}{\sqrt{1+{\lambda_1}^2}} & \frac{\lambda_2}{\sqrt{1+{\lambda_2}^2}} \\ \frac{1}{\sqrt{1+{\lambda_1}^2}} & \frac{1}{\sqrt{1+{\lambda_2}^2}} \end{pmatrix}$$

Now, by applying theorem (2.1) the orthogonal transformation of the symmetric matrices

$$\mathbb{y}^n = ND^nN^T$$

can be established by

$$\begin{pmatrix} y_{n+1} & y_n \\ y_n & y_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1}{\sqrt{1+{\lambda_1}^2}} & \frac{\lambda_2}{\sqrt{1+{\lambda_2}^2}} \\ \frac{1}{\sqrt{1+{\lambda_1}^2}} & \frac{1}{\sqrt{1+{\lambda_2}^2}} \end{pmatrix} \begin{pmatrix} {\lambda_1}^n & 0 \\ 0 & {\lambda_2}^n \end{pmatrix} \begin{pmatrix} \frac{\lambda_1}{\sqrt{1+{\lambda_1}^2}} & \frac{\lambda_2}{\sqrt{1+{\lambda_2}^2}} \\ \frac{1}{\sqrt{1+{\lambda_1}^2}} & \frac{1}{\sqrt{1+{\lambda_2}^2}} \end{pmatrix}^T$$

Simplifying the right-hand side of the equation and equating the $(1,2)^{\text{th}}$ entry on both sides, the generalized form of Cheldhiya sequence is estimated by

$$y_n = \frac{1}{2\sqrt{k^2+1}}\left[\left(k+\sqrt{k^2+1}\right)^n - \left(k-\sqrt{k^2+1}\right)^n\right] \text{ where } n = 0,1,2,3, \ldots.$$

Remark:

The above identity can also be written as $y_n = \frac{{\lambda_1}^n - {\lambda_2}^n}{\lambda_1 - \lambda_2}$ where $n = 0,1,2,3, \ldots.$

Theorem 2.3

If $\{y_n\}$ is the Cheldhiya sequence, then the sum of its first n terms is given by

$$\sum_{m=0}^{n-1} y_m = \frac{y_n + y_{n-1} - 1}{2k}$$

Proof:

$$\sum_{m=0}^{n-1} y_m = \sum_{m=0}^{n-1} \left(\frac{{\lambda_1}^m - {\lambda_2}^m}{\lambda_1 - \lambda_2} \right)$$

$$= \frac{1}{\lambda_1 - \lambda_2} \sum_{m=0}^{n-1} ({\lambda_1}^m - {\lambda_2}^m)$$

$$= \frac{1}{\lambda_1 - \lambda_2} \left(\frac{1 - {\lambda_1}^n}{1 - \lambda_1} - \frac{1 - {\lambda_2}^n}{1 - \lambda_2} \right)$$

$$= \frac{1}{\lambda_1 - \lambda_2} \left(\frac{-({\lambda_1}^n - {\lambda_2}^n) - ({\lambda_1 {\lambda_2}^n} - {\lambda_1}^n \lambda_2) + (\lambda_1 - \lambda_2)}{(1 - \lambda_1)(1 - \lambda_2)} \right)$$

$$= \frac{y_n + y_{n-1} - 1}{2k}$$

Hence,

$$\sum_{m=0}^{n-1} y_m = \frac{y_n + y_{n-1} - 1}{2k}$$

Theorem 2.4

If $G(x) = \sum_{n=0}^{\infty} y_n x^n$ is the Generating function, then the corresponding function for Cheldhiya Sequence is $G(x) = \frac{x}{1 - 2kx - x^2}$

Proof:

$$G(x) = \sum_{n=0}^{\infty} y_n x^n$$

$$= y_0 x^0 + y_1 x + \sum_{n=2}^{\infty} y_n x^n$$

$$= x + \sum_{n=2}^{\infty} (2k y_{n-1} + y_{n-2}) x^n$$

$$= x + 2kx\, G(x) + x^2 G(x)$$

Hence,

$$G(x) = \frac{x}{1 - 2kx - {}^{2}}$$

Theorem 2.5

If y_n is the n[th] term of the Cheldhiya sequence, then ${y_n}^2 - y_{n-r}y_{n+r} = (-1)^{n-r}{y_r}^2$

Proof:

$$
\begin{aligned}
{y_n}^2 - y_{n-r}y_{n+r} &= \left(\frac{{\lambda_1}^n - {\lambda_2}^n}{\lambda_1 - \lambda_2}\right)^2 - \left(\frac{{\lambda_1}^{n-r} - {\lambda_2}^{n-r}}{\lambda_1 - \lambda_2}\right)\left(\frac{{\lambda_1}^{n+r} - {\lambda_2}^{n+r}}{\lambda_1 - \lambda_2}\right) \\
&= \frac{1}{(\lambda_1 - \lambda_2)^2}\left(-2(\lambda_1\lambda_2)^n + (\lambda_1\lambda_2)^n\left(\frac{\lambda_1}{\lambda_2}\right)^r + (\lambda_1\lambda_2)^n\left(\frac{\lambda_2}{\lambda_1}\right)^r\right) \\
&= \frac{(-1)^n}{(\lambda_1 - \lambda_2)^2}\left(-2 + \left(\frac{\lambda_1}{\lambda_2}\right)^r + \left(\frac{\lambda_2}{\lambda_1}\right)^r\right) \\
&= \frac{(-1)^n}{(\lambda_1 - \lambda_2)^2}\left[\frac{\left({\lambda_1}^r - {\lambda_2}^r\right)^2}{(\lambda_1\lambda_2)^r}\right] \\
&= (-1)^{n-r}{y_r}^2
\end{aligned}
$$

Hence,

$${y_n}^2 - y_{n-r}y_{n+r} = (-1)^{n-r}{y_r}^2$$

Theorem 2.6

If $\{x_n\}$ and $\{y_n\}$ are Cheldhiya Companion sequence and Cheldhiya sequence respectively, then

(i) $x_n = y_{n-1} + ky_n, n \geq 1$

(ii) $x_n y_n = \frac{1}{2(k^2+1)}(x_{2n-1} + kx_{2n})$

(iii) $\lim_{n\to\infty} \frac{x_n}{y_n} = \sqrt{k^2+1}$

(iv) $x_n + ky_n = y_{n+1}$

(v) $y_{n+1} + y_{n-1} = 2x_n$

(vi) $\frac{1}{k^2+1}(x_{n+1} + x_{n-1}) = 2y_n$

Proof:

(i) Define the Cheldhiya Companion sequence matrix as

$$\mathbb{x} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}$$

Then, the characteristic roots of $\mathbb{x}$ are determined by $\lambda_1 = k + \sqrt{k^2+1}$,

$\lambda_2 = k - \sqrt{k^2+1}$.

Note that, $\lambda_1\lambda_2 = -1$.

The closed form of the Cheldhiya Companion sequence is given by

$$x_n = c_1{\lambda_1}^n + c_2{\lambda_2}^n \qquad (2.2)$$

By applying the initial values $x_0 = 1, x_1 = k$, the linear system of equations is evaluated by $c_1 + c_2 = 1$ and $c_1\lambda_1 + c_2\lambda_2 = k$.

Thus, $c_1 = \frac{\lambda_2 - k}{\lambda_2 - \lambda_1}$ and $c_2 = \frac{k - \lambda_1}{\lambda_2 - \lambda_1}$

Then, (2.2) becomes

$$x_n = \frac{\lambda_2 - k}{\lambda_2 - \lambda_1}{\lambda_1}^n + \frac{k - \lambda_1}{\lambda_2 - \lambda_1}{\lambda_2}^n$$

$$= \frac{1}{\lambda_2 - \lambda_1}\left[\lambda_1\lambda_2\left({\lambda_1}^{n-1} - {\lambda_2}^{n-1}\right) - k({\lambda_1}^n - {\lambda_2}^n)\right]$$

$$= \frac{1}{\lambda_1 - \lambda_2}\left[\left({\lambda_1}^{n-1} - {\lambda_2}^{n-1}\right) + k({\lambda_1}^n - {\lambda_2}^n)\right]$$

$$\therefore x_n = y_{n-1} + ky_n$$

(ii) $x_n y_n = (y_{n-1} + ky_n)y_n$

$$= y_{n-1}y_n + k{y_n}^2$$

$$= \frac{{\lambda_1}^{n-1} - {\lambda_2}^{n-1}}{\lambda_1 - \lambda_2} \cdot \frac{{\lambda_1}^n - {\lambda_2}^n}{\lambda_1 - \lambda_2} + k\left(\frac{{\lambda_1}^n - {\lambda_2}^n}{\lambda_1 - \lambda_2}\right)^2$$

$$= \frac{1}{(\lambda_1-\lambda_2)^2}\left({\lambda_1}^{2n-1} + {\lambda_2}^{2n-1} + (\lambda_1\lambda_2)^n\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right) + k\left({\lambda_1}^{2n} + {\lambda_2}^{2n} - 2(\lambda_1\lambda_2)^n\right)\right)$$

$$= \frac{1}{(\lambda_1-\lambda_2)^2}\left(2x_{2n-1} - 2k(-1)^n + k(2x_{2n} - 2(-1)^n)\right)$$

$$= \frac{1}{4(k^2+1)}(2x_{2n-1} + 2kx_{2n})$$

$$\therefore x_n y_n = \frac{1}{2(k^2+1)}(x_{2n-1} + kx_{2n})$$

(iii) $$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{{\lambda_1}^n+{\lambda_2}^n}{2}.\frac{\lambda_1-\lambda_2}{{\lambda_1}^n-{\lambda_2}^n}$$

$$= \left(\frac{\lambda_1-\lambda_2}{2}\right)\lim_{n\to\infty}\left[\frac{1+\left(\frac{\lambda_2}{\lambda_1}\right)^n}{1-\left(\frac{\lambda_2}{\lambda_1}\right)^n}\right]$$

Since $\lambda_2 < \lambda_1, \left|\frac{\lambda_2}{\lambda_1}\right| < 1$. Therefore, $\lim_{n\to\infty}\left(\frac{\lambda_2}{\lambda_1}\right)^n \to 0$.

Hence,

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \sqrt{k^2+1}$$

(iv) $$x_n + ky_n = \frac{{\lambda_1}^n+{\lambda_2}^n}{2} + k\,\frac{{\lambda_1}^n-{\lambda_2}^n}{\lambda_1-\lambda_2}$$

$$= \frac{(\lambda_1-\lambda_2)({\lambda_1}^n+{\lambda_2}^n)+2k({\lambda_1}^n-{\lambda_2}^n)}{2(\lambda_1-\lambda_2)}$$

$$= \frac{({\lambda_1}^{n+1}-{\lambda_2}^{n+1})}{(\lambda_1-\lambda_2)}$$

$$\therefore x_n + ky_n = y_{n+1}$$

(v) $$y_{n+1} + y_{n-1} = \frac{{\lambda_1}^{n+1}-{\lambda_2}^{n+1}}{\lambda_1-\lambda_2} + \frac{{\lambda_1}^{n-1}-{\lambda_2}^{n-1}}{\lambda_1-\lambda_2}$$

$$= \frac{1}{\lambda_1-\lambda_2}\left({\lambda_1}^{n+1} - {\lambda_2}^{n+1} + {\lambda_1}^{n-1} - {\lambda_2}^{n-1}\right)$$

$$= \frac{1}{\lambda_1-\lambda_2}\left({\lambda_1}^{n+1} - {\lambda_2}^{n+1} + \frac{{\lambda_1}^n\lambda_2-{\lambda_2}^n\lambda_1}{\lambda_1\lambda_2}\right)$$

$$= \frac{\mathbf{1}}{\lambda_1-\lambda_2}\left({\lambda_1}^{n+1} - {\lambda_2}^{n+1} - {\lambda_1}^n\lambda_2 + \lambda_1{\lambda_2}^n\right)$$

$$= {\lambda_1}^n + {\lambda_2}^n$$

$$\therefore y_{n+1} + y_{n-1} = 2x_n$$

(vi) $\frac{1}{k^2+1}(x_{n+1} + x_{n-1}) = \frac{1}{2(k^2+1)}\left({\lambda_1}^{n+1} - {\lambda_2}^{n+1} + {\lambda_1}^{n-1} + {\lambda_2}^{n-1}\right)$

$$= \frac{1}{2(k^2+1)}\left[{\lambda_1}^n\left(\frac{{\lambda_1}^2+1}{\lambda_1}\right) + {\lambda_2}^n\left(\frac{{\lambda_2}^2+1}{\lambda_2}\right)\right]$$

$$= \frac{{\lambda_1}^n - {\lambda_2}^n}{\sqrt{k^2+1}}$$

$$= 2\frac{{\lambda_1}^n - {\lambda_2}^n}{\lambda_1 - \lambda_2}$$

Hence,

$$\frac{1}{k^2+1}(x_{n+1} + x_{n-1}) = 2y_n$$

Remark:

In the above result, it is observed that

(i) $c_1 = \frac{\lambda_2 - k}{\lambda_2 - \lambda_1} = \frac{-\sqrt{k^2+1}}{-2\sqrt{k^2+1}} = \frac{1}{2}$

$c_2 = 1 - c_1 = \frac{1}{2}$

Hence,

$x_n = \frac{1}{2}({\lambda_1}^n + {\lambda_2}^n)$

(ii) $x_n y_n = \frac{1}{2}({\lambda_1}^n + {\lambda_2}^n).\left(\frac{{\lambda_1}^n - {\lambda_2}^n}{\lambda_1 - \lambda_2}\right)$

$$= \frac{1}{2}\left(\frac{{\lambda_1}^{2n} - {\lambda_2}^{2n}}{\lambda_1 - \lambda_2}\right)$$

$x_n y_n = \frac{1}{2}y_{2n}$

Theorem 2.7

If $\{x_n\}$ is a Cheldhiya Companion sequence, then for $m \geq 1$

(i) $x_{2m-1}x_{2m+1} + (k^2+1) = (x_{2m})^2$

(ii) $x_{2m}x_{2m+2} - (k^2+1) = (x_{2m+1})^2$

Proof:

(i) $x_{2m-1}x_{2m+1} + (k^2+1) = \frac{({\lambda_1}^{2m-1}+{\lambda_2}^{2m-1})}{2}\frac{({\lambda_1}^{2m+1}+{\lambda_2}^{2m+1})}{2} + (k^2+1)$

$= \frac{1}{4}\left({\lambda_1}^{4m} + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + {\lambda_2}^{4m}\right) + (k^2+1)$

$= \frac{1}{4}\left({\lambda_1}^{4m} - \left({\lambda_1}^2 + {\lambda_2}^2\right) + {\lambda_2}^{4m}\right) + (k^2+1)$

$= \frac{1}{4}\left({\lambda_1}^{4m} + 2 + {\lambda_2}^{4m}\right)$

$= \frac{1}{4}\left({\lambda_1}^{2m} + {\lambda_2}^{2m}\right)^2$

Hence,

$$x_{2m-1}x_{2m+1} + (k^2+1) = (x_{2m})^2$$

(ii) $x_{2m}x_{2m+2} - (k^2+1)$

$= \frac{({\lambda_1}^{2m}+{\lambda_2}^{2m})}{2}\frac{({\lambda_1}^{2m+2}+{\lambda_2}^{2m+2})}{2} - (k^2+1)$

$= \frac{1}{4}\left({\lambda_1}^{4m+2} + \left({\lambda_1}^2 + {\lambda_2}^2\right) + {\lambda_2}^{4m+2}\right) - (k^2+1)$

$= \frac{1}{4}\left({\lambda_1}^{4m+2} - 2 + {\lambda_2}^{4m+2}\right)$

$= \frac{1}{4}\left({\lambda_1}^{2m+1} + {\lambda_2}^{2m+1}\right)^2$

Hence,

$$x_{2m}x_{2m+2} - (k^2+1) = (x_{2m+1})^2$$

2.2 *Invention of Four Novel Sequences and their Properties*

In this section, four novel sequences named as Pan-San, Pan-San Buddy, Pan-San Comrade and Pan-San Mate sequences are discovered. Also, the recurrence relations, the general formulae for all sequences and some theorems are invented by exploiting basic concepts of matrices.

2.2.1 Pan-San and Pan-San Buddy Sequences

The values of C and D in the universal equation $D^2 - dC^2 = 1$ for certain non-zero square-free integer $d = k^2 + 2, k \in \mathcal{N} - \{1\}$ where $\mathcal{N}$ is the set of all-natural numbers propagate two fresh sequences $0, k, 2k(k^2+1), 4k(k^2+1) - k, 8k(k^2+1)^2 - 4k(k^2+1)$, etc and $1, k^2+1, 2(k^2+1)^2 - 1, 4(k^2+1)^3 - 3(k^2+1)$, etc named as Pan-San sequence and Pan-San Buddy sequence respectively. The n[th] term of the first sequence is interpreted by the recurrence relation

$$C_{n,k} = 2(k^2+1)C_{n-1,k} - C_{n-2,k}, \qquad k, n \in \mathcal{N} - \{1\}$$

where $C_{0,k} = 0, C_{1,k} = k$.

The n[th] term of the second sequence is standardized by the recurrence relation

$$D_{n,k} = 2(k^2+1)D_{n-1,k} - D_{n-2,k}, \qquad k, n \in \mathcal{N} - \{1\}$$

where $D_{0,k} = 1, D_{1,k} = k^2 + 1$.

Define the Pan-San sequence matrix as

$$\mathcal{M} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}$$

Now,

$$\mathcal{M}\begin{pmatrix} C_{1,k} \\ C_{0,k} \end{pmatrix} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k \\ 0 \end{pmatrix} = \begin{pmatrix} 2k(k^2+1) \\ k \end{pmatrix} = \begin{pmatrix} C_{2,k} \\ C_{1,k} \end{pmatrix}$$

Also,

$$\mathcal{M}\begin{pmatrix} C_{2,k} \\ C_{1,k} \end{pmatrix} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2k(k^2+1) \\ k \end{pmatrix} = \begin{pmatrix} 4k(k^2+1)^2 - k \\ 2k(k^2+1) \end{pmatrix} = \begin{pmatrix} C_{3,k} \\ C_{2,k} \end{pmatrix}$$

More generally,

$$\mathcal{M}\begin{pmatrix} C_{n,k} \\ C_{n-1,k} \end{pmatrix} = \begin{pmatrix} C_{n+1,k} \\ C_{n,k} \end{pmatrix}$$

Theorem 2.8

If $\mathcal{M} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}$ is a Pan-San sequence matrix, then the n[th] term of the Pan-San sequence is generalized by

$C_{n,k} = \frac{1}{2\sqrt{k^2+2}}\left[\left((k^2+1)+k\sqrt{k^2+2}\right)^n - \left((k^2+1)-k\sqrt{k^2+2}\right)^n\right]$ where $n \in \mathcal{W}$, the set of all whole numbers.

Proof:

Given

$$\mathcal{M} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}$$

The characteristic equation $\lambda^2 - 2(k^2+1)\lambda + 1 = 0$ of $\mathcal{M}$ reveals two distinct eigen values $\sigma = (k^2+1)+k\sqrt{k^2+2}$ and $\tau = (k^2+1)-k\sqrt{k^2+2}$.

Also, $\lambda^2 = 2(k^2+1)\lambda - 1$

$\lambda^3 = \lambda^2.\lambda$

$$= (4(k^2+1)^2 - 1)\lambda - 2(k^2+1) = \frac{1}{k}\left(C_{3,k}\lambda - C_{2,k}\right)$$

$$\Rightarrow k\lambda^3 = C_{3,k}\lambda - C_{2,k}$$

$$\lambda^4 = [8(k^2+1)^3 - 4(k^2+1)]\lambda - [4(k^2+1) - 1] = \frac{1}{k}\left(C_{4,k}\lambda - C_{3,k}\right)$$

$$\Rightarrow k\lambda^4 = C_{4,k}\lambda - C_{3,k}$$

In general, $k\lambda^n = C_{n,k}\lambda - C_{n-1,k}$ (2.3)

Since both σ and τ are the characteristic values, they must satisfy (2.3). Hence

$$k\sigma^n = \left(C_{n,k}\sigma - C_{n-1,k}\right) \text{ and } k\tau^n = \left(C_{n,k}\tau - C_{n-1,k}\right)$$

The above equations provide the following expression

$$k(\sigma^n - \tau^n) = C_{n,k}(\sigma - \tau)$$

$$\Rightarrow C_{n,k} = \frac{k(\sigma^n - \tau^n)}{(\sigma - \tau)}$$

Consequently, $C_{n,k} = \frac{1}{2\sqrt{k^2+2}}\left[\left((k^2+1) + k\sqrt{k^2+2}\right)^n - \left((k^2+1) - k\sqrt{k^2+2}\right)^n\right]$

where $n \in W$, the set of all whole numbers.

Theorem 2.9

If $\{D_{n,k}\}$ and $\{C_{n,k}\}$ are Pan-San Buddy sequence and Pan-San sequence respectively, then

(i) $kD_{n,k} = (k^2+1)C_{n,k} - C_{n-1,k}$

(ii) $2D_{n,k}C_{n,k} = C_{2n,k}$

(iii) $D_{n+1,k} - D_{n-1,k} = 2k(k^2+2)C_{n,k}$

(iv) $\left(C_{n,k} + C_{n-1,k}\right)^2 + 1 = D_{2n-1,k}$

(v) $C_{n+1,k} - C_{n-1,k} = 2kD_{n,k}$

(vi) $D_{n+1,k} - D_{n-1,k} = 2k(k^2+2)C_{n,k}$

Proof:

(i) By using the characteristic values of the Pan-San sequence as delivered in theorem 2.7, their product is given by $\sigma\tau = 1$.

The closed form of the Pan-San Buddy sequence is specified by

$$D_{n,k} = A\sigma^n + B\tau^n \qquad (2.4)$$

The fundamental values $D_{0,k} = 1, D_{1,k} = k^2 + 1$ provides the subsequent system of linear equations

$$A + B = 1$$

$$A\sigma + B\tau = k^2 + 1.$$

Precisely, $A = \frac{(k^2+1)-\tau}{\sigma-\tau}$ and $B = \frac{\sigma-(k^2+1)}{\sigma-\tau}$

Thus, the specific value of $D_{n,k}$ is pointed out by

$$D_{n,k} = \frac{(k^2+1)-\tau}{\sigma-\tau}\,\sigma^n + \frac{\sigma-(k^2+1)}{\sigma-\tau}\tau^n$$

$$= \frac{1}{\sigma-\tau}[(k^2+1)(\sigma^n - \tau^n) + \sigma\tau(\tau^{n-1} - \sigma^{n-1})]$$

$$= \frac{(k^2+1)}{k}C_{n,k} - \frac{1}{k}C_{n-1,k}$$

Consequently, $kD_{n,k} = (k^2+1)C_{n,k} - C_{n-1,k}$

(ii) The alternative forms of the above values of A and B are epitomized by

$$A = \frac{(k^2+1)-\tau}{\sigma-\tau} = \frac{k\sqrt{k^2+2}}{2k\sqrt{k^2+2}} = \frac{1}{2}$$

$$B = 1 - \mathrm{A} = \frac{1}{2}$$

The equivalent values of the general term of the Pan-San Buddy sequence are noted as

$$D_{n,k} = \frac{1}{2}(\sigma^n + \tau^n)$$

Hence,

$$D_{n,k}C_{n,k} = \left(\frac{\sigma^n+\tau^n}{2}\right)\left(\frac{k(\sigma^n-\tau^n)}{(\sigma-\tau)}\right)$$

$$= \frac{k}{2(\sigma-\tau)}(\sigma^{2n} - \tau^{2n}) = \frac{1}{2}C_{2n,k}$$

Hence,

$$2D_{n,k}C_{n,k} = C_{2n,k}$$

(iii) $\frac{D_{n+1,k}-D_{n-1,k}}{C_{n,k}} = \frac{\frac{1}{2}(\sigma^{n+1}+\tau^{n+1})-\frac{1}{2}(\sigma^{n-1}+\tau^{n-1})}{\frac{k(\sigma^n-\tau^n)}{(\sigma-\tau)}}$

$$= \frac{(\sigma-\tau)}{2k}\frac{(\sigma^{n+1}+\tau^{n+1}-\sigma^{n-1}-\tau^{n-1})}{(\sigma^n-\tau^n)}$$

$$= \frac{(\sigma-\tau)}{2k}\frac{(\sigma^{n+1}+\tau^{n+1}-\sigma^n\tau-\sigma\tau^n)}{(\sigma^n-\tau^n)}$$

$$= \frac{(\sigma-\tau)}{2k}(\sigma - \tau)$$

$$= 2k(k^2 + 2)$$

This implies that $D_{n+1,k} - D_{n-1,k} = 2k(k^2 + 2)C_{n,k}$

(iv) $\left(C_{n,k} + C_{n-1,k}\right)^2 + 1 = \left[\frac{k(\sigma^n-\tau^n)}{(\sigma-\tau)} + \frac{k(\sigma^{n-1}-\tau^{n-1})}{(\sigma-\tau)}\right]^2 + 1$

$$= \left\{\frac{k}{(\sigma-\tau)}[\sigma^n - \tau^n + \sigma^n\tau - \sigma\tau^n]\right\}^2 + 1$$

$$= \left\{\frac{k}{(\sigma-\tau)}[\sigma^n(1+\tau) - \tau^n(1+\sigma)]\right\}^2 + 1$$

$$= \frac{k^2}{(\sigma-\tau)^2}[\sigma^{2n}(1+\tau)^2 + \tau^{2n}(1+\sigma)^2 - 2\sigma^n\tau^n(1+\tau)(1+\sigma)] + 1$$

$$= \frac{k^2}{(\sigma-\tau)^2}[\sigma^{2n} + \tau^{2n} + \sigma^2\tau^2(\sigma^{2n-2} + \tau^{2n-2}) +$$

$$2\sigma\tau(\sigma^{2n-1} + \tau^{2n-1}) - 4(k^2 + 2)] + 1$$

$$= \frac{1}{4(k^2+2)}\left[2D_{2n,k} + 2D_{2n-2,k} + 4D_{2n-1,k}\right]$$

$= \frac{1}{4(k^2+2)} 4(k^2+2)D_{2n-1,k}$

Hence, $\left(C_{n,k} + C_{n-1,k}\right)^2 + 1 = D_{2n-1,k}$

(v) $C_{n+1,k} - C_{n-1,k} = 2(k^2+1)C_{n,k} - C_{n-1,k} - C_{n-1,k}$

$= 2(k^2+1)C_{n,k} - 2C_{n-1,k}$

Therefore, $C_{n+1,k} - C_{n-1,k} = 2kD_{n,k}$

(vi) $D_{n+1,k} - D_{n-1,k} = \frac{1}{2}(\sigma^{n+1} + \tau^{n+1}) - \frac{1}{2}(\sigma^{n-1} + \tau^{n-1})$

$= \frac{1}{2}[\sigma^{n+1} + \tau^{n+1} - \sigma^n\tau - \sigma\tau^n]$

$= \frac{1}{2}[\sigma^n(\sigma - \tau) - \tau^n(\sigma - \tau)]$

$= \frac{1}{2}[(\sigma^n - \tau^n)(\sigma - \tau)]$

$= \frac{\left(2k\sqrt{k^2+2}\right)^2}{2k} C_{n,k}$

Subsequently, $D_{n+1,k} - D_{n-1,k} = 2k(k^2+2)C_{n,k}$

2.2.2 Pan-San Comrade and Pan-San Mate Sequences

The values of R and S in the world-wide equation $S^2 - dR^2 = 1$ for an appropriate non-zero square-free integer $d = k^2 - 2,\ k \in \mathcal{N} - \{1\}$, $\mathcal{N}$ is the set of all natural numbers create two handsome sequences $0, k, 2k(k^2-1), 4k(k^2-1)^2 - k, 8k(k^2-1)^3 - 4k(k^2-1)$, etc and $1, k^2-1, 2(k^2-1)^2 - 1, 4(k^2-1)^3 - 3\,(k^2-1)$, etc called as Pan-San Comrade and Pan-San Mate Sequences.

The n[th] term of the earlier sequence is construed by the relation

$R_{n,k} = 2(k^2-1)R_{n-1,k} - R_{n-2,k}$, where $R_{0,k} = 0,\ R_{1,k} = k,\quad k \in \mathcal{N} - \{1\}$

The n[th] term of the later sequence is inferred by the relation

$S_{n,k} = 2(k^2-1)S_{n-1,k} - S_{n-2,k}$, where $S_{0,k} = 1, S_{1,k} = k^2-1$, $k \in \mathcal{N} - \{1\}$ and $n \in \mathcal{W}$.

Define the Pan-San Comrade sequence matrix as

$$\mathfrak{M} = \begin{pmatrix} 2(k^2-1) & -1 \\ 1 & 0 \end{pmatrix}$$

Now,

$$\mathfrak{M}\begin{pmatrix} R_{1,k} \\ R_{0,k} \end{pmatrix} = \begin{pmatrix} 2(k^2-1) & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k^2-1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2(k^2-1)^2-1 \\ k^2-1 \end{pmatrix} = \begin{pmatrix} R_{2,k} \\ R_{1,k} \end{pmatrix}$$

Also,

$$\mathfrak{M}\begin{pmatrix} R_{2,k} \\ R_{1,k} \end{pmatrix} = \begin{pmatrix} 2(k^2-1) & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2(k^2-1)^2-1 \\ k^2-1 \end{pmatrix}$$

$$= \begin{pmatrix} 4(k^2-1)^3 - 3(k^2-1) \\ 2(k^2-1)^2-1 \end{pmatrix}$$

$$= \begin{pmatrix} R_{3,k} \\ R_{2,k} \end{pmatrix}$$

In general, $\mathfrak{M}\begin{pmatrix} R_{n,k} \\ R_{n-1,k} \end{pmatrix} = \begin{pmatrix} R_{n+1,k} \\ R_{n,k} \end{pmatrix}$

As in section 2.1, it is enabled to prove the following theorems.

Theorem 2.10

If $\mathfrak{M} = \begin{pmatrix} 2(k^2-1) & -1 \\ 1 & 0 \end{pmatrix}$ is a Pan-San Comrade sequence matrix, then the n[th] term of the Pan-San Comrade Sequence is hypothesized by

$$R_{n,k} = \frac{1}{2\sqrt{k^2-2}}\left[\left((k^2-1)+k\sqrt{k^2-2}\right)^n - \left((k^2-1)-k\sqrt{k^2-2}\right)^n\right]$$

where $n = 0,1,2,3, \ldots$.

Theorem 2.11

If $\{R_{n,k}\}$ and $\{S_{n,k}\}$ are Pan-San Comrade and Pan-San Mate sequences respectively, then

(i) $kS_{n,k} = (k^2 - 1)R_{n,k} - R_{n-1,k}$

(ii) $S_{n,k}R_{n,k} = \frac{1}{2}R_{2n,k}$

(iii) $S_{n+1,k} - S_{n-1,k} = 2k(k^2 - 2)R_{n,k}$

(iv) $\left(R_{n,k} - R_{n-1,k}\right)^2 - 1 = S_{2n-1,k}$

(v) $R_{n+1,k} - R_{n-1,k} = 2kS_{n,k}$

(vi) $S_{n+1,k} - S_{n-1,k} = 2k(k^2 - 2)$

Chapter – III

Diophantine Triples involving Special Sequences

CHAPTER - III

Diophantine Triples involving Special Sequences

This chapter is divided into two sections, 3.1 and 3.2.

In Section 3.1, the patterns of Diophantine triples $\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \{a_3, a_4, a_5\}$, etc reside in Cheldhiya companion sequence with splendid properties $D\left(\pm(k^2+1)\right)$, $k \in \mathcal{N}$ are investigated.

In Section 3.2, two disparate arrangements of triples $\{\alpha, \beta, \gamma\}, \{\beta, \gamma, \delta\}, \{\gamma, \delta, \varepsilon\}$ etc where in one of each module is a Pan-San number and in the other it is a Pan-San Comrade number composed with the condition that the multiplication of any two modules added with k^2, $k \in \mathcal{N} - \{1\}$ is again a square of an integer are explored.

3.1 The Patterns of Diophantine Triples Engross Cheldhiya Companion Sequence with Inspiring Properties

Presume that

$a_1 = x_{2m}, a_2 = x_{2m+2},\ m \in \mathcal{N}$ where

$x_n = \frac{1}{2}\left(\left(k+\sqrt{k^2+1}\right)^n + \left(k-\sqrt{k^2+1}\right)^n\right), k \in \mathcal{N}$

be any two integers such that $a_1a_2 - (k^2+1)$ is a perfect square.

Let a_3 be another positive integer which satisfy the consequent provision

$$a_1a_3 - (k^2+1) = \phi^2 \tag{3.1}$$

$$a_2a_3 - (k^2+1) = \psi^2 \tag{3.2}$$

Resolving (3.1) and (3.2), the value of a_3 is attained by

$$a_3 = \frac{\phi^2-\psi^2}{a_1-a_2} \tag{3.3}$$

By utilizing (3.3) in (3.2), the relation to be perceived is

$$a_2\phi^2 - a_1\psi^2 = (k^2+1)(a_1-a_2) \tag{3.4}$$

Create the succeeding linear alterations

$$\phi = X + a_1T \tag{3.5}$$

$$\psi = X + a_2T \tag{3.6}$$

Restoring the above values of ϕ and ψ in (3.4), the quadratic equation with two unknowns is estimated by

$$X^2 - (a_1a_2)T^2 = -(k^2+1) \tag{3.7}$$

Selecting the least solution to (3.7) as

$$X_0 = x_{2m+1}, T_0 = 1$$

and the implementation this solution in (3.5) and (3.6) endow with the relations that

$$\phi = x_{2m+1} + a_1$$

$$\psi = x_{2m+1} + a_2$$

Exchanging the above said suitable modifications in (3.3), the third element in an essential pattern which assure the postulation is specified by

$$a_3 = x_{2m} + 2x_{2m+1} + x_{2m+2}$$

Hence,

$\{x_{2m},\ x_{2m+2},\ x_{2m} + 2x_{2m+1} + x_{2m+2}\}$ is a Diophantine triple with the property $D\left(-(k^2+1)\right), k \in \mathcal{N}$

Let a_4 be a new-fangled positive integer such that

$$a_2 a_4 - (k^2+1) = {\phi_1}^2 \quad (3.8)$$

$$a_3 a_4 - (k^2+1) = {\psi_1}^2 \quad (3.9)$$

Subtracting (3.9) from (3.8) and make a simple computation, the significant value of a_4 is determined by

$$a_4 = \frac{{\phi_1}^2 - {\psi_1}^2}{a_2 - a_3} \quad (3.10)$$

Now, choose a_5 be a positive integer which satisfies the conditions that

$$a_3 a_5 - (k^2+1) = {\phi_2}^2 \quad (3.11)$$

$$a_4a_5 - (k^2+1) = \psi_2{}^2 \tag{3.12}$$

By exploiting a plain numerical calculation in (3.11) and (3.12), it is to be noticed that

$$a_5 = \frac{\phi_2{}^2 - \psi_2{}^2}{a_3 - a_4} \tag{3.13}$$

Suppose that

$$a_4a_6 - (k^2+1) = \phi_3{}^2 \tag{3.14}$$

$$a_5a_6 - (k^2+1) = \psi_3{}^2 \tag{3.15}$$

where $a_6 \in Z - \{0\}$

Following the prior process in (3.14) and (3.15), the equivalent value of the factor a_6 in the sequence is established by

$$a_6 = \frac{\phi_3{}^2 - \psi_3{}^2}{a_4 - a_5} \tag{3.16}$$

Since the objective is to accomplish appropriate integer values for the parameters in the crucial patterns, make use of the subsequent transformations

$$\phi_1 = P_{k+1}x_{2m+1} + x_{2m} + a_2$$

$$\psi_1 = P_{k+1}x_{2m+1} + x_{2m} + a_3$$

$$\phi_2 = x_{2m} + 3x_{2m+1} + 2x_{2m+2} + a_3$$

$$\psi_2 = x_{2m} + 3x_{2m+1} + 2x_{2m+2} + a_4$$

$$\phi_3 = 2x_{2m} + 7x_{2m+1} + 6x_{2m+2} + a_4$$

$$\psi_3 = 2x_{2m} + 7x_{2m+1} + 6x_{2m+2} + a_5$$

where $\{P_n\} = \{2n-1\}, n \in \mathcal{N}$ is the sequence of odd numbers.

Proceeding the same mechanism as explained above from (3.8) to (3.16), the elements in the necessary patterns with the suitable property are studied by

$$a_4 = 3x_{2m} + (2 + 2P_{k+1})x_{2m+1} + 2x_{2m+2}$$

$$a_5 = 6x_{2m} + (10 + 2P_{k+1})x_{2m+1} + 7x_{2m+2}$$

$$a_6 = 13x_{2m} + (26 + 4P_{k+1})x_{2m+1} + 21x_{2m+2}$$

Thus,

$\{x_{2m},\ x_{2m+2},\ x_{2m} + 2x_{2m+1} + x_{2m+2}\}$,

$\{x_{2m+2},\ x_{2m} + 2x_{2m+1} + x_{2m+2},\ 3x_{2m} + (2 + 2P_{k+1})x_{2m+1} + 2x_{2m+2}\}$,

$\{x_{2m} + 2x_{2m+1} + x_{2m+2}, 3x_{2m} + (2 + 2P_{k+1})x_{2m+1} + 2x_{2m+2}, 6x_{2m} + (10 + 2P_{k+1})x_{2m+1} + x_{2m+2}\}$,

$\{3x_{2m} + (2 + 2P_{k+1})x_{2m+1} + 2x_{2m+2},\quad 6x_{2m} + (10 + 2P_{k+1})x_{2m+1} + 7x_{2m+2}, 13x_{2m} + (26 + 4P_{k+1})x_{2m+1} + 21x_{2m+2}\}$ …

are patterns of Diophantine triples concerning Cheldhiya companion sequence such that the product of any two of them decreased by $(k^2 + 1)$ is a perfect square where k is a natural number.

Hence, the patterns of Diophantine triples $\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \{a_3, a_4, a_5\}$, etc in which the factors are Cheldhiya companion sequence with the property $D\left(-(k^2 + 1)\right)$ where $k \in \mathcal{N}$ are evaluated.

Examples for the numerical replacement of the above patterns of Diophantine triples with the property $D\left(-(k^2+1)\right)$ are specified in table 3.1.

Table 3.1

k	m	k^2+1	$\{a_1, a_2, a_3\}$	$\{a_2, a_3, a_4\}$	$\{a_3, a_4, a_5\}$	$\{a_4, a_5, a_6\}$
1	1	2	{3,17,34}	{17,34,99}	{34,99,249}	{99,249, 662}
2	1	5	{9,161,246}	{161,246,805}	{246,805,1941}	{805,1941,5246}
3	3	10	$\begin{Bmatrix} 27379, \\ 1039681, \\ 1404494 \end{Bmatrix}$	$\begin{Bmatrix} 1039681, \\ 1404494, \\ 4860971 \end{Bmatrix}$	$\begin{Bmatrix} 1404494, \\ 4860971, \\ 11491249 \end{Bmatrix}$	$\begin{Bmatrix} 4860971, \\ 11491249, \\ 31299946 \end{Bmatrix}$

Remark:

Applying the similar procedure as enlightened above, it is pointed out the consequent patterns of Diophantine triples in which every element is a Cheldhiya companion sequence such that the product of any two of them increased by (k^2+1) is a perfect square.

$\{x_{2m-1},\ x_{2m+1}, x_{2m-1}+2x_{2m}+x_{2m+1}\}$,

$\{x_{2m+1},\ x_{2m-1}+2x_{2m}+x_{2m+1},\ x_{2m-1}+4x_{2m}+4x_{2m+1}\}$,

$\{x_{2m-1}+2x_{2m}+x_{2m+1},\ x_{2m-1}+4x_{2m}+4x_{2m+1},\ 4x_{2m-1}+12x_{2m}+9x_{2m+1}\}$,

$\{x_{2m-1}+4x_{2m}+4x_{2m+1},\ 4x_{2m-1}+12x_{2m}+9x_{2m+1},\ 9x_{2m-1}+30x_{2m}+25x_{2m+1}\}$,…

A small number of numerical cases for the above sequences of Diophantine triples with the property $D(k^2+1)$ are stated in table 3.2.

Table 3.2

k	m	k^2+1	$\{a_1, a_2, a_3\}$	$\{a_2, a_3, a_4\}$	$\{a_3, a_4, a_5\}$	$\{a_4, a_5, a_6\}$
1	1	2	{1,7,14}	{7,14,41}	{14,41,103}	{41,103,274}
2	1	5	{2,38,58}	{38,58,190}	{58,190,458}	{190,458,1238}
3	2	10	{117,4443, 6002}	{4443,6002, 20773}	{6002,20773, 49107}	{20773,49107, 133758}

Verification of the numerical examples for all values of m is displayed by the ensuing C program.

```
#include < stdio.h >
#include < conio.h >
#include < math.h >
void main()
{
int m, ca, k, n, p;
char ch;
long long int x(int n, int k), a, b, c, d, e, f, A;
clrscr();
do
{
printf("\nEnter the value of k and m\n");
```

$scanf("\%d\%d", \&k, \&m);$

$printf("\nEnter\ your\ choice\ 1\ or\ 2\ for\ D(-(k\hat{}2+1))\ or\ D(k\hat{}2+1)\n");$

$scanf("\%d", \&ca);$

$switch\ (ca)$

{

$case\ 1:$

$a = x(2*m, k);$

$b = x(2*m+2, k);$

$A = x(2*m+1, k);$

$p = 2*k+1;$

$c = a+2*A+b;$

$d = 3*a+(2+2*p)*A+2*b;$

$e = 6*a+(10+2*p)*A+7*b;$

$f = 13*a+(26+4*p)*A+21*b;$

$break;$

$case\ 2:$

$a = x(2*m-1, k);$

$b = x(2*m+1, k);$

$A = x(2*m, k);$

$c = a+2*A+b;$

$d = a+4*A+4*b;$

$e = 4*a+12*A+9*b;$

$f = 9*a+30*A+25*b;$

```
break;
}
printf(\n(%lld,%lld,%lld),(%lld,%lld,%lld),(%lld,%lld,%lld),
        (%lld,%lld,%lld),...",a,b,c,b,c,d,c,d,e,d,e,f);
printf("\nDo you want to continue for different m and k (y/n)?\n");
ch=getche();
}while(ch=='y'||ch=='Y');
getch();
}
long long int x(int n,int k)
{
long long int x[50],y;
x[0]=1;
x[1]=k;
int i;
for(i=2;i<=n;i++)
x[i]=2*k*x[i-1]+x[i-2];
y=x[i-1];
return y;
}
```

3.2 Demonstration of Two Disparate Structures of Integer Triples Concerning Pan-San and Pan-San Comrade Numbers

Hypothesize that

$\alpha = C_{n-1,k}, \beta = C_{n+1,k}, n \in \mathcal{N}$ where $C_{n,k} = 2(k^2+1)C_{n-1,k} - C_{n-2,k}, k, n \in \mathcal{N} - \{1\}$

be two conflicting Pan-San numbers such that $\alpha\beta + k^2$ is a number with power raised to two.

Let γ be an additional positive integer that accomplishes the ensuing consequences

$$\alpha\gamma + k^2 = a^2 \tag{3.17}$$

$$\beta\gamma + k^2 = b^2 \tag{3.18}$$

The resolution of (3.17) and (3.18) provides the possibility of γ by

$$\gamma = \frac{a^2 - b^2}{\alpha - \beta} \tag{3.19}$$

The collaboration of (3.19) in (3.18) interprets the relationship in terms of α and β as

$$\beta a^2 - \alpha b^2 = -k^2(\alpha - \beta) \tag{3.20}$$

To achieve the necessary condition, let us generate the following linear expansions

$$a = X - \alpha T \tag{3.21}$$

$$b = X - \beta T \tag{3.22}$$

The standard quadratic equation in X and T is projected by restoring the overhead values of a and b in (3.20) as below

$$X^2 - \alpha\beta T^2 = k^2 \tag{3.23}$$

Making a choice for the necessity of establishing (3.23) in the codes

$$X_0 = C_{n,k}, T_0 = 1$$

and enforcing these cryptographs in the previous held equations (3.21) and (3.22) vintages that

$$a = C_{n,k} - \alpha$$

$$b = C_{n,k} - \beta$$

The third element of conventional triple which pledge the assertion by swap the above appropriate outcomes of a and b in (3.19) is identified by

$$\gamma = 2k^2 C_{n,k}$$

Hence, it is clinched that

$\{C_{n-1,k},\ C_{n+1,k}, 2k^2 C_{n,k}\}$ is an integer triple with the property $D(k^2), k \in N - \{1\}$

Let δ be the next positive integer together with the statements that

$$\beta\delta + k^2 = c^2 \tag{3.24}$$

$$\gamma\delta + k^2 = d^2 \tag{3.25}$$

Deducting (3.25) from (3.24), the substantial value of δ is determined by

$$\delta = \frac{c^2 - d^2}{\beta - \gamma} \tag{3.26}$$

The partnership of (3.25) and (3.26) construes the succeeding bond as

$$\gamma c^2 - \beta d^2 = -k^2(\gamma - \delta) \tag{3.27}$$

Contemplate the fresh rectilinear modifications for c and d as

$$c = X + \beta T \tag{3.28}$$

$$d = X + \gamma T \tag{3.29}$$

Reestablishing the above values of c and d in (3.27), the orthodox second- degree equation is appraised by

$$X^2 - \gamma\delta T^2 = k^2 \tag{3.30}$$

Captivating $X_0 = C_{n+1,k} - C_{n,k}, T_0 = 1$ and imposing them in the expressions (3.28) and (3.29) produces the selections of c and d as

$$c = C_{n+1,k} - C_{n,k} + \beta$$

$$d = C_{n+1,k} - C_{n,k} + \gamma$$

In sight of (3.26), an essential option of δ is calculated by

$$\delta = 3C_{n+1,k} + 2C_{n,k}(k^2 - 1)$$

Hence, $\{C_{n+1,k}, 2k^2C_{n,k}, 3C_{n+1,k} + 2C_{n,k}(k^2 - 1)\}$ is a required triple in Pan-San numbers in which the product of two elements in the set added with a square other than 1 is a number with exponent two.

Now, pick ε to be some other integer that meets the following requirements

$$\gamma\varepsilon + k^2 = e^2 \tag{3.31}$$

$$\delta\varepsilon + k^2 = f^2 \tag{3.32}$$

By implementing a simple analysis in (3.31) and (3.32), it is interesting to emphasize that

$$\varepsilon = \frac{e^2 - f^2}{\gamma - \delta} \tag{3.33}$$

Suppose that

$$\delta\zeta + k^2 = g^2 \tag{3.34}$$

$$\varepsilon\zeta + k^2 = h^2 \tag{3.35}$$

where $\zeta \in Z - \{0\}$

Ensuing the erstwhile course in (3.34) and (3.35), the corresponding value of the factor ζ in the sequence is predicted by

$$\zeta = \frac{g^2 - h^2}{\delta - \varepsilon} \tag{3.36}$$

Since the mission is to deliver the exact integer values for the criteria in the vital patterns, let us use the following conversions

$$e = C_{n-1,k} + 2C_{n+1,k} - 3C_{n,k} - \gamma$$

$$f = C_{n-1,k} + 2C_{n+1,k} - 3C_{n,k} - \delta$$

$$g = 2C_{n-1,k} + 6C_{n+1,k} - 7C_{n,k} + \delta$$

$$h = 2C_{n-1,k} + 6C_{n+1,k} - 7C_{n,k} + \varepsilon$$

and subsequently the elements with the requisite forms of triples by the relevant resource in the same structure as outlined above are analyzed by

. $$\varepsilon = 2C_{n-1,k} + 7C_{n+1,k} + 4C_{n,k}(k^2 - 2)$$

$$\zeta = 6C_{n-1,k} + 22C_{n+1,k} + 6C_{n,k}(k^2 - 4)$$

Accordingly,

$\{C_{n-,k},\ C_{n+1,k}, 2k^2C_{n,k}\}$, $\{C_{n+1,k}, 2k^2C_{n,k}, 3C_{n+1,k} + 2C_{n,k}(k^2-1)\}$,

$\{2k^2C_{n,k}, 3C_{n+1,k} + 2C_{n,k}(k^2-1), 2C_{n-1,k} + 7C_{n+1,k} + 4C_{n,k}(k^2-2)\}$,

$\{3C_{n+1,k} + 2C_{n,k}(k^2-1), 2C_{n-1,k} + 7C_{n+1,k} + 4C_{n,k}(k^2-2), 6C_{n-1,k} +$

$22C_{n+1,k} + 6C_{n,k}(k^2-4)\}$, etc

are shapes of triples concerning Pan-San sequence whereas the multiplication of two barebones upgraded by k^2 is a perfect square where k is a natural number other than 1. Hence, the patterns of integer triples $\{\alpha,\beta,\gamma\},\{\beta,\gamma,\delta\},\{\gamma,\delta,\varepsilon\}$ etc in which the factors filling the above proclamation are assessed.

Elucidations for the numerical replacements of the above patterns of triples are demarcated in table 3.3.

Table 3.3

k	n	$\{\alpha,\beta,\gamma\}$	$\{\beta,\gamma,\delta\}$	$\{\gamma,\delta,\varepsilon\}$	$\{\delta,\varepsilon,\zeta\}$
2	2	{20,1960,1584}	{1960,1584,7068}	{1584,7068,15344}	{7068,15344, 43240}
3	1	{3,1197,1080}	{1197,1080,4551}	{1080,4551,10065}	{4551,10065,28152}
4	1	{4,4620,4352}	{4620,4352,17940}	{4352,17940,39964}	{17940,39964,111456}

Remark:

By smearing the identical technique as above, the following proposals of triples in which every component belong to Pan-San Comrade sequence such that the product of any two components enlarged by k^2 is a number with exponent two are designated.

$\{R_{n-,k},\ R_{n+1,k}, 2k^2R_{n,k}\}$, $\{R_{n+1,k}, 2k^2R_{n,k}, 3R_{n+1,k} + 2R_{n,k}(k^2+1)\}$,

$\{2k^2R_{n,k}, 3R_{n+1,k} + 2R_{n,k}(k^2+1), 2R_{n-1,k} + 7R_{n+1,k} + 4R_{n,k}(k^2+2)\}$,

$\{3R_{n+1,k} + 2R_{n,k}(k^2+1), 2R_{n-1,k} + 7R_{n+1,k} + 4R_{n,k}(k^2+2),$

$6R_{n-1,k} + 22R_{n+1,k} + 6R_{n,k}(k^2+4)\}$

where $R_{n,k} = 2(k^2-1)R_{n-1,k} - R_{n-2,k}$ and $R_{0,k} = 0, R_{1,k} = k$, $k \in \mathcal{N} - \{1\}$.

A limited number of numerical cases for the above sequences of triples are offered in table 3.4.

Table 3.4

k	n	$\{\alpha, \beta, \gamma\}$	$\{\beta, \gamma, \delta\}$	$\{\gamma, \delta, \varepsilon\}$	$\{\delta, \varepsilon, \zeta\}$
2	2	{12,408,560}	{408,560,1924}	{560,1924,4560}	{1924,4560,12408}
3	1	{3,765,864}	{765,864,3255}	{864,3255,7473}	{3255,7473,20592}
4	1	{4,3596,3840}	{3596,3840,14868}	{3840,14868,33820}	{14868,33820,93536}

Substantiation of the numerical examples is unveiled by the subsequent C program.

```
#include < stdio.h >
#include < conio.h >
#include < math.h >
void main( )
{
int ca, n, k;
char ch;
long long int C(int n, int k), a, b, c, d, e, f, int R(int n, int k);
```

```
clrscr( );
do
{
printf("\nEnter the value of k and n\n");
scanf("%d%d",&k,&n);
printf("\nEnter your choice 1 or 2 for Pan – San or Pan –
    San Comrade Sequence\n");
scanf("%d",&ca);
switch (ca)
{
case 1:
a = C(n – 1,k);
b = C(n + 1,k);
c = 2 * k * k * C(n,k);
d = 3 * b + 2 * C(n,k) * (k * k – 1);
e = 7 * b + 2 * a + 4 * C(n,k) * (k * k * –2);
f = 6 * a + 22 * b + 6 * C(n,k) * (k * k – 4);
break;
case 2:
a = R(n – 1,k);
b = R(n + 1,k);
c = 2 * k * k * R(n,k);
d = 3 * b + 2 * R(n,k) * (k * k + 1);
```

$e = 7 * b + 2 * a + 4 * R(n,k) * (k * k + 2);$

$f = 6 * a + 22 * b + 6 * R(n,k) * (k * k * +4);$

$break;$

}

$printf("\n(\%lld, \%lld, \%lld), (\%lld, \%lld, \%lld), (\%lld, \%lld, \%lld),$

$(\%lld, \%lld, \%lld), \ldots ", a, b, c, b, c, d, c, d, e, d, e, f);$

$printf("\nDo\ you\ want\ to\ continue\ for\ different\ n\ and\ k\ (y/n)?\n");$

$ch = getche();$

$\}while\ (ch == 'y' || ch == 'Y');$

$getch();$

}

$long\ long\ int\ C(int\ n, int\ k)$

{

$long\ long\ C[50], y;$

$C[0] = 0;$

$C[1] = k;$

$int\ i;$

$for(i = 2; i <= n; i + +)$

$C[i] = 2 * (k * k + 1) * C[i - 1] - C[i - 2];$

$y = C[i - 1];$

$return\ y;$

}

$long\ long\ R(int\ n, int\ k)$

```
{
long long R[50], y;
R[0] = 0;
R[1] = k;
int i;
for(i = 2, i <= n; i++)
R[i] = 2 * (k * k − 1) * R[i − 1] − R[i − 2];
y = R[i − 1];
return y;
}
```

Chapter – IV

Artwork of Integer Quadruple and Quintuple with Unique Properties

CHAPTER – IV

Artwork of Integer Quadruple and Quintuple with Unique Properties

This chapter comprises two sections, section 4.1 and 4.2.

In Section 4.1, an elegant non-zero distinct integer quadruple (a, b, c, d) in which addition of any three of them is a cubical integer is determined by exploiting the general solutions to a meticulous cubic Diophantine equation.

In Section 4.2, an incomparable integer quintuple (p, q, r, s, t) in such a way that the components with the renowned property in algebra named as arithmetic progression with the postulation that the addition of three consecutive terms shows a perfect square is established.

4.1 Fabrication of Gorgeous Integer Quadruple

Let a, b, c, d be four non-zero distinct integers such that addition of any three of them is a perfect cube.

Consider

$$a + b + c = p^3 \tag{4.1}$$

$$a + b + d = q^3 \tag{4.2}$$

$$a + c + d = r^3 \tag{4.3}$$

$$b + c + d = s^3 \tag{4.4}$$

together with the following condition

$$3(a + b + c + d) = (p + q + r + s)z^3 \tag{4.5}$$

Solving the system of equations from (4.1) to (4.4), the corresponding values of a, b, c, d are pointed out by

$$a = \frac{1}{3}(p^3 + q^3 + r^3 - 2s^3) \tag{4.6}$$

$$b = \frac{1}{3}(p^3 + q^3 + s^3 - 2r^3) \tag{4.7}$$

$$c = \frac{1}{3}(p^3 + r^3 + s^3 - 2q^3) \tag{4.8}$$

$$d = \frac{1}{3}(q^3 + r^3 + s^3 - 2p^3) \tag{4.9}$$

Adding (4.6), (4.7), (4.8) and (4.9), an interesting combination is enumerated by

$$3(a + b + c + d) = p^3 + q^3 + r^3 + s^3 \tag{4.10}$$

Comparison of (4.5) and (4.10) provides that

$$(p + q + r + s)z^3 = p^3 + q^3 + r^3 + s^3 \tag{4.11}$$

Employing the following linear transformations

$$p = x + 2y, q = 2x + y, r = 2y - x, s = y - 2x$$

where x and y are non-zero integers, from (4.6) to (4.9) gives

$$a = 8x^3 + 6xy^2 + 5y^3 \quad (4.12)$$

$$b = x^3 + 6x^2y + 12xy^2 - 2y^3 \quad (4.13)$$

$$c = -8x^3 - 6xy^2 + 5y^3 \quad (4.14)$$

$$d = -x^3 + 6x^2y - 12xy^2 - 2y^3 \quad (4.15)$$

Substitution of the same transformations reduce (4.11) to the quadratic equation with three unknowns as

$$6x^2 + 3y^2 = z^3 \quad (4.16)$$

Applying three different procedures of solving (4.16), the determination of an attractive integer quadruple satisfying the condition that the sum of any three quantities is a cubical integer is explained as follows.

Procedure (i):

The choice of $z = 6m^2 + 3n^2$ where $m, n \in \mathcal{Z} - \{0\}$ leads (4.16) to

$$\left(\sqrt{6}x\right)^2 + \left(\sqrt{3}y\right)^2 = \left(\left(\sqrt{6}m\right)^2 + \left(\sqrt{3}n\right)^2\right)^3$$

which implies that

$$\left(\sqrt{6}x + i\sqrt{3}y\right)\left(\sqrt{6}x - i\sqrt{3}y\right) = \left(\left(\sqrt{6}m + i\sqrt{3}n\right)\left(\sqrt{6}m - i\sqrt{3}n\right)\right)^3$$

Escalating the right-hand side of the above equation and equating real and imaginary parts on both the sides, it is to be noted that

$$x = 6m^3 - 9mn^2$$

$$y = 18m^2n - 3n^3$$

$$z = 6m^2 + 3n^2$$

Substituting the above values of x, y, z in (4.12), (4.13), (4.14) and (4.15), the values of a, b, c, d satisfying our assumption are deliberated by

$$a = 8(6m^3 - 9mn^2)^3 + 6(18m^2n - 3n^3)^2(6m^3 - 9mn^2) + 5(18m^2n - 3n^3)^3$$

$$b = (6m^3 - 9mn^2)^3 + 6(6m^3 - 9mn^2)^2(18m^2n - 3n^3) + 12(6m^3 - 9mn^2) \times (18m^2n - 3n^3)^2 - 2(18m^2n - 3n^3)^3$$

$$c = -8(6m^3 - 9mn^2)^3 - 6(6m^3 - 9mn^2)(18m^2n - 3n^3)^2 + 5(18m^2n - 3n^3)^3$$

$$d = -(6m^3 - 9mn^2)^3 + 6(6m^3 - 9mn^2)^2(18m^2n - 3n^3) - 12(6m^3 - 9mn^2) \times (18m^2n - 3n^3)^2 - 2(18m^2n - 3n^3)^3$$

Some numerical examples satisfying the hypothesis are specified in table 4.1.

Table 4.1

m	n	a	b	c	d	$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$
1	1	12609	-14067	21141	2187	27^3	9^3	33^3	21^3
2	1	2715525	1456542	569565	-2025378	168^3	129^3	108^3	9^3
3	2	165419721	9726264	104580288	-107228664	654^3	408^3	546^3	192^3

Procedure (ii):

Treating (4.16) as

$$6x^2 + 3y^2 = 1^2.z^3 \tag{4.17}$$

Assuming that

$$z = \left(\sqrt{6}m\right)^2 + \left(\sqrt{3}n\right)^2$$

and re-establish 1 by

$$1 = \frac{(\sqrt{6}+i\sqrt{3})(\sqrt{6}-i\sqrt{3})}{9}$$

in (4.17), it becomes

$$\left(\sqrt{6}x\right)^2 + \left(\sqrt{3}y\right)^2 = \left(\frac{(\sqrt{6}+i\sqrt{3})(\sqrt{6}-i\sqrt{3})}{9}\right)^2.\left(\left(\sqrt{6}m\right)^2 + \left(\sqrt{3}n\right)^2\right)^3$$

which is equivalent to

$$\left(\sqrt{6}x + i\sqrt{3}y\right)\left(\sqrt{6}x - i\sqrt{3}y\right) = \left(\frac{(\sqrt{6}+i\sqrt{3})(\sqrt{6}-i\sqrt{3})}{9}\right)^2.\left(\left(\sqrt{6}m + i\sqrt{3}n\right)\left(\sqrt{6}m - i\sqrt{3}n\right)\right)^3 \tag{4.18}$$

Equating the positive parts on both sides of (4.18) and comparing the like terms, it is examined that

$$x = 2m^3 - 12m^2n - 3mn^2 + 2n^3$$

$$y = 8m^3 + 6m^2n - 12mn^2 - n^3$$

In view of (4.12), (4.13), (4.14) and (4.15), the options of a, b, c, d are estimated by

$$a = 8(2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 + 6(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times$$

$$(8m^3 + 6m^2n - 12mn^2 - n^3)^2 + 5(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

$$b = (2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 + 6(2m^3 - 12m^2n - 3mn^2 + 2n^3)^2 \times$$

$$(8m^3 + 6m^2n - 12mn^2 - n^3) + 12(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times$$

$$(8m^3 + 6m^2n - 12mn^2 - n^3)^2 - 2(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

$$c = -8(2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 - 6(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times$$

$$(8m^3 + 6m^2n - 12mn^2 - n^3)^2 + 5(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

$$d = -(2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 + 6(2m^3 - 12m^2n - 3mn^2 + 2n^3)^2 \times$$

$$(8m^3 + 6m^2n - 12mn^2 - n^3) - 12(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times$$

$$(8m^3 + 6m^2n - 12mn^2 - n^3)^2 - 2(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

Some numerical examples satisfying the propositions are specified in table 4.2.

Table 4.2

m	*n*	*a*	*b*	*c*	*d*	$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$
1	1	-10709	-739	10719	2187	$(-9)^3$	$(-21)^3$	13^3	23^3
2	1	19683	-1771470	2480787	1751058	90^3	$(-9)^3$	162^3	135^3
3	2	-55002032	-46632952	105976512	94647096	162^3	$(-192)^3$	526^3	536^3

Procedure (iii):

Consider an alternative solution to (4.16) as

$$x = \frac{1}{\sqrt{6}}(k^3 + kl^2)$$

$$y = \frac{1}{\sqrt{3}}(l^3 + lk^2)$$

$$z = k^2 + l^2$$

Case (i):

Since the target is to evaluate integral values for the variables, it is observed that the subsequent two parametric choices of $k = \sqrt{6}m$ and $l = \sqrt{3}n$ provides the values of x and y in integers.

Then, the integral solutions to (4.16) are calculated by

$$x = 6m^3 + 3mn^2$$

$$y = 3n^3 + 6nm^2$$

$$z = 6m^2 + 3n^2$$

Substituting the above quantities in (4.12), (4.13), (4.14) and (4.15), the appropriate values of a, b, c, d are discovered by

$$a = 1728m^9 + 3888m^7n^2 + 1080m^6n^3 + 3240m^5n^4 + 1620m^4n^5$$
$$+1188m^3n^6 + 810m^2n^7 + 162mn^8 + 135n^9$$

$$b = 216m^9 + 1296m^8n + 2916m^7n^2 + 1512m^6n^3 + 4050m^5n^4$$
$$+324m^4n^5 + 1971m^3n^6 - 162m^2n^7 + 324mn^8 - 54n^9$$

$$c = -1728m^9 - 3888m^7n^2 + 1080m^6n^3 - 3240m^5n^4 + 1620m^4n^5$$
$$-1188m^3n^6 + 810m^2n^7 - 162mn^8 + 135n^9$$

$$d = -216m^9 + 1296m^8n - 2916m^7n^2 + 1512m^6n^3 - 4050m^5n^4$$
$$+324m^4n^5 - 1971m^3n^6 - 162m^2n^7 - 324mn^8 - 54n^9$$

Some numerical examples satisfying our assumption are précised in table 4.3.

Table 4.3

m	n	a	b	c	d	$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$
1	1	13851	12393	-6561	-6561	27^3	27^3	9^3	9^3
2	1	1594323	1062882	-1397493	-196830	108^3	135^3	0^3	$(-81)^3$
3	2	94298688	75611448	-71299008	-22712184	462^3	528^3	66^3	$(-264)^3$

Case (ii):

As in case (i), the single parametric choices of $k = \sqrt{6}t$ and $l = \sqrt{3}t$ offers the values of x and y in integers.

Thus,

$$x = 9t^3$$

$$y = 9t^3$$

$$z = 9t^2$$

Substituting the above magnitudes in (4.12), (4.13), (4.14) and (4.15), it is determined by

$$a = 13851\ t^9$$

$$b = 12393\ t^9$$

$$c = -6561\ t^9$$

$$d = -6561\ t^9$$

Some numerical examples satisfying the hypothesis are exemplified in table 4.4.

Table 4.4

k	a	b	c	d	$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$
1	13851	12393	-6561	-6561	27^3	27^3	9^3	$(-9)^3$
2	7091712	6345216	-3359232	-3359232	216^3	216^3	72^3	$(-72)^3$
3	272629233	243931419	-129140163	-129140163	729^3	729^3	243^3	$(-243)^3$

The C Program for numerical examples satisfying our hypotheses are illustrated below.

```
#include < stdio.h >
#include < conio.h >
#include < math.h >
void main( )
{
char ch;
int m, n, ca;
signed long int x, y, a, b, c, d, cup, cuq, cur, cus, cup1, cuq1, cur1, cus1, p, q, r, s;
clrscr();
do
{
printf("\nEnter m and n values\n");
scanf("%d%d", &m, &n);
printf("\nEnter your choice case 1 or 2 or 3 or 4\n");
```

$scanf("\%d", \&ca);$

$switch(ca)$

$\{$

$case\ 1:$

$x = (6 * m * m * m) - (9 * m * n * n);$

$y = (18 * m * m * n) - (3 * n * n * n);$

$break;$

$case\ 2:$

$x = (2 * m * m * m) - (12 * m * m * n) - (3 * m * n * n) + (2 * n * n * n);$

$y = (8 * m * m * m) + (6 * m * m * n) - (12 * m * n * n) - (n * n * n);$

$break;$

$case\ 3:$

$x = (6 * m * m * m) + (3 * m * n * n);$

$y = (3 * n * n * n) + (6 * n * m * m);$

$break;$

$case\ 4:$

$x = 9 * m * m * m;$

$y = 9 * m * m * m;$

$break;$

$\}$

$a = (8 * x * x * x) + (6 * x * y * y) + (5 * y * y * y);$

$b = (x * x * x) + (6 * x * x * y) + (12 * x * y * y) - (2 * y * y * y);$

$c = (-8 * x * x * x) - (6 * x * y * y) + (5 * y * y * y);$

$d = -(x * x * x) + (6 * x * x * y) - (12 * x * y * y) - (2 * y * y * y);$

$cup = a + b + c;$

$cuq = a + b + d;$

$cur = a + c + d;$

$cus = b + c + d;$

$if(cup < 0)$

{

$cup1 = -1 * cup;$

$p = pow(cup1,1.0/3.0);$

}

$else$

{

$p = pow(cup, 1.0/3.0);$

}

$If(p == 0)$

$p = 0;$

$else$

$p++;$

$if(cup < 0)$

$p = -p;$

$if(cuq < 0)$

{

$cuq1 = -1 * cuq;$

$q = pow(cuq1,1.0/3.0);$

$\}$

$else$

$\{$

$q = pow(cuq, 1.0/3.0);$

$\}$

$If(q == 0)$

$q = 0;$

$else$

$q++;$

$if(cuq < 0)$

$q = -q;$

$if(cur < 0)$

$\{$

$cur1 = -1 * cur;$

$p = pow(cur1,1.0/3.0);$

$\}$

$else$

$\{$

$r = pow(cur, 1.0/3.0);$

$\}$

$If(r == 0)$

$r = 0;$

```
else
r++;
if(cur<0)
r=-r;
if(cus<0)
{
cus1=-1*cus;
s=pow(cus1,1.0/3.0);
}
else
{
s=pow(cus,1.0/3.0);
}
if(s==0)
s=0;
else
s++;
if(cus<0)
s=-s;
printf("\nm=%d,n=%d,a=%ld,b=%ld,b=%ld,c=%ld,
        d=%ld\n",m,n,a,b,c,d);
printf("\na+b+c=(%ld)^3,a+b+d=(%ld)^3,a+c+d=(%ld)^3,"
        "b+c+d=(%ld)^3",p,q,r,s);
```

$printf("Do\ you\ want\ to\ continue\ (y/n)?");$

$ch = getche();$

$\}\ while\ (ch == 'Y'\ ||\ ch == 'y');$

$getch();$

}

4.2 Incomparable Integer Quintuple in Arithmetic Progression with Prominent Condition

Presume that p, q, r, s, t be five non-zero separate integers such that the elements in the quintuple (p, q, r, s, t) materialize in Arithmetic Progression.

To symbolize this proclamation, let a and d be two non-zero integers such that

$$p = a - 2d, q = a - d, r = a, s = a + d, t = a + 2d$$

Consider that the sum of three consecutive elements in the already assumed quintuple is a square of an integer.

The above declaration is replicated by the subsequent equations

$$p + q + r = 3a - 3d = \varphi^2 \quad (4.19)$$

$$q + r + s = 3a = \eta^2 \quad (4.20)$$

$$r + s + t = 3a + 3d = \chi^2 \quad (4.21)$$

Addition of (4.19) and (4.21) endow with the proportion that

$$a = \frac{\varphi^2 + \chi^2}{6} \quad (4.22)$$

Similarly, subtraction of (4.19) from (4.21) bestow as in the succeeding fraction

$$d = \frac{\chi^2 - \varphi^2}{6} \quad (4.23)$$

Elucidation of (4.20) and (4.22) yields the following equation

$$\eta^2 = \frac{\varphi^2+\chi^2}{2} \tag{4.24}$$

To convert the above said value of η as in integer, launch the novel conversions

$$\eta = 3\lambda, \varphi = 6\mu, \chi = 6\omega \tag{4.25}$$

These translations imitate (4.23) and (4.24) as follows

$$d = 6(\omega^2 - \mu^2) \tag{4.26}$$

$$\lambda^2 = 2(\mu^2 + \omega^2) \tag{4.27}$$

The elements in the required quintuple are making into integers with the property looking for is portrayed by the three procedures as below.

Procedure (i):

Decode the parameter λ as

$$\lambda = u^2 + v^2$$

Then, the equation (4.27) can be altered by

$$(u^2 + v^2)^2 = 2(\mu^2 + \omega^2)$$

$$\Longrightarrow (u + iv)^2(u - iv)^2 = (1 + i)(1 - i)(\mu + i\omega)(\mu - i\omega)$$

Then equating real and imaginary parts after escalating and balancing positive terms on both sides, the resulting equations are revealed by

$$\mu - \omega = u^2 - v^2$$

$$\mu + \omega = 2uv$$

Resolving the above equations, the most plausible values of μ and ω are demonstrated by

$$\mu = \frac{1}{2}(u^2 - v^2 + 2uv)$$

$$\omega = \frac{1}{2}(v^2 - u^2 + 2uv)$$

The parametric values of λ, μ and ω in integers are created by selecting the options of $u = 2U$ and and $v = 2V$ as follows

$$\lambda = 4(U^2 + V^2)$$

$$\mu = 2(U^2 - V^2 + 2UV)$$

$$\omega = 2(V^2 - U^2 + 2UV)$$

The replacement of the above value of λ in (4.25), endow with the value of η as

$$\eta = 12(U^2 + V^2)$$

According to (4.20) and (4.26), the components in an essential quintuple are offered by

$$a = 48(U^2 + V^2)^2$$

$$d = 192UV(V^2 - U^2)$$

Subsequently, the necessary quintuple in which the elements form an Arithmetic progression is discovered by

$$\begin{aligned}(p, q, r, s, t) = (&48(U^2 + V^2)^2 - 384UV(V^2 - U^2),\\ &48(U^2 + V^2)^2 - 192UV(V^2 - U^2), 48(U^2 + V^2)^2,\\ &48(U^2 + V^2)^2 + 192UV(V^2 - U^2),\\ &48(U^2 + V^2)^2 + 384UV(V^2 - U^2)\big)\end{aligned}$$

Logical postulation is checked for certain values of U and V as in table 4.5.

Table 4.5

U	V	(p, q, r, s, t)	$p+q+r$	$q+r+s$	$r+s+t$
2	1	(3504, 2352, 1200,48, -1104)	84^2	60^2	12^2
5	7	(-59712, 101568, 262848, 424128, 585408)	552^2	888^2	1128^2
1	3	(-4416, 192, 4800, 9408, 14016)	24^2	120^2	168^2

Procedure (ii):

The same conversion of $\lambda = u^2 + v^2$ supplies the alternative appearance of (4.27) as

$$(u+iv)^2(u-iv)^2 = \frac{(7+i)(7-i)}{25}(\mu+i\omega)(\mu-i\omega)$$

Replicate the same course of action as mentioned in procedure (i), the corresponding values of μ and ω satisfying the double equations $7\mu - \omega = 5(u^2 - v^2)$, $\mu + 7\omega = 10uv$ are appraised by

$$\mu = \frac{1}{10}(7(u^2 - v^2) + 2uv)$$

$$\omega = \frac{1}{10}(v^2 - u^2 + 14uv)$$

The chances of λ, μ and ω in integers by picking $u = 10U$ and $v = 10V$ are produced by

$$\lambda = 100(U^2 + V^2)$$

$$\mu = 10(7U^2 - 7V^2 + 2UV)$$

$$\omega = 10(V^2 - U^2 + 14UV)$$

Renovate the value of λ in (4.25), the value of η is calculated by

$$\eta = 300(U^2 + V^2)$$

In sight of (4.20) and (4.26), the equivalent choices of a and d are pointed out by

$$a = 30000(U^2 + V^2)^2$$

$$d = -4800(6U^4 + 6V^4 + 7U^3V - 7UV^3 - 36U^2V^2)$$

Hence, the needed quintuple with desired property is exposed by

$$\begin{aligned}(p,q,r,s,t) = (&30000(U^2 + V^2)^2 + 9600(6U^4 + 6V^4 + 7U^3V - 7UV^3 - 36U^2V^2),\\ &30000(U^2 + V^2)^2 + 4800(6U^4 + 6V^4 + 7U^3V - 7UV^3 - 36U^2V^2),\\ &30000(U^2 + V^2)^2,\\ &30000(U^2 + V^2)^2 - 4800(6U^4 + 6V^4 + 7U^3V - 7UV^3 - 36U^2V^2),\\ &30000(U^2 + V^2)^2 - 9600(6U^4 + 6V^4 + 7U^3V - 7UV^3 - 36U^2V^2)\big)\end{aligned}$$

Presumption is verified for definite values of U and V in table 4.6.

Table 4.6

U	V	(p,q,r,s,t)	$p+q+r$	$q+r+s$	$r+s+t$
0	1	(87600, 58800, 30000, 1200, -27600)	420^2	300^2	60^2
1	2	(-56400, 346800, 750000, 1153200, 1556400)	1020^2	1500^2	1860^2
1	1	(-110400, 4800, 120000, 235200, 350400)	120^2	600^2	840^2

Procedure (iii):

Commencement of the fresh renovation $\lambda = 2A$ in (4.27) declare the same equation as

$$2\mathrm{A}^2 = \mu^2 + \omega^2$$

$$\Longrightarrow \mathrm{A}^2 - \mu^2 = \omega^2 - \mathrm{A}^2$$

$$\Longrightarrow (\mathrm{A} + \mu)(\mathrm{A} - \mu) = (\omega + \mathrm{A})(\omega - \mathrm{A})$$

$$\Longrightarrow \left(1 + \frac{\mu}{\mathrm{A}}\right)\left(1 - \frac{\mu}{\mathrm{A}}\right) = \left(\frac{\omega}{\mathrm{A}} + 1\right)\left(\frac{\omega}{\mathrm{A}} - 1\right) \quad (4.28)$$

Again, make use of the transformations $\frac{\mu}{\mathrm{A}} = \theta, \frac{\omega}{\mathrm{A}} = \rho$ in (4.28) produces the proportion as

$$\frac{(1+\theta)}{(1+\rho)} = \frac{(\rho-1)}{(1-\theta)} = \frac{m}{n}, n \neq 0 \quad (4.29)$$

Hereafter, calculate the values of θ and ρ from (4.29) by the process of cross multiplication and then substituting these values in the ultimate transformation, it is determined by

$$\mathrm{A} = m^2 + n^2 \Longrightarrow \lambda = 2(m^2 + n^2) \quad (4.30)$$

$$\mu = m^2 + 2mn - n^2$$

$$\omega = n^2 + 2mn - m^2$$

Interpretation of (4.20) and (4.26) offers the relevant values of a and d as presented in the equations scripted below.

$$a = 12(m^2 + n^2)^2$$

$$d = 48mn(n^2 - m^2)$$

Hence, the necessary quintuple in which the elements in Arithmetic progression is rendered by

$$(p, q, r, s, t) = \{12(m^2 + n^2)^2 - 96mn(n^2 - m^2), 12(m^2 + n^2)^2 - 48mn(n^2 - m^2),$$
$$12(m^2 + n^2)^2, 12(m^2 + n^2)^2 + 48mn(n^2 - m^2),$$
$$2(m^2 + n^2)^2 + 96mn(n^2 - m^2)\}$$

Supposition is authenticated for specific values of m and n in the following table 4.7.

Table 4.7

m	n	(p, q, r, s, t)	$p+q+r$	$q+r+s$	$r+s+t$
2	1	(-276, 12, 300, 588, 876)	6^2	30^2	42^2
5	7	(-14928, 25392, 65712, 106032, 146352)	276^2	444^2	564^2
1	3	(-1104, 48, 1200, 2352, 3504)	12^2	60^2	84^2

The emerging C software shows verification of the numerical samples:

```
#include <stdio.h>
#include <conio.h>
#include <math.h>
void main()
{
char ch;
clrscr();
do
{
long long int x,u,v,m,n;
long long int U,V,M,N,a,d,p,q,r,s,t,A,B,C,E,F,G;
printf("\nEnter the case 1 or 2 or 3\n");
scanf("%lld",&x);
```

$switch(x)$

{

$case\ 1$:

$printf("\nEnter\ integer\ values\ for\ u\ and\ v\n");$

$scanf("\%lld\%lld", \&u, \&v);$

$U = u * u;$

$V = v * v;$

$a = 48 * (U + V) * (U + V);$

$d = 192 * u * v * (V - U);$

$p = a - 2 * d;$

$q = a - d;$

$r = a;$

$s = a + d;$

$t = a + 2 * d;$

$break;$

$case\ 2$:

$printf("\n\ Enter\ integer\ values\ for\ u\ and\ v\n");$

$scanf("\%lld\%lld", \&u, \&v);$

$U = u * u;$

$V = v * v;$

$a = 30000 * (U + V) * (U + V);$

$d = -4800 * (6 * U * U + 6 * V * V + 7 * U * u * v - 7 * u * v * V - 36 * U * V);$

$p = a - 2 * d;$

$q = a - d;$

$r = a;$

$s = a + d;$

$t = a + 2 * d;$

$break;$

$case\ 3:$

$printf("\n\ interger\ values\ for\ m\ and\ n\n");$

$scanf("\%lld\%lld", \&m, \&n);$

$M = m * m;$

$N = n * n;$

$a = 12 * (M + N) * (M + N);$

$d = 48 * m * n * (N - M);$

$p = a - 2 * d;$

$q = a - d;$

$r = a;$

$s = a + d;$

$t = a + 2 * d;$

$break;$

}

$A = p + q + r;$

$B = q + r + s;$

$C = r + s + t;$

$E = sqrt(A);$

$F = sqrt(B);$

$G = sqrt(C);$

$printf("\np + q + r = \%lld = \%lld\^2\nq + r + s = \%lld = \%lld\^2\n$

$r + s + t = \%lld = \%lld\^2", A, E, B, F, C, G);$

$printf("\nDo\ you\ want\ to\ continue\ for\ different\ cases\ (y/n)?");$

$ch = getche( );$

$}while\ (ch == 'y' || ch == 'Y');$

$getch( );$

}

Chapter – V

A State of the-Art of Sums, Congruence Relations and Divisibility Properties of Pell and Pell-Lucas Numbers

CHAPTER – V

A State of the-Art of Sums, Congruence Relations and Divisibility Properties of Pell and Pell-Lucas Numbers

This chapter encompasses two sections, 5.1 and 5.2.

In Section 5.1, several new-fangled identities regarding Pell and Pell-Lucas numbers enable to provide certain congruence relations for those numbers are deliberated.

In Section 5.2, divisibility properties of Pell and Pell-Lucas numbers are revealed by means of the derived congruence relations detailed in section 5.1.

5.1 Sums and Congruences of Pell and Pell-Lucas Numbers

In this section, some novel identities concerning Pell and Pell-Lucas numbers allow to offer certain congruence relations for such numbers are reflected.

Theorem 5.1

If X is a square matrix with $X^2 = 2X + I$, then $X^n = P_n X + P_{n-1} I$ for every integer n.

Proof:

Let $Z[\alpha] = \{A\alpha + B; A, B \in Z\}$ and $Z[X] = \{AX + BI; A, B \in Z\}$

Define a function $f: Z[\alpha] \to Z[X]$ by $f(A\alpha + B) = AX + BI$.

Then f is a ring isomorphism. Moreover, it is clear that $f(\alpha) = X$ and $f(Q_m) = Q_m I$.

Therefore, $X^n = \big(f(\alpha)\big)^n = f(\alpha^n) = f(P_n\alpha + P_{n-1}) = P_n X + P_{n-1} I$

Corollary 5.1.1

If $M = \begin{bmatrix} 1 & 4 \\ 1/2 & 1 \end{bmatrix}$, then $M^n = \begin{bmatrix} \frac{Q_n}{2} & 4P_n \\ \frac{P_n}{2} & \frac{Q_n}{2} \end{bmatrix}$.

Proof:

Since, $M^2 = 2M + I$, it follows from theorem (5.1) that

$$M^n = P_n M + P_{n-1} I$$

$$M^n = \begin{bmatrix} P_n + P_{n-1} & 4Pn \\ \frac{P_n}{2} & P_n + P_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{Q_n}{2} & 4P_n \\ \frac{P_n}{2} & \frac{Q_n}{2} \end{bmatrix}.$$

Remark:

From the fact that $f: Z[\alpha] \to Z[M]$, defined by $f(A\alpha + B) = AM + BI$ is a ring isomorphism, it is observed that

$$\alpha^{2m} - Q_m\alpha^m + (-1)^m = 0 \tag{5.1}$$

and $$\alpha^{2m} - 2\sqrt{2}P_m\alpha^m - (-1)^m = 0 \tag{5.2}$$

Applying the function f on each side of (5.1) and (5.2), the relations in matrix as mentioned in the theorem discovered are pointed out by

$$M^{2m} - Q_m M^m + (-1)^m I = 0 \tag{5.3}$$

and $$M^{2m} - KP_m M^m - (-1)^m I = 0 \tag{5.4}$$

where $K = f\left(2\sqrt{2}\right) = f(2\alpha - 2) = 2M - 2I = \begin{bmatrix} 0 & 8 \\ 1 & 0 \end{bmatrix}$.

Theorem 5.2

If $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$, then

$$Q_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i Q_{mi+k}$$

and $$P_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i P_{mi+k}$$

Proof:

From (5.3), it is noted that

$$M^{2m} = Q_m M^m - (-1)^m I \tag{5.5}$$

Raising n^{th} power on both sides of (5.5).

Then, $M^{2mn} = (Q_m M^m - (-1)^m I)^n = (Q_m M^m + (-1)^{m+1} I)^n$

$$= \sum_{i=o}^{n} \binom{n}{i} ((-1)^{m+1} I)^{n-i} (Q_m M^m)^i$$

$$= (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i M^{mi}$$

Therefore, $M^{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i M^{mi+k}$

It comprehends from corollary 5.1.1 that

$$Q_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i Q_{mi+k}$$

and $$P_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i P_{mi+k}$$

Corollary 5.2.1

$$Q_{2mn+k} \equiv (-1)^{(m+1)n} Q_k \ (mod\ Q_m) \tag{5.6}$$

and $$P_{2mn+k} \equiv (-1)^{(m+1)n} P_k \ (mod\ Q_m) \tag{5.7}$$

for every $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$.

Remark:

(i) Since $K = 2M - 2I = M + M^{-1},\ M^m K = K M^m, \forall m \in \mathcal{Z}$

(ii) $K^2 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = 8I$ and $\begin{bmatrix} 0 & 8 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 8c & 8d \\ a & b \end{bmatrix}$.

Theorem 5.3

If $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$, then

$$Q_{2mn+k} = (-1)^{mn} \left\{ \sum_{i=o}^{\lfloor n/2 \rfloor} \binom{n}{2i} 8^i {P_m}^{2i} Q_{2mi+k} + \right.$$

$$\left. \sum_{i=o}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} (-1)^{mi} 8^i {P_m}^{2i+1} P_{2mi+m+k} \right\}$$

and

$$P_{2mn+k} = (-1)^{mn}\left\{\sum_{i=o}^{\lfloor n/2\rfloor}\binom{n}{2i}8^i\,{P_m}^{2i}P_{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2\rfloor}\binom{n}{2i+1}(-1)^{mi}8^i\,{P_m}^{2i+1}Q_{2mi+m+k}\right\}$$

Proof:

From (5.4), it follows that

$$M^{2m} = KP_mM^m+(-1)^mI$$

Therefore, $M^{2mn+k} = (KP_mM^m+(-1)^mI)^nM^k$

$$= \left[\sum_{i=o}^{n}\binom{n}{i}((-1)^mI)^{n-i}\,(KP_mM^m)^i\right]M^k$$

$$= (-1)^{mn}\sum_{i=o}^{n}\binom{n}{i}(-1)^{mi}K^i{P_m}^i\,M^{mi+k}$$

$$= (-1)^{mn}\left\{\sum_{i=o}^{\lfloor n/2\rfloor}\binom{n}{2i}K^{2i}\,{P_m}^{2i}M^{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2\rfloor}\binom{n}{2i+1}(-1)^{mi}K^{2i+1}\,{P_m}^{2i+1}M^{2mi+m+k}\right\}$$

$$= (-1)^{mn}\left\{\sum_{i=o}^{\lfloor n/2\rfloor}\binom{n}{2i}8^i\,{P_m}^{2i}M^{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2\rfloor}\binom{n}{2i+1}(-1)^{mi}8^i\,KP_m{}^{2i+1}M^{2mi+m+k}\right\}$$

The required results are accomplished by trading the matrices K and M on both sides and equating the same entries.

Corollary 5.3.1

If $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$, then

$$Q_{2mn+k} \equiv (-1)^{mn}Q_k\ (mod\ P_m) \qquad (5.8)$$

and $$P_{2mn+k} \equiv (-1)^{mn}P_k\ (mod\ P_m) \qquad (5.9)$$

5.2 Divisibility Properties of Pell and Pell-Lucas Numbers

To begin, it is established two well-known theorems in a novel manner by exploiting the congruences postulated in Corollaries 5.2.1 and 5.3.1. Regarding the divisibility of Pell and Pell-Lucas numbers, readers will investigate the formulae and learn how to use them efficiently to resolve problems. Thus, this section explains the fundamental divisibility for Pell and Pell-Lucas numbers.

Remark:

From the identity $2(-1)^n = Q_n P_{n-1} - Q_{n-1} P_n$, it can be seen that $gcd(Q_n, P_n) = 1$ or $gcd(Q_n, P_n) = 2$. Furthermore,

$$Q_n{}^2 - 8P_n{}^2 = 4(-1)^n \tag{5.10}$$

From equation (5.8), it is seen that $Q_{8q+r} \equiv Q_r(mod 12)$ and therefore $12 \nmid Q_n$, for every natural number n.

Now, some Pell-Lucas numbers identities that will be needed in the sequel are deliberated below.

$$Q_{2n} = Q_n{}^2 - 2(-1)^n \tag{5.11}$$

$$Q_{3n} = Q_n\left(Q_n{}^2 - 3(-1)^n\right) \tag{5.12}$$

Theorem 5.4

The necessary and sufficient conditions for $Q_m | Q_n$ are

(i) $m | n$ and

(ii) $\frac{n}{m}$ is an odd integer

for all $m, n \in \mathcal{N}$ and $m \geq 2$.

Proof:

Presume that $Q_m|Q_n$

Suppose $m \nmid n$, then by fundamental property of divisibility, n can be expressed as $n = mq + r, 0 \leq r < m.$

If q is an even integer, then $q = 2s$ for some $s \in \mathcal{Z}$.

From (5.6), it tracks that

$$Q_n = Q_{2ms+r} \equiv (-1)^{(m+1)s} Q_r \ (mod\ Q_m)$$

Since $Q_m|Q_n$, $Q_m|Q_r$. This is a contradiction since $Q_r < Q_m$ as $r < m$. Hence, q is an odd integer. Sustain $q = 2s + 1$ for some $s \in \mathcal{Z}$. So,

$$Q_n = Q_{2ms+m+r} \equiv (-1)^{(m+1)s} Q_{m+r} \ (mod\ Q_m)$$

Also, since $Q_m|Q_n, Q_m|Q_{m+r}$.

To prove: $r = 0$

Suppose $r > 0$. By the identity $Q_{m+r} = Q_m P_{r-1} + P_r Q_{m+1}$, the above implies that $Q_m|P_r Q_{m+1}$.

Since $(Q_m, Q_{m+1}) = 1$, it follows that $Q_m|P_r$. This is a contradiction to the fact that if $r < m$, then $P_r \leq P_m < Q_m$. As a result, it is determined that $r = 0$.

Thus, $n = mq$, with q being an odd integer.

Conversely, suppose that $m|n$ and $\frac{n}{m}$ is an odd integer,

That is, $n = m(2s + 1)$, for some integer s. Then it is procured that,

$$Q_n = Q_{2ms+m} \equiv (-1)^{(m+1)s} Q_m \ (mod\ Q_m)$$

This is true only when $Q_m|Q_n$.

Hence, the theorem.

Theorem 5.5

Let $m, n \in \mathcal{N}$ and $m \geq 2$. Then $Q_m|P_n$ if and only if $m|n$ and $\frac{n}{m}$ is an even integer.

Proof:

Suppose that $Q_m|P_n$ and $m \nmid n$.

This assumption means that $n = mq + r, 0 \leq r < m$ where $m \geq 2$.

If q is an odd integer, it may phrase $q = 2s + 1$ for some integer s.

From (5.7), it is distinguished that

$$P_n = P_{2ms+m+r} \equiv (-1)^{(m+1)s} P_{m+r} \ (mod\ Q_m)$$

Then, $Q_m|P_{m+r}$ and hence $Q_m|8P_{m+r}$.

Then, from the identity that $8P_{m+r} = Q_m Q_{r-1} + Q_r Q_{m+1}$ then $Q_m|Q_r Q_{m+1}$. Since Q_m and Q_{m+1} are relatively prime, the only possibility is $Q_m|Q_r$. But $r < m$ delivers $Q_r < Q_m$. So, $Q_m \nmid Q_r$. This conflict befalls as a result of our erroneous assumption about q being an odd number. Therefore, q is an even integer. Thus, it may have $q = 2s$ for some integer s.

Now, (5.7) condenses to $P_n = P_{2ms+r} \equiv (-1)^{(m+1)s} P_r \ (mod\ Q_m)$

Since $Q_m|P_n$, $Q_m|P_r$.

However, this cannot be true since $r < m$ and hence $P_r \leq P_m < Q_m$. This contributes that $r = 0$. So, it can be concluded that $n = mq$, q is an even integer.

Conversely, suppose that $m|n$ and $n = 2ms$ for some $s \in \mathcal{Z}$. Then, it is acquired from (5.7) that

$$P_n = P_{2ms} \equiv (-1)^{(m+1)s} P_0 \ (mod\ Q_m)$$

It follows that $Q_m|P_n$.

Theorem 5.6

For all $m, n \in \mathcal{N}$ and $m \geq 3$, $P_m|P_n$ if and only if $m|n$.

Proof:

Initially consider that $P_m|P_n$ but $m \nmid n$. Then $n = mq + r$ with $0 \leq r < m$. Now, suppose that q is an even integer, then this may be taken as $q = 2s$ for any integer s.

Hence, (5.9) provides the succeeding identity

$$P_n = P_{2ms+r} \equiv (-1)^{ms} P_r \ (mod\ P_m)$$

Since $P_m|P_n$, by applying the above identity, $P_m|P_r$. Since, if $0 \leq r < m$ and $m \geq 3$, it leads to $P_r < P_m$. Hence, q must be an odd integer. Then $q = 2s + 1$, for some $s \in \mathcal{Z}$. Thus, (5.9) becomes

$$P_n = P_{2ms+m+r} \equiv (-1)^{ms} P_{m+r} \ (mod\ P_m)$$

Since $P_m|P_n$, it follows that $P_m|P_{m+r}$. By the identity, $P_{m+r} = P_{m+1}P_r + P_m P_{r-1}$, it is noted that $P_m|P_{m+1}P_r$. Due to the fact that $(P_m, P_{m+1}) = 1$, it is received that $P_m|P_r$

which is a contradiction. This emerges as a consequence of $P_r < P_m$ as $r < m$ and $m \geq 3$. As a result, $r = 0$ and subsequently $n = mq$, resulting in $m|n$.

Conversely, pretend that $m|n$. Then, the conclusion is $n = mq$ for some natural number q. As an outcome,

$$P_n = P_{mq} = \sum_{i=0}^{q} \binom{q}{i} {P_m}^i {P_{m-1}}^{q-i} P_i$$

Hence, it is realized that $P_m|P_n$.

Theorem 5.7

Let $r \geq 1$, be an odd number and $m > 1$. Then, there is no Pell-Lucas number Q_n such that $Q_n = Q_{2r}Q_m x^2$

Proof:

Assume that $Q_n = Q_{2r}Q_m x^2$ and r is an odd number. Then $Q_{2r}|Q_n$ and $Q_m|Q_n$. Then, $n = 2rt$ and $n = mk$ for some odd natural number t & k by theorem 5.4, this implies that $2|n \Rightarrow n = mk \Rightarrow 2|m$. It is thus obvious that $m = 2v$, for some odd $v \in \mathcal{N}$. Since $2|n$ and $\frac{n}{2}$ is an odd natural number, it can be written as $n = 8q + s$ with $s = 2,6$ and $q \geq 0$. Hence,

$$Q_n = Q_{8q+s} \equiv Q_s (mod\ 12)$$

$$\Rightarrow Q_n \equiv Q_2, Q_6\ \ (mod 12)$$

$$\Rightarrow Q_n \equiv 6\ (mod\ 12)$$

Similarly, it can be seen that $Q_m \equiv 6\ (mod\ 12)$.

Since, r is an odd natural number, it is attained that $Q_{2r} \equiv 6(mod\ 12)$. Then it follows that

$$Q_n = Q_{2r}Q_m x^2 \equiv 6Q_m x^2 (mod\ 12)$$

Moreover, $6x^2 \equiv 0,6\ (mod\ 12)$ and $Q_m \equiv 6\ (mod\ 12)$, $Q_n \equiv 0(mod\ 12)$ which contradicts the fact that $Q_n \equiv 6\ (mod\ 12)$. This concludes the proof.

Theorem 5.8

$Q_{2^k t} \equiv 2,10\ (mod\ 12)$ for every $k \geq 2$ and for every odd natural number t.

Proof:

Assume that t is an odd natural number, then $t \equiv \pm 1, \pm 3, \pm 5, \pm 7\ (mod\ 8)$.

Moreover, it can be proved by induction that $2^k \equiv 0, \pm 4\ (mod\ 8)$ for $k \geq 2$ and $2^k t \equiv 0, \pm 4\ (mod\ 8)$.

Therefore, $2^k t = 8q$ or $2^k t = 8q \pm 4$ for $q \geq 0$. Then it seeks that

$$Q_{2^k t} = Q_{8q} \equiv Q_0\ (mod\ P_4)$$

Or

$$Q_{2^k t} = Q_{8q \pm 4} \equiv Q_{\pm 4}\ (mod\ P_4)$$

Thus, $Q_{2^k t} \equiv 2,10\ (mod\ 12), k \geq 2$.

Now, it is possible to generalize theorem as follows:

Theorem 5.9

Let $m > 1, k \geq 2$ and t be an odd natural number. Then there is no Pell-Lucas number Q_n such that $Q_n = Q_{2^k t} Q_m x^2$.

Proof:

Assume that $Q_n = Q_{2^k t} Q_m x^2$ and t is an odd natural number. Since $Q_{2^k t} | Q_n$ and $Q_m | Q_n$, there exist two odd natural numbers u and v such that $n = 2^k tu$ and $n = mv$ by theorem (5.4). Thus, $m = 2^k r$, for some $r \in \mathcal{N}$ because $n = 2^k tu = mv$ and t, u, v are odd natural numbers. Then, it is pursued that 4 divides both m and n by the fact that $k \geq 2$. Hence, $n = 8q + s$ with $s = 0,4,8,12$. Thus,

$$Q_n = Q_{8q+s} \equiv Q_s \ (mod\ 12)$$

Since $s \in \{0,4,8,12\}$, it follows that

$$Q_n \equiv 2,10 \ (mod\ 12)$$

It may be showed in a similar manner that

$$Q_m \equiv 2,10 \ (mod\ 12)$$

On the other hand, $Q_{2^k t} \equiv 2,10 \ (mod\ 12)$ by theorem (5.8).

If $Q_{2^k t} \equiv 2 \ (mod\ 12)$, then $Q_n = Q_{2^k t} Q_m x^2 \equiv 2 Q_m x^2 \ (mod\ 12)$.

Since $2x^2 \equiv 0,2,6,8 \ mod(12)$ and $Q_m \equiv 2,10 \ (mod\ 12)$, $Q_n \equiv 0,4,8 \ (mod\ 12)$, which is a contradiction to the fact that $Q_n \equiv 2,10 \ (mod\ 12)$.

Therefore, $Q_{2^k t} \equiv 10 \ (mod\ 12)$. Then $Q_n = Q_{2^k t} Q_m x^2 \equiv 10 Q_m x^2 \ (mod\ 12)$.

Since $10x^2 \equiv 0,4,6,10 \ mod(12)$ and $Q_m \equiv 2,10 \ (mod\ 12)$, $Q_n \equiv 0,4,8 \ (mod\ 12)$, which denies the fact that $Q_n \equiv 2,10 \ (mod\ 12)$. Hence the proof.

Theorem 5.10

If m and r are odd natural numbers, then there is no Pell-Lucas number Q_n such that $Q_n = Q_m Q_r$.

Proof:

Assume that $Q_n = Q_m Q_r$, for $m > 1$ and $r > 1$ and are odd numbers. Since $Q_m | Q_n$ and $Q_r | Q_n$, there exists two odd natural numbers u and v such that $n = mu$ and $n = rv$.

Hence, $u = 4k \pm 1$ for some $k \geq 1$.

Therefore, $n = mu = m(4k \pm 1) = 4km \pm m$.

$$Q_n = Q_{4km \pm m} \equiv (-1)^k Q_{\pm m} \ (mod\ Q_{2m})$$

i.e., $$Q_r Q_m \equiv \pm Q_m \ (mod\ Q_{2m}) \qquad (5.13)$$

Similarly, it is conquered that

$$Q_m Q_r \equiv \pm Q_r \ (mod\ Q_{2r}) \qquad (5.14)$$

Suppose that $Q_m | Q_{2r}$ then $\frac{2r}{m}$ = an odd integer which is not possible. Hence $Q_m \nmid Q_{2r}$ which implies that $gcd(Q_m, Q_{2r}) = 2$. Then (5.13) and (5.14) yields that

$$Q_r \equiv \pm 1 \ (mod\ \tfrac{Q_{2m}}{2}) \text{ and } Q_m \equiv \pm 1 \ (mod\ \tfrac{Q_{2r}}{2})$$

$$\Rightarrow Q_{2m} \leq 2Q_r \pm 2 \text{ and } Q_{2r} \leq 2Q_m \pm 2$$

$$\Rightarrow Q_{2m} + Q_{2r} \leq 2Q_r + 2Q_m \pm 4$$

Also, from (5.11) the inequalities derived are

$${Q_m}^2 \pm 2 + {Q_r}^2 \pm 2 \leq 2Q_r + 2Q_m \pm 4$$

$$Q_m(Q_m - 2) + Q_r(Q_r - 2) \leq 0,$$

which is a contradiction. This completes the proof of the theorem.

Corollary 5.10.1

There is no Pell-Lucas number Q_n such that $Q_n = Q_m Q_r$, for any $m > 1$ and $r > 1$.

Proof:

If $r > 1$ and even, then ther proof follows from theorems (5.7) and (5.9).

If m and r are odd natural numbers, then it is proved in theorem (5.10).

Chapter – VI

Quadratic Diophantine Equations with Solutions as Familiar Numbers

CHAPTER – VI

Quadratic Diophantine Equations with Solutions as Familiar Numbers

This chapter consists of two sections 6.1 and 6.2.

In Section 6.1, the widespread solutions in rapports with Pell and Pell-Lucas numbers for restricted number of an unambiguous polynomial equations of degree two in two variables are exposed.

Section 6.2 unveils patterns of positive integer solutions for limited number of explicit binary quadratic equations encompassing Jacobsthal and Jacobsthal-Lucas numbers by means of the pertinent features connecting these two numbers and the notions of divisibility.

6.1 Assessment of solutions in Pell and Pell – Lucas numbers to Disparate Polynomial Equations of degree two

Needed Theorems:

Theorem [I]

If $\omega = 1 + \sqrt{2}$. Then the numbers $\pm\omega^{n}, \pm\omega^{-n}$ are the only unities of $k(\sqrt{2})$ where $k(\sqrt{2})$ is a quadratic field. See [66]

Theorem [II]

If positive integers x, y, k and the integer m with $gcd(x, m) = 1$ satisfy the equations $x^2 - kxy + y^2 \mp mx = 0$, then $x = u^2$ and $y = uv$ for some positive integers u and v. If positive integers x, y, k and the integer m with $gcd(y, m) = 1$ satisfy the equations $x^2 - kxy - y^2 \mp my = 0$, then $y = u^2$ and $x = uv$ for some positive integers u and v. See [82]

Theorem [III] Fundamental theorem of arithmetic

For each integer $n > 1$, there exists primes $p_1 \leq p_2 \leq \cdots \leq p_r$ such that $n = p_1 p_2 \ldots p_r$, this factorization is unique.

Theorem [IV]

If positive integers x, y, a, b, c with $gcd(x, c) = 1$ satisfying the equations $x^2 - axy - by^2 \pm cx = 0$, then $x = u^2$ and $y = uv$ for some positive integers u and v.

If positive integers x, y, p, q, r with $gcd(y, r) = 1$ satisfying the equations $x^2 - axy - by^2 \pm cy = 0$, then $y = u^2$ and $x = uv$ for some positive integers u and v.

Primary Results:

The n^{th} Pell number labelled by P_n is demarcated by $P_0 = 0, P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. If α, β be the roots of the equation $x^2 - 2x - 1 = 0$, then $\alpha = 1 + \sqrt{2}, \beta = 1 - \sqrt{2}$ where $\alpha\beta = -1$ and $\alpha + \beta = 2$. In addition, it is well-known and simple to demonstrate the identities that $\alpha^n = \alpha P_n + P_{n-1}$ and $\beta^n = \beta P_n + P_{n-1}$ for every $n \in \mathcal{Z}$, the set of all integers. In the other hand, it could be perceived by induction that

$$P_n{}^2 - 2P_nP_{n-1} - P_{n-1}{}^2 = (-1)^{n+1} \quad \forall n \in \mathcal{Z}. \tag{6.1}$$

The n^{th} Pell-Lucas number Q_n is characterized as $Q_0 = Q_1 = 2$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$ .The associations between Pell and Pell-Lucas numbers are agreed as follows

(i) $Q_n = P_n + P_{n+1}$ for every $n \in \mathcal{Z}$

(ii) $Q_n{}^2 - 2Q_nQ_{n-1} - Q_{n-1}{}^2 = 8(-1)^n$ for every $n \in \mathcal{Z}$ (6.2)

Theorem 6.1

The necessary and sufficient condition for all non-negative integer solutions to the second-degree equation in two variables $X^2 - 2XY - Y^2 = (-1)^{n+1}$ is $(X, Y) = (P_n, P_{n-1})$ with $n \geq 1$.

Proof:

If $(X, Y) = (P_n, P_{n-1})$, then from identity (6.1), it seeks that $X^2 - 2XY - Y^2 = (-1)^{n+1}$.

Conversely suppose that $X^2 - 2XY - Y^2 = \mp 1$ for some positive integers X and Y.

Then by theorem (I), $(Y + \alpha X)(Y + \beta X) = \pm 1 \Rightarrow (Y + \alpha X) \in k(\sqrt{2})$.

Thus, $Y + \alpha X = \alpha^n = \alpha P_n + P_{n-1}$ and hence $(X, Y) = (P_n, P_{n-1}), n \geq 1$.

Corollary 6.1.1

The feasible solutions of the quadratic polynomial equation $X^2 - 2XY - Y^2 = 1$ are specified by $(X, Y) = (P_{2m+1}, P_{2m})$ with $m \geq 0$.

Proof:

If n is odd such that $n = 2m + 1$ in theorem 6.1, then the apt integer solutions to $X^2 - 2XY - Y^2 = 1$ is obtained as $(X, Y) = (P_{2m+1}, P_{2m}), m \geq 0$.

Corollary 6.1.2

Every possible solution in Pell numbers of the quadratic equation $X^2 - 2XY - Y^2 = -1$ are stated by $(X, Y) = (P_{2m}, P_{2m-1})$ with $m \geq 1$.

Proof:

If n is even such that $n = 2m$, $m \geq 1$ in theorem 6.1, then the appropriate solutions to the equation $X^2 - 2XY - Y^2 = -1$ are $(X, Y) = (P_{2m}, P_{2m-1})$.

Theorem 6.2

The probable integer solutions to the second-degree polynomial equation $X^2 - 6XY + Y^2 = 4$ are attained by $(X, Y) = (P_{2n+2}, P_{2n})$ with $n \geq 0$.

Proof:

Assume that $X^2 - 6XY + Y^2 = 4$ for some positive integers X and Y.

Without loss of generality, suppose that $X > Y$.

Then, $\left(\frac{X-Y}{2}\right)^2 - 2\left(\frac{X-Y}{2}\right)Y - Y^2 = 1 \Rightarrow X^2 - 6XY + Y^2 = 4$. By corollary 6.1.1, it should have $\frac{X-Y}{2} = P_{2n+1}$ and $Y = P_{2n}$ and therefore $(X - Y, Y) = (2P_{2n+1}, P_{2n})$

Consequently, $X = P_{2n+2}$ and $Y = P_{2n}$, $n \geq 0$.

Theorem 6.3

The positive integer roots of the binary quadratic equation $X^2 - 6XY + Y^2 = -4$ are conquered by $(X, Y) = (P_{2n+1}, P_{2n-1})$ with $n \geq 1$.

Proof:

The proof is equivalent to Theorem 6.2.

Theorem 6.4

Let $\mathcal{N}$ be the set of all-natural numbers and $X, Y \in \mathcal{N}$ sustaining the particular equation of the form $X^2 - 2XY - Y^2 \pm X = 0$, then $X = A^2$ and $Y = AB$ where $A, B \in \mathcal{N}$.

Proof:

If $X, Y \in N$ are satisfying the corresponding equation in the statement, then it follows that $X|Y^2$ and hence $Y^2 = XZ$ for some $Z \in \mathcal{N}$.

Suppose that $p|X$ and $p|Z$ for some prime number p.

Then, $p|Y$ which leads the implicit equation to $X - 2Y - Z \pm 1 = 0$

This ensure that $p|1$ which is absurd.

In what follows that $\gcd(X, Z) = 1$.

Then by theorem [III], $X = A^2$ and $Z = B^2$ for some positive integers A and B where $gcd(A, B) = 1$.

Hence, $Y^2 = XZ = A^2B^2 \Rightarrow Y = AB$.

Theorem 6.5

If two positive integers X, Y be such that $X^2 - 2XY - Y^2 \pm Y = 0$, then $X = AB$ and $Y = A^2$ for some positive integers A and B with $gcd(A, B) = 1$.

Proof:

The proof is analogous to Theorem 6.4.

Corollary 6.5.1

The conceivable Pell values of X, Y in the equation $X^2 - 2XY - Y^2 + Y = 0$ are achieved by $(X, Y) = \left(P_{2n}\, P_{2n-1}\, , {P_{2n-1}}^2\right)$, $n \geq 1$.

Corollary 6.5.2

The plausible solutions in Pell numbers to the equation $X^2 - 2XY - Y^2 - Y = 0$ are specified by $(X, Y) = \left(P_{2n+1}\, P_{2n}\, , {P_{2n}}^2\right)$, $n \geq 0$.

Theorem 6.6

Let $X, Y \in \mathcal{Z}^+$. Then $X^2 - 2XY - Y^2 + X = 0$ if and only if $(X, Y) = \left({P_{2n}}^2, P_{2n}\, P_{2n-1}\right)$ where $n \geq 1$.

Proof:

Assume that $X^2 - 2XY - Y^2 + X = 0$ where $X, Y \in \mathcal{Z}^+$

Then by theorem [II], $X = A^2$ and $Y = AB$ for some $A, B \in \mathcal{Z}^+$

Subsequently $A^2 - 2AB - B^2 + 1 = 0$

Again, by corollary 6.1.2, it is noticed that $(A, B) = (P_{2n}, P_{2n-1})$

This implies that $(X, Y) = \left({P_{2n}}^2, P_{2n}\, P_{2n-1}\right)$ where $n \geq 1$.

Conversely, if $(X, Y) = \left({P_{2n}}^2, P_{2n}\, P_{2n-1}\right)$ with $n \geq 1$, then

$$X^2 - 2XY - Y^2 + X = {P_{2n}}^4 - 2{P_{2n}}^3\, P_{2n-1} - {P_{2n}}^2\, {P_{2n-1}}^2 + {P_{2n}}^2$$

$$= {P_{2n}}^2 \left\{{P_{2n}}^2 - 2P_{2n}\, P_{2n-1} - {P_{2n-1}}^2 + 1\right\}$$

By corollary 6.1.2, it is pursued that

$$X^2 - 2XY - Y^2 + X = 0.$$

Theorem 6.7

The sequence of several positive integer solutions for the quadradic equation $X^2 - 6XY + Y^2 + 4X = 0$ are epitomized by $(X, Y) = \left({P_{2n-1}}^2,\ P_{2n-1}P_{2n+1}\right)$ where $n \geq 1$.

Proof:

Let $X, Y \in \mathcal{Z}^+$ be such that $X^2 - 6XY + Y^2 + 4X = 0$.

Then $X | Y^2$ and hence $Y^2 = XZ$ for some $Z \in \mathcal{Z}^+$.

If $p|X$ and $p|Z$ for some prime number p, then $p|Y$ and also the relation $X - 6Y + Z + 4 = 0$ is true for all $X, Y \in \mathcal{Z}^+$.

Thus, p divides 4.

It is evident that, the possibility of such p is $p = 2$.

This condition offers that $X = 2X_1, Y = 2Y_1$ for some $X_1, Y_1 \in \mathcal{Z}^+$ and obviously the projected quadratic equation becomes ${X_1}^2 - 6X_1Y_1 + {Y_1}^2 + 2X_1 = 0$. Then $X_1|{Y_1}^2$ and hence ${Y_1}^2 = X_1Z_1$ for some $Z_1 \in \mathcal{Z}^+$.

Again, if $p|X_1$ and $p|Z_1$ for some prime number p, then $p|Y_1$ and the relation $X_1 - 6Y_1 + Z_1 + 2 = 0$ is true for all $X_1, Y_1 \in \mathcal{Z}^+$.

Clearly, the chance of such p is $p = 2$. This diagnosis provides that $X_1 = 2X_2, Y = 2Y_2$ implying that $X = 4X_1, Y = 4Y_1$.

These contributions state that ${X_2}^2 - 6X_2Y_2 + {Y_2}^2 + X_2 = 0$, which has no positive integer solution [98]. Hence, our assumption that X and Z have common divisors is wrong. This proposes that $\gcd(X, Z) = 1$. Thus, by the fact that the product two coprime numbers should be a perfect square if and only if each of them is a perfect square, $X = R^2$ and $Z = S^2$ for some positive integers R and S and $\gcd(R, S) = 1$. These choices of X and Z affords $Y = RS$ and afterward the desired equation can be modified into $R^2 - 6RS + S^2 + 4 = 0$.

By theorem 6.3, $(R, S) = (P_{2n+1}, P_{2n-1})$ and hence $(X, Y) = \left({P_{2n+1}}^2, P_{2n+1}\, P_{2n-1}\right)$, $n \geq 1$.

Theorem 6.8

Every solution in Pell-Lucas numbers for two dissimilar quadratic equations $X^2 - 2XY - Y^2 = 8$ and $X^2 - 2XY - Y^2 = -8$ are offered by $(X, Y) = (Q_{2n},\ Q_{2n-1})$, $n \geq 1$ and $(X, Y) = (Q_{2n+1},\ Q_{2n}), n \geq 0$ respectively.

Theorem 6.9

Let $X, Y \in N$, the set of natural numbers.

(i) If $X^2 - 6XY + Y^2 = 32$, then $(X, Y) = (Q_{2n+1},\ Q_{2n-1})$, $n \geq 1$.

(ii) If $X^2 - 6XY + Y^2 = -32$, then $(X, Y) = (Q_{2n+2},\ Q_{2n})$, $n \geq 0$.

Theorem 6.10

If X , Y be any two positive integers such that $X^2 - 2XY - Y^2 + 8X = 0$, then either $(X, Y) = \left(8{P_{2n}}^2,\ 8P_{2n}P_{2n-1}\right)$, $n \geq 1$ or $(X, Y) = \left({Q_{2n+1}}^2,\ Q_{2n+1}Q_{2n}\right)$, $n \geq 0$.

Proof:

Consider that $X^2 - 2XY - Y^2 + 8X = 0$ for some positive integers X and Y.

If $8|X$, then $8|Y \Rightarrow X = 8A$ and $Y = 8B$ for some A and B belong to the set of all positive integers Therefore, the original equation in two variables X and Y is converted into $A^2 - 2AB - B^2 + A = 0$.

By theorem 6.5, $(A, B) = \left({P_{2n}}^2,\ P_{2n}P_{2n-1}\right) \Rightarrow (X, Y) = \left(8{P_{2n}}^2,\ 8P_{2n}P_{2n-1}\right)$.

If $8 \nmid X$, then by theorem [II], $X = A^2$ and $Y = AB$.

Then, the original equation becomes $A^2 - 2AB - B^2 + 8 = 0$.

By applying theorem 6.8, it is concluded that

$(A,B) = (Q_{2n+1},\ Q_{2n}) \implies (X,Y) = \left({Q_{2n+1}}^2,\ Q_{2n+1}Q_{2n}\right)$ with $n \geq 0$.

Conversely, if $(X,Y) = \left(8{P_{2n}}^2,\ 8P_{2n}P_{2n-1}\right)$, then

$$X^2 - 2XY - Y^2 + 8X = \left(8{P_{2n}}^2\right)^2 - 2\left(8{P_{2n}}^2\right)(8P_{2n}P_{2n-1}) - (8P_{2n}P_{2n-1})^2 + 8\left(8{P_{2n}}^2\right)$$

$$= 64{P_{2n}}^2\left\{{P_{2n}}^2 - 2P_{2n}P_{2n-1} - {P_{2n-1}}^2 + 1\right\} = 0,$$

by the implementation of corollary 6.1.2.

Similarly, the same equation could be satisfied for $(X,Y) = \left({Q_{2n+1}}^2,\ Q_{2n+1}Q_{2n}\right)$.

Theorem 6.11

Let X , Y be any two positive integers.

(i) If $X^2 - 2XY - Y^2 - 8X = 0$, then either $(X,Y) = \left(8{P_{2n+1}}^2, 8P_{2n+1}P_{2n}\right)$, $n \geq 0$ or $(X,Y) = \left({Q_{2n}}^2,\ Q_{2n}Q_{2n-1}\right)$, $n \geq 1$.

(ii) If $X^2 - 2XY - Y^2 + 8Y = 0$, then either $(X,Y) = \left(\ 8P_{2n}P_{2n-1}, 8{P_{2n-1}}^2\right)$, $n \geq 1$ or $(X,Y) = \left(Q_{2n}Q_{2n+1}, {Q_{2n+1}}^2\right)$, $n \geq 0$.

(iii) If $X^2 - 2XY - Y^2 - 8Y = 0$, then either $(X,Y) = \left(\ 8P_{2n+1}P_{2n}, 8{P_{2n}}^2\right)$, $n \geq 0$ or $(X,Y) = \left(Q_{2n}Q_{2n-1}, {Q_{2n-1}}^2\right)$, $n \geq 1$.

Theorem 6.12

Let $X, Y \in \mathcal{Z}^+$, the set of all positive integer. Then

(i) The positive integer solutions to the equation $X^2 - 6XY + Y^2 + 32X = 0$ are either $(X,Y) = \left(8{P_{2n+1}}^2,\ 8P_{2n+1}P_{2n-1}\right)$, $n \geq 1$ or $(X,Y) = \left({Q_{2n+2}}^2,\ Q_{2n+2}Q_{2n}\right), n \geq 0$

(ii) The complete solutions in Pell numbers and Pell-Lucas numbers of the equation $X^2 - 6XY + Y^2 - 32X = 0$ are either

$(X, Y) = \left(8{P_{2n+2}}^2,\ 8P_{2n}P_{2n+2}\right), n \geq 0$ or

$(X, Y) = \left({Q_{2n+1}}^2,\ Q_{2n+1}Q_{2n-1}\right), n \geq 1$

6.2. Conception of positive integer solutions relating Jacobsthal and Jacobsthal – Lucas numbers to restricted number of quadratic equations with double variables

Primary Consequences:

The n^{th} Jacobsthal number designated by J_n is delineated by $J_n = J_{n-1} + 2J_{n-2}$ for $n \geq 2$ where $J_0 = 0, J_1 = 1$. If α, β are two roots of the equation $x^2 - x - 2 = 0$, then $\alpha = -1, \beta = 2$ such that $\alpha\beta = -2$ and $\alpha + \beta = 2$. Furthermore, it is well- recognized and modest to reveal the characteristics that $\alpha^{n-1} = J_n - \beta J_{n-1}$ and $\beta^{n-1} = J_n - \alpha J_{n-1}$ for every $n \in \mathcal{Z}$, the set of all integers. Also, it might be declared by Mathematical induction that

$${J_n}^2 - J_nJ_{n-1} - 2{J_{n-1}}^2 = (-2)^{n-1} \ \forall n \in \mathcal{Z} \qquad (6.3)$$

Similarly, the n^{th} Jacobsthal-Lucas number j_n is described as $j_n = j_{n-1} + 2j_{n-2}$ for $n \geq 2$ and $j_0 = 2, j_1 = 1$. The interrelation between Jacobsthal and Jacobsthal-Lucas numbers are approved as follows

(i) $j_n = J_{n+1} + 2J_{n-1}$ for every $n \in \mathcal{Z}$

(ii) ${j_n}^2 - j_nj_{n-1} - 2{j_{n-1}}^2 = -9(-2)^{n-1}$ for every $n \in \mathcal{Z}$ (6.4)

Theorem 6.13

The constitutive criterion for all non-negative integer solutions to the specific second-degree equation involving two variables $X^2 - XY - 2Y^2 = C(-1)^{n-1}$ is $(X, Y, C) = (J_n, J_{n-1}, 2^{n-1})$ with $n \geq 1$.

Proof:

If $(X, Y, C) = (J_n, J_{n-1}, 2^{n-1})$, then it follows from identity (6.4) that

$$X^2 - 2XY - Y^2 = C(-1)^{n-1}.$$

Conversely suppose that $X^2 - XY - 2Y^2 = C(-1)^{n-1}$ for some positive integers X , Y and $C = 2^{n-1}$.

Then,

$$(X - \alpha Y)(X - \beta Y) = (\alpha\beta)^{n-1}$$

$$\Rightarrow (X - \alpha Y)(X - \beta Y) = (J_n - \beta J_{n-1})(J_n - \alpha J_{n-1}).$$

Thus, $X - \alpha Y = J_n - \alpha J_{n-1}$ and hence $(X, Y, C) = (J_n, J_{n-1}, 2^{n-1}), n \geq 1$.

Corollary 6.13.1

The viable solutions to the certain quadratic equation $X^2 - XY - 2Y^2 = C$ are enumerated by $(X, Y, C) = (J_{2m+1}, J_{2m}, 2^{2m})$, $m \geq 0$.

Corollary 6.13.2

Every conceivable solution in Jacobsthal numbers of the equation $X^2 - XY - 2Y^2 = -C$ are quantified by $(X, Y, C) = (J_{2m}, J_{2m-1}, 2^{2m-1})$ with $m \geq 1$.

Theorem 6.14

The trustworthy integer solutions to the exact equation $X^2 - 5XY + 4Y^2 = C$ are conquered by $(X, Y, C) = (J_{2n+2}, J_{2n}, 2^{2n})$ with $n \geq 0$.

Proof:

For our convenience, let us choose $X > 2Y$

Then, $X^2 - 5XY + 4Y^2 = C \Rightarrow (X - 2Y)^2 - (X - 2Y)Y - Y^2 = C$

By corollary 6.13.1, it should be $X - 2Y = J_{2n+1}$, $Y = J_{2n}$ and $C = 2^{2n}$

The first two of the above equations yields the value of X as $X = J_{2n+2}$

Hence, the solutions to the required equation are mentioned by

$$(X, Y, C) = (J_{2n+2}, J_{2n}, 2^{2n}), n \geq 0.$$

Corollary 6.14.1

The infinitely many positive integer solutions to the equation $X^2 - 5XY + 4Y^2 = -C$ are attained by $(X, Y, C) = (J_{2n+1}, J_{2n-1}, 2^{2n-1})$ with $n \geq 1$.

Theorem 6.15

Let X, Y be any two natural numbers sustaining the equation $X^2 - XY - 2Y^2 \pm CX = 0$. Then $X = U^2$ and $Y = UV$ where $U, V \in N$.

Proof:

Modify the original equation as $X(X - Y \pm C) = 2Y^2$

It is easy to see that $\frac{X}{Y^2}$ and hence $Y^2 = XZ$ for some natural number Z.

If p is any prime number such that $p|X$ and $p|$Z, then $p|Y$

This affords the expression $X - Y - 2Z \pm C = 0$ which guarantees that $p|C$.

Here, the only possible value of p is $p = 2$ which implies that $X = 2X_1, Y = 2Y_1$

Again, it grasps that ${X_1}^2 - X_1Y_1 - 2{Y_1}^2 \pm C_1X_1 = 0$ where $C_1 = \frac{C}{2}$.

Enduring the same method as enlightened above till the constant C vanishes, it is found that ${X_n}^2 - X_nY_n - 2{Y_n}^2 \pm X_n = 0$

It follows that $X_n|{Y_n}^2$ and hence ${Y_n}^2 = X_nZ_n$ for some positive integer Z_n

If a prime number p satisfying the conditions $p|X_n$ and $p|Z_n$, then $p|Y_n$

Then, it is detected that $X_n - Y_n - 2Z_n \pm 1 = 0$.

This equation infers that $p|1$ which is not possible.

Therefore, $gcd(X, Z) = 1$.

By the needed theorem [III] stated above, it is noted that $X = U^2$ and $Z = V^2$ for some positive integers U and V where $\gcd(U, V) = 1$.

Hence, it is concluded that $Y^2 = XZ = U^2V^2 \Rightarrow Y = UV$.

Corollary 6.15.1

The probable values of X, Y in the equation $X^2 - XY - 2Y^2 + CX = 0$ are given by $(X, Y, C) = \left({J_{2n}}^2, J_{2n}\, J_{2n-1}, 2^{2n-1}\right)$, $n \geq 1$.

Corollary 6.15.2

The realistic solutions in Jacobsthal numbers to the equation $X^2 - XY - 2Y^2 - CX = 0$ are computed by $(X, Y, C) = \left({J_{2n+1}}^2, J_{2n+1}\, J_{2n}, 2^{2n}\right)$, $n \geq 0$.

Theorem 6.16

If X, Y be any two positive integers such that $X^2 - XY - 2Y^2 \pm CY = 0$, then $X = UV$ and $Y = U^2$ for some positive integers U and V with $\gcd(U, V) = 1$.

Proof:

The proof is analogous to Theorem 6.15.

Corollary 6.16.1

The convincing integer values of X, Y in the equation $X^2 - XY - 2Y^2 + CY = 0$ are resolved by $(X, Y, C) = \left(J_{2n}\, J_{2n-1}, {J_{2n-1}}^2, 2^{2n-1}\right)$, $n \geq 1$.

Corollary 6.16.2

The conventional solutions to the quadratic equation $X^2 - XY - 2Y^2 - CY = 0$ are particularized by $(X, Y, C) = \left(J_{2n+1}\, J_{2n}\, , {J_{2n}}^2, 2^{2n}\right)$, $n \geq 0$.

Theorem 6.17

The patterns of non-negative integer solutions to the equation $X^2 - 5XY + 4Y^2 + CX = 0$ are exemplified by $(X, Y, C) = \left({J_{2n+1}}^2,\ J_{2n-1}J_{2n+1}, 2^{2n-1}\right)$ where $n \geq 1$.

Proof:

Let X, Y be two non-negative integers such that $X^2 - 5XY + 4Y^2 + CX = 0$.

The alteration of the above equation $4Y^2 = X(5Y - X - C)$ ensures that X divides $Y^2$and henceforth $Y^2 = XZ$ for some non-negative integer Z.

Suppose that a certain prime number p divides both X and Z.

Then $p|Y$ and also the relation $X - 6Y + 4Z + C = 0$ holds for all $X, Y \in \mathcal{Z}^+$, the set of all positive integers.

Thus, p divides C and the chance of such p is $p = 2$.

This condition confirms that $X = 2X_1, Y = 2Y_1$ for some $X_1, Y_1 \in \mathcal{Z}^+$ and perceptibly the equation in which solutions to be evaluated is converted into

$${X_1}^2 - 5X_1Y_1 + 4{Y_1}^2 + C_1X_1 = 0 \text{ where } C_1 = \frac{C}{2}.$$

By the argument as explained above, $X_1|{Y_1}^2$ and hence ${Y_1}^2 = X_1Z_1$ for some $Z_1 \in \mathcal{Z}^+$. Again, if $p|X_1$ and $p|Z_1$, then $p|Y_1$ and the precise relation $X_1 - 5Y_1 + 4Z_1 + C_1 = 0$ is also true for all $X_1, Y_1 \in \mathcal{Z}^+$.

Carrying on this procedure till the equation ${X_n}^2 - 5X_nY_n + 4{Y_n}^2 + X_n = 0$ is reached. Further if $p|X_n$ and $p|Z_n$, then $p|Y_n$ and the accurate equation $X_n - 5Y_n + 4Z_n + 1 = 0$ is detected for all $X_n, Y_n \in \mathcal{Z}^+$.

Finally, p divides 1 which is impossible.

As a result, our supposition that X and Z have common divisors is erroneous. This shows that that $gcd(X, Z) = 1$.

Thus, by the necessary and sufficient condition that the product two coprime numbers should be a perfect square if and only if each of them is a perfect square, $X = P^2$ and $Z = Q^2$ where $P, Q \in \mathcal{Z}^+$ and $\gcd(P, Q) = 1$.

These adoptions of X and Z provides that $Y = PQ$ and subsequently the essential equation can be developed into $P^2 - 5PQ + 4R^2 + C = 0$.

By Corollary 6.14.1, the values of P, Q and C are searched by $(P, Q, C) = (J_{2n+1}, J_{2n-1}, 2^{2n-1})$.

Therefore $(X, Y, C) = \left({J_{2n+1}}^2,\ J_{2n-1}J_{2n+1}, 2^{2n-1}\right)$, $n \geq 1$.

Corollary 6.17.1

The non-negative integer solutions for the equation $X^2 - 5XY + 4Y^2 - CX = 0$ are symbolized by $(X, Y, C) = \left({J_{2n+2}}^2,\ J_{2n+2}J_{2n}, 2^{2n}\right)$ where $n \geq 0$.

Theorem 6.18

(i) The patterns of positive integer solutions to the equation $X^2 - 5XY + 4Y^2 + CY = 0$ are incarnated by the Jacobsthal numbers $(X, Y, C) = \left(J_{2n-1}J_{2n+1}, {J_{2n-1}}^2, 2^{2n-1}\right)$ where $n \geq 1$.

(ii) The infinitely several positive integer solutions to the equation $X^2 - 5XY + 4Y^2 - CY = 0$ are signified by $(X, Y, C) = \left(J_{2n+2}\, J_{2n}, {J_{2n}}^2, 2^{2n}\right)$ where $n \geq 0$.

Theorem 6.19

The feasible solution in Jacobsthal-Lucas numbers for two unlike binary quadratic equations $X^2 - XY - 2Y^2 = 9C$ and $X^2 - XY - 2Y^2 = -9C$ are presented by $(X, Y, C) = (j_{2n},\ j_{2n-1}, 2^{2n-1})$, $n \geq 1$ and $(X, Y, C) = (j_{2n+1},\ j_{2n}, 2^{2n}), n \geq 0$ respectively.

Theorem 6.20

Let X, Y be two distinct natural numbers.

(i) If $X^2 - 5XY + 4Y^2 = 9C$, then $(X, Y, C) = (j_{2n+1}, j_{2n-1}, 2^{2n-1})$, $n \geq 1$.

(ii) If $X^2 - 5XY + 4Y^2 = -9C$, then $(X, Y, C) = (j_{2n+2}, j_{2n}, 2^{2n})$, $n \geq 0$.

Theorem 6.21

If X , Y be any two non-negative integers such that $X^2 - XY - 2Y^2 + 9CX = 0$, then either

$(X, Y, C) = \left(9{J_{2n}}^2, 9J_{2n}J_{2n-1}, 2^{2n}\right)$, $n \geq 1$ or

$(X, Y, C) = \left({j_{2n+1}}^2, j_{2n+1}j_{2n}, 2^{2n-1}\right)$, $n \geq 0$.

Proof:

Assume that $X^2 - XY - 2Y^2 + 9CX = 0$ for some non-negative integers X and Y.

If $9|X$, then $9|Y \Rightarrow X = 9U$ and $Y = 9V$ for some $U, V \in \mathcal{Z}^+$

Therefore, the needed equation in two unknowns X and Y is enhanced in terms of U and V as $U^2 - UV - 2V^2 + CU = 0$.

By corollary 6.15.1,

$$(U, V, C) = \left({J_{2n}}^2, J_{2n}J_{2n-1}, 2^{2n-1}\right) \Rightarrow (X, Y, C) = \left(9{J_{2n}}^2, 9J_{2n}J_{2n-1}, 2^{2n-1}\right).$$

If $9 \nmid X$, then by theorem [IV], $X = U^2$ and $Y = UV$.

These choices of U and V simplifies the considered equation into the following one $U^2 - UV - 2V^2 + 9C = 0$.

By theorem 6.19, it is resolved that

$(U, V, C) = (j_{2n+1},\ j_{2n}, 2^{2n}) \Longrightarrow (X, Y, C) = \left({j_{2n+1}}^2,\ j_{2n+1}j_{2n}, 2^{2n}\right)$ where $n \geq 0$.

Conversely if $(X, Y, C) = \left(9{J_{2n}}^2, 9J_{2n}J_{2n-1}, 2^{2n}\right)$, then by the implementation of corollary 6.13.2

$$X^2 - XY - 2Y^2 + 9CX = \left(9{J_{2n}}^2\right)^2 - \left(9{J_{2n}}^2\right)(9J_{2n}J_{2n-1}) - 2(9J_{2n}J_{2n-1})^2 + 9C\left(9{J_{2n}}^2\right)$$

$$= 81{J_{2n}}^2\left\{{J_{2n}}^2 - J_{2n}J_{2n-1} - 2{J_{2n-1}}^2 + C\right\} = 0,$$

Likewise, the very same equation might well be fulfilled for $(X, Y, C) = \left(j_{2n+1}{}^2, j_{2n+1}\, j_{2n}, 2^{2n}\right)$.

Theorem 6.22

Let $X, Y \in \mathcal{Z}^+$, the set of all positive integers.

(i) If $X^2 - XY - 2Y^2 - 9CX = 0$, then the solution is either of the form $(X, Y, C) = \left(9J_{2n+1}{}^2, 9J_{2n+1}\, J_{2n}, 2^{2n}\right), n \geq 0$ or of the form $(X, Y, C) = \left(j_{2n}{}^2, j_{2n}\, j_{2n-1}, 2^{2n-1}\right), n \geq 1$.

(ii) If $X^2 - XY - 2Y^2 + 9CY = 0$, then the solution in Jacobsthal sequence is $(X, Y, C) = \left(9J_{2n}\, J_{2n-1}, 9J_{2n-1}{}^2, 2^{2n-1}\right), n \geq 1$ or in Jacobsthal-Lucas sequence is $(X, Y, C) = \left(j_{2n}\, j_{2n+1}, j_{2n+1}{}^2, 2^{2n}\right), n \geq 0$.

(iii) If $X^2 - XY - 2Y^2 - 9CY = 0$, then the solution of the equation is either in Jacobsthal numbers $(X, Y, C) = \left(9J_{2n+1}\, J_{2n}, 9J_{2n}{}^2, 2^{2n}\right), n \geq 0$ or in Jacobsthal-Lucas numbers $(X, Y, C) = \left(j_{2n}\, j_{2n-1}, j_{2n-1}{}^2, 2^{2n-1}\right), n \geq 1$.

Theorem 6.23

Let X, Y be two distinct non-negative integers. Then

(i) The two different sets ofs non-negative integer solutions to the equation $X^2 - 5XY + 4Y^2 + 9CX = 0$ are discovered by Jacobsthal numbers $(X, Y, C) = \left(9J_{2n+1}{}^2, 9J_{2n+1}\, J_{2n-1}, 2^{2n-1}\right), n \geq 1$ and by Jacobsthal-Lucas numbers $(X, Y, C) = \left(j_{2n+2}{}^2, j_{2n+2}\, j_{2n}, 2^{2n}\right), n \geq 0$.

(ii) All possible solutions in Jacobsthal and Jacobsthal-Lucas numbers to the equation $X^2 - 5XY + 4Y^2 - 9CX = 0$ are determined by

$(X, Y, C) = \left(9{J_{2n+2}}^2, 9J_{2n}\, J_{2n+2}, 2^{2n}\right), n \geq 0$ and

$(X, Y, C) = \left({j_{2n+1}}^2, j_{2n+1}\, j_{2n-1}, 2^{2n-1}\right), n \geq 1.$

(iii) If $X^2 - 5XY + 4Y^2 + 9CY = 0$, then two sequences of solutions in integers are presented by $(X, Y, C) = \left(9J_{2n-1}\, J_{2n+1}, 9{J_{2n-1}}^{2,2^{2n-1}}\right), n \geq 0$ and

$(X, Y, C) = \left(j_{2n}\, j_{2n+2}, {j_{2n}}^2, 2^{2n-1}\right), n \geq 1.$

(iv) If $X^2 - 5XY + 4Y^2 - 9CY = 0$, then one of the following two solutions exists: $(X, Y, C) = \left(\ 9J_{2n}\, J_{2n+2}, 9{J_{2n}}^2, 2^{2n}\right), n \geq 1$ or

$(X, Y, C) = \left(j_{2n-1}\, j_{2n+1}, {j_{2n-1}}^2, 2^{2n-1}\right), n \geq 0$

Chapter – VII

Mordell Type Diophantine Equations

CHAPTER – VII

Mordell Type Diophantine Equations

Chapter VII entails of two sections 7.1 and 7.2.

In Section 7.1, three unrivalled forms of Mordell's Diophantine equations $B^2 = A^3 + K$, $K = U^3 - V^2, U^3 - 2V^2$, $2V^2 + U^3$ where U, V are integers together with some prime number p satisfying certain congruence relations are examined and proved that all such equations have no solution in integer.

In Section 7.2, four types of Mordell equations $Y^2 = X^3 + C$, $C = \pm 9, -16, 36$ are considered and showed that two of the equations $Y^2 = X^3 - 9, Y^2 = X^3 - 16$ have no integer solutions and the remaining two equations $Y^2 = X^3 + 9, Y^2 = X^3 + 36$ have restricted number of integer solutions by mainly using the perceptions of properties of congruences.

7.1 Methodology of Proving no Solutions to three Categories of Mordell Type Diophantine Equations

Definition

The Legendre symbols are well-defined by

(i) $\left(\frac{-1}{p}\right) = \begin{cases} 1 \; if \;\; p \equiv 1 \; (mod \; 4) \\ -1 \; if \;\; p \equiv 3 \; (mod \; 4) \end{cases}$

(ii) $\left(\frac{-2}{p}\right) = \begin{cases} 1 \; if \;\; p \equiv 1 \; or \; 3 \; (mod \; 8) \\ -1 \; if \;\; p \equiv 7 \; or \; 5 \; (mod \; 8) \end{cases}$

In this section, three different Mordell kind Diophantine Equations $B^2 = A^3 + K$, $K = U^3 - V^2$, $U^3 - 2V^2$, $2V^2 + U^3$, $U, V \in \mathcal{Z}$ are considered and proved that all these equations have no solutions by the following theorems.

Theorem 7.1

Let U and V be integers such that $U \equiv 2 \; (mod \; 4)$, $V \equiv 3 \; (mod 4)$ and for any prime number p divides V implies that $p \equiv 1 \; (mod \; 4)$. Then, the equation $B^2 = A^3 + K$ where $K = U^3 - V^2$ has no integer solution (A, B).

Proof:

Suppose that it happens a solution (A, B) in integers to an explored equation.

Since $K = U^3 - V^2 \equiv -1 \; (mod \; 4)$, it is noticed that $B^2 \equiv A^3 - 1 \; (mod \; 4)$

Hence, $A \not\equiv 0 \; (mod \; 2)$ and $A \not\equiv 3 \; (mod \; 4)$ and so $A \equiv 1 \; (mod \; 4)$.

Now, the deliberated equation for the preferred choice of K can be adapted into

$$B^2 + V^2 = A^3 + U^3 = (A + U)(A^2 - AU + U^2) \qquad (7.1)$$

As $A \equiv 1 \; (mod \; 4) \; and \; U \equiv 2 (mod \; 4)$, it must be

$$(A^2 - AU + U^2) \equiv 3 (mod \; 4)$$

Hence, $(A^2 - AU + U^2)$ is odd and by (7.1) it has a prime factor p_1, $p_1 \equiv 3 \; (mod \; 4)$.

Thus, $B^2 \equiv -V^2 \ (mod\ p_1)$.

By our assertion, $p_1 \nmid V$. Hence,

$$\left(-\frac{1}{p_1}\right) = \left(\frac{-V^2}{p_1}\right) = \left(\frac{B^2}{p_1}\right) = 1$$

which is true only when $p_1 \equiv 1 \ (mod\ 4)$ denies the fact that $p_1 \equiv 3 \ (mod\ 4)$.

Therefore, our assumption is wrong.

This proves that the examined equation $B^2 = A^3 + K$ with the considered K has no solution.

Theorem 7.2

Let U and V be integers sustaining the conditions $U \equiv 3 \ (mod\ 4),\ V \equiv 0, 2 (mod\ 4)$. If a prime number p divides $V/2$ implies that $p \equiv 1 \ (mod\ 4)$, then the equation $B^2 = A^3 + K$ where $K = U^3 - V^2$ has no solution (A, B) in integers.

Proof:

Suppose that (A, B) is an integer solution of the inventive equation with an apt K.

Since, $U^3 - V^2 \equiv 3 \ (mod\ 4)$, it is achieved by $B^2 \equiv A^3 + 3 \ (mod\ 4)$.

Hence, $A \equiv 1 \ (\text{mod}\ 4)$.

Now, the original equation is reformed for the prescribed value of K as

$$B^2 + V^2 = A^3 + U^3 = (A + U)(A^2 - AU + U^2) \qquad (7.2)$$

Since $A \equiv 1 \ (mod\ 4)$ and $U \equiv 3 \ (mod\ 4)$, it is found that

$$A^2 - AU + U^2 \equiv 3 \ (mod\ 4)$$

Hence, $A^2 - AU + U^2$ is odd and by (7.2) it has a prime factor p_2, $p_2 \equiv 3 \ (mod\ 4)$.

Thus, $B^2 \equiv -V^2 \ (mod\ p_2)$.

By our assumption, $p_2 \nmid \left(\frac{V}{2}\right)$ and hence $p_2 \nmid V$

But, $\left(\frac{-1}{p_2}\right) = \left(\frac{-V^2}{p_2}\right) = \left(\frac{B^2}{p_2}\right) = 1$.

This is correct only for $p_2 \equiv 1(mod\ 4)$ contradicts the chance that $p_2 \equiv 3\ (mod\ 4)$.

Hence, the Diophantine equation $B^2 = A^3 + K$ for the selected K has no solution.

Theorem 7.3

Let U and V be integers nourishing with the conditions $U \equiv 2\ (mod\ 8)$, $V \equiv 1\ (mod\ 2)$. If p is a prime such that $p \equiv 1, 3\ (mod\ 8)$ and p divides V, then the equation $B^2 = A^3 + K$ where $K = U^3 - 2V^2$ has no integral solution.

Proof:

For the choice $K = U^3 - 2V^2 \equiv 2\ (mod\ 4)$, it must be

$$B^2 \equiv A^3 + 2\ (mod\ 4)$$

Therefore, $A \not\equiv 0\ (mod\ 2)$, $A \not\equiv 1\ (mod\ 4)$ and consequently $A \equiv 3\ (mod\ 4)$

Hence, $A \equiv 3$ or $7\ (mod\ 8)$

Moreover, $K \equiv -2\ (mod\ 8)$

So that $A \not\equiv 7\ (mod\ 8) \Rightarrow A \equiv 3\ (mod\ 8)$

Now, $B^2 + 2V^2 = A^3 + U^3 = (A + U)(A^2 - UA + U^2)$

As $A \equiv 3\ (mod\ 8)$ and $U \equiv 2\ (mod\ 8)$, it is seen that $A^2 - UA + U^2 \equiv 7\ (mod\ 8)$ and $A + U \equiv 5\ (mod\ 8)$

$\therefore B^2 + 2V^2$ has a prime factor p_3 such that $p_3 \equiv 5\ or\ 7\ (mod\ 8)$

By our hypothesis, $p_3 \nmid V$ and $B^2 \equiv -2V^2\ (mod\ p_3)$

Also, $\left(-\frac{2}{p_3}\right) = \left(-\frac{2V^2}{p_3}\right) = \left(\frac{B^2}{p_3}\right) = 1$ contrasting $p_3 \equiv 5\ or\ 7\ (mod\ p)$

Hence, it is resolved that there exists no solution for the Diophantine equation $B^2 = A^3 + K$ when $K = U^3 - 2V^2$.

Theorem 7.4

Assume $U, V \in Z$ and $U \equiv 6\ (mod\ 8)$, $V \equiv 1\ (mod 4)$. Let p be a prime number such that $p \mid V$ and $p \equiv \pm 1\ (mod\ 8)$. Then, the equation $B^2 = A^3 + K$, where $K = 2V^2 + U^3$ does not embrace any integer solution.

Proof:

Since, $K = 2V^2 + U^3 \equiv 2\ (\ mod\ 8)$, it is detected that $B^2 \equiv A^3 + 2\ (mod\ 8)$

Then, $A \not\equiv 0\ (mod\ 2), A \not\equiv 1\ (mod\ 4)$ and subsequently $A \equiv 3\ (mod\ 4)$

Hence $A \equiv 3\ or\ 7\ (mod\ 8)$

If $A \equiv 3\ (mod\ 8)$, then $B^2 \equiv 5\ (mod\ 8)$ which is not possible.

Thus, $A \equiv 7\ (mod\ 8)$.

Now, $B^2 - 2V^2 = (A + U)(A^2 - UA + U^2)$

$$\Rightarrow A^2 - UA + U^2 \equiv 3\ (mod\ 8)$$

Therefore $A^2 - UA + U^2$ is odd and is divisible by an odd prime p_4 with $p_4 \equiv 3\ (mod\ 8)$

$$\Rightarrow B^2 \equiv 2V^2\ (mod\ p_4).$$

By our postulation, $p_4 \nmid V$

$$\therefore \left(\frac{2}{p_4}\right) = \left(\frac{2V^2}{p_4}\right) = \left(\frac{B^2}{p_4}\right) = 1$$

disagreeing $p_1 \equiv 3\ (mod\ 8)$

Hence the proof.

7.2 Attesting Finite Number of Integer Solutions or No Integer Solutions to Four Mordell Kind Equations

The intention of each phase is to treasure comprehensive solutions for Mordell type Diophantine equations of the form $Y^2 = X^3 + C$ where $C = \pm 9, -16, 36$.

Applicable Theorem I

In [136], "Let p denote a prime. Then $x^2 = -1(mod\ p)$ has solutions if and only if $p \equiv 2\ (mod\ 4)$ or $p \equiv 1\ (mod\ 4)$"

Theorem 7.5

If $X, Y \in \mathcal{Z}$, then there is no solution to $Y^2 = X^3 - 9$.

Proof:

Initially let us assume that $Y^2 = X^3 - 9$ has an integer solution (X, Y).

If $X \equiv 0\ (mod\ 2)$ and $X \equiv 3\ (mod\ 4)$, then $Y^2 \equiv -1\ (mod\ 4)$ and $Y^2 \equiv 2\ (mod\ 4)$ respectively. But, both of them are impossible.

Hence, it is possible that $X \equiv 1\ (mod\ 4)$ because it leads to $Y^2 \equiv 0\ (mod\ 4)$.

Now, the implicit equation can be considered as

$$Y^2 + 1 = X^3 - 8 = (X - 2)(X^2 + 2X + 4).$$

As $X \equiv 1\ (mod\ 4)$, $X^2 + 2X + 4 \equiv 3\ (mod\ 4)$.

This implies that $Y^2 + 1$ is divisible by a prime number p such that $p \equiv 3\ (mod\ 4)$

That is, $Y^2 \equiv -1\ (mod\ p)$ where $p \equiv 3\ (mod\ 4)$.

This is a contradiction to theorem I.

Hence, $Y^2 = X^3 - 9$ has no integer solution.

Theorem 7.6

The feasible integral solutions to the particular Mordell equation $Y^2 = X^3 + 9$ are $(X, Y) = \{(0, \pm 3), (3, \pm 6), (6, \pm 15), (40, \pm 253), (-2, \pm 1)\}$.

Proof:

Rewrite the proposed equation as $X^3 = Y^2 - 9 = (Y + 3)(Y - 3)$ (7.3)

If X is even, then Y is odd. If X is odd, then Y is even. If d is the common divisor of $(Y+3)$ and $(Y-3)$, then d also divides their difference $(Y+3)-(Y-3)=6$. Therefore, d must be any one of the values 1,2,3,6.

Case 1: Suppose Y is even

Then, $(Y+3)$ and $(Y-3)$ are both odd. So, $gcd(Y+3, Y-3)$ is either 1 or 3.

If $gcd(Y+3, Y-3)=1$, then they are relatively prime. Since their product is a cube, they both are cube.

That is $(Y+3)=a^3$ and $(Y-3)=b^3$

$$\Rightarrow a^3-b^3=6$$

However, no two odd cubes produce a difference 6.

Hence, $gcd(Y+3, Y-3)=3$

Since Y is even, $Y \equiv 0\ (mod\ 4)$ or $Y \equiv 2\ (mod\ 4)$

Subcase 1.1: Suppose $Y \equiv 0\ (mod\ 4)$

Then, $Y+3 \equiv 3\ (mod\ 4)$ and $Y-3 \equiv 1\ (mod\ 4)$.

Dividing (7.3) by 27, it is emblazoned as

$$\left(\frac{X}{3}\right)^3=\left(\frac{Y+3}{3}\right)\left(\frac{Y-3}{9}\right)$$

Due to the fact that division of each component by a multiple of $3=gcd(Y+3, Y-3)$, the right-hand side of the preceding equation comprehends relatively prime factors and therefore each factor is a cube.

That is, $\frac{Y+3}{3}=a^3$ and $\frac{Y-3}{9}=b^3$

$$\Rightarrow 3a^3-3=9b^3+3$$

$$\Rightarrow a^3-3b^3=2$$

This is factual only for $a=-1, b=-1$

If $a = -1$ or $b = -1$, then $Y = -6$ and $X = 3$.

Therefore, an integral solution to 7.3 is $(X, Y) = (3, -6)$

Subcase 1.2: Suppose $Y \equiv 2\ (mod\ 4)$

However, $Y + 3 \equiv 1\ (mod\ 4)$ and $Y - 3 \equiv 3\ (mod\ 4)$.

By dividing (7.3) by 27, it is converted into

$$\left(\frac{X}{3}\right)^3 = \left(\frac{Y+3}{9}\right)\left(\frac{Y-3}{3}\right)$$

As the description specified in subcase 1.1, it is possible to designate the components on the right-hand side of the previous equation as

$$\frac{Y+3}{9} = a^3 \text{ and } \frac{Y-3}{3} = b^3$$

$$\Rightarrow 9a^3 - 3 = 3b^3 + 3$$

$$\Rightarrow 3a^3 - b^3 = 2$$

The only possibility of the above equation is $a = 1$, $b = 1$.

Either of the above choices of a and b delivers $Y = 6$ and $X = 3$.

Therefore, in this circumstance the comprehensive solution is $(X, Y) = (3,6)$

As an effect, this case grants two integral solutions $(X, Y) = (3, \pm 6)$.

Case 2: Presume Y is odd

Then, both the factors in the right-hand side of (7.3) are even and $gcd(Y + 3, Y - 3)$ is either 2 or 6.

Also, $Y \equiv 1(mod\ 4)$ or $Y \equiv 3\ (mod\ 4)$

Subcase 2.1:

If $gcd(Y + 3, Y - 3) = 2$ and $Y \equiv 1\ (mod\ 4)$, then $Y + 3 \equiv 0\ (mod\ 4)$ and $Y - 3 \equiv 2\ (mod\ 4)$.

Divide both sides of (7.3) by 8, it is received that

$$\left(\frac{X}{2}\right)^3 = \left(\frac{Y+3}{4}\right)\left(\frac{Y-3}{2}\right)$$

As in case 1, it is taken as

$$\frac{Y+3}{4} = a^3 \text{ and } \frac{Y-3}{2} = b^3$$

$$\Rightarrow 4a^3 - 3 = 2b^3 + 3$$

$$\Rightarrow 2a^3 - b^3 = 3$$

The preparable chances of a and b are declared by

$$a = 1, b = -1 \text{ and } a = 4, b = 5$$

If $a = 1$ or $b = -1$, then $Y = 1$ and $X = -2$.

If $a = 4$ or $b = 5$, then $Y = 253$ and $X = 40$.

Therefore, the two integral solutions to (7.3) achieved in this instance are $(X, Y) = \{(-2,1), (40,253)\}$.

Subcase 2.2:

If $Y \equiv 3\ (mod\ 4)$, then $Y + 3 \equiv 2\ (mod\ 4)$ and $Y - 3 \equiv 0\ (mod\ 4)$ and proceeding as in case 1, it is concluded that $(X, Y) = \{(-2, -1), (40, -253), \}$.

Similarly, if $gcd(Y + 3, Y - 3) = 6$, then integer solutions $(X, Y) = (0, \pm 3)$ of (7.3) are accomplished for both selections of $Y \equiv 1\ (mod\ 4)$ and $Y \equiv 3\ (mod\ 4)$ by means of the same methodology in case 1.

Case 3:

In all the above two cases, the fact about the unique factorization domain in the ring of integers is considered. But one of the factors in the right-hand side of (7.3) divided by d^2 can be stretched to a fractional number.

That is, $\frac{Y+3}{d^2}$ or $\frac{Y-3}{d^2}$ may be a fractional number. The product of an integer and a fractional number is a cube of an integer means that it fulfils the following conditions.

$\frac{Y+3}{d^2} = \frac{a}{b}$ and $\frac{Y-3}{d} = a^2b$ or $\frac{Y+3}{d^2} = \frac{a^2}{b}$ and $\frac{Y-3}{d} = ab$

These choices afford the succeeding equations

$a^2b^2 - da - b = 0$ or $ab^2 - da^2 - b = 0$

Solving them for b, it is acquired that

$b = \frac{1 \pm \sqrt{1+4a^2d}}{2a^2}$ or $b = \frac{1 \pm \sqrt{1+4a^2d}}{2a}$

In these two equations, the discriminant is a positive integer if and only if $a = 1, d = 6$.

Implementing this condition, a diverse solution is obtained as follows:

$\frac{Y+3}{36} = \frac{1}{b}$ and $\frac{Y-3}{6} = b$

$\Rightarrow \frac{36}{b} - 3 = 6b + 3$

$\Rightarrow b^2 + b - 6 = 0$

$\Rightarrow b = 2\ or - 3$

$\Rightarrow Y = \pm 15$ and hence $X = 6$.

So, all the solutions assimilated for the preferred equation are

$(X, Y) = \{(0, \pm 3), (3, \pm 6), (6, \pm 15), (40, \pm 253), (-2, \pm 1)\}$

Theorem 7.7

If $X, Y \in \mathcal{Z}$, then the equation $Y^2 = X^3 - 16$ has no solution.

Proof:

Redraft the suggested equation as

$$X^3 = Y^2 + 16 = (Y + 4i)(Y - 4i) \qquad (7.4)$$

Since $Y^2 \equiv X^3\ (mod\ 4)$, either X and Y are both even or they are both odd.

Case 1: Suppose X and Y are both odd

If a is a common divisor of $(Y + 4i)$ and $(Y - 4i)$, then a divides their difference

$$(Y + 4i) - (Y - 4i) = 8i.$$

Hence, there exists some $b \in Z[i]$ such that $a = b \times 8i$.

Since norm is multiplicative in $Z[i]$, the norm of $'a'$ denoted by $N(a)$is provided by

$$N(a) = N(b \times 8i) = N(b)N(8i)$$

This implies that $N(a)$ divides $N(8i) = 64$ where $N(x + iy) = x^2 + y^2$ is the norm of the Gaussian integer $x + iy$.

As well as $N(a)$ divides $N(Y + 4i) = Y^2 + 16 = X^3$ which is odd.

Then, $N(a) = 1$ and thus a is a unit in $Z[i]$. It is well known that every unit is a cube in $Z[i]$.

This means that $(Y + 4i)$ and $(Y - 4i)$ are relatively prime.

From (7.4), both $(Y + 4i)$ and $(Y - 4i)$ are cubes.

Contemplate

$$Y + 4i = (M + Ni)^3$$

$$\Rightarrow Y = M(M^2 - 3N^2) \text{ and } 4 = N(3M^2 - N^2)$$

From the second of the above equations, N must be ± 1 or ± 2 or ± 4. None of these values of N gives an integer value for M.

Hence, the projected equation has no solution when X and Y are both odd.

Case 2: Suppose X and Y are both even.

Transcribe $X = 2X'$ and $Y = 2Y'$

Then, (7.4) can be swotted into the following equation

$$Y'^2 = 2X'^3 - 4 \qquad (7.5)$$

$\Rightarrow Y'$ is even

If X' is odd, then $Y'^2 \equiv 2\ (mod\ 4)$. This is not possible, since even squares are always congruent to 0 modulo 4.

Hence, X' is even.

Now, designate $X' = 2X''$ and $Y' = 2Y''$.

Engaging these two alternations in (7.5), it can be extolled by

$Y''^2 = 4X''^3 - 1$ implies that $Y''$is odd

If X'' is odd or even, then $Y''^2 \equiv 3\ (mod\ 4)$ which is impossible because $Y''^2 \equiv 1\ (mod\ 4)$.

Hence, this case offers no solution to the predictable equation.

Theorem 7.8

The limited number of integral solutions to the definite type of Mordell equation $Y^2 = X^3 + 36$ are $(X, Y) = \{(0, \pm 6), (4, \pm 10), (12, \pm 42), (-3, \pm 3)\}$.

Proof:

The considered equation can be rephrased as

$$X^3 = Y^2 - 36 = (Y + 6)(Y - 6) \qquad (7.6)$$

Here both X and Y are either even or odd. If $(Y + 6)$ and $(Y - 6)$ have a common divisor d', then their difference $(Y + 6) - (Y - 6) = 12$ is also divided by d'. Therefore, d' essentially be any one of the values 1,2,3,4,6,12.

Case 1: Suppose X and Y are even

Then, $(Y + 6)$ and $(Y - 6)$ are together even and also $gcd(Y + 6, Y - 6) = 2\ or\ 4\ or\ 6\ or\ 12$.

If $gcd(Y + 6, Y - 6) = 2$, then by dividing (7.6) by 8, the following conjectures can be made

$$\left(\frac{X}{2}\right)^3 = \left(\frac{Y+6}{2}\right)\left(\frac{Y-6}{4}\right) = a^3b^3 \text{ or } \left(\frac{X}{2}\right)^3 = \left(\frac{Y+6}{4}\right)\left(\frac{Y-6}{2}\right) = a^3b^3$$

$$\Rightarrow a^3 - 2b^3 = 6 \text{ or } 2a^3 - b^3 = 6$$

It is scrutinized that both equations do not have solution in integers.

When $gcd(Y+6, Y-6) = 4$, then dividing (7.6) by 64 and utilizing the prior technique as revealed earlier, the subsequent equations are grasped

$$a^3 - 4b^3 = 3 \text{ or } 4a^3 - b^3 = 3$$

$$\Rightarrow a = -1, b = -1 \text{ or } a = 1, b = 1$$

If $a = -1$ and $b = -1$, then $(X, Y) = (4, -10)$.

If $a = 1$ and $b = 1$, then $(X, Y) = (4,10)$

Hence, the needed solution is $(X, Y) = (4, \pm 10)$

If $gcd(Y+6, Y-6) = 6$, then dividing (7.6) by 6^3 and using the identical method as acknowledged in earlier theorems, it is scrutinized that the ensuing equations

$$a^3 - 6b^3 = 2 \text{ or } 6a^3 - b^3 = 2$$

procure no integer values for both a and b.

If $gcd(Y+6, Y-6) = 12$, then dividing (7.6) by 12^3 and by the similar procedures as given above, the succeeding equations are detected

$$a^3 - 12b^3 = 1 \text{or } 12a^3 - b^3 = 1$$

$$\Rightarrow a = 1, b = 0 \text{ or } a = 0, b = -1$$

If $a = 1$ and $b = 0$, then $(X, Y) = (0,6)$.

If $a = 0$ and $b = -1$, then $(X, Y) = (0, -6)$.

Hence, the necessary solution is $(X, Y) = (0, \pm 6)$

This scenario thus provides the ideal solutions $(X, Y) = \{(4, \pm 10), (0, \pm 6)\}$.

Case 2: Suppose X and Y are odd

Then both the factors on the right-hand side of (7.6) are odd and $gcd(Y+6, Y-6)$ is either 1 or 3.

If $gcd(Y+6, Y-6) = 1$, then it is evidenced that

$$Y + 6 = a^3 \text{ and } Y - 6 = b^3$$

$$\Rightarrow a^3 - b^3 = 12$$

But the difference is not 12 between any two odd cubes.

Hence, $gcd(Y+6, Y-6) = 3$. Then for both selections $Y \equiv 1\ (mod\ 4)$ and $Y \equiv 3\ (mod\ 4)$ and by utilizing the undistinguishable approach, it is noted that

$$a^3 - 3b^3 = 4 \text{ or } 3a^3 - b^3 = 4$$

$$\Rightarrow a = 1, b = -1$$

$$\Rightarrow Y = \pm 3, X = -3$$

Henceforth, the outcome in this case, is $(X, Y) = (-3, \pm 3)$.

Case 3:

Appeal the concept of case 3 in theorem 7.5, an exclusive solution is perceived when $a = 1, d = 12$ as follows

$$\frac{Y+6}{144} = \frac{1}{b} \text{ and } \frac{Y-6}{12} = b$$

$$\Rightarrow \frac{144}{b} - 6 = 12b + 6$$

$$\Rightarrow b^2 + b - 12 = 0$$

$$\Rightarrow b = -4\ or 3 \Rightarrow Y = \pm 42 \text{ and hence } x = 12.$$

The solution obtained in this case is $(X, Y) = (12, \pm 42)$.

Chapter – VIII

Exponential Diophantine Equations

CHAPTER - VIII

Exponential Diophantine Equations

This chapter is composed of three sections, 8.1 to 8.3.

In Section 8.1, an exclusive exponential Diophantine equation $p^x + (p+1)^y = z^2$ such that the sum of integer powers x and y of two consecutive prime numbers engrosses a square is examined for estimating enormous integer solutions by exploiting the fundamental notions of Mathematics and the speculation of divisibility for all possibilities of $x + y = 1, 2, 3, 4$.

In Section 8.2, an inimitable Diophantine equation with variables as exponents $n^x + (n+1)^y = z^2, n \in \mathcal{N}$, the collection of all-natural numbers in order to estimate immense non-negative integer solutions by implementing ultimate moralities of Mathematics and the formulae for solutions of acquainted Pell equations is investigated for whole categories of two exponents x and y such that $x + y = 1, 2, 3, 4$.

In Section 8.3, a particular kind of an exponential Diophantine equation ${p_1}^x + {p_2}^y + {p_3}^z = M^2$ where (p_1, p_2, p_3) is a prime triplet of the forms $(p, p+2, p+6) = (4n+1, 4n+3, 4n+7)$ or $(4n+3, 4n+5, 4n+9)$ and $(p, p+4, p+6) = (4n+1, 4n+5, 4n+7)$ or $(4n+3, 4n+7, 4n+9)$ for certain $n \in \mathcal{N}$, the combination of x, y, z takes the values either 1 or 2 is investigated with the help of MATLAB program and basic concepts of Mathematics.

8.1 Exploration of Solutions for an Exponential Diophantine Equation $p^x + (p+1)^y = z^2$

In this section, the possible integer solutions to an exponential Diophantine equation $p^x + (p+1)^y = z^2$ are explored in the following theorem.

Theorem 8.1

If p is any prime and x, y and z are integers persuading the condition that $x + y = 1,2,3,4$, then all feasible integer solutions to an exponential Diophantine equation $p^x + (p+1)^y = z^2$ are given by

$(p, x, y, z) = \{(2,0,1,2), (3,1,0,2), (3,2,2,5)\}$ when $p = 2, 3$ and

$(p, x, y, z) = (4n^2 + 4n - 1, 0, 1, 2n + 1)$ where $n \in \mathcal{N}$ for $p > 3$

Proof:

The equation for performing solutions in integer is taken as

$$p^x + (p+1)^y = z^2 \tag{8.1}$$

All doable predilection of the supposition $x + y = 1,2,3,4$ is carried out by eight cases for assessing solutions in integers.

Case 1: $x = 0$, $y = 1$

The equation (8.1) to explore solutions in integers trims down by

$$p + 2 = z^2 \tag{8.2}$$

If $p = 2$, then $z = 2$

Hence, the one and only one integer solution is communicated as $(p, x, y, z) = (2,0,1,2)$

If p is an odd prime, then $p + 2$ is an odd number.

This means that z^2 is an odd number and consequently z is also an odd number.

If $z = 1$, then $p + 2 = 1$ which is impossible.

As a result, $z \geq 3$.

Describe $z = 2n + 1, n \in \mathcal{N}$ (8.3)

The square of the selection of z in (8.3) can be characterized by

$$z^2 = 4n^2 + 4n + 1, n \in \mathcal{N}$$

Insight of (8.2), the promising value of an odd prime complied with the specified equation is distinguished by

$$p = 4n^2 + 4n - 1, n \in \mathcal{N}$$

Hence, the enormous solutions to (8.1) is $(p, x, y, z) = (4n^2 + 4n - 1,0,1,2n + 1)$ where $n \in \mathcal{N}$

Case 2: $x = 1, y = 0$

The inventive equation (8.1) is diminished as

$$p + 1 = z^2 \quad (8.4)$$

If $p = 2$, then $z^2 = 3$ which is not possible for the integer value of z.

If p is an odd prime, then $p + 1$ is an even number which can be articulated by

$$p + 1 = 2n, n \in \mathcal{N}$$

Match up the above equation with (8.4), $2n$ is a perfect square only if $n = 2m^2$ where $m \in \mathcal{N}$.

Thus, $p = (2m)^2 - 1$.

If $m = 1$, then $p = 3$

Therefore, the solution belongs to the set Z of integers is $(p, x, y, z) = (3,1,0,2)$

If $m \neq 1$, then $p = (2m - 1)(2m + 1)$

If p divides $(2m - 1)$, then $2m - 1 = ap$ and as a consequence $2m + 1 = ap + 2$

Thus, $p = ap(ap + 2)$ and leads to the ensuing equation

$$1 = a(a + 2)$$

But the above equation is not true for any integer value of a.

If p divides $(2m + 1)$, then $2m + 1 = bp$ and from now $2m - 1 = bp - 2$

Therefore, $p = bp(bp - 2)$ and consequently $1 = b(b + 2)$ which is not factual for any integer options for b.

Hence, in this case there exists a unique solution to (8.1) given by $(p, x, y, z) = (3,1,0,2)$.

Case 3: $x = 1$, $y = 1$

The creative equation (8.1) is adjusted by

$$2p + 1 = z^2$$

Since z^2 is an odd number for all selections of p, it follows that

$$z^2 \equiv 1 \ (mod\ 4)$$

$\Longrightarrow 2p + 1 \equiv 1\ (mod\ 4)$

$\Longrightarrow 2p \equiv 0\ (mod\ 4)$

Capture that $2p = 4k$ which means that $p = 2k$ for some positive integer k

This declaration is possible only when $k = 1$

Then $p = 2$, and $2p + 1 = 5$ which is not a perfect square of an integer.

Hence, in this case, there is no integer solution to the presupposed equation.

Case 4 : $x = 1, y = 2$

The resourceful equation (8.1) is reconstructed as

$$p^2 + 3p = z^2 - 1$$

$$\Longrightarrow p(p + 3) = (z - 1)(z + 1) \qquad (8.5)$$

If $p|(z - 1)$, then $z - 1 = kp$, and $z + 1 = kp + 2$

Executions of these two equations in (8.5) go along with the subsequent quadratic equation in k

$$pk^2 + 2k - (p + 3) = 0$$

which consent the value of $k = \frac{-1 \pm \sqrt{p(p+3)+1}}{p}$

It is deeply monitored that no prime number p provides an integer value for k.

An alternative vision of $p|(z + 1)$ reveals that $z + 1 = lp$ and $z - 1 = lp - 2$

By making use of these two equations in (8.5) espouse the second-degree equation in l as

$$pl^2 - 2l - (p + 3) = 0$$

which yields $l = \frac{1\pm\sqrt{1-p(p+3)}}{p}$.

The above value of l is a complex number for any prime p.

Hence, the ultimate result is no integer solutions to the most wanted equation (8.1).

Case 5 : $x = 2, y = 1$

The quick-witted equation (8.1) is restructured as

$$p^2 + p = z^2 - 1$$

$$\Longrightarrow p(p+1) = (z-1)(z+1) \qquad (8.6)$$

It follows from equation (8.6) that p must divide any one of the values $(z-1)$ or $(z+1)$.

If $p|(z-1)$, then $z-1 = mp$ and $z+1 = mp+2$ for some integer m.

Then, equation (8.6) makes available with the value of p as

$$p = \frac{1-2m}{m^2-1} \qquad (8.7)$$

Accordingly, one can easily notice that the right-hand side of (8.7) can never be a prime number for $m \in Z$.

This circumstance offers that $p \nmid (z-1)$.

If $p|(z+1)$, then $z+1 = np$ and $z-1 = np-2$ for some integer n.

Then, equation (8.6) endows with the value of p as

$$p = \frac{1+2n}{n^2-1} \qquad (8.8)$$

None of the value of $n \in \mathcal{Z}$ in the right-hand side of (8.8) supplies the prime number establish that

$p \nmid (z+1)$.

Hence, this case does not grant an integer solution for equation (8.1).

Case 6 : $x = 1, y = 3$

For these choices of x and y, the well-groomed equation (8.1) be converted into

$$(p+1)^3 + p = z^2 \tag{8.9}$$

If $p = 2$, $z^2 = 29$ which make sure that z cannot be an integer.

If p is any odd prime, then p takes any one of the forms $4N+1$ or $4N+3$.

If $p = 4N+1$ and the perception that z^2 must be odd reduces (8.9) to

$$64N^3 + 96N^2 + 52N + 9 = (2T-1)^2$$

$$\Longrightarrow 16N^3 + 24N^2 + 13N + 8 = T(T-1) \tag{8.10}$$

It is perceived that none of the values of N ensures that the left-hand side of (8.10) as the product of two consecutive integers.

Similarly, the chance of $p = 4N+3$, and the discernment that z^2 is an odd integer reduces (8.9) to

$$64N^3 + 192N^2 + 196N + 67 = (2T-1)^2$$

$$\Longrightarrow 2(32N^3 + 96N^2 + 94N + 33) = 4T(T-1)$$

The above equality does not hold since the left-hand side is twice an odd number and the right-hand side is a multiple of 4.

Hence, in this case there does not exist an integer solution.

Case 7: $x = 2, y = 2$

These preferences of x and y altered the well-designed equation (8.1) into

$$p^2 + (p+1)^2 = z^2$$

$$\Longrightarrow 2p\,(p+1) + 1 = z^2 \qquad (8.11)$$

Since z^2 is an odd number, $z^2 \equiv 1\ (mod\ 4)$

Then, $2p\,(p+1) \equiv 0\ (mod\ 4)$

Hence, either p or $p+1$ is a multiple of 2.

If p is a multiple of 2, then p must be 2.

Implementation of this value of p in (8.7) furnishes $z^2 = 13$ which does not enable as an integer for z.

If $p+1$ is a multiple of 2, then $p+1 = 2A$ for some $A \in \mathcal{Z}$.

The only odd prime satisfying all the above conditions is 3 and the corresponding value of $z = 3$

Consequently, the only integer solution to (8.1) is $(p, x, y, z) = (3, 2, 2, 5)$

Case 8: $x = 3, y = 1$

The original equation (8.1) can be written as

$$p^3 + p + 1 = z^2$$

If $p = 2$, then $z^2 = 11$ which cannot acquiesce an integer for z.

Also, $z^2 \equiv 1\ (mod\ 4)$ and $p(p^2+1) \equiv 0\ (mod\ 4)$ which implies that either $4|p$ or $4|(p^2+1)$

For the reason that p is an odd prime, $4 \nmid p$ and so $p^2 + 1 = 4n$

This is not possible since p can take either of the forms $4N+1, N \geq 1$ or $4N+3$,

$N \geq 0$.

Hence, the conclusion of this case is there cannot discover an integer solution to (8.1).

8.2 Tactics of achieving Non-Negative Integer Solutions to an Exponential Equation with the base as Natural Numbers

Theorem: 8.2

If $n \in \mathcal{N}$, the set of all-natural numbers and x, y and z are integers such that $x + y = 1,2,3,4$, then all viable non-negative integer solutions to an exponential Diophantine equation $n^x + (n+1)^y = z^2$ are prearranged by

$$(n,x,y,z) = \{(4m^2 + 4m - 1,0,1,2m+1), (4m^2 - 2,0,1,2m), (4\mathrm{m}^2 - 1,1,0,2\mathrm{m}),$$

$$(4m(m+1),1,0,2m+1), (2m(m+1),1,1,2m+1), (1,0,3,3),$$

$$(2,3,0,3), (1,1,3,3), (12,1,3,47), (n_{m+1},2,2,z_{m+1}), (72,3,1,611)\}$$

where $n_{m+1} = \frac{1}{2}(3X_m + 4z_m - 1)$, $z_{m+1} = 2X_m + 3z_m$, $m = 0,1,2\ldots$with $X_0 = 7$, $z_0 = 5$.

Proof:

The equation for executing integer solutions is measured as

$$n^x + (n+1)^y = z^2, n \in \mathcal{N} \tag{8.12}$$

The succeeding categories are analyzed for receiving non-negative integer solutions for all effective inclinations of the estimations $x + y = 1,2,3,4$.

Case 1: $x = 0$, $y = 1$

The equation (8.12) for looking solutions in integers is curtailed by

$$n + 2 = z^2 \qquad (8.13)$$

If n is an odd number, then z^2 and hence z is also an odd number.

Presume that $z = 2m + 1, m \in \mathcal{N}$

In the vision of (8.13), the choice of n is then led by

$$n = 4m^2 + 4m - 1$$

Foremost if n is an even number, then z^2 also ensures that z is an even number.

Adopt that $z = 2m, m \in \mathcal{N}$

Then, (8.13) leads to the already mentioned value of n as

$$n = 4m^2 - 2$$

Hence, the massive solutions to (8.12) is embodied by

$(n, x, y, z) = (4m^2 + 4m - 1,0,1,2m + 1)$ and $(n, x, y, z) = (4m^2 - 2,0,1,2m)$

where $m \in \mathcal{N}$.

Case 2: $x = 1, y = 0$

The desirable equation (8.12) to discover integer solutions is revised by

$$n + 1 = z^2 \qquad (8.14)$$

If n is an odd number, then z^2 and also z are even numbers.

Let us select $z = 2m, m \in \mathcal{N}$

Replacing this choice of z in (8.14), the equivalent option for the selected natural number n is given by

$$n = 4m^2 - 1$$

On the other hand, if n is an even number then z^2 as well as z is an odd number.

For easy accessibility, it is denoted by $z = 2m + 1$ where $m \in \mathcal{N}$

Exchanging the precise value of z in (8.14), the corresponding possibility of n is attained by

$$n = 4m(m + 1)$$

Hence, in this case all probable solutions to (8.12) are decided by

$(n, x, y, z) = (4m^2 - 1, 1, 0, 2m)$ and $(n, x, y, z) = (4m(m + 1), 1, 0, 2m + 1)$ where $m \in \mathcal{N}$

Case 3: $x = 0$, $y = 2$

The above chosen values of x and y diminished (8.12) to the successive equation

$$n^2 + 1 = z^2$$

Since any square of an integer added with one can never be a square of an integer, z and subsequently n cannot be an integer.

Case 4: $x = 2, y = 0$

These selections of x and y shortened (8.12) to

$$n^2 = z^2 - 1$$

Since any square of an integer reduced by one is not at all a square of an integer, z and then n are not integers.

Hence, for Case 3 and Case 4, it is impossible to treasure an integer value for all necessary variables.

Case 5: $x = 1$, $y = 1$

The structure of an inventive equation (8.12) is symbolized by

$$2n + 1 = z^2$$

Since the left-hand side is an odd number for all $n \in \mathcal{N}$, the value of z^2 implying that z is also an odd number.

Perceive that $z = 2m + 1$ for all $m \in \mathcal{N}$. Clearly, $n = 2m(m + 1)$.

As a result, the solution is detected by $(n, x, y, z) = (2m(m + 1), 1,1,2m + 1)$ where $m \in \mathcal{N}$.

Case 6: $x = 0$, $y = 3$

The ingenious equation (8.12) is revamped as

$$(n + 1)^3 + 1 = z^2$$

$$\Longrightarrow (n + 2)(n^2 + n + 1) = z^2 \qquad (8.15)$$

If z is odd, then $(n + 2) = z$ and $(n^2 + n + 1) = z$ implying that $n = \pm 1$

Our presumption cannot acknowledge $n = -1$.

As an effect, $n = 1$ and thus $z = 3$

If z is even, then $(n + 2)(n^2 + n + 1) = 2z\left(\frac{z}{2}\right)$.

Obviously, $2z = n^2 + n + 1$ and $\frac{z}{2} = n + 2$ together gives $n^2 - 3n - 7 = 0$

But, the roots of this quadratic equation do not belong to $\mathcal{Z}$, the set of all integers.

Thus, $z \notin \mathcal{Z}$.

As a consequence, the end outcome is $(n, x, y, z) = (1,0,3,3)$.

Case 7: $x = 1, y = 2$

The original equation (8.12) is renovated as

$$n + (n+1)^2 = z^2$$

$$\Longrightarrow n = \big(z + (n+1)\big)(z - (n+1)) \tag{8.16}$$

Then, both $\big(z + (n+1)\big)$ and $\big(z - (n+1)\big)$ are factors of n.

This is feasible for no integer value of z.

Hence, no integer solution exists for the scheme of $x = 1$ and $y = 2$.

Case 8: $x = 2, y = 1$

The shrewd equation (8.12) is reorganized as

$$n^2 + n + 1 = z^2$$

$$\Longrightarrow n + 1 = (z + n)(z - n) \tag{8.17}$$

This equation is not achievable irrespective of z.

Eventually, this case does not contribute an integer solution for equation (8.12).

Case 9: $x = 3, y = 0$

The proficient equation (8.12) is reassembled as

$$n^3 = z^2 - 1$$

$$\Longrightarrow n^3 = (z-1)(z+1) \tag{8.18}$$

If $(z-1)$ and $(z+1)$ are both odd, then they are coprime.

Thus, $z-1=r^3$ and $z+1=s^3$ where $r,s \in \mathcal{N}$ and $gcd(r,s)=1$ which stretches (8.18) as

$$s^3-r^3=2$$

But no two natural numbers please this condition.

If $(z-1)$ and $(z+1)$ are both even, then (8.18) becomes $n^3=2k(2k+2), k \in \mathcal{N}$

That is, $n^3=2^2k(k+1)$ (8.19)

If either k or $k+1$ is an even number, then they are relatively prime. Using the well-known fact that the factors of a cubic number are always a cubic number, equation (8.19) is valid only for $k=1$. None other values of k confirm the above statement which suggests that $n=2$ and $z=3$.

Hence, the imminent result is $(n,x,y,z)=(2,3,0,3)$.

Case 10: $x=0, y=4$

The well-made equation (8.12) be transmuted into

$$(n+1)^4=z^2-1$$

$$\Longrightarrow (n+1)(n+1)^3=(z-1)(z+1) \quad (8.20)$$

Suppose that from equation (8.20), $(n+1)=(z-1)$ and $(n+1)^3=(z+1)$

A simple analysis of these assumptions produces the equation $n^3+3n^2+2n-2=0$

It is examined that none of the integer values of n assures the preceding third-degree equation.

Hence, for this case, there does not occur an integer solution.

Case 11: $x = 1, y = 3$

These espousals of x and y convert equation (8.12) into

$$n + (n+1)^3 = z^2 \tag{8.21}$$

It is analyzed that, the prereferral equation (8.21) is satisfied only for two couples $(n, z) = (1,3)$ and $(n, z) = (12, 47)$.

This can be established by the subsequent MATLAB Program:

```
clear all;

clc;

n = input('Enter a natural number to check the condition

n + (n + 1)^3 = z^2\n');

z2 = n + (n + 1). ^3;

z = sqrt(z2);

if (mod(z, 1) == 0)

fprintf('\nn = %d,\tz = %d', n, z)

else

fprintf('The given natural number n =

%d is not satistying the condition', n)

end
```

Hence, the anticipated solutions are $(n, x, y, z) = \{(1,1,3,3), (12,1,3,47)\}$

Case 12: $x = 2, y = 2$

These appropriate picks of x and y reformed the well-framed equation (8.12) into the standard equation

$$X^2 - 2z^2 = -1 \tag{8.22}$$

where $X = 2n + 1$ (8.23)

The primary values obeying (8.22) are $X_0 = 7, z_0 = 5$.

The recurrence relations to the equation (8.22) are specified by

$$X_{m+1} = 3X_m + 4z_m \tag{8.24}$$

$$z_{m+1} = 2X_m + 3z_m, m = 0,1,2, \ldots \tag{8.25}$$

From (8.23) and (8.24), the sequence of necessary values of n is noted by

$$n_{m+1} = \frac{1}{2}(3X_m + 4z_m - 1), m = 0,1,2, \ldots. \tag{8.26}$$

Consequently, the patterns of positive integer solutions to (8.12) are executed by

$(n, x, y, z) = (n_{m+1}, 2,2, z_{m+1})$ with $n_0 = 3, z_0 = 5$ and $m = 0,1,2, \ldots$

The successive values of n and z can be evaluated by (8.25) and (8.26).

Case 13: $x = 3, y = 1$

The equation (8.12) can be scratched as

$$n^3 + n + 1 = z^2$$

This is satisfied only for the pair $(n, z) = (72, 611)$.

This can be done through the following MATLAB Program:

$clear\ all;$

$clc;$

$n = input('Enter\ a\ natural\ number\ to\ check\ the\ condition$

$n\text{^}3 + n + 1 = z\text{^}2\backslash n');$

$z2 = n.\text{^}3 + (n + 1);$

$z = sqrt(z2);$

$if\ (mod(z, 1) == 0)$

$fprintf('\backslash nn = \%d,\backslash tz = \%d', n, z)$

$else$

$fprintf('The\ given\ natural\ number\ n =$

$\%d\ is\ not\ satistying\ the\ condition', n)$

end

Therefore, the essential non- negative integer solution is $(n, x, y, z) = (72,3,1,611)$.

Case 14: $x = 4, y = 0$

The equation (8.12) can be engraved as

$$n^4 + 1 = z^2$$

$$\Longrightarrow n^4 = (z + 1)(z - 1)$$

Proceeding similarly as in case 10, it is not predicted integer solutions to (8.12).

Hence, the theorem.

8.3 Investigation of Solutions to an Exponential Diophantine Equation $p_1{}^x + p_2{}^y + p_3{}^z = M^2$

The 3-tuples consisting three distinct prime numbers such that the difference between the biggest and smallest prime numbers is six, then the 3-tuple is called a prime triplet. For example, the 3-tuple $(p, p + 2, p + 6)$ or $(p, p + 4, p + 6)$ where p is neither 2 nor 3 are prime triplets.

The approach of existence of integer solutions to an equation $p_1{}^x + p_2{}^y + p_3{}^z = M^2$ where (p_1, p_2, p_3) is a prime triplet of the forms

$$(p, p + 2, p + 6) = (4n + 1, 4n + 3, 4n + 7) \text{ or } (4n + 3, 4n + 5, 4n + 9)$$

and $(p, p + 4, p + 6) = (4n + 1, 4n + 5, 4n + 7)$ or $(4n + 3, 4n + 7, 4n + 9)$

for certain $n \in \mathcal{N}$, the collection of x, y, z are either 1 or 2 is analyzed in the following theorems.

Theorem 8.3

If $x, y, z \in \{1,2\}$ and $(p, p + 2, p + 6)$ is a prime triplet of the form $(4n + 1, 4n + 3, 4n + 7)$ for fixed $n \in \mathcal{N}$, then an equation $p^x + (p + 2)^y + (p + 6)^z = M^2$ has no solution.

Proof:

The theorem is proved by considering the following eight cases.

Case 1: $x = 1, y = 1, z = 1$

Then,

$$p^x + (p + 2)^y + (p + 6)^z = M^2$$

$$\Rightarrow 4n + 1 + 4n + 3 + 4n + 7 = M^2$$

$$\Rightarrow 12n + 11 = M^2$$

It is scrutinized that the expression $12n + 11$ is not a perfect square for any $n \in \mathcal{N}$.

The following MATLAB Program demonstrates the statement given above.

$clc;\ clear\ all;$

$n = input('Enter\ a\ natural\ number\ n');$

$for\ i = 1{:}n$

$p1 = 4 * i + 1;\ p2 = 4 * i + 3;\ p3 = 4 * i + 7;$

$if(isprime(p1) == 1\ \&\ isprime(p2) == 1\ \&\ isprime(p3) == 1)$

$MS = 12 * n + 11;$

$M = sqrt(MS);$

$if(rem(M, 1) == 0)$

$fprintf('p1 = \%d, p2 = \%d, p3 = \%d, M = \%d', p1, p2, p3, M)$

end

end

end

Case 2: $x = 2, y = 1, z = 1$

The equation to analyze solutions in integers can be written as

$$(4n+1)^2+(4n+3)+(4n+7)=M^2$$

$$\Rightarrow 16n^2+16n+11=M^2$$

$$\Rightarrow (4n+2)^2+7=M^2$$

$$\Rightarrow M^2-(4n+2)^2=7$$

This is possible only when $M=4$ and $4n+2=3$. But no such $n\in\mathcal{N}$ satisfies the equation $4n+2=3$.

Case 3: $x=1,y=2,z=1$

For these choices of x,y,z, the original equation is reduced into

$$(4n+1)+(4n+3)^2+(4n+7)=M^2$$

$$\Rightarrow 16n^2+32n+17=M^2$$

$$\Rightarrow (4n+4)^2+1=M^2$$

$$\Rightarrow M^2-(4n+4)^2=1$$

It is well-known that the difference of two square numbers cannot be 1.

Case 4 : $x=1,y=1,z=2$

The selected values of the variables convert the given equation as follows

$$(4n+1)+(4n+3)+(4n+7)^2=M^2$$

$$\Rightarrow 16n^2+64n+53=M^2$$

$$\Rightarrow (4n+8)^2-11=M^2$$

$$\Rightarrow (4n+8)^2-M^2=11$$

This is true only if $M = 5$ and $4n + 8 = 6$. But for any $n \in N$, $4n + 8 = 6$ is not valid.

Case 5 : $x = 2, y = 2, z = 1$

The desired equation becomes

$$(4n + 1)^2 + (4n + 3)^2 + (4n + 7) = M^2$$

$$\Rightarrow 32n^2 + 36n + 17 = M^2$$

Case 6 : $x = 2, y = 1, z = 2$

The considered equation becomes

$$(4n + 1)^2 + (4n + 3) + (4n + 7)^2 = M^2$$

$$\Rightarrow 32n^2 + 68n + 53 = M^2$$

Case 7: $x = 1, y = 2, z = 2$

Then, $p^x + (p + 2)^y + (p + 6)^z = M^2$

$$\Rightarrow (4n + 1) + (4n + 3)^2 + (4n + 7)^2 = M^2$$

$$\Rightarrow 32n^2 + 84n + 59 = M^2$$

Case 8: $x = 2, y = 2, z = 2$

The given equation can be written as

$$(4n + 1)^2 + (4n + 3)^2 + (4n + 7)^2 = M^2$$

$$\Rightarrow 48n^2 + 88n + 59 = M^2$$

It is a well-known fact that if $b^2 = 4ac$, the quadratic polynomial $ax^2 + bx + c$ is a perfect square.

But, the quadratic equation in n mentioned above from case 5 to case 8 does not meet this criterion.

Consequently, none of the choices of x, y, z considered from case 5 to case 8 provides solutions to an equation.

As a conclusion, all the cases are not providing possible solutions to the equation.

Theorem 8.4

A solution to the equation $p^x + (p+2)^y + (p+6)^Z = M^2$ where $x, y, z \in \{1,2\}$ is inconceivable if $(p, p+2, p+6)$ is a prime triplet of the form $(4n+3, 4n+5, 4n+9)$ for suitable $n \in \mathcal{N}$.

Proof:

This theorem is showed by the succeeding eight cases as in theorem 8.3.

Case 1: $x = 1, y = 1, z = 1$

Then, $p^x + (p+2)^y + (p+6)^Z = M^2$

$$\Rightarrow 4n + 3 + 4n + 5 + 4n + 9 = M^2$$

$$\Rightarrow 12n + 17 = M^2$$

This is not true for any $n \in \mathcal{N}$. This statement is confirmed by the succeeding MATLAB Program.

$clc;\ clear\ all;$

$n = input('Enter\ a\ natural\ number\ n');$

$for\ i = 1{:}n$

$p1 = 4 * i + 3;\ p2 = 4 * i + 5;\ p3 = 4 * i + 9;$

$if(isprime(p1) == 1\ \&\ isprime(p2) == 1\ \&\ isprime(p3) == 1)$

$MS = 12 * n + 11;$

$M = sqrt(MS);$

$if(rem(M, 1) == 0)$

$fprintf('p1 = \%d, p2 = \%d, p3 = \%d, M = \%d', p1, p2, p3, M)$

end

end

end

Case 2: $x = 2, y = 1, z = 1$

The required equation to be solved becomes

$$(4n + 3)^2 + (4n + 5) + (4n + 9) = M^2$$

$$\Rightarrow 16n^2 + 32n + 23 = M^2$$

$$\Rightarrow (4n + 4)^2 + 7 = M^2$$

$$\Rightarrow M^2 - (4n + 4)^2 = 7$$

This declaration is true only when $M = 4$ and $4n + 4 = 3$. But there is no $n \in \mathcal{N}$ sustaining the condition $4n + 4 = 3$.

Case 3: $x = 1, y = 2, z = 1$

The developed equation can be modified into

$$(4n + 3) + (4n + 5)^2 + (4n + 9) = M^2$$

$$\Rightarrow 16n^2 + 48n + 37 = M^2$$

$$\Rightarrow (4n + 6)^2 + 1 = M^2$$

$$\Rightarrow M^2 - (4n + 6)^2 = 1$$

As is case 2 of theorem 2.1, this is impossible.

Case 4 : $x = 1, y = 1, z = 2$

The given equation can be rewritten as

$$(4n + 3) + (4n + 5) + (4n + 9)^2 = M^2$$

$$\Rightarrow 16n^2 + 80n + 89 = M^2$$

$$\Rightarrow (4n + 10)^2 - 11 = M^2$$

$$\Rightarrow (4n + 10)^2 - M^2 = 11$$

$\Rightarrow M = 5$ and $4n + 10 = 6$

But for any $n \in N$, $4n + 10 = 6$ is not possible.

Case 5 : $x = 2, y = 2, z = 1$

The stated equation becomes

$$(4n + 3)^2 + (4n + 5)^2 + (4n + 9) = M^2$$

$$\Rightarrow 32n^2 + 68n + 43 = M^2$$

Case 6 : $x = 2, y = 1, z = 2$

The considered equation is converted into

$$(4n + 3)^2 + (4n + 5) + (4n + 9)^2 = M^2$$

$$\Rightarrow 32n^2 + 100n + 95 = M^2$$

Case 7: $x = 1, y = 2, z = 2$

These options of the variables reduce the scrutinized equation into

$$(4n + 3) + (4n + 5)^2 + (4n + 9)^2 = M^2$$

$$\Rightarrow 32n^2 + 116n + 109 = M^2$$

Case 8: $x = 2, y = 2, z = 2$

The equation in which solutions to be discovered becomes

$$(4n + 3)^2 + (4n + 5)^2 + (4n + 9)^2 = M^2$$

$$\Rightarrow 48n^2 + 136n + 115 = M^2$$

As in theorem 8.3, in this theorem also case 5 to case 8 does not yield the solution to an equation. Hence, there exists no solution in the integer to the given equation.

Theorem 8.5

There are infinitely many solutions to the equation $p^x + (p + 4)^y + (p + 6)^z = M^2$ if $(p, p + 4, p + 6)$ is a prime triplet the form $(4n + 1, 4n + 5, 4n + 7)$ for selected $n \in \mathcal{N}$, x, y, z are either of 1 or 2.

Proof:

The theorem is proved as in previous two theorems.

Case 1: $x = 1, y = 1, z = 1$

Then, $p^x + (p + 4)^y + (p + 6)^z = M^2$

$$\Rightarrow 4n + 1 + 4n + 5 + 4n + 7 == M^2$$

$$\Rightarrow 12n + 13 = M^2$$

It is observed from the following MATLAB Program, there are enormous prime triplets can be extracted. For instance, if $n = 3, 9, 69, 153$ provides the prime triplets $(13, 17, 19), (31, 41, 43), (277, 281, 283), (613, 617, 619)$ as solutions to the designated equation.

$clc;\ clear\ all;$

$n = input('Enter\ a\ natural\ number\ n');$

$for\ i = 1{:}n$

$p1 = 4 * i + 1;\ p2 = 4 * i + 5;\ p3 = 4 * i + 7;$

$if(isprime(p1) == 1\ \&\ isprime(p2) == 1\ \&\ isprime(p3) == 1)$

$MS = 12 * n + 13;$

$M = sqrt(MS);$

$if(rem(M, 1) == 0)$

$fprintf('p1 = \%d, p2 = \%d, p3 = \%d, M = \%d', p1, p2, p3, M)$

end

end

end

Case 2: $x = 2, y = 1, z = 1$

The assumed equation becomes

$$(4n + 1)^2 + (4n + 5) + (4n + 7) = M^2$$

$$\Rightarrow 16n^2 + 16n + 13 = M^2$$

$$\Rightarrow (4n + 2)^2 + 9 = M^2$$

$$\Rightarrow M^2 - (4n + 2)^2 = 9$$

This is achievable only when $M = 5$ and $4n + 2 = 4$. However, for every $n \in \mathcal{N}$, the equation $4n + 2 = 4$ is invalid.

Case 3: $x = 1, y = 2, z = 1$

The elected choices of x, y, z minimizes the given equation as

$$(4n + 1) + (4n + 5)^2 + (4n + 7) = M^2$$

$$\Rightarrow 6n^2 + 48n + 33 = M^2$$

$\Rightarrow (4n+6)^2 - 3 = M^2$

$\Rightarrow (4n+6)^2 - M^2 = 3$

Therefore, $4n+6=2$ and $M=1$ are the only values that enable the above equation to be accomplished. But $4n+6=2$ is not conceivable for any $n \in \mathcal{N}$.

Case 4 : $x=1, y=1, z=2$

For these options of x, y, z, the equation to be resolved is

$(4n+1)+(4n+5)+(4n+7)^2 = M^2$

$\Rightarrow 16n^2 + 64n + 55 = M^2$

$\Rightarrow (4n+8)^2 - 9 = M^2$

$\Rightarrow (4n+8)^2 - M^2 = 9$

The only values which attain the above condition are $4n+8=5$ and $M=4$.

But for any $n \in \mathcal{N}$, $4n+8=5$ is not possible.

Case 5 : $x=2, y=2, z=1$

Therefore, the original equation is converted into the quadratic equation as follows

$32n^2 + 52n + 33 = M^2$

Case 6 : $x=2, y=1, z=2$

Then, the equation is altered into the quadratic equation in n as given below.

$32n^2 + 68n + 55 = M^2$

Case 7: $x=1, y=2, z=2$

The similar form of the given equation is

$32n^2 + 100n + 75 = M^2$

Case 8: $x=2, y=2, z=2$

The identical from of the considered equation is

$$48n^2 + 104n + 75 = M^2$$

As the explanation given in theorem 8.3, there is no solution in integers for the cases listed above from 5 to 8.

Hence, the combinations of all the cases of x, y, z are not solutions to an equation.

Theorem 8.6

Any $n \in \mathcal{N}$ such that $p = 4n + 3$ is a prime number and $(p, p + 4, p + 6)$ is a prime triplet, then $p^x + (p + 4)^y + (p + 6)^z = M^2$ has no solution when x, y, z are either 1 or 2.

Proof:

The proof is analogous to theorem 8.3

Remarks:

(i) If $(p_1, p_2, p_3) = (2, 3, 5)$, then the possible solutions of $2^x + 3^y + 5^z = M^2$ are $(x, y, z, M) = (1, 2, 1, 4)$ and $(1, 2, 2, 6)$.

(ii) If $(p_1, p_2, p_3) = (3, 5, 7)$, then there is no solution to the proposed equation $3^x + 5^y + 7^z = M^2$.

Chapter- IX

Application of Linear Diophantine Equation in Chemistry

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Application of Linear Diophantine Equation in Chemistry

In Section 9.1, it is exhibited with few examples of how to use the linear Diophantine equation to contract the molecular formulae of organic or inorganic chemical compounds in order to determine their structure.

9.1 Usage of Linear Diophantine Equation in the Resolution of Molecular Formulae for Various Chemical Substances

Needed Theorem [1]

The linear Diophantine equation $ax + by = c$ has a solution if and only if d divides c where $d = gcd(a, b)$. Furthermore, if (x_0, y_0) is a solution of this equation, then the set of solutions of the equation consists of all pairs (x, y) where $x = x_0 + t\,{}^{b}/_{d}$ and $y = y_0 - t\,{}^{a}/_{d}$, $t \in \mathcal{Z}$.

9.1.1 Determination of Chemical Molecular Formulae

Enabling the possibility that a chemical substance with a molecular weight W encompasses elements A_1, A_2 and A_3 with atomic weights a_1, a_2 and a_3 respectively and that the numbers X, Y and Z represent the number of atoms of elements A_1, A_2 and A_3 visible in each of the elements' molecules. Then, it is obtained that

$$a_1X + a_2Y + a_3Z = W \tag{9.1}$$

Let α_1, α_2 and α_3 constitutes the integers closest to the values a_1, a_2 and a_3 and let w signify the integer closest to the value W.

Then the similar form of linear Diophantine equation (9.1) to be solved is represented by

$$\alpha_1X + \alpha_2Y + \alpha_3Z = w \tag{9.2}$$

If a limit is imposed on the integers X, Y and Z in (9.1), then (9.2) can be solved under a restriction

$$|(a_1 - \alpha_1)X + (a_2 - \alpha_2)Y + (a_3 - \alpha_3)Z| < |W - w| \tag{9.3}$$

If more solutions of (9.2) are retrieved, then the genuine values can be found by substituting them in (9.1) and assessing which satisfies (9.2) with the least significant deviation from W.

The process of finding molecular formulae for three chemical substances using the linear Diophantine equation is enlightened as follows.

9.1.2 Molecular Formula for Substance 1

Consider substance 1 as the chemical compound comprising Carbon, Hydrogen, and Oxygen has a molecular weight of 342.2965 g/mol.

Let X, Y and Z stand for the number of atoms of Carbon, Hydrogen and Oxygen respectively. Consider the first-degree Diophantine equation as

$$12.0107X + 1.00784Y + 15.999Z = 342.2965 \quad (9.4)$$

where $12.0107u, 1.00784u$ and $15.999u$ are the atomic weights of Carbon, Hydrogen and Oxygen respectively.

Next, it is clear that $\alpha_1 = 12, \alpha_2 = 1, \alpha_3 = 16$ and $w = 342$.

Furthermore, the corresponding linear form of (9.4) to discover molecular formula is converted into

$$12X + Y + 16Z = 34 \quad (9.5)$$

subject to the constraint

$$|0.0107X + 0.00784Y - 0.001Z| < 0.2965$$

which provides that $X \leq 12, Y \leq 23, Z \leq 13$

Modify (9.5) as in the following form

$12X + Y = 342 + 16T$ where $Z = -T$

Then, its common solution is given by

$$X = 28 + K$$

$$Y = 6 + 16T - 12K,\ K, T \in \mathcal{Z}.$$

It is assured that the values of X, Y, Z must be greater than 0. Hence, it is enabled to discover the ranges for T and K.

Now,

$$X > 0 \Rightarrow 28 + K > 0 \Rightarrow K > -28$$

$$Y > 0 \Rightarrow 6 + 16T - 12K > 0 \Rightarrow 8T - 6K > -3$$

$$Z > 0 \Rightarrow -T > 0 \Rightarrow T < 0$$

In particular, if $T = -1$, then $8T - 6K > -3$ leads to $K < 1$.

Thus, the range for K should be $-28 < K < 1$.

Continuing the process for $T = -2, T = -3, \ldots, T = -21$, it is received that $K < 1$. But for $T = -22$, $K < -29$ contradicting $-28 < K < 1$.

Thus, the range for T should be $-22 < T \leq -1$.

Therefore, there exist 588 solutions in combinations of T and K.

Eliminate the solutions which violating the conditions that $0 < X < 13, 0 < Y < 24$ and $0 < Z < 14$ by the succeeding MATLAB Program:

```
clear all; clc;
for t = -21:-1
  for k = -27:0
    x = 28 + k;
    y = 6 + 16 * t - 12 * k;
    z = -t;
```

```
    if (x < 13 && y < 24 && z < 14)
      fprintf('x = %d, y = %d, z = %d\n', x, y, z)
    end
  end
end
```

The residual solutions which satisfy the necessary conditions are listed in table 9.1.

Table 9.1

X	***Y***	***Z***
10	14	13
11	2	13
11	18	12
12	6	12
12	22	11

Note that the last two solutions represent the compounds $C_{12}H_6O_{12}$ (Mellitic acid) and $C_{12}H_{22}O_{11}$ (Sucrose or Table Sugar) with molar mass 342.16 g/mol and 342.2965 g/mol respectively.

Therefore, the exact solution is $C_{12}H_{22}O_{11}$.

9.1.3 Molecular Formula for Substance 2

Let us choose substance 2 as the chemical compound with a molecular weight of 98.079 g/mol and a mixture of Hydrogen, Sulphur, and Oxygen.

Let X, Y and Z be the number of atoms of Hydrogen, Sulphur, and Oxygen with respective atomic mass $1.00784u, 32.065u$ and $15.999u$ respectively.

As in Section 9.1.2, choose the linear Diophantine equation in three variables as

$$1.00784X + 32.065Y + 15.999Z = 98.079$$

Clearly, $\alpha_1 = 1, \alpha_2 = 32, \alpha_3 = 16$ and $w = 98$.

Consequently, let us solve the ensuing linear Diophantine equation

$$X + 32Y + 16Z = 98 \quad (9.6)$$

subject to the restriction

$$|0.0078X + 0.065Y - 0.001Z| < 0.079$$

The upper limit for the choices of X, Y and Z are noted by

$$X \leq 3, Y \leq 1, Z \leq 4 \quad (9.7)$$

From (9.6), $X = 98 - 32Y - 16Z > 0, Y > 0$ and $Z > 0$. (9.8)

All the possibilities of X, Y and Z supporting (9.8) are evaluated by $\{(50,1,1), (18,2,1), (34,1,2), (2,2,2), (18,1,3), (2,1,4)\}$

The only choice of (X, Y, Z) that satisfies (9.7) is $(2,1,4)$.

Hence, the component is H_2SO_4, which is Sulphuric Acid with a molar mass 98.079 g/mol.

9.1.4 Molecular Formula for Substance 3

Consider Substance 3 is a combination of Zinc, Sulphur, and Oxygen having a molecular weight of 161.47 g/mol.

Let X, Y and Z be the number of atoms of Zinc, Sulphur, and Oxygen with respective atomic mass $65.38u, 32.065u$ and $15.999u$ respectively.

As in the previous two sections, the equation to be resolved is

$$65.38\,X + 32.065\,Y + 15.999\,Z = 161.47 \quad (9.9)$$

With the same notations as in section 9.1.2, $\alpha_1 = 65, \alpha_2 = 32, \alpha_3 = 16$ and $w = 161$.

Then, an equivalent form of (9.9) to be solved is taken as

$$65X + 32Y + 16Z = 161 \quad (9.10)$$

together with the condition that

$$|0.38X + 0.065Y - 0.001Z| < 0.47$$

The options of such X, Y and Z in (9.10) are viewed by

$$X \leq 1, Y \leq 1, Z \leq 444$$

Since X, Y and Z are positive, the only possibility of X and Y are pointed out by

$$X = 1, Y = 1$$

Now, rearrange (9.10) in the form as given below

$$65X + 16U = 161 \quad (9.11)$$

where $U = 2Y + Z$ (9.12)

In (9.11), $gcd(65,16) = 1$ and 1 divide 161.

Also, the least solution to (9.11) is taken as $X_0 = 161$ and $U_0 = -644$

Hence by theorem [I], there exists infinitely many integer solutions to (9.11) which are represented by

$$X = 161 + 16T \quad (9.13)$$

$$U = -644 - 65T \quad (9.14)$$

where $T \in \mathcal{Z}$

Since $X = 1$, the chance of T is evaluated from (9.13) as

$$T = -10$$

Note that (9.12) is satisfied by $Y_0 = U$ and $Z_0 = -U$. Also, $gcd\ (2,1)\ = 1$

Again, by Theorem [I] the infinitely many solutions to (9.12) are received by

$$Y = U + K = -644 - 65T + K \quad (9.15)$$

$$Z = -U - 2K = 644 - 65T - 2K,\ K \in \mathcal{Z} \quad (9.16)$$

Since $Y = 1$, the value of K is calculated from (9.15) by

$$K = -5$$

Substituting the values of T and K in (9.16), it is determined that

$$Z = 4$$

The only solution that satisfies (9.10) is $(X, Y, Z) = (1,1,4)$.

Hence, the component is $ZnSO_4$, which is Zinc Sulphate with a molar mass 161.47 g/mol.

Conclusion

CONCLUSION

In this dissertation, peculiar patterns of numbers baptized as Cheldhiya, Cheldhiya Companion, Pan-San, Pan-San Comrade, Pan-San Buddy, and Pan-San Mate sequences are being emphasized by assigning specific values for a square free integer d in the worldwide recognized Pell equation $y^2 = dx^2 \pm 1$ and topographies of all these sequences are investigated by utilizing the normalizing technique in Matrices.

Variety of triples which comprise all the above consequent sequences sustaining numerous characteristics are explored. Conclusions have also been made for integer quadruples and quintuples that have exclusive properties.

The congruence relationship and divisibility properties between Pell and Pell Lucas numbers has also been illustrated. Certain theorems have been successfully proved by employing these congruence relations.

In addition to this, procedures for acquiring an infinitely large number of non-zero integer solutions in terms of Pell numbers, Pell-Lucas numbers, Jacobsthal as well as Jacobsthal-Lucas numbers for various second-degree Diophantine equations consisting two variables are assessed.

Some Mordell-type Diophantine equations of the form $y^2 = x^3 + k$ for selected values of k have been displayed and proved that some of them have limited number of integer solutions, few of them have no solutions.

Analyses for equations having finite number of integer solutions are also being done on exponential Diophantine equations that embrace natural numbers and prime numbers with the assistance of elementary concepts of Mathematics.

Investigations are being carried out on the possibility of applications of linear Diophantine equations subject to certain restrictions in Chemistry especially for determining the molecular formulae of chemical compounds.

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Appendix

UGC-CARE List

You searched for "**mathematics**". Total Journals : **98**

Search:

Sr.No.	Journal Title	Publisher	ISSN	E-ISSN	Action
81	Sarajevo Journal of Mathematics	Department of Natural Sciences and Mathematics, Academy of Sciences and Arts of Bosnia and Herzegovina	1840-0655	2233-1964	View
82	Scientific Studies and Research, Series Mathematics and Informatics	Vasile Alecsandri University of Bacau	2457-497X	2067-3566	Discontinued from July 2021
83	Sema Journal	Springer	2254-3902	2281-7875	Indexed in Scopus
84	Serdica Mathematical Journal	Bulgarian Academy of Sciences, Institute of Mathematics and Informatics	1310-6600	NA	View
85	South East Asian Journal of Mathematics and Mathematical Sciences	Ramanujan Society of Mathematics and Mathematical Sciences	0972-7752	NA	View
86	Southeast Asian Bulletin of Mathematics	Department of Mathematics, Yunnan University	0129-2021	0219-175X	View
87	Statistica	Department of Statistical Sciences Paolo Fortunato, University of Bologna	0390-590X	1973-2201	View
88	Statistics and Applications	Society of Statistics, Computer and Applications	NA	2454-7395	View
89	Stochastic Modeling and Applications	MUK Publications and Distributions	0972-3641	NA	View
90	The Electronic International Journal Advanced Modeling and Optimization	Research Institute for Informatics	NA	1841-4311	Discontinued from Oct. 2020

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MORDELL'S EQUATION WHICH HAS NO SOLUTION FOR CERTAIN SELECTION OF K

P. SANDHYA AND V. PANDICHELVI

ABSTRACT

This article examines an incomparable Diophantine equation $B^2 = A^3 + C$, $C \in Z$, the set of all integers and demonstrates for which values of C, no solution in integer has been provided in the suggested equation

Keywords: *Diophantine equation, integer solutions, Mordell's equation, Bacht's equation.*

I. INTRODUCTION

The equation $y^2 = x^3 + k$, for $k \in Z$, is alluded to as Mordell's equation due to Mordell's deep passion in it. Mordell [4] established in 1920 that the equation $y^2 = x^3 + k$ has an infinite number of integral solutions for any $k \in \mathcal{Z}$. Michael A. Bennett and Amir Ghadermarzi [2] cast-off the traditional link between Mordell and cubic Thue equations to solve the Diophantine problem $y^2 = x^3 + k$ for all non-zero integers$k$ with $|k| \leq 10^7$.

In this artefact,an unrivalled Diophantine equation $B^2 = A^3 + C$, $C \in Z$, the set of all integers is studied and it is exposed that, for which values of C in theprofessed equation, no integer solution was supplied by using some classical congruence relations and Legendre symbols.

II. MAIN RESULTS

Congruence relations and Legendre symbolsare exploited to demonstrate that $B^2 = A^3 + C$does not have an integer solution for certain values of$C$.

Theorem: 2.1

Let U & V be integers such that $U \equiv 2 \pmod 4$, $V \equiv 3 \pmod 4$and Let$p \mid V$ and $p \equiv 1 \pmod 4$, where p is a prime number.Then the equation $B^2 = A^3 + C$, where $C = U^3 - V^2$ has no integer solution (A, B).

Proof:

Suppose that there exists a solution (A, B)in integers.

As$C = U^3 - V^2 \equiv -1 \pmod 4$& we have$B^2 \equiv A^3 - 1 \pmod 4$

Hence, $A \not\equiv 0 \pmod 2$and $A \not\equiv 3 \pmod 4$ and so $A \equiv 1 \pmod 4$.

Now,$B^2 + V^2 = A^3 + U^3 = (A + U)(A^2 - AU + U^2)$ (1)

As$A \equiv 1 \pmod 4$ *and* $U \equiv 2 \pmod 4$, it should be

$(A^2 - AU + U^2) \equiv 3 \pmod 4$.

Hence,$(X^2 - XU + U^2)$is odd and by (1) it has a prime factorp_1, $p_1 \equiv 3 \pmod 4$. Thus,$B^2 \equiv -V^2 \pmod{p_1}$.

By our assertion, $p_1 \nmid V$. Hence,

$$\left(-\frac{1}{p_1}\right) = \left(\frac{-N^2}{p_1}\right) = \left(\frac{Y^2}{p_1}\right) = 1 ,$$

denying to $p_1 \equiv 3 \pmod 4$.

This proves that the Diophantine equation $Y^2 = X^3 + K$ has no solution.

Theorem: 2.2

Let U and V be integers satisfying$U \equiv 3 \pmod 4$, $V \equiv 0, 2 \pmod 4$. If a prime number p divides $V/2$ and $p \equiv 1 \pmod 4$, then the Diophantine equation $B^2 = A^3 + C$ where $C = U^3 - V^2$ has no solution (A, B) in integers.

Proof:

Suppose that $(A, B)$is an integer solution of the equation $B^2 = A^3 + C, C = U^3 - V^2$

Since,$U^3 - V^2 \equiv 3 \pmod 4$, it is attained by$B^2 \equiv A^3 + 3 \pmod 4$.

Hence,$A \equiv 1 \pmod 4$.

Now, the original equation is converted for the prescribe value of C as

$B^2 + V^2 = A^3 + U^3 = (A + U)(A^2 - AU + U^2)$ (2)

Since $A \equiv 1\ (mod\ 4)$& $U \equiv 3\ (mod\ 4)$,

$$A^2 - AU + U^2 \equiv 3\ (mod\ 4)$$

Hence,$A^2 - AU + U^2$is odd and by (2) it has a prime factorp_1, $p_1 \equiv 3\ (mod\ 4)$. Thus,$B^2 \equiv -V^2\ (mod\ p_1)$.

By our assertion, $p_1 \nmid V/2$. Hence,$p_1 \nmid V$.

$\left(\frac{-1}{p_1}\right) = \left(\frac{-V^2}{p_1}\right) = \left(\frac{B^2}{p_1}\right) = 1$,

denying$p_1 \equiv 3\ (mod\ 4)$.

Hence, the Diophantine equation $B^2 = A^3 + C$ has no solution.

Theorem: 2.3

Let U and V be integers nourishing with the conditions$U \equiv 2\ (mod\ 8), V \equiv 1\ (mod\ 2)$. If$p$is a prime such that$p \equiv 1, 3\ (mod\ 8)$ and p divides V, then the equation $B^2 = A^3 + C$, where $C = U^3 - 2V^2$has no integral solution.

Proof:

Since,$C = U^3 - 2V^2 \equiv 2\ (mod\ 4)$, it must be

$$B^2 \equiv A^3 + 2\ (mod\ 4)$$

Therefore, $A \not\equiv 0\ (mod\ 2)$, $A \not\equiv 1\ (mod\ 4)$and consequently$A \equiv 3\ (mod\ 4)$

Hence,$A \equiv 3$ or $7\ (mod\ 8)$

Moreover,$C \equiv -2\ (mod\ 8)$.

So that $A \not\equiv 7\ (mod\ 8) \Rightarrow A \equiv 3\ (mod\ 8)$

Now,$B^2 + 2V^2 = A^3 + U^3 = (A + U)(A^2 - UA + U^2)$

As $A \equiv 3\ (mod\ 8)$ and $U \equiv 2\ (mod\ 8)$, it is seen that$A^2 - UA + U^2 \equiv 7\ (mod\ 8)$ and $A + U \equiv 5\ (mod\ 8)$

$\therefore B^2 + 2V^2$has a prime factorp_1 such that$p_1 \equiv 5\ or\ 7\ (mod\ 8)$

By our assumption,$p_1 \nmid V$ and $B^2 \equiv -2V^2\ (mod\ p_1)$

$$\left(-\frac{2}{p_1}\right) = \left(-\frac{2V^2}{p_1}\right) = \left(\frac{B^2}{p_1}\right) = 1$$

contrasting$p_1 \equiv 5\ or\ 7\ (mod\ p)$

Hence no solution for the Diophantine equation $B^2 = A^3 + C$, if $C = U^3 - 2V^2$.

Theorem: 2.4

Assume $U, V \in \mathcal{Z}$ and$U \equiv 6\ (mod\ 8)$, $V \equiv 1\ (mod 4)$. Let$p$ be a prime number such that $p \mid V$ and $p \equiv \pm 1\ (mod\ 8)$. Then the equation $B^2 = A^3 + C$, where $C = 2V^2 + U^3$ does not embrace any integer solution.

Proof:

Since, $C = 2V^2 + U^3 \equiv 2\ (\ mod\ 8)$, then $B^2 \equiv A^3 + 2\ (mod\ 8)$

$\Rightarrow A \not\equiv 0\ (mod\ 2)$, $A \not\equiv 1\ (mod\ 4)$and hence$A \equiv 3\ (mod\ 4)$

Hence $A \equiv 3\ or\ 7\ (mod\ 8)$

If $\equiv 3\ (mod\ 8)$, then $B^2 \equiv 5\ (mod\ 8)$which is not possible.

Thus,$A \equiv 7\ (mod\ 8)$.

Now, $B^2 - 2V^2 = (A + U)(A^2 - UA + U^2)$

$$\Rightarrow A^2 - UA + U^2 \equiv 3\ (mod\ 8.)$$

Therefore$A^2 - UA + U^2$ is odd and is divisible by an odd prime p_1 with $p_1 \equiv 3\ (mod\ 8)$

$\Rightarrow B^2 \equiv 2V^2 \ (mod\ p_1)$.

By our postulation, $p_1 \nmid V$

$$\therefore \left(\frac{2}{p_1}\right) = \left(\frac{2V^2}{p_1}\right) = \left(\frac{B^2}{p_1}\right) = 1$$

disagreeing$p_1 \equiv 3 \ (mod\ 8)$

Hence the proof.

III. CONCLUSION

As a result, using certain traditional mathematical tools like as congruence and Legendre symbols, it is sensed that the hypothesized equation does not have a solution in integers for some specific integer values of C. Thus, one may ascertain the values of C for which the Mordell's equation has no integer solution by following the steps described above.

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82	Scientific Studies and Research, Series Mathematics and Informatics	Vasile Alecsandri University of Bacau	2457-497X	2067-3566	Discontinued from July 2021
83	Sema Journal	Springer	2254-3902	2281-7875	Indexed in Scopus
84	Serdica Mathematical Journal	Bulgarian Academy of Sciences, Institute of Mathematics and Informatics	1310-6600	NA	View
85	South East Asian Journal of Mathematics and Mathematical Sciences	Ramanujan Society of Mathematics and Mathematical Sciences	0972-7752	NA	View
86	Southeast Asian Bulletin of Mathematics	Department of Mathematics, Yunnan University	0129-2021	0219-175X	View
87	Statistica	Department of Statistical Sciences Paolo Fortunato, University of Bologna	0390-590X	1973-2201	View
88	Statistics and Applications	Society of Statistics, Computer and Applications	NA	2454-7395	View
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A STATE OF THE-ART OF SUMS, CONGRUENCE RELATIONS AND DIVISIBILITY PROPERTIES OF PELL AND PELL-LUCAS NUMBERS

P. SANDHYA AND V. PANDICHELVI

ABSTRACT
In this document, several new-fangled identities regarding Pell and Pell-Lucas numbers enable to provide certain congruence relations for those numbers are deliberated. Also, divisibility properties of Pell and Pell-Lucas numbers are revealed by means of these derived congruence relations.

Keywords: *Pell numbers, Pell-Lucas numbers, congruence relations*

I. INTRODUCTION

$P_0 = 0, P_1 = 1$,and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ establish the Pell sequence $\{P_n\}$. P_n is referenced to the n^{th}Pell number. The Pell-Lucas sequence Q_n is defined as $Q_n = P_{n-1} + P_{n+1}$. For each $n \in Z$, $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$and $Q_{n-1} + Q_{n+1} = 8P_n$. For more information on the Pell and Pell-Lucas sequences, see [1]. Numerous well-known relationships exist among the Pell and Pell-Lucas numbers. Typically, these relations are achieved using Binet's formula, which is signified by $P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$and $Q_n = \alpha^n + \beta^n$, for any $n \in Z$, where $\alpha = 1 + \sqrt{2}$and $\beta = 1 - \sqrt{2}$. Additionally, the most well-known formulas for Pell numbers are $\alpha^n = \alpha P_n + P_{n-1}$and $\beta^n = \beta P_n + P_{n-1}$, for $n \in Z$.

Numerous sums incorporating Pell and Pell-Lucas numbers are provided in this study. Following that, certain congruences relating Pell and Pell-Lucas numbers are elaborated. These congruences enable one to establish a number of previously known characteristics. Additionally, with the use these congruences, many additional theorems are acclaimed.

II. SUMS AND CONGRUENCESOF PELL AND PELL-LUCAS NUMBERS

Theorem:2.1
If X is a square matrix with $X^2 = 2X + I$, then $X^n = P_n X + P_{n-1} I$for every integer n.

Proof:
Let$Z[\alpha] = \{A\alpha + B; A, B \in Z\}$ and $Z[X] = \{AX + BI; A, B \in Z\}$

Define a function $f: Z[\alpha] \to Z[X]$by $f(A\alpha + B) = AX + BI$.

Then $f$is a ring isomorphism. Moreover, it is clear that $f(\alpha) = X$ and $f(Q_m) = Q_m I$.

Therefore, $X^n = \big(f(\alpha)\big)^n = f(\alpha^n) = f(P_n\alpha + P_{n-1}) = P_n X + P_{n-1} I$

Corollary: 1.1

If$M = \begin{bmatrix} 1 & 4 \\ 1/2 & 1 \end{bmatrix}$, then $M^n = \begin{bmatrix} \frac{Q_n}{2} & 4P_n \\ \frac{P_n}{2} & \frac{Q_n}{2} \end{bmatrix}$.

Proof:
Since, $M^2 = 2M + I$, it follows from theorem (1) that

$$M^n = P_n M + P_{n-1} I$$

$$M^n = \begin{bmatrix} P_n + P_{n-1} & 4Pn \\ \frac{P_n}{2} & P_n + P_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{Q_n}{2} & 4P_n \\ \frac{P_n}{2} & \frac{Q_n}{2} \end{bmatrix}.$$

Remark:
From the fact that $f: Z[\alpha] \to Z[M]$, defined by $f(A\alpha + B) = AM + BI$ is a ring isomorphism, it is observed that

$$\alpha^{2m} - Q_m \alpha^m + (-1)^m = 0 \quad (1)$$

and $$\alpha^{2m} - 2\sqrt{2} P_m \alpha^m - (-1)^m = 0 \quad (2)$$

Applying the function f on each side of (1) and (2), the relations discovered are pointed out by:

$M^{2m} - Q_m M^m + (-1)^m I = 0$ (3)

and $M^{2m} - KP_m M^m - (-1)^m I = 0$ (4)

where $K = f(2\sqrt{2}) = f(2\alpha - 2) = 2M - 2I = \begin{bmatrix} 0 & 8 \\ 1 & 0 \end{bmatrix}$.

Theorem: 2.2
If $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$, then

$$Q_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i Q_{mi+k}$$

and $P_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i P_{mi+k}$

Proof:
From (3), it is noted that

$M^{2m} = Q_m M^m - (-1)^m I$ (5)

Raising n^{th} power on both sides of (5).

Then, $M^{2mn} = (Q_m M^m - (-1)^m I)^n = (Q_m M^m + (-1)^{m+1} I)^n$

$$= \sum_{i=o}^{n} \binom{n}{i} ((-1)^{m+1} I)^{n-i} (Q_m M^m)^i$$

$$= (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i M^{mi}$$

Therefore, $M^{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i M^{mi+k}$

It comprehends from corollary 1.1 that

$$Q_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i Q_{mi+k}$$

and $P_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i P_{mi+k}$

Corollary 2.2.1:
$Q_{2mn+k} \equiv (-1)^{(m+1)n} Q_k (mod\ Q_m)$ (6)

and $P_{2mn+k} \equiv (-1)^{(m+1)n} P_k (mod\ Q_m)$ (7)

for every $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$.

Remark:

i. Since $K = 2M - 2I = M + M^{-1}, \therefore M^m K = K M^m, \forall m \in Z$

ii. $K^2 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = 8I$ and $\begin{bmatrix} 0 & 8 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 8c & 8d \\ a & b \end{bmatrix}$.

Theorem2.3:
If $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$, then

$$Q_{2mn+k} = (-1)^{mn} \left\{ \sum_{i=o}^{\lfloor n/2 \rfloor} \binom{n}{2i} 8^i {P_m}^{2i} Q_{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} (-1)^{mi} 8^i {P_m}^{2i+1} P_{2mi+m+k} \right\}$$

and

$$P_{2mn+k} = (-1)^{mn}\left\{\sum_{i=o}^{\lfloor n/2\rfloor}\binom{n}{2i}8^i\,{P_m}^{2i}P_{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2\rfloor}\binom{n}{2i+1}(-1)^{mi}8^i\,{P_m}^{2i+1}Q_{2mi+m+k}\right\}$$

Proof:
From (4), it follows that

$$M^{2m} = KP_mM^m+(-1)^mI$$

Therefore, $M^{2mn+k} = (KP_mM^m+(-1)^mI)^nM^k$

$$= \left[\sum_{i=o}^{n}\binom{n}{i}((-1)^mI)^{n-i}\,(KP_mM^m)^i\right]M^k$$

$$= (-1)^{mn}\sum_{i=o}^{n}\binom{n}{i}(-1)^{mi}K^i{P_m}^i\,M^{mi+k}$$

$$= (-1)^{mn}\left\{\sum_{i=o}^{\lfloor n/2\rfloor}\binom{n}{2i}K^{2i}\,{P_m}^{2i}M^{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2\rfloor}\binom{n}{2i+1}(-1)^{mi}K^{2i+1}\,{P_m}^{2i+1}M^{2mi+m+k}\right\}$$

$$= (-1)^{mn}\left\{\sum_{i=o}^{\lfloor n/2\rfloor}\binom{n}{2i}8^i\,{P_m}^{2i}M^{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2\rfloor}\binom{n}{2i+1}(-1)^{mi}8^i\,K{P_m}^{2i+1}M^{2mi+m+k}\right\}$$

We will get the results by trading the matrices K and M on both sides and equating the same entries.

Corollary 2.3.1:
If $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$, then

$$Q_{2mn+k} \equiv (-1)^{mn}Q_k(mod\ P_m) \quad (8)$$

and $$P_{2mn+k} \equiv (-1)^{mn}P_k(mod\ P_m) \quad (9)$$

III. DIVISIBILITY PROPERTIES OF PELL AND PELL-LUCAS NUMBERS

To begin, it is established two well-known theorems in a novel manner by exploiting the congruences postulated in Corollaries 2.2.1 and 2.3.1. Regarding the divisibility of Pell and Pell-Lucas numbers, readers will investigate the formulae and learn how to use them efficiently to resolve problems. Thus, this article explains the fundamental divisibility for Pell and Pell-Lucas numbers.

Theorem 3.1:
The necessary and sufficient conditions for $Q_m|Q_n$ are

i. $m|n$ and

ii. $\frac{n}{m}$ is an odd integer

for all $m, n \in N$ and $m \geq 2$.

Proof:
Presume that $Q_m|Q_n$

Suppose $m \nmid n$, then by fundamental property of divisibility, n can be expressed as$n = mq + r, 0 \leq r < m$.

If q is an even integer, then$q = 2s$ for some $s \in Z$.

From (6), it follows that

$$Q_n = Q_{2ms+r} \equiv (-1)^{(m+1)s}Q_r(mod\ Q_m)$$

Since $Q_m|Q_n$, $Q_m|Q_r$. This is a contradiction since $Q_r < Q_m$as $r < m$. Hence, q is an odd integer. Sustain$q = 2s + 1$ for some $s \in Z$. So,

$$Q_n = Q_{2ms+m+r} \equiv (-1)^{(m+1)s} Q_{m+r} (mod\ Q_m)$$

Also, since $Q_m | Q_n, Q_m | Q_{m+r}$.

To prove: $r = 0$

Suppose $r > 0$. By the identity $Q_{m+r} = Q_m P_{r-1} + P_r Q_{m+1}$, the above implies that $Q_m | P_r Q_{m+1}$.

Since $(Q_m, Q_{m+1}) = 1$, it follows that $Q_m | P_r$. This is a contradiction to the fact that if $r < m$, then $P_r \leq P_m < Q_m$. As a result, it is resolved that $r = 0$.

Thus, that $n = mq$, with q being an odd integer.

Conversely, suppose that $m | n$ and $\frac{n}{m}$ is an odd integer,

That is, $n = m(2s + 1)$, for some integer s. Then it isprocured that,

$$Q_n = Q_{2ms+m} \equiv (-1)^{(m+1)s} Q_m (mod\ Q_m)$$

$\Rightarrow Q_m | Q_n$.

Hence, the result.

Theorem 3.2:

Let $m, n \in N$ and $m \geq 2$. Then $Q_m | P_n$ if and only if $m | n$ and $\frac{n}{m}$ is an even integer.

Proof:

Suppose that $Q_m | P_n$ and $m \nmid n$.This assumption means that $n = mq + r, 0 \leq r < m$ where $m \geq 2$.

If q is an odd integer, it may phrase $q = 2s + 1$ for some integer s.

From (7), it is pointed out that

$$P_n = P_{2ms+m+r} \equiv (-1)^{(m+1)s} P_{m+r} (mod\ Q_m)$$

Then, $Q_m | P_{m+r}$ and hence $Q_m | 8P_{m+r}$. It is well-known that $8P_{m+r} = Q_m Q_{r-1} + Q_r Q_{m+1}$, then $Q_m | Q_r Q_{m+1}$. Since Q_m and Q_{m+1} are relatively prime, the only possibility is $Q_m | Q_r$. But $r < m$ delivers $Q_r < Q_m$. So, $Q_m \nmid Q_r$. This conflict befalls as a result of our erroneous assumption about q being an odd number. Therefore, q is an even integer. Thus, it may have $q = 2s$ for some integer s. Hence,

Form (7), $P_n = P_{2ms+r} \equiv (-1)^{(m+1)s} P_r (mod\ Q_m)$

Since $Q_m | P_n$, $Q_m | P_r$. However, this cannot be true since $r < m$ and hence $P_r \leq P_m < Q_m$. This contributes that $r = 0$. So, it can be concluded that $n = mq$, q is an even integer.

Conversely, suppose that $m | n$ and $n = 2ms$ for some $s \in Z$. Then,it isacquired from (7) that

$$P_n = P_{2ms} \equiv (-1)^{(m+1)s} P_0 (mod\ Q_m)$$

It follows that $Q_m | P_n$.

Theorem3.3:

For all $m, n \in N$ and $m \geq 3$, $P_m | P_n$ if and only if $m | n$.

Proof:

Initially consider that $P_m | P_n$ but $m \nmid n$. Then $n = mq + r$ with $0 \leq r < m$. Now, suppose that q is an even integer, then this may be taken as $q = 2s$ for any integer s.

Hence, (9) provides the succeeding identity

$$P_n = P_{2ms+r} \equiv (-1)^{ms} P_r (mod\ P_m)$$

Since $P_m | P_n$, by applying the above identity, $P_m | P_r$. Since, if $0 \leq r < m$ and $m \geq 3$, it leads to

$P_r < P_m$. Hence, q must be an odd integer. Then $q = 2s + 1$, for some $s \in Z$. Thus, (9) becomes

$$P_n = P_{2ms+m+r} \equiv (-1)^{ms} P_{m+r} (mod\ P_m)$$

Since $P_m|P_n$, it follows that $P_m|P_{m+r}$. By the identity, $P_{m+r} = P_{m+1}P_r + P_mP_{r-1}$, it is noted that $P_m|P_{m+1}P_r$. Due to the fact that$(P_m, P_{m+1}) = 1$, it isreceived that$P_m|P_r$, which is a contradiction. This emerges as a consequence of$P_r < P_m$ as $r < m$ and $m \geq 3$. As a result,$r = 0$ and subsequently$n = mq$, resulting in$m|n$.

Conversely, pretend that $m|n$. Then, the conclusion is$n = mq$ for some natural number q. As an outcome,

$$P_n = P_{mq} = \sum_{i=0}^{q} \binom{q}{i} {P_m}^i {P_{m-1}}^{q-i} P_i$$

Hence, it is realized that $P_m|P_n$.

IV. MAIN THEOREMS

From the identity $2(-1)^n = Q_nP_{n-1} - Q_{n-1}P_n$, it can be seen that $\gcd(Q_n, P_n) = 1$or $\gcd(Q_n, P_n) = 2$. Furthermore,

$${Q_n}^2 - 8{P_n}^2 = 4(-1)^n \qquad (10)$$

From equation (8), it is seen that $Q_{8q+r} \equiv Q_r(mod 12)$ and therefore $12 \nmid Q_n$, for every natural number n.

Now, we'll go over some Pell-Lucas numbers identities that will be necessary in the sequel:

$$Q_{2n} = {Q_n}^2 - 2(-1)^n \qquad (11)$$

$$Q_{3n} = Q_n\left({Q_n}^2 - 3(-1)^n\right) \qquad (12)$$

Theorem4.1:
Let $r \geq 1$, be an odd number and $m > 1$. Then, there is no Pell-Lucas number $Q_n$such that $Q_n = Q_{2r}Q_mx^2$

Proof:
Assume that $Q_n = Q_{2r}Q_mx^2$ and r is an odd number. Then $Q_{2r}|Q_n$ and $Q_m|Q_n$. Then,$n = 2rt$ and $n = mk$ for some odd natural number t & k by theorem 2.4,$\Rightarrow 2|n \Rightarrow n = mk \Rightarrow 2|m$. It is thus obvious that$m = 2v$, for some odd $v \in \mathcal{N}$. Since $2|n$ and $\frac{n}{2}$ is an odd natural number, it can be written as$n = 8q + s$ with $s = 2,6$ and $q \geq 0$. Hence,

$$Q_n = Q_{8q+s} \equiv Q_s(mod\ 12)$$
$$\Rightarrow Q_n \equiv Q_2, Q_6\ (mod 12)$$
$$\Rightarrow Q_n \equiv 6\ (mod\ 12)$$

Similarly, it can be seen that $Q_m \equiv 6\ (mod\ 12)$.

Since, r is an odd natural number, it is obtained that $Q_{2r} \equiv 6(mod\ 12)$. Then it follows that

$$Q_n = Q_{2r}Q_mx^2 \equiv 6Q_mx^2(mod\ 12)$$

Moreover, $6x^2 \equiv 0,6\ (mod\ 12)$and $Q_m \equiv 6\ (mod\ 12)$,

$Q_n \equiv 0(mod\ 12)$ which contradicts the fact that $Q_n \equiv 6\ (mod\ 12)$. This concludes the proof.

Theorem4.2:
$Q_{2^kt} \equiv 2,10\ (mod\ 12)$ for every $k \geq 2$ and for every odd natural number t.

Proof:
Assume that t is an odd natural number, then $t \equiv \pm1, \pm3, \pm5, \pm7\ (mod\ 8)$. Moreover, it can be proved by induction that $2^k \equiv 0, \pm4\ (mod\ 8)$ for $k \geq 2$. $2^kt \equiv 0, \pm4\ (mod\ 8)$.

Therefore, $\Rightarrow 2^kt = 8q$ or $2^kt = 8q \pm 4$ for $q \geq 0$. Then it seeks that

$$Q_{2^kt} = Q_{8q} \equiv Q_0\ (mod\ P_4)$$

Or

$$Q_{2^kt} = Q_{8q\pm4} \equiv Q_{\pm4}\ (mod\ P_4)$$

Thus, $Q_{2^kt} \equiv 2,10\ (mod\ 12), k \geq 2$.

Now, it is possible to generalize theorem as follows:

Theorem4.3:
Let $m > 1, k \geq 2$ and t be an odd natural number. Then there is no Pell-Lucas number Q_n such that $Q_n = Q_{2^k t} Q_m x^2$.

Proof:
Assume that $Q_n = Q_{2^k t} Q_m x^2$ and t is an odd natural number. Since $Q_{2^k t} | Q_n$ and $Q_m | Q_n$, there exist two odd natural numbers u and v such that $n = 2^k tu$ and $n = mv$ by theorem 3.1. Thus, we have $m = 2^k r$, for some $r \in N$, because$n = 2^k tu = mv$ and t, u, v are odd natural numbers. Then, we have 4 divides both m & n, by the fact that $k \geq 2$. Hence, $n = 8q + s$ with $s = 0,4,8,12$. Thus,

$$Q_n = Q_{8q+s} \equiv Q_s \ (mod\ 12)$$

Since $s \in \{0,4,8,12\}$, it follows that

$$Q_n \equiv 2,10 \ (mod\ 12)$$

It may be observed in a similar manner that

$$Q_m \equiv 2,10 \ (mod\ 12)$$

On the other hand, $Q_{2^k t} \equiv 2,10 \ (mod\ 12)$ by theorem (8).

If$Q_{2^k t} \equiv 2 \ (mod\ 12)$, then $Q_n = Q_{2^k t} Q_m x^2 \equiv 2 Q_m x^2 (mod\ 12)$.

Since $2x^2 \equiv 0,2,6,8\ mod(12)$ and $Q_m \equiv 2,10 \ (mod\ 12)$, $Q_n \equiv 0,4,8 \ (mod\ 12)$, which is a contradiction to the fact that $Q_n \equiv 2,10 \ (mod\ 12)$.

Therefore,$Q_{2^k t} \equiv 10 \ (mod\ 12)$. Then

$Q_n = Q_{2^k t} Q_m x^2 \equiv 10 Q_m x^2 (mod\ 12)$.

Since $10x^2 \equiv 0,4,6,10\ mod(12)$ and $Q_m \equiv 2,10 \ (mod\ 12)$, $Q_n \equiv 0,4,8 \ (mod\ 12)$, which denies the fact that $Q_n \equiv 2,10 \ (mod\ 12)$. Hence the proof.

Theorem4.4:
If m and r are odd natural numbers, then there is no Pell-Lucas number Q_n such that $Q_n = Q_m Q_r$.

Proof:
Assume that $Q_n = Q_m Q_r$, for $m > 1$ and $r > 1$ and are odd numbers. Since $Q_m | Q_n$ and $Q_r | Q_n$, there exist two odd natural numbers u and v such that $n = mu$ and $n = rv$.

Hence, we have $u = 4k \pm 1$ for some $k \geq 1$. Therefore, we get $n = mu = m(4k \pm 1) = 4km \pm m$.

$$Q_n = Q_{4km \pm m} \equiv (-1)^k Q_{\pm m} \ (mod\ Q_{2m})$$

i.e., $Q_r Q_m \equiv \pm Q_m \ (mod\ Q_{2m})$ (13)

Similarly, it can be obtained that

$Q_m Q_r \equiv \pm Q_r \ (mod\ Q_{2r})$ (14)

Suppose that $Q_m | Q_{2r}$ then $\frac{2r}{m}$ =an odd integer which is not possible. Hence $Q_m \nmid Q_{2r}$ which implies that $gcd(Q_m, Q_{2r}) = 2$. Then by equations (13) and (14),

$Q_r \equiv \pm 1 \ (mod\ \frac{Q_{2m}}{2})$ and $Q_m \equiv \pm 1 \ (mod\ \frac{Q_{2r}}{2})$

$\Rightarrow Q_{2m} \leq 2Q_r \pm 2$ and $Q_{2r} \leq 2Q_m \pm 2$

$$\Rightarrow Q_{2m} + Q_{2r} \leq 2Q_r + 2Q_m \pm 4$$

By equation (11),

$$Q_m{}^2 \pm 2 + Q_r{}^2 \pm 2 \leq 2Q_r + 2Q_m \pm 4$$

$Q_m(Q_m - 2) + Q_r(Q_r - 2) \leq 0$,

which is a contradiction. This completes the proof of the theorem.

Corollary 4.4.1:

There is no Pell-Lucas number Q_n such that $Q_n = Q_m Q_r$, for any $m > 1$ and $r > 1$.

Proof:
If $r > 1$ and even, then ther proof follows from theorems (7) and (9).

If m and r are odd natural numbers, then it is proved in theorem (10).

V. CONCLUSION

In this research, various quantities by means of the Pell and Pell-Lucas numbers are presented. Then, some specific congruences concerning the Pell and Pell-Lucas numbers have been provided. These analogues allowed to govern a number of previously known features. These congruences have also been utilized to prove many other theorems.

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Manifestation of Two Tremendous Sequences Cheldhiya and Cheldhiya Companion Sequences

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Abstract- In this paper, two peculiar sequences named as Cheldhiya sequence and Cheldhiya Companion sequence are discovered. The general formula for Cheldhiya sequence is enumerated by using the special property called as normalization of the matrix. Also, few theorems involving these sequences are elucidated.

Index Terms-Cheldhiya sequence; Cheldhiya Companion sequence;eigenvalues;eigenvectors.

1. INTRODUCTION

The Pell Equation is a quadratic Diophantine equation of the form $x^2 - \mathrm{d}y^2 = 1$ where d is a positive square-free integer. The equation $x^2 - \mathrm{d}y^2 = 1$ has infinitely many solutions whereas the negative Pellequation $x^2 - \mathrm{d}y^2 = -1$ does not always have a solution. In this communication the sequences of the solution to the equation$x^2 - \mathrm{d}y^2 = \pm 1$, for certain d,are developed as Cheldhiya sequence and Cheldhiya Companion sequence. Also, a few theorems are proved based on these sequences.

2. MAIN RESULTS

The y values of the equation$x^2 - \mathrm{d}y^2 = \pm 1$ for certain non-zero square-free integer d can be

$$x_n = 2kx_{n-1} + x_{n-2}, \qquad k = 1,2,3,\dots, n \geq 1$$

with initial values $x_0 = 1, x_1 = k$.

Here$k$ and $d$can be related as$d = k^2 + 1$where k indicates the order of the sequence while n indicates the number of terms in the k^{th}order sequence.

Define the Cheldhiya sequence matrix as

$$\mathbb{y} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}$$

Now,

$$\mathbb{y}\begin{pmatrix} y_1 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2k \\ 1 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}$$

Also,

generally sequenced as $0, 1, 2k, 4k^2 + 1, 8k^3 + 4k, \dots$ called as **Cheldhiya sequence**. The nth term of this sequence is generalized by the recurrence relation

$$y_n = 2ky_{n-1} + y_{n-2}, \qquad k = 1,2,3,\dots, n \geq 1.$$

with initial values $y_0 = 0, y_1 = 1$.

The x values of the equation$x^2 - \mathrm{d}y^2 = \pm 1$ for certain non-zero square-free integer d can be generally sequenced as $1, k, 2k^2 + 1, 4k^3 + 3k, \dots$ called as **Cheldhiya Companion Sequence.** The nth term of this sequence is generalized by the recurrence relation

$$\mathbb{y}\begin{pmatrix} y_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2k \\ 1 \end{pmatrix} = \begin{pmatrix} 4k^2 + 1 \\ 2k \end{pmatrix} = \begin{pmatrix} y_3 \\ y_2 \end{pmatrix}$$

In general,

$$\mathbb{y}\begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ y_n \end{pmatrix}$$

Theorem:2.1

If$\mathbb{y} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}$ is a Cheldhiya sequence matrix, then

(i) $\mathbb{y}^n = \begin{pmatrix} y_{n+1} & y_n \\ y_n & y_{n-1} \end{pmatrix}$ for all $n \in z^+$.

(ii) $\mathbb{y}^n \begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} y_{k+n} \\ y_{k+n-1} \end{pmatrix}$ for all $n, k \in z^+$.

Proof:

This theorem can be proved by using the principle of mathematical induction on n.

(i) Since $y_0 = 0, y_1 = 1, y_2 = 2k$, the excvlusive Cheldhiya sequence matrix is interpreted by

$$\mathbb{y} = \begin{pmatrix} y_2 & y_1 \\ y_1 & y_0 \end{pmatrix}$$

Therefore, the theorem is valid for $n = 1$.

Assume that the result is true for $n = k$.

That is

$$\mathbb{y}^k = \begin{pmatrix} y_{k+1} & y_k \\ y_k & y_{k-1} \end{pmatrix}$$

Now,

$$\mathbb{y}^{k+1} = \mathbb{y}^k\mathbb{y} = \begin{pmatrix} 2ky_{k+1} + y_k & y_{k+1} \\ 2ky_k + y_{k-1} & y_k \end{pmatrix} = \begin{pmatrix} y_{k+2} & y_{k+1} \\ y_{k+1} & y_k \end{pmatrix}$$

Thus, the theorem is valid for $n = k + 1$.

Hence, the conclusion of the theorem is perceived by$\mathbb{y}^n = \begin{pmatrix} y_{n+1} & y_n \\ y_n & y_{n-1} \end{pmatrix}$

(ii) Since,

$$\mathbb{y}\begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} y_{k+1} \\ y_k \end{pmatrix}$$

the theorem is valid for $n = 1$.

Assume that the theorem is true for $n = t$.

That is,

$$\mathbb{y}^t\begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} y_{k+t} \\ y_{k+t-1} \end{pmatrix}$$

Now, consider

$$\mathbb{y}^{t+1}\begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \mathbb{y}.\mathbb{y}^t\begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_{t+1} & y_t \\ y_t & y_{t-1} \end{pmatrix}\begin{pmatrix} y_{k+1} \\ y_k \end{pmatrix} = \begin{pmatrix} y_{k+t+1} \\ y_{k+t} \end{pmatrix}$$

Hence,

$$\mathbb{y}^n\begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} y_{k+n} \\ y_{k+n-1} \end{pmatrix}$$

Theorem: 2.2 Generalization of Cheldhiya sequence

If $\mathbb{y} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}$ is a Cheldhiya sequence matrix, then the n[th] term of the Cheldhiya Sequence is generalized by

$$y_n = \frac{1}{2\sqrt{k^2+1}}\left[\left(k + \sqrt{k^2+1}\right)^n - \left(k - \sqrt{k^2+1}\right)^n\right],$$ where $n = 0,1,2,3, \ldots.$

Proof:

Given

$$\mathbb{y} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}$$

The two eigenvalues of the above matrix can be derived from the characteristic equation

$$|\mathbb{y} - \lambda I| = 0$$

as$\lambda_1 = k + \sqrt{k^2+1}$ and $\lambda_2 = k - \sqrt{k^2+1}$.

The eigenvectors of $\mathbb{y}$ are given by

$$(\mathbb{y} - \lambda I)V = 0$$

which implies that

$$\begin{pmatrix} 2k - \lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \qquad (1)$$

Case1: If $\lambda_1 = k + \sqrt{k^2+1}$, then one of the eigen vector of $\mathbb{y}$ is performed from Eq. (1) by

$$V_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$$

Case 2: If $\lambda_2 = k - \sqrt{k^2+1}$, then the other eigen vector of $\mathbb{y}$ is computed from Eq. (1) by

$$V_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

If the Diagonal matrix of $\mathbb{y}$ is given by,

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then$D^n = \begin{pmatrix} {\lambda_1}^n & 0 \\ 0 & {\lambda_2}^n \end{pmatrix}$

Let the Normalized eigenvector matrix be

$$N = \begin{pmatrix} \frac{\lambda_1}{\sqrt{1+{\lambda_1}^2}} & \frac{\lambda_2}{\sqrt{1+{\lambda_2}^2}} \\ \frac{1}{\sqrt{1+{\lambda_1}^2}} & \frac{1}{\sqrt{1+{\lambda_2}^2}} \end{pmatrix}$$

Now, by applying theorem (1) the orthogonal transformation of the symmetric matrices

$$\mathbb{y}^n = ND^nN^T$$

can be established by

$$\begin{pmatrix} y_{n+1} & y_n \\ y_n & y_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1}{\sqrt{1+{\lambda_1}^2}} & \frac{\lambda_2}{\sqrt{1+{\lambda_2}^2}} \\ \frac{1}{\sqrt{1+{\lambda_1}^2}} & \frac{1}{\sqrt{1+{\lambda_2}^2}} \end{pmatrix} \begin{pmatrix} {\lambda_1}^n & 0 \\ 0 & {\lambda_2}^n \end{pmatrix} \begin{pmatrix} \frac{\lambda_1}{\sqrt{1+{\lambda_1}^2}} & \frac{\lambda_2}{\sqrt{1+{\lambda_2}^2}} \\ \frac{1}{\sqrt{1+{\lambda_1}^2}} & \frac{1}{\sqrt{1+{\lambda_2}^2}} \end{pmatrix}^T$$

Simplifying the right-hand side of the equation and equating the $(1,2)^{\text{th}}$ entry on both sides, the generalized form of Cheldhiya sequence is estimated by

$y_n = \frac{1}{2\sqrt{k^2+1}}\left[\left(k+\sqrt{k^2+1}\right)^n - \left(k-\sqrt{k^2+1}\right)^n\right]$
where $n = 0,1,2,3,\ldots.$

Remark:

The above identity can also be written as$y_n = \frac{{\lambda_1}^n - {\lambda_2}^n}{\lambda_1 - \lambda_2}$ where $n = 0,1,2,3,\ldots.$

Theorem: 2.3

If $\{y_n\}$ is the Cheldhiya sequence, then the sum of its first n terms is given by

$$\sum_{m=0}^{n-1} y_m = \frac{y_n + y_{n-1} - 1}{2k}$$

Proof:

$$\sum_{m=0}^{n-1} y_m = \sum_{m=0}^{n-1} \left(\frac{{\lambda_1}^m - {\lambda_2}^m}{\lambda_1 - \lambda_2}\right)$$

$$= \frac{1}{\lambda_1 - \lambda_2} \sum_{m=0}^{n-1} ({\lambda_1}^m - {\lambda_2}^m)$$

$$= \frac{1}{\lambda_1 - \lambda_2}\left(\frac{1-{\lambda_1}^n}{1-\lambda_1} - \frac{1-{\lambda_2}^n}{1-\lambda_2}\right)$$

$$= \frac{1}{\lambda_1 - \lambda_2}\left(\frac{-({\lambda_1}^n - {\lambda_2}^n) - (\lambda_1{\lambda_2}^n - {\lambda_1}^n\lambda_2) + (\lambda_1 - \lambda_2)}{(1-\lambda_1)(1-\lambda_2)}\right)$$

$$= \frac{y_n + y_{n-1} - 1}{2k}$$

Hence,

$$\sum_{m=0}^{n-1} y_m = \frac{y_n + y_{n-1} - 1}{2k}$$

Theorem: 2.4

If $G(x) = \sum_{n=0}^{\infty} y_n x^n$ is the Generating function, then the corresponding function forCheldhiya Sequence is$G(x) = \frac{x}{1-2kx-x^2}$

Proof:

$$G(x) = \sum_{n=0}^{\infty} y_n x^n$$

$$= y_0x^0 + y_1x + \sum_{n=2}^{\infty} y_n x^n$$

$$= x + \sum_{n=2}^{\infty} (2ky_{n-1} + y_{n-2})x^n$$

$$= x + 2kx\,G(x) + x^2G(x)$$

Hence,

$$G(x) = \frac{x}{1 - 2kx - x^2}$$

Theorem: 2.5

If y_n is the n[th] term of the Cheldhiya sequence, then $y_n{}^2 - y_{n-r}y_{n+r} = (-1)^{n-r}y_r{}^2$

Proof:

$$y_n{}^2 - y_{n-r}y_{n+r} = \left(\frac{\lambda_1{}^n - \lambda_2{}^n}{\lambda_1 - \lambda_2}\right)^2 - \left(\frac{\lambda_1{}^{n-r} - \lambda_2{}^{n-r}}{\lambda_1 - \lambda_2}\right)\left(\frac{\lambda_1{}^{n+r} - \lambda_2{}^{n+r}}{\lambda_1 - \lambda_2}\right)$$

$$= \frac{1}{(\lambda_1 - \lambda_2)^2}\left(-2(\lambda_1\lambda_2)^n + (\lambda_1\lambda_2)^n\left(\frac{\lambda_1}{\lambda_2}\right)^r + (\lambda_1\lambda_2)^n\left(\frac{\lambda_2}{\lambda_1}\right)^r\right)$$

$$= \frac{(-1)^n}{(\lambda_1 - \lambda_2)^2}\left(-2 + \left(\frac{\lambda_1}{\lambda_2}\right)^r + \left(\frac{\lambda_2}{\lambda_1}\right)^r\right)$$

$$= \frac{(-1)^n}{(\lambda_1 - \lambda_2)^2}\left[\frac{(\lambda_1{}^r - \lambda_2{}^r)^2}{(\lambda_1\lambda_2)^r}\right]$$

$$= (-1)^{n-r}y_r{}^2$$

Hence,

$$y_n{}^2 - y_{n-r}y_{n+r} = (-1)^{n-r}y_r{}^2$$

Theorem: 2.6

If $\{x_n\}$ and $\{y_n\}$ are CheldhiyaCompanion sequence and Cheldhiya sequence respectively, then

[1] $x_n = y_{n-1} + ky_n, n \geq 1$

[2] $x_n y_n = \frac{1}{2(k^2+1)}(x_{2n-1} + kx_{2n})$

[3] $\lim_{n\to\infty}\frac{x_n}{y_n} = \sqrt{k^2+1}$

[4] $x_n + ky_n = y_{n+1}$

[5] $y_{n+1} + y_{n-1} = 2x_n$

[6] $\frac{1}{k^2+1}(x_{n+1} + x_{n-1}) = 2y_n$

Proof:

[1] Define the Cheldhiya Companion sequence matrix as

$$\mathbb{X} = \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix}$$

Then, the characteristic roots of $\mathbb{X}$ are determined by $\lambda_1 = k + \sqrt{k^2+1}, \lambda_2 = k - \sqrt{k^2+1}$.

Note that, $\lambda_1\lambda_2 = -1$.

The closed form the Cheldhiya Companion sequence is given by

$$x_n = c_1\lambda_1{}^n + c_2\lambda_2{}^n \qquad (2)$$

By applying the initial values $x_0 = 1, x_1 = k$, the linear system of equations are evaluated by$c_1 + c_2 = 1$ and $c_1\lambda_1 + c_2\lambda_2 = k$.

Thus, $c_1 = \frac{\lambda_2 - k}{\lambda_2 - \lambda_1}$ and $c_2 = \frac{k - \lambda_1}{\lambda_2 - \lambda_1}$

Then, Eq. (2) becomes

$$x_n = \frac{\lambda_2 - k}{\lambda_2 - \lambda_1}\lambda_1{}^n + \frac{k - \lambda_1}{\lambda_2 - \lambda_1}\lambda_2{}^n$$

$$= \frac{1}{\lambda_2 - \lambda_1}\left[\lambda_1\lambda_2\left(\lambda_1{}^{n-1} - \lambda_2{}^{n-1}\right) - k(\lambda_1{}^n - \lambda_2{}^n)\right]$$

$$= \frac{1}{\lambda_1 - \lambda_2}\left[\left(\lambda_1{}^{n-1} - \lambda_2{}^{n-1}\right) + k(\lambda_1{}^n - \lambda_2{}^n)\right]$$

$\therefore \boldsymbol{x_n = y_{n-1} + ky_n}$

[2] $x_n y_n = (y_{n-1} + ky_n)y_n$

$$= y_{n-1}y_n + ky_n{}^2$$

$$= \frac{\lambda_1{}^{n-1} - \lambda_2{}^{n-1}}{\lambda_1 - \lambda_2}\cdot\frac{\lambda_1{}^n - \lambda_2{}^n}{\lambda_1 - \lambda_2} + k\left(\frac{\lambda_1{}^n - \lambda_2{}^n}{\lambda_1 - \lambda_2}\right)^2$$

$$= \frac{1}{(\lambda_1-\lambda_2)^2}\left({\lambda_1}^{2n-1} + {\lambda_2}^{2n-1} + (\lambda_1\lambda_2)^n\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right) + k\left({\lambda_1}^{2n} + {\lambda_2}^{2n} - 2(\lambda_1\lambda_2)^n\right)\right)$$

$$= \frac{1}{(\lambda_1-\lambda_2)^2}\left(2x_{2n-1} - 2k(-1)^n + k(2x_{2n} - 2(-1)^n)\right)$$

$$= \frac{1}{4(k^2+1)}(2x_{2n-1} + 2kx_{2n})$$

$$\therefore \boldsymbol{x_n y_n = \frac{1}{2(k^2+1)}(x_{2n-1} + kx_{2n})}$$

[3] $\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{{\lambda_1}^n+{\lambda_2}^n}{2}.\frac{\lambda_1-\lambda_2}{{\lambda_1}^n-{\lambda_2}^n}$

$$= \left(\frac{\lambda_1-\lambda_2}{2}\right)\lim_{n\to\infty}\left[\frac{1+\left(\frac{\lambda_2}{\lambda_1}\right)^n}{1-\left(\frac{\lambda_2}{\lambda_1}\right)^n}\right]$$

Since $\lambda_2 < \lambda_1, \left|\frac{\lambda_2}{\lambda_1}\right| < 1.$ Therefore, $\lim_{n\to\infty}\left(\frac{\lambda_2}{\lambda_1}\right)^n \to 0.$

Hence,

$$\boldsymbol{\lim_{n\to\infty}\frac{x_n}{y_n} = \sqrt{k^2+1}}$$

[4] $x_n + ky_n = \frac{{\lambda_1}^n+{\lambda_2}^n}{2} + k\,\frac{{\lambda_1}^n-{\lambda_2}^n}{\lambda_1-\lambda_2}$

$$= \frac{(\lambda_1-\lambda_2)({\lambda_1}^n+{\lambda_2}^n) + 2k({\lambda_1}^n-{\lambda_2}^n)}{2(\lambda_1-\lambda_2)}$$

$$= \frac{\left({\lambda_1}^{n+1}-{\lambda_2}^{n+1}\right)}{(\lambda_1-\lambda_2)}$$

$$\therefore \boldsymbol{x_n + ky_n = y_{n+1}}$$

[5] $y_{n+1} + y_{n-1}$

$$= \frac{{\lambda_1}^{n+1}-{\lambda_2}^{n+1}}{\lambda_1-\lambda_2} + \frac{{\lambda_1}^{n-1}-{\lambda_2}^{n-1}}{\lambda_1-\lambda_2}$$

$$= \frac{1}{\lambda_1-\lambda_2}\left({\lambda_1}^{n+1}-{\lambda_2}^{n+1}+{\lambda_1}^{n-1}-{\lambda_2}^{n-1}\right)$$

$$= \frac{1}{\lambda_1-\lambda_2}\left({\lambda_1}^{n+1}-{\lambda_2}^{n+1}+\frac{{\lambda_1}^n\lambda_2-{\lambda_2}^n\lambda_1}{\lambda_1\lambda_2}\right)$$

$$= \frac{\boldsymbol{1}}{\lambda_1-\lambda_2}\left({\lambda_1}^{n+1}-{\lambda_2}^{n+1}-{\lambda_1}^n\lambda_2+\lambda_1{\lambda_2}^n\right)$$

$$= {\lambda_1}^n + {\lambda_2}^n$$

$$\therefore \boldsymbol{y_{n+1} + y_{n-1} = 2x_n}$$

[6] $\frac{1}{k^2+1}(x_{n+1} + x_{n-1})$

$$= \frac{1}{2(k^2+1)}\left({\lambda_1}^{n+1}-{\lambda_2}^{n+1}+{\lambda_1}^{n-1}+{\lambda_2}^{n-1}\right)$$

$$= \frac{1}{2(k^2+1)}\left[{\lambda_1}^n\left(\frac{{\lambda_1}^2+1}{\lambda_1}\right)+{\lambda_2}^n\left(\frac{{\lambda_2}^2+1}{\lambda_2}\right)\right]$$

$$= \frac{{\lambda_1}^n-{\lambda_2}^n}{\sqrt{k^2+1}}$$

$$= 2\frac{{\lambda_1}^n-{\lambda_2}^n}{\lambda_1-\lambda_2}$$

Hence,

$$\frac{1}{k^2+1}(x_{n+1} + x_{n-1}) = 2y_n$$

Remark:

In the above result, it is observed that

[1] $c_1 = \frac{\lambda_2-k}{\lambda_2-\lambda_1} = \frac{-\sqrt{k^2+1}}{-2\sqrt{k^2+1}} = \frac{1}{2}$

$$c_2 = 1 - c_1 = \frac{1}{2}$$

Hence,

$$\boldsymbol{x_n = \frac{1}{2}({\lambda_1}^n + {\lambda_2}^n)}$$

[2] $x_n y_n = \frac{1}{2}({\lambda_1}^n + {\lambda_2}^n).\left(\frac{{\lambda_1}^n-{\lambda_2}^n}{\lambda_1-\lambda_2}\right)$

$$= \frac{1}{2}\left(\frac{{\lambda_1}^{2n}-{\lambda_2}^{2n}}{\lambda_1-\lambda_2}\right)$$

$$\boldsymbol{x_n y_n = \frac{1}{2}y_{2n}}$$

Theorem: 2.7

If $\{x_n\}$ is a Cheldhiya Companion sequence, then for $m \geq 1$

[1] $x_{2m-1}x_{2m+1} + (k^2+1) = (x_{2m})^2$
[2] $x_{2m}x_{2m+2} - (k^2+1) = (x_{2m+1})^2$

Proof:

[1] $$x_{2m-1}x_{2m+1} + (k^2+1) = \frac{\left(\lambda_1{}^{2m-1} + \lambda_2{}^{2m-1}\right)}{2}\frac{\left(\lambda_1{}^{2m+1} + \lambda_2{}^{2m+1}\right)}{2} + (k^2+1)$$

$$= \frac{1}{4}\left(\lambda_1{}^{4m} + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + \lambda_2{}^{4m}\right) + (k^2+1)$$

$$= \frac{1}{4}\left(\lambda_1{}^{4m} - \left(\lambda_1{}^2 + \lambda_2{}^2\right) + \lambda_2{}^{4m}\right) + (k^2+1)$$

$$= \frac{1}{4}\left(\lambda_1{}^{4m} + 2 + \lambda_2{}^{4m}\right)$$

$$= \frac{1}{4}\left(\lambda_1{}^{2m} + \lambda_2{}^{2m}\right)^2$$

Hence,

$$x_{2m-1}x_{2m+1} + (k^2+1) = (x_{2m})^2$$

[2] $$x_{2m}x_{2m+2} - (k^2+1)$$

$$= \frac{\left(\lambda_1{}^{2m} + \lambda_2{}^{2m}\right)}{2}\frac{\left(\lambda_1{}^{2m+2} + \lambda_2{}^{2m+2}\right)}{2} - (k^2+1)$$

$$= \frac{1}{4}\left(\lambda_1{}^{4m+2} + \left(\lambda_1{}^2 + \lambda_2{}^2\right) + \lambda_2{}^{4m+2}\right) - (k^2+1)$$

$$= \frac{1}{4}\left(\lambda_1{}^{4m+2} - 2 + \lambda_2{}^{4m+2}\right)$$

$$= \frac{1}{4}\left(\lambda_1{}^{2m+1} + \lambda_2{}^{2m+1}\right)^2$$

Hence,

$$x_{2m}x_{2m+2} - (k^2+1) = (x_{2m+1})^2$$

3. CONCLUSION

In this paper, the general solution to the Pell equation $x^2 - \mathrm{d}y^2 = \pm 1$for some particular positive values of d are developed as Cheldhiya and Cheldhiya Companion sequences. Based on these sequences some interesting results are provided. In this manner, one can develop various sequences for any other values of d and investigate some other results.

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INVENTION OF FOUR NOVEL SEQUENCES AND THEIR PROPERTIES

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Abstract- In this paper, four novel sequences named as Pan-San, Pan-San Buddy, Pan-San Comrade and Pan-San Mate sequences are discovered. Also, the recurrence relations, the general formulae for all sequences and some theorems are invented by exploiting basic concepts of matrices.

Keywords- Pan-San sequence, Pan-San Buddy sequence, Pan-San Comrade sequence and Pan-San Mate sequence, characteristic equation, eigenvalues.

1. INTRODUCTION

Assume that $d \neq 1$ is any positive square free integer. Then the equation $x^2 - dy^2 = 1$ is called as the classical Pell equation. There are numerous integer solutions (x_n, y_n) for $n \geq 0$, for this Pell equation. Many authors such as Lenstra [4], Matthews [5], Techan [9] and others take some certain Pell equations and solutions into account. Pandichelvi .V, Sandhya .P [7] discovered two tremendous sequences Cheldhiya and Cheldhiya companion sequences by taking $d = k^2 + 1$ in the Pell equation $x^2 - dy^2 = \pm 1$. The Pan-San, Pan-San Buddy, Pan-San Comrade and Pan-San Mate sequences are

formed in this communication for the solution to the equation $x^2 - dy^2 = 1$ for some specific d. A few theorems are also supported using these sequences.

2. PAN-SAN AND PAN-SAN BUDDY SEQUENCES

The C and D values in the universal equation $D^2 - dC^2 = 1$ for certain non-zero square-free integer $d = k^2 + 2, k \in N - \{1\}$ propagates two novel sequences $0, k, 2k(k^2+1), 4k(k^2+1)^2 - k, 8k^2(k^2+1)^3 - 4k(k^2+1)^2, \ldots$ and $1, k^2+1, 2(k^2+1)^2 - 1, 4(k^2+1)^3 - 3(k^2+1), \ldots$ and named as Pan-San sequence and Pan-San Buddy sequence respectively. The nth term of the first sequence is interpreted by the recurrence relation

$$C_{n,k} = 2(k^2+1)C_{n-1,k} - C_{n-2,k}, \qquad k, n \in N - \{1\}$$

where $C_{0,k} = 0, C_{1,k} = k$.

The nth term of the second sequence is standardized by the recurrence relation

$$D_{n,k} = 2(k^2+1)D_{n-1,k} - D_{n-2,k}, \qquad k, n \in N - \{1\}$$

where $D_{0,k} = 1, D_{1,k} = k^2 + 1$.

Define the Pan-San sequence matrix as

$$\mathcal{M} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}$$

Now,

$$\mathcal{M}\begin{pmatrix} C_{1,k} \\ C_{0,k} \end{pmatrix} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k \\ 0 \end{pmatrix} = \begin{pmatrix} 2k(k^2+1) \\ k \end{pmatrix} = \begin{pmatrix} C_{2,k} \\ C_{1,k} \end{pmatrix}$$

Also,

$$\mathcal{M}\begin{pmatrix} C_{2,k} \\ C_{1,k} \end{pmatrix} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2k(k^2+1) \\ k \end{pmatrix} = \begin{pmatrix} 4k(k^2+1)^2 - k \\ 2k(k^2+1) \end{pmatrix} = \begin{pmatrix} C_{3,k} \\ C_{2,k} \end{pmatrix}$$

More generally,

$$\mathcal{M}\begin{pmatrix} C_{n,k} \\ C_{n-1,k} \end{pmatrix} = \begin{pmatrix} C_{n+1,k} \\ C_{n,k} \end{pmatrix}$$

Theorem: 2.1

If $\mathcal{M} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}$ is a Pan-San sequence matrix, then the n[th] term of the Pan-San sequence is generalized by

$C_{n,k} = \frac{1}{2\sqrt{k^2+2}}\left[\left((k^2+1)+k\sqrt{k^2+2}\right)^n - \left((k^2+1)-k\sqrt{k^2+2}\right)^n\right]$, where $n \in W$, the set of all whole numbers.

Proof:

Given

$$\mathcal{M} = \begin{pmatrix} 2(k^2+1) & -1 \\ 1 & 0 \end{pmatrix}$$

The characteristic equation $\lambda^2 - 2(k^2+1)\lambda + 1 = 0$ of $\mathcal{M}$ reveals two distinct eigen values $\sigma = (k^2+1)+k\sqrt{k^2+2}$ and $\tau = (k^2+1)-k\sqrt{k^2+2}$.

Also, $\lambda^2 = 2(k^2+1)\lambda - 1$

$$\lambda^3 = \lambda^2.\lambda$$

$$= (4(k^2+1)^2 - 1)\lambda - 2(k^2+1) = \frac{1}{k}\left(C_{3,k}\lambda - C_{2,k}\right)$$

$$k\lambda^3 = C_{3,k}\lambda - C_{2,k}$$

$$\lambda^4 = [8(k^2+1)^3 - 4(k^2+1)]\lambda - [4(k^2+1) - 1] = \frac{1}{k}\left(C_{4,k}\lambda - C_{3,k}\right)$$

$$k\lambda^4 = C_{4,k}\lambda - C_{3,k}$$

In general, $k\lambda^n = C_{n,k}\lambda - C_{n-1,k}$ (1)

Since both σ and τ are the characteristic values, they must satisfy (1), hence

$k\sigma^n = \left(C_{n,k}\sigma - C_{n-1,k}\right)$ and $k\tau^n = \left(C_{n,k}\tau - C_{n-1,k}\right)$

$\Rightarrow k(\sigma^n - \tau^n) = C_{n,k}(\sigma - \tau)$

Therefore, $C_{n,k} = \frac{k(\sigma^n - \tau^n)}{(\sigma - \tau)}$

$C_{n,k} = \frac{1}{2\sqrt{k^2+2}}\left[\left((k^2+1) + k\sqrt{k^2+2}\right)^n - \left((k^2+1) - k\sqrt{k^2+2}\right)^n\right]$, where $n \in W$, the set of all whole numbers.

Theorem: 2.2

If $\{D_{n,k}\}$ and $\{C_{n,k}\}$ are Pan-San Buddy sequence and Pan-San sequence respectively, then

i. $kD_{n,k} = (k^2+1)C_{n,k} - C_{n-1,k}$

ii. $D_{n,k}C_{n,k} = \frac{1}{2}C_{2n,k}$

iii. $D_{n+1,k} - D_{n-1,k} = 2k(k^2+2)C_{n,k}$

iv. $\left(C_{n,k} + C_{n-1,k}\right)^2 + 1 = D_{2n-1,k}$

v. $C_{n+1,k} - C_{n-1,k} = 2kD_{n,k}$

vi. $D_{n+1,k} - D_{n-1,k} = 2k(k^2+2)C_{n,k}$

Proof:

i. By using the characteristic values of the Pan-San sequence as explained in theorem 2.1, their product is given by $\sigma\tau = 1$.

The closed form the Pan-San Buddy sequence is specified by

$$D_{n,k} = A\sigma^n + B\tau^n \quad (2)$$

The fundamental values $D_{0,k} = 1, D_{1,k} = k^2 + 1$ provides the subsequence system of linear equations $A + B = 1$

$$A\sigma + B\tau = k^2 + 1.$$

Precisely, $A = \frac{(k^2+1)-\tau}{\sigma-\tau}$ and $B = \frac{\sigma-(k^2+1)}{\sigma-\tau}$

Consequently, the specific value of $D_{n,k}$ is pointed out by

$$D_{n,k} = \frac{(k^2+1)-\tau}{\sigma-\tau}\,\sigma^n + \frac{\sigma-(k^2+1)}{\sigma-\tau}\tau^n$$

$$= \frac{1}{\sigma-\tau}[(k^2+1)(\sigma^n - \tau^n) + \sigma\tau(\tau^{n-1} - \sigma^{n-1})]$$

$$= \frac{(k^2+1)}{k}C_{n,k} - \frac{1}{k}C_{n-1,k}$$

$$\therefore \boldsymbol{kD_{n,k} = (k^2+1)C_{n,k} - C_{n-1,k}}$$

ii. The alternative forms of the above values of A and B are epitomized by

$$A = \frac{(k^2+1)-\tau}{\sigma-\tau} = \frac{k\sqrt{k^2+2}}{2k\sqrt{k^2+2}} = \frac{1}{2}$$

$$B = 1 - \mathrm{A} = \frac{1}{2}$$

The equivalent values of the general term of the Pan-San Buddy sequence is noted as

$$\boldsymbol{D_{n,k} = \frac{1}{2}(\sigma^n + \tau^n)}$$

Hence,

$$D_{n,k}C_{n,k} = \left(\frac{\sigma^n+\tau^n}{2}\right)\left(\frac{k(\sigma^n-\tau^n)}{(\sigma-\tau)}\right)$$

$$= \frac{k}{2(\sigma-\tau)}(\sigma^{2n} - \tau^{2n})$$

$$\boldsymbol{D_{n,k}C_{n,k} = \frac{1}{2}C_{2n,k}}$$

iii. $$\frac{D_{n+1,k}-D_{n-1,k}}{C_{n,k}} = \frac{\frac{1}{2}(\sigma^{n+1}+\tau^{n+1})-\frac{1}{2}(\boldsymbol{\sigma^{n-1}+\tau^{n-1}})}{\frac{k(\sigma^n-\tau^n)}{(\sigma-\tau)}}$$

$$= \frac{(\sigma-\tau)}{2k}\frac{(\sigma^{n+1}+\tau^{n+1}-\sigma^{n-1}-\tau^{n-1})}{(\sigma^n-\tau^n)}$$

$$= \frac{(\sigma-\tau)}{2k}\frac{(\sigma^{n+1}+\tau^{n+1}-\sigma^n\tau-\sigma\tau^n)}{(\sigma^n-\tau^n)}$$

$$= \frac{(\sigma-\tau)}{2k}(\sigma - \tau)$$

$$= 2k(k^2 + 2)$$

$$\boldsymbol{D_{n+1,k} - D_{n-1,k} = 2k(k^2 + 2)C_{n,k}}$$

iv. $$\left(C_{n,k} + C_{n-1,k}\right)^2 + 1 = \left[\frac{k(\sigma^n-\tau^n)}{(\sigma-\tau)} + \frac{k(\sigma^{n-1}-\tau^{n-1})}{(\sigma-\tau)}\right]^2 + 1$$

$$= \left\{\frac{k}{(\sigma-\tau)}[\sigma^n - \tau^n + \sigma^n\tau - \sigma\tau^n]\right\}^2 + 1$$

$$= \left\{\frac{k}{(\sigma-\tau)}[\sigma^n(1+\tau) - \tau^n(1+\sigma)]\right\}^2 + 1$$

$$= \frac{k^2}{(\sigma-\tau)^2}[\sigma^{2n}(1+\tau)^2 + \tau^{2n}(1+\sigma)^2 - 2\sigma^n\tau^n(1+\tau)(1+\sigma)] + 1$$

$$= \frac{k^2}{(\sigma-\tau)^2}[\sigma^{2n} + \tau^{2n} + \sigma^2\tau^2(\sigma^{2n-2} + \tau^{2n-2}) + 2\sigma\tau(\sigma^{2n-1} + \tau^{2n-1}) - 4(k^2+2)] + 1$$

$$= \frac{k^2}{(\sigma-\tau)^2}[(\sigma^{2n} + \tau^{2n}) + \sigma^2\tau^2(\sigma^{2n-2} + \tau^{2n-2}) + 2\sigma\tau(\sigma^{2n-1} + \tau^{2n-1}) - 4(k^2+2)] + 1$$

$$= \frac{1}{4(k^2+2)}\left[2D_{2n,k} + 2D_{2n-2,k} + 4D_{2n-1,k}\right]$$

$$= \frac{1}{4(k^2+2)}4(k^2+2)D_{2n-1,k}$$

$$\boldsymbol{\left(C_{n,k} + C_{n-1,k}\right)^2 + 1 = D_{2n-1,k}}$$

v. $C_{n+1,k} - C_{n-1,k} = 2(k^2+1)C_{n,k} - C_{n-1,k} - C_{n-1,k}$

$$= 2(k^2+1)C_{n,k} - 2C_{n-1,k}$$

$$\boldsymbol{C_{n+1,k} - C_{n-1,k} = 2kD_{n,k}}$$

vi. $D_{n+1,k} - D_{n-1,k} = \frac{1}{2}(\sigma^{n+1} + \tau^{n+1}) - \frac{1}{2}(\sigma^{n-1} + \tau^{n-1})$

$$= \frac{1}{2}[\sigma^{n+1} + \tau^{n+1} - \sigma^n\tau - \sigma\tau^n]$$

$$= \frac{1}{2}[\sigma^n(\sigma-\tau) - \tau^n(\sigma-\tau)]$$

$$= \frac{1}{2}[(\sigma^n - \tau^n)(\sigma-\tau)]$$

$$= \frac{\left(2k\sqrt{k^2+2}\right)^2}{2k} C_{n,k}$$

$$\boldsymbol{D_{n+1,k} - D_{n-1,k} = 2k(k^2+2)C_{n,k}}$$

3. PAN-SAN COMRADE AND PAN-SAN MATE SEQUENCES

The values of R and S in the world-wide equation $S^2 - dR^2 = 1$ for a firm non-zero square-free integer $d = k^2 - 2,\ k \in N - \{1\}$ creates two handsome sequences $0, k, 2k(k^2-1), 4k(k^2-1)^2 - k, 8k(k^2-1)^3 - 4k(k^2-1), \ldots$ and $1, k^2-1, 2(k^2-1)^2 - 1, 4(k^2-1)^3 - 3(k^2-1), \ldots$ And Called As Pan-San Comrade And Pan-San Mate Sequences. The nth term of the earlier sequence is construed by the relation

$R_{n,k} = 2(k^2-1)R_{n-1,k} - R_{n-2,k}$, where $R_{0,k} = 0, R_{1,k} = k, \quad k \in N - \{1\}$ and N is the set of all-natural numbers.

The nth term of the later sequence is inferred by the recurrence relation

$S_{n,k} = 2(k^2-1)S_{n-1,k} - S_{n-2,k}$, where $S_{0,k} = 1, S_{1,k} = k^2 - 1,\ k \in N - \{1\}$ and $n \in W$, the set of whole numbers.

Define the Pan-San Comrade sequence matrix as

$$\mathfrak{M} = \begin{pmatrix} 2(k^2-1) & -1 \\ 1 & 0 \end{pmatrix}$$

Now,

$$\mathfrak{M}\begin{pmatrix} R_{1,k} \\ R_{0,k} \end{pmatrix} = \begin{pmatrix} 2(k^2-1) & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k^2-1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2(k^2-1)^2 - 1 \\ k^2-1 \end{pmatrix} = \begin{pmatrix} R_{2,k} \\ R_{1,k} \end{pmatrix}$$

Also,

$$\mathfrak{M}\begin{pmatrix}R_{2,k}\\R_{1,k}\end{pmatrix}=\begin{pmatrix}2(k^2-1) & -1\\1 & 0\end{pmatrix}\begin{pmatrix}2(k^2-1)^2-1\\k^2-1\end{pmatrix}=\begin{pmatrix}4(k^2-1)^3-3(k^2-1)\\2(k^2-1)^2-1\end{pmatrix}=\begin{pmatrix}R_{3,k}\\R_{2,k}\end{pmatrix}$$

In general, $\mathfrak{M}\begin{pmatrix}R_{n,k}\\R_{n-1,k}\end{pmatrix}=\begin{pmatrix}R_{n+1,k}\\R_{n,k}\end{pmatrix}$

As to section, it is possible to prove the following theorem.

Theorem: 3.1

If $\mathfrak{M}=\begin{pmatrix}2(k^2-1) & -1\\1 & 0\end{pmatrix}$ is a Pan-San Comrade sequence matrix, then the n[th] term of the Pan-San Comrade Sequence is hypothesized by

$R_{n,k}=\frac{1}{2\sqrt{k^2-2}}\left[\left((k^2-1)+k\sqrt{k^2-2}\right)^n-\left((k^2-1)-k\sqrt{k^2-2}\right)^n\right]$, where $n=0,1,2,3,\ldots.$

Theorem:3.2

If $\{R_{n,k}\}$ and $\{S_{n,k}\}$ are Pan-San Comrade and Pan-San Mate sequences respectively, then

i. $kS_{n,k}=(k^2-1)R_{n,k}-R_{n-1,k}$

ii. $S_{n,k}R_{n,k}=\frac{1}{2}R_{2n,k}$

iii. $S_{n+1,k}-S_{n-1,k}=2k(k^2-2)R_{n,k}$

iv. $\left(R_{n,k}-R_{n-1,k}\right)^2-1=S_{2n-1,k}$

v. $R_{n+1,k}-R_{n-1,k}=2kS_{n,k}$

vi. $S_{n+1,k}-S_{n-1,k}=2k(k^2-2)R_{n,k}$

4. CONCLUSION

In this paper, four disparate sequences and their recurrence relations named as Pan-San, Pan-San Buddy, Pan-San Comrade and Pan-San Mate sequences are established by utilizing the generalized

solutions (x,y) to the universal equation called as Pell equation for two non-zero square-free integers $d = k^2 + 2, d = k^2 - 2$ where $k \in N - \{1\}$. Also, the general formulae and few theorems are proved involving such sequences for distinct values of d and can analyze the corresponding results.

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THE PATTERNS OF DIOPHANTINE TRIPLES ENGROSS CHELDHIYA COMPANION SEQUENCE WITH INSPIRING PROPERTIES

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I. Abstract:

In this manuscript, the patterns of Diophantine triples $\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \{a_3, a_4, a_5\}, \ldots$ reside in Cheldhiya companion sequence with splendid properties $D\left(\pm(k^2+1)\right)$, $k \in N$ are investigated.

Keywords: Cheldhiya companion sequence, Diophantine triples, Pellian equation.

II. Introduction:

A Diophantine m-tuple with property $D(n)$ is a set of m positive integers $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + n$ is a perfect square for all $i \neq j$ in $\{1,2, \ldots m\}$. Diophantus has already researched how to locate these and he found the rational quadruple $\{1/16, 33/16, 68/16, 105/16\}$ with the property $D(1)$ (see[1]). Fermat has discovered the first integer quadruple $\{1,3,8,120\}$ with the same property. Euler gave the solution $\{a, b, a+b+2r, 4r(r+a)(r+b)\}$, where $ab+1=r^2$ (see[2]). For all-embracing review of a variety of articles one may refer [3 - 14]. In this communication, the patterns enclosing three elements each of which is a Cheldhiya companion sequence $\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \{a_3, a_4, a_5\}, \ldots$ with impressive properties $D\left(\pm(k^2+1)\right), k \in N$ are examined.

III. Method of Analysis:

Presume that

$$a_1 = x_{2m}, a_2 = x_{2m+2},\ m \in N \text{ where } x_n = \frac{1}{2}\left(\left(k+\sqrt{k^2+1}\right)^n + \left(k-\sqrt{k^2+1}\right)^n\right), k \in N$$

be any two integers such that $a_1 a_2 - (k^2+1)$ is a perfect square.

Let a_3 be another positive integer which satisfy the consequent provision

$$a_1 a_3 - (k^2+1) = \phi^2 \qquad (1)$$

$$a_2a_3 - (k^2+1) = \psi^2 \quad (2)$$

Resolving (1) and (2), the value of a_3 is attained by

$$a_3 = \frac{\phi^2 - \psi^2}{a_1 - a_2} \quad (3)$$

By utilizing (3) in (2), the relation to be perceived is

$$a_2\phi^2 - a_1\psi^2 = (k^2+1)(a_1 - a_2) \quad (4)$$

Create the succeeding linear alterations

$$\phi = X + a_1T \quad (5)$$

$$\psi = X + a_2T \quad (6)$$

Restoring the above values of ϕ and ψ in (4), the quadratic equation with two unknowns is estimated by

$$X^2 - (a_1a_2)T^2 = -(k^2+1) \quad (7)$$

Selecting the least solution to (7) as

$$X_0 = x_{2m+1}, T_0 = 1$$

and the implementation this solution in (5) and (6) endow with the relations that

$$\phi = x_{2m+1} + a_1$$

$$\psi = x_{2m+1} + a_2$$

Exchanging the above said suitable modifications in (3), the third element in the essential patterns which assure the postulation is specified by

$$a_3 = x_{2m} + 2x_{2m+1} + x_{2m+2}$$

Hence,

$\{x_{2m},\ x_{2m+2},\ x_{2m} + 2x_{2m+1} + x_{2m+2}\}$ is a Diophantine triple with the property $D\left(-(k^2+1)\right), k \in N$

Let a_4 be a new-fangled positive integer such that

$$a_2a_4 - (k^2+1) = {\phi_1}^2 \quad (8)$$

$$a_3a_4 - (k^2+1) = {\psi_1}^2 \quad (9)$$

Subtracting (9) from (8) and make a simple computation, the significant value of a_4 is determined by

$$a_4 = \frac{{\phi_1}^2 - {\psi_1}^2}{a_2 - a_3} \quad (10)$$

Now, choose a_5 be a positive integer which satisfies the conditions that

$$a_3a_5 - (k^2+1) = {\phi_2}^2 \tag{11}$$

$$a_4a_5 - (k^2+1) = {\psi_2}^2 \tag{12}$$

By exploiting a plain numerical calculation in (11) and (12), it is to be noticed that

$$a_5 = \frac{{\phi_2}^2 - {\psi_2}^2}{a_3 - a_4} \tag{13}$$

Suppose that

$$a_4a_6 - (k^2+1) = {\phi_3}^2 \tag{14}$$

$$a_5a_6 - (k^2+1) = {\psi_3}^2 \tag{15}$$

where $a_6 \in Z - \{0\}$

Following the prior process in (14) and (15), the equivalent value of the factor a_4 in the sequence is established by

$$a_6 = \frac{{\phi_3}^2 - {\psi_3}^2}{a_4 - a_5} \tag{16}$$

Since the objective is to accomplish the appropriate integer values for the parameters in the crucial patterns, make use of the subsequent transformations

$$\phi_1 = P_{k+1}x_{2m+1} + x_{2m} + a_2$$

$$\psi_1 = P_{k+1}x_{2m+1} + x_{2m} + a_3$$

$$\phi_2 = x_{2m} + 3x_{2m+1} + 2x_{2m+2} + a_3$$

$$\psi_2 = x_{2m} + 3x_{2m+1} + 2x_{2m+2} + a_4$$

$$\phi_3 = 2x_{2m} + 7x_{2m+1} + 6x_{2m+2} + a_4$$

$$\psi_3 = 2x_{2m} + 7x_{2m+1} + 6x_{2m+2} + a_5$$

where $\{P_n\} = \{2n-1\}, n \in N$ is the sequence of odd numbers from (8) to (16) and proceeding the same mechanism as explained above, the elements in the necessary patterns with the suitable property are studied by

$$a_4 = 3x_{2m} + (2 + 2P_{k+1})x_{2m+1} + 2x_{2m+2}$$

$$a_5 = 6x_{2m} + (10 + 2P_{k+1})x_{2m+1} + 7x_{2m+2}$$

$$a_6 = 13x_{2m} + (26 + 4P_{k+1})x_{2m+1} + 21x_{2m+2}$$

Thus,

$$\{x_{2m},\ x_{2m+2},\ x_{2m} + 2x_{2m+1} + x_{2m+2}\},$$

$\{x_{2m+2},\ x_{2m}+2x_{2m+1}+x_{2m+2},\ 3x_{2m}+(2+2P_{k+1})x_{2m+1}+2x_{2m+2}\}$,

$\{x_{2m}+2x_{2m+1}+x_{2m+2}, 3x_{2m}+(2+2P_{k+1})x_{2m+1}+2x_{2m+2}, 6x_{2m}+(10+2P_{k+1})x_{2m+1}+x_{2m+2}\}$,

$\{3x_{2m}+(2+2P_{k+1})x_{2m+1}+2x_{2m+2},\quad 6x_{2m}+(10+2P_{k+1})x_{2m+1}+7x_{2m+2}, 13x_{2m}+(26+4P_{k+1})x_{2m+1}+21x_{2m+2}\}$ …

are patterns of Diophantine triples concerning Cheldhiya companion sequence such that the product of any two of them decreased by (k^2+1) is a perfect square where k is a natural number.

Hence, the patterns of Diophantine triples $\{a_1,a_2,a_3\},\{a_2,a_3,a_4\},\{a_3,a_4,a_5\}$, ... in which the factors are Cheldhiya companion sequence with the property $D\left(-(k^2+1)\right)$ where $k \in N$ are evaluated.

Examples for the numerical replacement of the above patterns of Diophantine triples with the property $D\left(-(k^2+1)\right)$ are specified in the tabular form as follows.

k	m	k^2+1	$\{a_1,a_2,a_3\}$	$\{a_2,a_3,a_4\}$	$\{a_3,a_4,a_5\}$	$\{a_4,a_5,a_6\}$
1	1	2	{3,17,34}	{17,34,99}	{34,99,249}	{99,249,662}
2	1	5	{9,161,246}	{161,246,805}	{246,805,1941}	{805,1941,5246}
3	3	10	{27379, 1039681, 1404494}	{1039681, 1404494, 4860971}	{1404494, 4860971, 11491249}	{4860971, 11491249, 31299946}

Remark:

Applying the similar procedure as enlightened above, it is pointed out the consequent patterns of Diophantine triples in which every element is a Cheldhiya companion sequence such that the product of any two of them increased by (k^2+1) is a perfect square.

$\{x_{2m-1}, x_{2m+1}, x_{2m-1}+2x_{2m}+x_{2m+1}\}$, $\{x_{2m+1}, x_{2m-1}+2x_{2m}+x_{2m+1}, x_{2m-1}+4x_{2m}+4x_{2m+1}\}$,

$\{x_{2m-1}+2x_{2m}+x_{2m+1},\ x_{2m-1}+4x_{2m}+4x_{2m+1},\ 4x_{2m-1}+12x_{2m}+9x_{2m+1}\}$,

$\{x_{2m-1}+4x_{2m}+4x_{2m+1},\ 4x_{2m-1}+12x_{2m}+9x_{2m+1},\ 9x_{2m-1}+30x_{2m}+25x_{2m+1}\}$,…

A small number of numerical cases for the above sequences of Diophantine triples with the property $D(k^2+1)$ are stated below

k	m	k^2+1	$\{a_1,a_2,a_3\}$	$\{a_2,a_3,a_4\}$	$\{a_3,a_4,a_5\}$	$\{a_4,a_5,a_6\}$
1	1	2	{1,7,14}	{7,14,41}	{14,41,103}	{41,103,274}
2	1	5	{2,38,58}	{38,58,190}	{58,190,458}	{190,458,1238}
3	2	10	{117,4443,6002}	{4443,6002,20773}	{6002,20773,49107}	{20773,49107,133758}

Verification of the numerical examples is displayed by the ensuing **C** program.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
int m,ca,k,n,p;
char ch;
long long int x(int n,int k),a,b,c,d,e,f,A;
clrscr();
do
{
printf("\nEnter the value of k and m\n");
scanf("%d%d",& k,& m);
printf("\nEnter your choice 1 or 2 for D(-(k^2 + 1)) or D(k^2 + 1)\n");
scanf("%d",&ca);
switch (ca)
{
case 1:
a = x(2 * m, k);
b = x(2 * m + 2, k);
A = x(2 * m + 1, k);
p = 2 * k + 1;
c = a + 2 * A + b;
d = 3 * a + (2 + 2 * p) * A + 2 * b;
e = 6 * a + (10 + 2 * p) * A + 7 * b;
f = 13 * a + (26 + 4 * p) * A + 21 * b;
break;
case 2:
a = x(2 * m - 1, k);
b = x(2 * m + 1, k);
A = x(2 * m, k);
c = a + 2 * A + b;
d = a + 4 * A + 4 * b;
e = 4 * a + 12 * A + 9 * b;
f = 9 * a + 30 * A + 25 * b;
break;
}
printf("\n(%lld,%lld,%lld),(%lld,%lld,%lld),(%lld,%lld,%lld),(%lld,%lld,%lld),...",a,b,c,b,c,d,c,d,e,d,e,f);
printf("\nDo you want to continue for different m and k (y/n)?\n");
ch=getche();
}while (ch=='y'||ch=='Y');
getch();
}
long long int x(int n,int k)
{
long long int x[50], y;
x[0] = 1;
x[1] = k;
int i;
for (i = 2; i <= n; i + +)
x[i] = 2 * k * x[i - 1] + x[i - 2];
```

$y = x[i-1];$
return y;
}

IV. Conclusion:

In this article, the patterns of Diophantine triples comprising the Cheldhiya companion sequence satisfying certain intriguing characteristics are created. In this manner, different patterns of Diophantine triples, quadruples, quintuples, etc. can be look into the research with some other characteristics.

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Fabrication of Gorgeous Integer Quadruple

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Abstract: In this paper, an elegant non-zero distinct integer quadruple (a, b, c, d) in which addition of any three of them is a cubical integer is determined by exploiting the general solutions to a meticulous cubic Diophantine equation.

Keywords: Diophantine triples, Ternary quadratic Diophantine equation.

1.Introduction

Diophantus of Alexandria noted that the numbers $\frac{1}{16}, \frac{33}{16}, \frac{68}{16}, \frac{105}{16}$ had the property that the product of either of these two numbers increased by 1 is the square of a rational number. Sets of integers with a comparable property have been of concern for many years, and a sequence of non- negative integers, is verbalized to be a Diophantine m-tuple $\{a_1, a_2, \dots, a_m\}$ with property $D(n)$ if each $a_i a_j + n (i \neq j)$ is the square of an integer [1-7.10]. A variety of integer solutions to different Diophantine equations are analysed in [8,9]. In this communication, the Diophantine quadruple consisting of non-zero distinct integers where the sum of any three elements is a cubic of an integer is discovered.

2. Method of Analysis

Let a, b, c, d be four non-zero distinct integers such that the addition of any three of them is a perfect cube.

Consider

$$a + b + c = p^3 \quad (1)$$

$$a + b + d = q^3 \quad (2)$$

$$a + c + d = r^3 \quad (3)$$

$$b + c + d = s^3 \quad (4)$$

together with the following condition

$$3(a + b + c + d) = (p + q + r + s)z^3 \quad (5)$$

Solving the system of equations from (1) to (4), the corresponding values of a, b, c, d are pointed out by

$$a = \frac{1}{3}(p^3 + q^3 + r^3 - 2s^3) \quad (6)$$

$$b = \frac{1}{3}(p^3 + q^3 + s^3 - 2r^3) \quad (7)$$

$$c = \frac{1}{3}(p^3 + r^3 + s^3 - 2q^3) \quad (8)$$

$$d = \frac{1}{3}(q^3 + r^3 + s^3 - 2p^3) \quad (9)$$

Adding (6), (7), (8) and (9), an interesting combination is enumerated by

$$3(a + b + c + d) = p^3 + q^3 + r^3 + s^3 \quad (10)$$

Comparison of (5) & (10) provides that

$$(p + q + r + s)z^3 = p^3 + q^3 + r^3 + s^3 \quad (11)$$

Employing the following linear transformations

$p = x + 2y, q = 2x + y, r = 2y - x, s = y - 2x$, where x and y are non-zero integers

from (6) to (9) gives

$$a = 8x^3 + 6xy^2 + 5y^3 \quad (12)$$

$$b = x^3 + 6x^2y + 12xy^2 - 2y^3 \quad (13)$$

$$c = -8x^3 - 6xy^2 + 5y^3 \quad (14)$$

$$d = -x^3 + 6x^2y - 12xy^2 - 2y^3 \quad (15)$$

Substitution of the same transformations reduce (11) to the quadratic equation with three unknowns as

$$6x^2 + 3y^2 = z^3 \quad (16)$$

Applying four different procedures of solving (16), the evaluation of an attractive integer quadruple satisfying the condition that the sum of any three quantities is a cubical integer is explained as follows.

Procedure (i):

The choice of $z = 6m^2 + 3n^2$, where $m, n \in Z - \{0\}$, leads (16) to

$$\left(\sqrt{6}x\right)^2+\left(\sqrt{3}y\right)^2=\left(\left(\sqrt{6}m\right)^2+\left(\sqrt{3}n\right)^2\right)^3$$

which implies that

$$\left(\sqrt{6}x+i\sqrt{3}y\right)\left(\sqrt{6}x-i\sqrt{3}y\right)=\left(\left(\sqrt{6}m+i\sqrt{3}n\right)\left(\sqrt{6}m-i\sqrt{3}n\right)\right)^3$$

Escalating the right hand side of the above equation and equating real and imaginary parts on both the sides, it is to be noted that

$$x=6m^3-9mn^2$$

$$y=18m^2n-3n^3$$

$$z=6m^2+3n^2$$

Substituting the above values of x,y,z in (12), (13), (14) and (15), the values of a,b,c,d satisfying our assumption are deliberated by

$$a=8(6m^3-9mn^2)^3+6(18m^2n-3n^3)^2(6m^3-9mn^2)$$
$$+5(18m^2n-3n^3)^3$$

$$b=(6m^3-9mn^2)^3+6(6m^3-9mn^2)^2(18m^2n-3n^3)$$
$$+12(6m^3-9mn^2)\times(18m^2n-3n^3)^2-2(18m^2n-3n^3)^3$$

$$c=-8(6m^3-9mn^2)^3-6(6m^3-9mn^2)(18m^2n-3n^3)^2$$
$$+5(18m^2n-3n^3)^3$$

$$d=-(6m^3-9mn^2)^3+6(6m^3-9mn^2)^2(18m^2n-3n^3)$$
$$-12(6m^3-9mn^2)\times(18m^2n-3n^3)^2-2(18m^2n-3n^3)^3$$

Some numerical examples satisfying the hypothesis are specified below

m	n	a	b	c	d	$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$
1	1	12609	-14067	21141	2187	27^3	9^3	33^3	21^3
2	1	2715525	1456542	569565	-2025378	168^3	129^3	108^3	9^3
3	2	165419721	9726264	104580288	-107228664	654^3	408^3	546^3	192^3

Procedure (ii):

Treating (16) as

$$6x^2+3y^2=1^2.z^3 \qquad (17)$$

Assuming that

$$z=\left(\sqrt{6}m\right)^2+\left(\sqrt{3}n\right)^2$$

and re-establish 1 by

$$1 = \frac{(\sqrt{6}+i\sqrt{3})(\sqrt{6}-i\sqrt{3})}{9}$$

in (17), it becomes

$$(\sqrt{6}x)^2 + (\sqrt{3}y)^2 = \left(\frac{(\sqrt{6}+i\sqrt{3})(\sqrt{6}-i\sqrt{3})}{9}\right)^2 . \left((\sqrt{6}m)^2 + (\sqrt{3}n)^2\right)^3$$

which is equivalent to

$$(\sqrt{6}x + i\sqrt{3}y)(\sqrt{6}x - i\sqrt{3}y) = \left(\frac{(\sqrt{6}+i\sqrt{3})(\sqrt{6}-i\sqrt{3})}{9}\right)^2 . \left((\sqrt{6}m + i\sqrt{3}n)(\sqrt{6}m - i\sqrt{3}n)\right)^3 \quad (18)$$

Equating the positive parts on both sides of (18) and comparing the like terms, it is examined that

$$x = 2m^3 - 12m^2n - 3mn^2 + 2n^3$$

$$y = 8m^3 + 6m^2n - 12mn^2 - n^3$$

In view of (12), (13), (14) and (15), the options of a, b, c, d are estimated by

$$a = 8(2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 + 6(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times (8m^3 + 6m^2n - 12mn^2 - n^3)^2 + 5(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

$$b = (2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 + 6(2m^3 - 12m^2n - 3mn^2 + 2n^3)^2 \times (8m^3 + 6m^2n - 12mn^2 - n^3) + 12(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times (8m^3 + 6m^2n - 12mn^2 - n^3)^2 - 2(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

$$c = -8(2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 - 6(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times (8m^3 + 6m^2n - 12mn^2 - n^3)^2 + 5(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

$$d = -(2m^3 - 12m^2n - 3mn^2 + 2n^3)^3 + 6(2m^3 - 12m^2n - 3mn^2 + 2n^3)^2 \times (8m^3 + 6m^2n - 12mn^2 - n^3) - 12(2m^3 - 12m^2n - 3mn^2 + 2n^3) \times (8m^3 + 6m^2n - 12mn^2 - n^3)^2 - 2(8m^3 + 6m^2n - 12mn^2 - n^3)^3$$

Some numerical examples satisfying the propositions are specified below

m	n	a	b	c	d	$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$
1	1	-10709	-739	10719	2187	$(-9)^3$	$(-21)^3$	13^3	23^3
2	1	19683	-1771470	2480787	1751058	90^3	$(-9)^3$	162^3	135^3
3	2	-55002032	-46632952	105976512	94647096	162^3	$(-192)^3$	526^3	536^3

Procedure (iii):

Consider an alternative solutions to (16) as

$$x = \frac{1}{\sqrt{6}}(k^3 + kl^2)$$

$$y = \frac{1}{\sqrt{3}}(l^3 + lk^2)$$

$$z = k^2 + l^2$$

Case (i):

Since the target is to evaluate integral values for the variables, it is observed that the subsequent two parametric choices of $k = \sqrt{6}m$ and $l = \sqrt{3}n$ provides the values of x and y in integers.

Then, the integral solutions to (16) are calculated by

$$x = 6m^3 + 3mn^2$$

$$y = 3n^3 + 6nm^2$$

$$z = 6m^2 + 3n^2$$

Substituting the above quantities in (12), (13), (14) and (15), the appropriate values of a, b, c, d are discovered by

$$a = 1728m^9 + 3888m^7n^2 + 1080m^6n^3 + 3240m^5n^4 + 1620m^4n^5$$
$$+1188m^3n^6 + 810m^2n^7 + 162mn^8 + 135n^9$$

$$b = 216m^9 + 1296m^8n + 2916m^7n^2 + 1512m^6n^3 + 4050m^5n^4$$
$$+324m^4n^5 + 1971m^3n^6 - 162m^2n^7 + 324mn^8 - 54n^9$$

$$c = -1728m^9 - 3888m^7n^2 + 1080m^6n^3 - 3240m^5n^4 + 1620m^4n^5$$
$$-1188m^3n^6 + 810m^2n^7 - 162mn^8 + 135n^9$$

$$d = -216m^9 + 1296m^8n - 2916m^7n^2 + 1512m^6n^3 - 4050m^5n^4$$
$$+324m^4n^5 - 1971m^3n^6 - 162m^2n^7 - 324mn^8 - 54n^9$$

Some numerical examples satisfying our assumption are precised below

m	n	a	b	c	d	$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$
1	1	13851	12393	-6561	-6561	27^3	27^3	9^3	9^3
2	1	1594323	1062882	-1397493	-196830	108^3	135^3	0^3	$(-81)^3$
3	2	94298688	75611448	-71299008	-22712184	462^3	528^3	66^3	$(-264)^3$

Case (ii):

As in case (i), the single parametric choices of $k = \sqrt{6}t$ and $l = \sqrt{3}t$ offers the values of x and y in integers.

Thus,

$$x = 9t^3$$

$$y = 9t^3$$

$$z = 9t^2$$

Substituting the above magnitudes in (12), (13), (14) and (15), it is determined by

$$a = 13851\ t^9$$

$$b = 12393\ t^9$$

$$c = -6561\ t^9$$

$$d = -6561\ t^9$$

Some numerical examples satisfying the hypothesis are exemplified below

k	a	b	c	d	$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$
1	13851	12393	-6561	-6561	27^3	27^3	9^3	$(-9)^3$
2	7091712	6345216	-3359232	-3359232	216^3	216^3	72^3	$(-72)^3$
3	272629233	243931419	-129140163	-129140163	729^3	729^3	243^3	$(-243)^3$

The C Program for numerical examples satisfying our hypotheses are illustrated below.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
char ch;
intm,n,ca;
signed long int x,y,a,b,c,d,cup,cuq,cur,cus,cup1,cuq1,cur1,cus1,p,q,r,s;
clrscr();
do
{
printf("\nEnter m and n values\n");
scanf("%d%d",&m,&n);
printf("\nEnter your choice case 1 or 2 or 3 or 4\n");
scanf("%d",&ca);
switch(ca)
{
case 1:
x=(6*m*m*m)-(9*m*n*n);
y=(18*m*m*n)-(3*n*n*n);
break;
case 2:
x=(2*m*m*m)-(12*m*m*n)-(3*m*n*n)+(2*n*n*n);
y=(8*m*m*m)+(6*m*m*n)-(12*m*n*n)-(n*n*n);
break;
case 3:
x=(6*m*m*m)+(3*m*n*n);
y=(3*n*n*n)+(6*n*m*m);
break;
case 4:
x=9*m*m*m;
```

```
y=9*m*m*m;
break;
}
a=(8*x*x*x)+(6*x*y*y)+(5*y*y*y);
b=(x*x*x)+(6*x*x*y)+(12*x*y*y)-(2*y*y*y);
c=(-8*x*x*x)-(6*x*y*y)+(5*y*y*y);
d=-(x*x*x)+(6*x*x*y)-(12*x*y*y)-(2*y*y*y);
cup=a+b+c;
cuq=a+b+d;
cur=a+c+d;
cus=b+c+d;
if(cup<0)
{
cup1=-1*cup;
p=pow(cup1,1.0/3.0);
}
else
{
p=pow(cup,1.0/3.0);
}
If(p==0)
p=0;
else
p++;
if(cup<0)
p=-p;
if(cuq<0)
{
cuq1=-1*cuq;
q=pow(cuq1,1.0/3.0);
}
else
{
q=pow(cuq,1.0/3.0);
}
If(q==0)
q=0;
else
q++;
if(cuq<0)
q=-q;
if(cur<0)
{
cur1=-1*cur;
p=pow(cur1,1.0/3.0);
}
else
{
r=pow(cur,1.0/3.0);
}
```

```
If(r==0)
r=0;
else
r++;
if(cur<0)
r=-r;
if(cus<0)
{
cus1=-1*cus;
s=pow(cus1,1.0/3.0);
}
else
{
s=pow(cus,1.0/3.0);
}
if(s==0)
s=0;
else
s++;
if(cus<0)
s=-s;
printf("\nm=%d,n=%d,a=%ld,b=%ld,b=%ld,c=%ld,d=%ld\n",m,n,a,b,c,d);
printf("\na+b+c=(%ld)^3,a+b+d=(%ld)^3,a+c+d=(%ld)^3,b+c+d=(%ld)^3",p,q,r,s);
printf("Do you want to continue (y/n)?");
ch=getche();
}while (ch=='Y' || ch=='y');
getch();
}
```

3. Conclusion

In this communication, the quadruple (a, b, c, d) so that the sum of any three of them is a cubical integer is scrutinized. To conclude, one can search for different quintuples, sextuples, septuples etc satisfying some other properties.

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Incomparable integer quintuple in arithmetic progression with prominent condition

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Abstract

In this document, the incomparable integer quintuple (p,q,r,s,t) in such a way that the components with the renowned property in algebra named as arithmetic progression with the postulation that the addition of three consecutive terms shows a perfect square is established.

Keywords

Diophantine m-tuple, quintuple in arithmetic progression.

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Contents

1. Introduction

Let n be an integer. A set of positive integers $(a_1,a_2,a_3,\ldots a_m)$ is said to have the property $D(n)$ if a_ia_j+n is a perfect square for all $1\leq i<j\leq m$; such a set is called a Diophantine $m-$ tuple $[1-6]\cdot$ In [7], the authors were evaluated the triplesin Arithmetic Progression $(a-d,a,a+d)$ such that2 a - $d=\alpha^2,2a+d=\beta^2,2a=\chi^3$ and $2a-d=\alpha^2,2a+d=\beta^2,2a=\chi^4$ where a and d be two non- zero distinct integer.In [8] , triples were procured in Arithmetic Progression such that the sum of any two is a perfect square. In [9], the authors found the triples in Arithmetic Progression $(a-d,a,a+d)$ such that each of the expression $a^2-ad,2a+d,2a$ is a perfect square. In [10], the authors found the quadruples of the form (x,y,z,w) where the elements arein Arithmetic Progression satisfying the conditions $x+y=\alpha^2z+w=\beta^2$ and $x+y+z+w=\gamma^3$.

In this manuscript, three unlike integer quintuples with the elements in Arithmetic Progression rewarding the condition that that the sum of three consecutive integers indicates a perfect square is acquired.

2. Course of action for survey

Presume that p,q,r,s,t be five non-zero separate integers such that the elements in the quintuple (p,q,r,s,t) materialize in Arithmetic Progression.

To symbolize this proclamation, let a and d be two non-zero integers such that $p=a-2d,q=a-d,r=a,s=a+d,t=a+2d$.

For the exploration of the perception of the manuscript, imagine the sum of three consecutive elements in the already assumed quintuple is a square of an integer. The above declaration is replicated by the subsequent equations

$$p+q+r=3a-3d=\varphi^2 \tag{2.1}$$

$$q+r+s=3a=\eta^2 \tag{2.2}$$

$$r+s+t=3a+3d=\chi^2 \tag{2.3}$$

Addition of (2.1) and (2.3) endow with the proportion that

$$a=\frac{\varphi^2+\chi^2}{6} \tag{2.4}$$

Similarly, subtraction of (2.1) from (2.3) bestow as in the succeeding fraction

$$d=\frac{\chi^2-\varphi^2}{6} \tag{2.5}$$

Elucidationof (2.2) and (2.4) yields the following equation

$$\eta^2 = \frac{\varphi^2 + \chi^2}{2} \tag{2.6}$$

To convert the above said value of η as in integer, launch the novel conversions

$$\eta = 3\lambda, \varphi = 6\mu, \chi = 6\omega \tag{2.7}$$

These translations imitate (2.5) and (2.6) as follows

$$d = 6\left(\omega^2 - \mu^2\right) \tag{2.8}$$
$$\lambda^2 = 2\left(\mu^2 + \omega^2\right) \tag{2.9}$$

The elements in the required quintuple are making into integers with the property looking for is portrayed by the three procedures as below.

Procedure 1: Decode the parameter λ as

$$\lambda = u^2 + v^2$$

Then, the equation (2.9) can be altered by

$$\left(u^2 + v^2\right)^2 = 2\left(\mu^2 + \omega^2\right)$$
$$\Rightarrow (u+iv)^2(u-iv)^2 = (1+i)(1-i)(\mu+i\omega)(\mu-i\omega)$$

By escalating and balancing positive terms and thenequating real and imaginary parts on both sides, the resulting equations are revealed by

$$\mu - \omega = u^2 - v^2$$
$$\mu + \omega = 2uv$$

Resolving the above equations the most plausible values of μ and ω are demonstrated by

$$\mu = \frac{1}{2}\left(u^2 - v^2 + 2uv\right)$$
$$\omega = \frac{1}{2}\left(v^2 - u^2 + 2uv\right)$$

The parametric values of λ, μ and ω in integers are created by selecting the options of $u = 2U$ and and $v = 2V$ as follows

$$\lambda = 4\left(U^2 + V^2\right)$$
$$\mu = 2\left(U^2 - V^2 + 2UV\right)$$
$$\omega = 2\left(V^2 - U^2 + 2UV\right)$$

The replacement of the above value of λ in (2.7), endow with the value of η as

$$\eta = 12\left(U^2 + V^2\right)$$

According to (2.2) and (2.8), the components in the essential quintuple are offered by

$$a = 48\left(U^2 + V^2\right)^2$$
$$d = 192UV\left(V^2 - U^2\right)$$

Subsequently, the necessary guintuple in which the elements form an Arithmetic progression is discovered by

$$(p,q,r,s,t) = \left\{48\left(U^2+V^2\right)^2 - 384UV\left(V^2-U^2\right)\right.$$
$$48\left(U^2+V^2\right)^2 - 192UV\left(V^2-U^2\right), 48\left(U^2+V^2\right)^2$$
$$48\left(U^2+V^2\right)^2 + 192UV\left(V^2-U^2\right), 48\left(U^2+V^2\right)^2$$
$$\left.+384UV\left(V^2-U^2\right)\right\}$$

2.1 Logical postulation is checked for certain values of U and V as tabulated below

Table 1.

U	V	(p, q, r, s, t)	p+q+r	q+r+s	r+s+t
2	1	(3504,2352,1200, 48,-1104)	84^2	60^2	12^2
5	7	(-59712,101568, 262848,424128, 585408)	552^2	888^2	1128^2
1	3	(-4416,192,4800, 9408,14016)	24^2	120^2	168^2

Procedure 2:

The same conversion of $\lambda = u^2 + v^2$ supplies the alternative appearance of (2.9) as

$$(u+iv)^2(u-iv)^2 = \frac{(7+i)(7-i)}{25}(\mu+i\omega)(\mu-i\omega)$$

Replicate the same course of action as mentioned in procedure (2.1), the corresponding values of μ and ω satisfying the double equations $7\mu - \omega = 5\left(u^2 - v^2\right), \mu + 7\omega = 10uv$ are appraised by

$$\mu = \frac{1}{10}\left(7\left(u^2 - v^2\right) + 2uv\right)$$
$$\omega = \frac{1}{10}\left(v^2 - u^2 + 14uv\right)$$

The chances of λ, μ and ω in integers by picking $u = 10U$ and $v = 10Vare$ produced by

$$\lambda = 100\left(U^2 + V^2\right)$$
$$\mu = 10\left(7U^2 - 7V^2 + 2UV\right)$$
$$\omega = 10\left(V^2 - U^2 + 14UV\right)$$

Renovate the value of λ in (2.7), the value of η is calculated by

$$\eta = 300\left(U^2 + V^2\right)$$

In sight of (2.2) and (2.8), the equivalent choices of a and d are pointed out by

$$a = 30000\left(U^2 + V^2\right)^2$$
$$d = -4800\left(6U^4 + 6V^4 + 7U^3V - 7UV^3 - 36U^2V^2\right)$$

Hence, the needed quintuple with desired property is exposed by

$$\begin{aligned}(p,q,r,s,t) = \Big\{ & 30000\left(U^2+V^2\right)^2 + 9600(6U^4+6V^4 \\ & +7U^3V-7UV^3-36U^2V^2), 30000\left(U^2+V^2\right)^2 \\ & +4800\left(6U^4+6V^4+7U^3V-7UV^3-36U^2V^2\right) \\ & 30000\left(U^2+V^2\right)^2, 30000\left(U^2+V^2\right)^2 \\ & -4800\left(6U^4+6V^4+7U^3V-7UV^3-36U^2V^2\right), \\ & 30000\left(U^2+V^2\right)^2 - 9600(6U^4+6V^4 \\ & +7U^3V-7UV^3-36U^2V^2)\Big\}\end{aligned}$$

2.2 Presumption is verified for definite values of U and V in the table given below

Table 2.

U	V	(p, q, r, s, t)	p+q+r	q+r+s	r+s+t
0	1	(87600,58800, 30000,1200, -27600)	420^2	300^2	60^2
1	2	(-56400,346800, 750000,1153200, 1556400)	1020^2	1500^2	1860^2
1	1	(-110400,4800,120000, 235200,350400)	120^2	600^2	840^2

Procedure 3:

Commencement of the fresh renovation $\lambda = 2\,\mathrm{Ain}(9)$ declare the same equation as

$$\begin{aligned}2A^2 &= \mu^2+\omega^2 \\ &\Rightarrow A^2-\mu^2 = \omega^2-A^2 \\ &\Rightarrow (A+\mu)(A-\mu) = (\omega+A)(\omega-A) \\ &\Rightarrow \left(1+\frac{\mu}{A}\right)\left(1-\frac{\mu}{A}\right) = \left(\frac{\omega}{A}+1\right)\left(\frac{\omega}{A}-1\right)\end{aligned} \tag{2.10}$$

Again, make use of the transformations $\frac{\mu}{A} = \theta, \frac{\omega}{A} = \rho$ in (2.10) produces the proportion as

$$\frac{(1+\theta)}{(1+\rho)} = \frac{(\rho-1)}{(1-\theta)} = \frac{m}{n}, n \neq 0 \tag{2.11}$$

Hereafter, calculate the values of θ and ρ from (2.11) by the process of cross multiplication and then substituting these values in the ultimate transformation, it is determined by

$$\begin{aligned}A &= m^2+n^2 \Rightarrow \lambda = 2\left(m^2+n^2\right) \\ \mu &= m^2+2mn-n^2 \\ \omega &= n^2+2mn-m^2\end{aligned} \tag{2.12}$$

Interpretation (2.2) and (2.7) offers the relevant values of a and d as presented in the equations scripted below.

$$\begin{aligned}a &= 12\left(m^2+n^2\right)^2 \\ d &= 48mn\left(n^2-m^2\right)\end{aligned}$$

Hence, the essential quintuple in which the elements in Arithmetic progression is rendered by

$$\begin{aligned}&(p,q,r,s,t) \\ &= \Big\{12\left(m^2+n^2\right)^2 - 96mn\left(n^2-m^2\right), 12\left(m^2+n^2\right)^2 \\ &-48mn\left(n^2-m^2\right), 12\left(m^2+n^2\right)^2, 12\left(m^2+n^2\right)^2 \\ &+48mn\left(n^2-m^2\right), 12\left(m^2+n^2\right)^2 + 96mn\left(n^2-m^2\right)\Big\}\end{aligned}$$

2.3 Supposition is authenticated for specific values of U and V in the following table

Table 3.

m	n	(p, q, r, s, t)	p+q+r	q+r+s	r+s+t
2	1	(-276,12,300, 588,876)	6^2	30^2	42^2
5	7	(-14928,25392, 65712,106032, 146352)	276^2	444^2	564^2
1	3	(-1104,48,1200, 2352,3504)	12^2	60^2	84^2

The emerging C software shows verification of the numerical samples:

```
#include <stdio.h>
#include <conio.h>
#include <math.h>
void main()
{
char ch;
clrscr();
do {
long long int x,u,v,m,n;
long long int U,V,M,N,a,d,p,q,r,s,t,A,B,C,E,F,G;
printf("\n Enter the case 1 or 2 or 3\n");
scanf("%lld",&x);
switch(x)
{
case 1:
printf("\n Enter integer values for u and v \n");
scanf("%lld%lld",&u,&v);
U=u*u;
V=v*v;
a=48*(U+V)*(U+V);
d=192*u*v*(V-U);
p=a-2*d;
q=a-d;
r=a;
s=a+d;
t=a+2*d;
break;
case 2:
printf("\n Enter integer values for u and v \n");
scanf("%lld%lld",&u,&v);
U=u*u;
V=v*v;
```

```
a=30000*(U+V)*(U+V);
d=-4800*(6*U*U+6*V*V+7*U*u*v-7*u*v*V-36*U*V);
p=a-2*d;
q=a-d;
r=a;
s=a+d;
t=a+2*d;
break;
case 3:
printf("\n interger values for m and n \n");
scanf("%lld%lld",&m,&n);
M=m*m;
N=n*n;
a=12*(M+N)*(M+N);
d=48*m*n*(N-M);
p=a-2*d;
q=a-d;
r=a;
s=a+d;
t=a+2*d;
break;
}
A=p+q+r;
B=q+r+s;
C=r+s+t;
E=sqrt(A);
F=sqrt(B);
G=sqrt(C);
printf("\n p+q+r=%lld=%lld^2 \n q+r+s=%lld=%lld^2\nr+s+t
=%lld =%lld^2",A,E,B,F,C,G);
printf("\n Do you want to continue for different cases (y/n)?");
ch=getche();
}
while (ch=='y'——ch=='Y');
getch();
}
```

3. Conclusion

In this paper, an elegant integer quintuple (p,q,r,s,t) where the components make ensure in arithmetic progression with the conjecture that the sum of any three consecutive elements designates a perfect square is recognized. In this manner, one can search an integer quintuple (p,q,r,s,t) with elements in Geometric progression or Harmonic progression satisfying some other condition.

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A STATE OF THE-ART OF SUMS, CONGRUENCE RELATIONS AND DIVISIBILITY PROPERTIES OF PELL AND PELL-LUCAS NUMBERS

P. SANDHYA AND V. PANDICHELVI

ABSTRACT
In this document, several new-fangled identities regarding Pell and Pell-Lucas numbers enable to provide certain congruence relations for those numbers are deliberated. Also, divisibility properties of Pell and Pell-Lucas numbers are revealed by means of these derived congruence relations.

Keywords: *Pell numbers, Pell-Lucas numbers, congruence relations*

I. INTRODUCTION

$P_0 = 0, P_1 = 1$,and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ establish the Pell sequence $\{P_n\}$. P_n is referenced to the n^{th}Pell number. The Pell-Lucas sequence Q_n is defined as $Q_n = P_{n-1} + P_{n+1}$. For each $n \in Z$, $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$and $Q_{n-1} + Q_{n+1} = 8P_n$. For more information on the Pell and Pell-Lucas sequences, see [1]. Numerous well-known relationships exist among the Pell and Pell-Lucas numbers. Typically, these relations are achieved using Binet's formula, which is signified by $P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$and $Q_n = \alpha^n + \beta^n$, for any $n \in Z$, where $\alpha = 1 + \sqrt{2}$and $\beta = 1 - \sqrt{2}$. Additionally, the most well-known formulas for Pell numbers are $\alpha^n = \alpha P_n + P_{n-1}$and $\beta^n = \beta P_n + P_{n-1}$, for $n \in Z$.

Numerous sums incorporating Pell and Pell-Lucas numbers are provided in this study. Following that, certain congruences relating Pell and Pell-Lucas numbers are elaborated. These congruences enable one to establish a number of previously known characteristics. Additionally, with the use these congruences, many additional theorems are acclaimed.

II. SUMS AND CONGRUENCESOF PELL AND PELL-LUCAS NUMBERS

Theorem:2.1
If X is a square matrix with $X^2 = 2X + I$, then $X^n = P_n X + P_{n-1} I$for every integer n.

Proof:
Let$Z[\alpha] = \{A\alpha + B; A, B \in Z\}$ and $Z[X] = \{AX + BI; A, B \in Z\}$

Define a function $f: Z[\alpha] \to Z[X]$by $f(A\alpha + B) = AX + BI$.

Then $f$is a ring isomorphism. Moreover, it is clear that $f(\alpha) = X$ and $f(Q_m) = Q_m I$.

Therefore, $X^n = \big(f(\alpha)\big)^n = f(\alpha^n) = f(P_n\alpha + P_{n-1}) = P_n X + P_{n-1} I$

Corollary: 1.1

If$M = \begin{bmatrix} 1 & 4 \\ 1/2 & 1 \end{bmatrix}$, then $M^n = \begin{bmatrix} \frac{Q_n}{2} & 4P_n \\ \frac{P_n}{2} & \frac{Q_n}{2} \end{bmatrix}$.

Proof:
Since, $M^2 = 2M + I$, it follows from theorem (1) that

$$M^n = P_n M + P_{n-1} I$$

$$M^n = \begin{bmatrix} P_n + P_{n-1} & 4Pn \\ \frac{P_n}{2} & P_n + P_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{Q_n}{2} & 4P_n \\ \frac{P_n}{2} & \frac{Q_n}{2} \end{bmatrix}.$$

Remark:
From the fact that $f: Z[\alpha] \to Z[M]$, defined by $f(A\alpha + B) = AM + BI$ is a ring isomorphism, it is observed that

$$\alpha^{2m} - Q_m \alpha^m + (-1)^m = 0 \quad (1)$$

and $$\alpha^{2m} - 2\sqrt{2} P_m \alpha^m - (-1)^m = 0 \quad (2)$$

Applying the function f on each side of (1) and (2), the relations discovered are pointed out by:

$M^{2m} - Q_m M^m + (-1)^m I = 0$ (3)

and $M^{2m} - KP_m M^m - (-1)^m I = 0$ (4)

where $K = f(2\sqrt{2}) = f(2\alpha - 2) = 2M - 2I = \begin{bmatrix} 0 & 8 \\ 1 & 0 \end{bmatrix}$.

Theorem: 2.2
If $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$, then

$$Q_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i Q_{mi+k}$$

and $P_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i P_{mi+k}$

Proof:
From (3), it is noted that

$M^{2m} = Q_m M^m - (-1)^m I$ (5)

Raising n^{th} power on both sides of (5).

Then, $M^{2mn} = (Q_m M^m - (-1)^m I)^n = (Q_m M^m + (-1)^{m+1} I)^n$

$$= \sum_{i=o}^{n} \binom{n}{i} ((-1)^{m+1} I)^{n-i} (Q_m M^m)^i$$

$$= (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i M^{mi}$$

Therefore, $M^{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i M^{mi+k}$

It comprehends from corollary 1.1 that

$$Q_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i Q_{mi+k}$$

and $P_{2mn+k} = (-1)^{(m+1)n} \sum_{i=o}^{n} \binom{n}{i} (-1)^{(m+1)i} {Q_m}^i P_{mi+k}$

Corollary 2.2.1:
$Q_{2mn+k} \equiv (-1)^{(m+1)n} Q_k (mod\ Q_m)$ (6)

and $P_{2mn+k} \equiv (-1)^{(m+1)n} P_k (mod\ Q_m)$ (7)

for every $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$.

Remark:

i. Since $K = 2M - 2I = M + M^{-1}, \therefore M^m K = K M^m, \forall m \in Z$

ii. $K^2 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = 8I$ and $\begin{bmatrix} 0 & 8 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 8c & 8d \\ a & b \end{bmatrix}$.

Theorem2.3:
If$n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$, then

$$Q_{2mn+k} = (-1)^{mn} \left\{ \sum_{i=o}^{\lfloor n/2 \rfloor} \binom{n}{2i} 8^i {P_m}^{2i} Q_{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} (-1)^{mi} 8^i {P_m}^{2i+1} P_{2mi+m+k} \right\}$$

and

$$P_{2mn+k} = (-1)^{mn}\left\{\sum_{i=o}^{\lfloor n/2\rfloor}\binom{n}{2i}8^i\,{P_m}^{2i}P_{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2\rfloor}\binom{n}{2i+1}(-1)^{mi}8^i\,{P_m}^{2i+1}Q_{2mi+m+k}\right\}$$

Proof:
From (4), it follows that

$$M^{2m} = KP_mM^m+(-1)^mI$$

Therefore, $M^{2mn+k} = (KP_mM^m+(-1)^mI)^nM^k$

$$= \left[\sum_{i=o}^{n}\binom{n}{i}((-1)^mI)^{n-i}\,(KP_mM^m)^i\right]M^k$$

$$= (-1)^{mn}\sum_{i=o}^{n}\binom{n}{i}(-1)^{mi}K^i{P_m}^i\,M^{mi+k}$$

$$= (-1)^{mn}\left\{\sum_{i=o}^{\lfloor n/2\rfloor}\binom{n}{2i}K^{2i}\,{P_m}^{2i}M^{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2\rfloor}\binom{n}{2i+1}(-1)^{mi}K^{2i+1}\,{P_m}^{2i+1}M^{2mi+m+k}\right\}$$

$$= (-1)^{mn}\left\{\sum_{i=o}^{\lfloor n/2\rfloor}\binom{n}{2i}8^i\,{P_m}^{2i}M^{2mi+k} + \sum_{i=o}^{\lfloor (n-1)/2\rfloor}\binom{n}{2i+1}(-1)^{mi}8^i\,K{P_m}^{2i+1}M^{2mi+m+k}\right\}$$

We will get the results by trading the matrices K and M on both sides and equating the same entries.

Corollary 2.3.1:
If $n \in \mathcal{N}$ and $m, k \in \mathcal{Z}$, then

$$Q_{2mn+k} \equiv (-1)^{mn}Q_k(mod\ P_m) \quad (8)$$

and $$P_{2mn+k} \equiv (-1)^{mn}P_k(mod\ P_m) \quad (9)$$

III. DIVISIBILITY PROPERTIES OF PELL AND PELL-LUCAS NUMBERS

To begin, it is established two well-known theorems in a novel manner by exploiting the congruences postulated in Corollaries 2.2.1 and 2.3.1. Regarding the divisibility of Pell and Pell-Lucas numbers, readers will investigate the formulae and learn how to use them efficiently to resolve problems. Thus, this article explains the fundamental divisibility for Pell and Pell-Lucas numbers.

Theorem 3.1:
The necessary and sufficient conditions for $Q_m|Q_n$ are

i. $m|n$ and

ii. $\frac{n}{m}$ is an odd integer

for all $m, n \in N$ and $m \geq 2$.

Proof:
Presume that $Q_m|Q_n$

Suppose $m \nmid n$, then by fundamental property of divisibility, n can be expressed as$n = mq + r, 0 \leq r < m$.

If q is an even integer, then$q = 2s$ for some $s \in Z$.

From (6), it follows that

$$Q_n = Q_{2ms+r} \equiv (-1)^{(m+1)s}Q_r(mod\ Q_m)$$

Since $Q_m|Q_n$, $Q_m|Q_r$. This is a contradiction since $Q_r < Q_m$as $r < m$. Hence, q is an odd integer. Sustain$q = 2s + 1$ for some $s \in Z$. So,

$$Q_n = Q_{2ms+m+r} \equiv (-1)^{(m+1)s} Q_{m+r} (mod\ Q_m)$$

Also, since $Q_m | Q_n, Q_m | Q_{m+r}$.

To prove: $r = 0$
Suppose $r > 0$. By the identity $Q_{m+r} = Q_m P_{r-1} + P_r Q_{m+1}$, the above implies that $Q_m | P_r Q_{m+1}$.

Since $(Q_m, Q_{m+1}) = 1$, it follows that $Q_m | P_r$. This is a contradiction to the fact that if $r < m$, then $P_r \leq P_m < Q_m$. As a result, it is resolved that $r = 0$.

Thus, that $n = mq$, with q being an odd integer.

Conversely, suppose that $m | n$ and $\frac{n}{m}$ is an odd integer,

That is, $n = m(2s + 1)$, for some integer s. Then it isprocured that,

$$Q_n = Q_{2ms+m} \equiv (-1)^{(m+1)s} Q_m (mod\ Q_m)$$

$\Rightarrow Q_m | Q_n$.

Hence, the result.

Theorem 3.2:
Let $m, n \in N$ and $m \geq 2$. Then $Q_m | P_n$ if and only if $m | n$ and $\frac{n}{m}$ is an even integer.

Proof:
Suppose that $Q_m | P_n$ and $m \nmid n$.This assumption means that $n = mq + r, 0 \leq r < m$ where $m \geq 2$.

If q is an odd integer, it may phrase $q = 2s + 1$ for some integer s.

From (7), it is pointed out that

$$P_n = P_{2ms+m+r} \equiv (-1)^{(m+1)s} P_{m+r} (mod\ Q_m)$$

Then, $Q_m | P_{m+r}$ and hence $Q_m | 8P_{m+r}$. It is well-known that $8P_{m+r} = Q_m Q_{r-1} + Q_r Q_{m+1}$, then $Q_m | Q_r Q_{m+1}$. Since Q_m and Q_{m+1} are relatively prime, the only possibility is $Q_m | Q_r$. But $r < m$ delivers $Q_r < Q_m$. So, $Q_m \nmid Q_r$. This conflict befalls as a result of our erroneous assumption about q being an odd number. Therefore, q is an even integer. Thus, it may have $q = 2s$ for some integer s. Hence,

Form (7), $P_n = P_{2ms+r} \equiv (-1)^{(m+1)s} P_r (mod\ Q_m)$

Since $Q_m | P_n$, $Q_m | P_r$. However, this cannot be true since $r < m$ and hence $P_r \leq P_m < Q_m$. This contributes that $r = 0$. So, it can be concluded that $n = mq$, q is an even integer.

Conversely, suppose that $m | n$ and $n = 2ms$ for some $s \in Z$. Then,it isacquired from (7) that

$$P_n = P_{2ms} \equiv (-1)^{(m+1)s} P_0 (mod\ Q_m)$$

It follows that $Q_m | P_n$.

Theorem3.3:
For all $m, n \in N$ and $m \geq 3$, $P_m | P_n$ if and only if $m | n$.

Proof:
Initially consider that $P_m | P_n$ but $m \nmid n$. Then $n = mq + r$ with $0 \leq r < m$. Now, suppose that q is an even integer, then this may be taken as $q = 2s$ for any integer s.

Hence, (9) provides the succeeding identity

$$P_n = P_{2ms+r} \equiv (-1)^{ms} P_r (mod\ P_m)$$

Since $P_m | P_n$, by applying the above identity, $P_m | P_r$. Since, if $0 \leq r < m$ and $m \geq 3$, it leads to

$P_r < P_m$. Hence, q must be an odd integer. Then $q = 2s + 1$, for some $s \in Z$. Thus, (9) becomes

$$P_n = P_{2ms+m+r} \equiv (-1)^{ms} P_{m+r} (mod\ P_m)$$

Since $P_m|P_n$, it follows that $P_m|P_{m+r}$. By the identity, $P_{m+r} = P_{m+1}P_r + P_mP_{r-1}$, it is noted that $P_m|P_{m+1}P_r$. Due to the fact that$(P_m, P_{m+1}) = 1$, it isreceived that$P_m|P_r$, which is a contradiction. This emerges as a consequence of$P_r < P_m$ as $r < m$ and $m \geq 3$. As a result,$r = 0$ and subsequently$n = mq$, resulting in$m|n$.

Conversely, pretend that $m|n$. Then, the conclusion is$n = mq$ for some natural number q. As an outcome,

$$P_n = P_{mq} = \sum_{i=0}^{q} \binom{q}{i} {P_m}^i {P_{m-1}}^{q-i} P_i$$

Hence, it is realized that $P_m|P_n$.

IV. MAIN THEOREMS

From the identity $2(-1)^n = Q_nP_{n-1} - Q_{n-1}P_n$, it can be seen that $\gcd(Q_n, P_n) = 1$or $\gcd(Q_n, P_n) = 2$. Furthermore,

$${Q_n}^2 - 8{P_n}^2 = 4(-1)^n \qquad (10)$$

From equation (8), it is seen that $Q_{8q+r} \equiv Q_r(mod 12)$ and therefore $12 \nmid Q_n$, for every natural number n.

Now, we'll go over some Pell-Lucas numbers identities that will be necessary in the sequel:

$$Q_{2n} = {Q_n}^2 - 2(-1)^n \qquad (11)$$

$$Q_{3n} = Q_n\left({Q_n}^2 - 3(-1)^n\right) \qquad (12)$$

Theorem4.1:
Let $r \geq 1$, be an odd number and $m > 1$. Then, there is no Pell-Lucas number $Q_n$such that $Q_n = Q_{2r}Q_mx^2$

Proof:
Assume that $Q_n = Q_{2r}Q_mx^2$ and r is an odd number. Then $Q_{2r}|Q_n$ and $Q_m|Q_n$. Then,$n = 2rt$ and $n = mk$ for some odd natural number t & k by theorem 2.4,$\Rightarrow 2|n \Rightarrow n = mk \Rightarrow 2|m$. It is thus obvious that$m = 2v$, for some odd $v \in \mathcal{N}$. Since $2|n$ and $\frac{n}{2}$ is an odd natural number, it can be written as$n = 8q + s$ with $s = 2,6$ and $q \geq 0$. Hence,

$$Q_n = Q_{8q+s} \equiv Q_s(mod\ 12)$$
$$\Rightarrow Q_n \equiv Q_2, Q_6\ (mod 12)$$
$$\Rightarrow Q_n \equiv 6\ (mod\ 12)$$

Similarly, it can be seen that $Q_m \equiv 6\ (mod\ 12)$.

Since, r is an odd natural number, it is obtained that $Q_{2r} \equiv 6(mod\ 12)$. Then it follows that

$$Q_n = Q_{2r}Q_mx^2 \equiv 6Q_mx^2(mod\ 12)$$

Moreover, $6x^2 \equiv 0,6\ (mod\ 12)$and $Q_m \equiv 6\ (mod\ 12)$,

$Q_n \equiv 0(mod\ 12)$ which contradicts the fact that $Q_n \equiv 6\ (mod\ 12)$. This concludes the proof.

Theorem4.2:
$Q_{2^kt} \equiv 2,10\ (mod\ 12)$ for every $k \geq 2$ and for every odd natural number t.

Proof:
Assume that t is an odd natural number, then $t \equiv \pm1, \pm3, \pm5, \pm7\ (mod\ 8)$. Moreover, it can be proved by induction that $2^k \equiv 0, \pm4\ (mod\ 8)$ for $k \geq 2$. $2^kt \equiv 0, \pm4\ (mod\ 8)$.

Therefore, $\Rightarrow 2^kt = 8q$ or $2^kt = 8q \pm 4$ for $q \geq 0$. Then it seeks that

$$Q_{2^kt} = Q_{8q} \equiv Q_0\ (mod\ P_4)$$

Or

$$Q_{2^kt} = Q_{8q\pm4} \equiv Q_{\pm4}\ (mod\ P_4)$$

Thus, $Q_{2^kt} \equiv 2,10\ (mod\ 12), k \geq 2$.

Now, it is possible to generalize theorem as follows:

Theorem4.3:
Let $m > 1, k \geq 2$ and t be an odd natural number. Then there is no Pell-Lucas number Q_n such that $Q_n = Q_{2^k t} Q_m x^2$.

Proof:
Assume that $Q_n = Q_{2^k t} Q_m x^2$ and t is an odd natural number. Since $Q_{2^k t} | Q_n$ and $Q_m | Q_n$, there exist two odd natural numbers u and v such that $n = 2^k tu$ and $n = mv$ by theorem 3.1. Thus, we have $m = 2^k r$, for some $r \in N$, because$n = 2^k tu = mv$ and t, u, v are odd natural numbers. Then, we have 4 divides both m & n, by the fact that $k \geq 2$. Hence, $n = 8q + s$ with $s = 0,4,8,12$. Thus,

$$Q_n = Q_{8q+s} \equiv Q_s \ (mod\ 12)$$

Since $s \in \{0,4,8,12\}$, it follows that

$$Q_n \equiv 2,10 \ (mod\ 12)$$

It may be observed in a similar manner that

$$Q_m \equiv 2,10 \ (mod\ 12)$$

On the other hand, $Q_{2^k t} \equiv 2,10 \ (mod\ 12)$ by theorem (8).

If$Q_{2^k t} \equiv 2 \ (mod\ 12)$, then $Q_n = Q_{2^k t} Q_m x^2 \equiv 2 Q_m x^2 (mod\ 12)$.

Since $2x^2 \equiv 0,2,6,8 \ mod(12)$ and $Q_m \equiv 2,10 \ (mod\ 12)$, $Q_n \equiv 0,4,8 \ (mod\ 12)$, which is a contradiction to the fact that $Q_n \equiv 2,10 \ (mod\ 12)$.

Therefore,$Q_{2^k t} \equiv 10 \ (mod\ 12)$. Then

$Q_n = Q_{2^k t} Q_m x^2 \equiv 10 Q_m x^2 (mod\ 12)$.

Since $10x^2 \equiv 0,4,6,10 \ mod(12)$ and $Q_m \equiv 2,10 \ (mod\ 12)$, $Q_n \equiv 0,4,8 \ (mod\ 12)$, which denies the fact that $Q_n \equiv 2,10 \ (mod\ 12)$. Hence the proof.

Theorem4.4:
If m and r are odd natural numbers, then there is no Pell-Lucas number Q_n such that $Q_n = Q_m Q_r$.

Proof:
Assume that $Q_n = Q_m Q_r$, for $m > 1$ and $r > 1$ and are odd numbers. Since $Q_m | Q_n$ and $Q_r | Q_n$, there exist two odd natural numbers u and v such that $n = mu$ and $n = rv$.

Hence, we have $u = 4k \pm 1$ for some $k \geq 1$. Therefore, we get $n = mu = m(4k \pm 1) = 4km \pm m$.

$$Q_n = Q_{4km \pm m} \equiv (-1)^k Q_{\pm m} \ (mod\ Q_{2m})$$

i.e., $Q_r Q_m \equiv \pm Q_m \ (mod\ Q_{2m})$ (13)

Similarly, it can be obtained that

$Q_m Q_r \equiv \pm Q_r \ (mod\ Q_{2r})$ (14)

Suppose that $Q_m | Q_{2r}$ then $\frac{2r}{m}$ =an odd integer which is not possible. Hence $Q_m \nmid Q_{2r}$ which implies that $gcd(Q_m, Q_{2r}) = 2$. Then by equations (13) and (14),

$Q_r \equiv \pm 1 \ (mod\ \frac{Q_{2m}}{2})$ and $Q_m \equiv \pm 1 \ (mod\ \frac{Q_{2r}}{2})$

$\Rightarrow Q_{2m} \leq 2Q_r \pm 2$ and $Q_{2r} \leq 2Q_m \pm 2$

$$\Rightarrow Q_{2m} + Q_{2r} \leq 2Q_r + 2Q_m \pm 4$$

By equation (11),

$$Q_m{}^2 \pm 2 + Q_r{}^2 \pm 2 \leq 2Q_r + 2Q_m \pm 4$$

$Q_m(Q_m - 2) + Q_r(Q_r - 2) \leq 0$,

which is a contradiction. This completes the proof of the theorem.

Corollary 4.4.1:

There is no Pell-Lucas number Q_n such that $Q_n = Q_m Q_r$, for any $m > 1$ and $r > 1$.

Proof:
If $r > 1$ and even, then ther proof follows from theorems (7) and (9).

If m and r are odd natural numbers, then it is proved in theorem (10).

V. CONCLUSION

In this research, various quantities by means of the Pell and Pell-Lucas numbers are presented. Then, some specific congruences concerning the Pell and Pell-Lucas numbers have been provided. These analogues allowed to govern a number of previously known features. These congruences have also been utilized to prove many other theorems.

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ASSESSMENT OF SOLUTIONS IN PELL AND PELL – LUCAS NUMBERS TO DISPARATE POLYNOMIAL EQUATIONS OF DEGREE TWO

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ABSTRACT:
In this paper, the widespread solutions in rapports with Pell and Pell-Lucas numbers for a restricted number of unambiguous polynomial equations of degree two in two variables are exposed.
KEYWORDS: Binary quadratic Diophantine equations, Pell numbers, Pell-Lucas numbers.

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INTRODUCTION:

In [3], Keskin, Refik, and Bahar Demirtürk discovered the solutions of some Diophantine equations using generalized Fibonacci and Lucas sequences. In [8], A. Marlewski, P. Zarzycki studied the particular Diophantine equation $x^2 - kxy + y^2 + x = 0$. In [11], Pingzhi Yuan, Yongzhong discussed the Diophantine equation $x^2 - kxy + y^2 + lx = 0, l \in \{1, 2, 4\}$. For an all-embracing review, one can refer [1-2,4-7,9-10,12].

In this paper, the solutions in Pell and Pell-Lucas numbers for some explicit polynomial equations of degree two in two variables $x^2 - 2xy - y^2 = \pm k$ when $k = 1, 8$, $x^2 - 6xy + y^2 = \pm l$ when $l = 4, 32$, $x^2 - 2xy - y^2 \pm x = 0$, $x^2 - 2xy - y^2 \pm y = 0$, $x^2 - 2xy - y^2 \pm 8x = 0$, $x^2 - 6xy + y^2 \pm 4x = 0$ and $x^2 - 6xy + y^2 \pm 32x = 0$ are investigated.

Needed Theorems:

Theorem: [I]

The numbers $\pm\omega^n, \pm\omega^{-n}$ where $\omega = 1 + \sqrt{2}$ are the only unities of $k(\sqrt{2})$, where $k(\sqrt{2})$ is a quadratic field. See [1]

Theorem: [II]

If positive integers x, y, k and the integer m with $\gcd(x, m) = 1$ satisfy the equations $x^2 - kxy + y^2 \mp mx = 0$ then $x = u^2$ and $y = uv$ for some positive integers u and v. If positive integers x, y, k and the integer m with $\gcd(y, m) = 1$ satisfy the equations $x^2 - kxy - y^2 \mp my = 0$ then $y = u^2$ and $x = uv$ for some positive integers u and v. See [4]

Theorem: [III] Fundamental theorem of arithmetic

For each integer $n > 1$, there exists primes $p_1 \leq p_2 \leq \cdots \leq p_r$ such that $n = p_1 p_2 \ldots p_r$,this factorization is unique.

MAIN RESULTS:

The n^{th} Pell number labelled by P_n is demarcated by $P_0 = 0, P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$. If α, β be the roots of the equation $x^2 - 2x - 1 = 0$, then $\alpha = 1 + \sqrt{2}, \beta = 1 - \sqrt{2}$ where $\alpha\beta = -1$ and $\alpha + \beta = 2$. In addition, it is well- known and simple to demonstrate the identities that $\alpha^n = \alpha P_n + P_{n-1}$ and $\beta^n = \beta P_n + P_{n-1}$ for every $n \in \mathbb{Z}$, the set of all integers. In the other hand, it could be perceived by induction that ${P_n}^2 - 2P_n P_{n-1} + {P_{n-1}}^2 = (-1)^{n+1} \ \forall n \in \mathbb{Z}$.

(1)

The n^{th} Pell-Lucas number Q_n is characterized as $Q_0 = Q_1 = 2$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$ .The associations between Pell and Pell-Lucas numbers are agreed as follows

1. $Q_n = P_n + P_{n+1}$ for every $n \in \mathbb{Z}$.
2. ${Q_n}^2 - 2Q_n Q_{n-1} - {Q_{n-1}}^2 = 8(-1)^n$ for every $n \in \mathbb{Z}$

(2)

Theorem: 1

The necessary and sufficient condition for all non-negative integer solutions to the second-degree equation in two variables $X^2 - 2XY - Y^2 = (-1)^{n+1}$ is $(X, Y) = (P_n, P_{n-1})$ with $n \geq 1$.

Proof:

If $(X, Y) = (P_n, P_{n-1})$, then from identity (1), it seeks that $X^2 - 2XY - Y^2 = (-1)^{n+1}$.

Conversely suppose that $X^2 - 2XY - Y^2 = \mp 1$ for some positive integers X and Y.

Then by theorem (I), $(Y + \alpha X)(Y + \beta X) = \pm 1 \Rightarrow (Y + \alpha X) \in k(\sqrt{2})$.

Thus, $Y + \alpha X = \alpha^n = \alpha P_n + P_{n-1}$ and hence $(X, Y) = (P_n, P_{n-1}), n \geq 1$.

Corollary: 1.1

The feasible solutions of the quadratic polynomial equation $X^2 - 2XY - Y^2 = 1$ are specified by $(X, Y) = (P_{2m+1}, P_{2m})$ with $m \geq 0$.

Proof:

If n is odd such that $n = 2m + 1$, then the apt integer solutions to $X^2 - 2XY - Y^2 = 1$ is obtained as $(X, Y) = (P_{2m+1}, P_{2m}), m \geq 0$.

Corollary: 1.2

Every possible solution in Pell numbers of the quadratic equation $X^2 - 2XY - Y^2 = -1$ are stated by $(X, Y) = (P_{2m}, P_{2m-1})$ with $m \geq 1$.

Proof:

If n is even such that $n = 2m$, $m \geq 1$, then the appropriate solutions to $X^2 - 2XY - Y^2 = -1$ is $(X, Y) = (P_{2m}, P_{2m-1})$.

Theorem: 2

The probable integer solutions to the second-degree polynomial equation $X^2 - 6XY + Y^2 = 4$ are attained by $(X, Y) = (P_{2n+2}, P_{2n})$ with $n \geq 0$.

Proof:

Assume that $X^2 - 6XY + Y^2 = 4$ for some positive integers X and Y.

Without loss of generality, suppose that $X > Y$.

Then, $\left(\frac{X-Y}{2}\right)^2 - 2\left(\frac{X-Y}{2}\right)Y - Y^2 = 1 \Rightarrow X^2 - 6XY + Y^2 = 4$. By corollary 1.1, it should have $\frac{X-Y}{2} = P_{2n+1}$ and $Y = P_{2n}$ and therefore $(X - Y, Y) = (2P_{2n+1}, P_{2n})$

Consequently, $X = P_{2n+2}$ and $Y = P_{2n}$, $n \geq 0$.

Theorem: 3

The positive integer roots of the binary quadratic equation $X^2 - 6XY + Y^2 = -4$ are conquered by $(X, Y) = (P_{2n+1}, P_{2n-1})$ with $n \geq 1$.

Proof:

The proof is equivalent to Theorem 2.

Theorem: 4

Let $\mathbb{N}$ be the set of all-natural numbers and $X, Y \in \mathbb{N}$ sustaining the particular equation of the form $X^2 - 2XY - Y^2 \pm X = 0$, then $X = A^2$ and $Y = AB$ where , $B \in \mathbb{N}$.

Proof:

If $X, Y \in \mathbb{N}$ are satisfying the corresponding equation in the statement, then it follows that $X|Y^2$ and hence $Y^2 = XZ$ for some $Z \in \mathbb{N}$.

Suppose that $p|X$ and $p|Z$ for some prime number p.

Then, $p|Y$ which leads the implicit equation to $X - 2Y - Z \pm 1 = 0$

This ensure that $p|1$ which is absurd.

In what follows that $\gcd(X, Z) = 1$.

Then by fundamental theorem of arithmetic, $X = A^2$ and $Z = B^2$ for some positive integers A and B where $\gcd(A, B) = 1$.

Then $Y^2 = XZ = A^2B^2 \Rightarrow Y = AB$.

Theorem: 5

If two positive integers X, Y be such that $X^2 - 2XY - Y^2 \pm Y = 0$, then $X = AB$ and $Y = A^2$ for some positive integers A and B with $\gcd(A, B) = 1$.

Proof:

The proof is analogous to Theorem 4.

Corollary: 5.1

The conceivable Pell values of X, Y in the equation $X^2 - 2XY - Y^2 + Y = 0$ are achieved by $(X, Y) = \left(P_{2n}\, P_{2n-1}\,, {P_{2n-1}}^2\right)$, $n \geq 1$.

Corollary: 5.2

The plausible solutions in Pell numbers to the equation $X^2 - 2XY - Y^2 - Y = 0$ are specified by $(X, Y) = \left(P_{2n+1}\, P_{2n}\, , {P_{2n}}^2\right)$, $n \geq 0$.

Theorem: 6

Let $X, Y \in \mathbb{Z}^+$.Then $X^2 - 2XY - Y^2 + X = 0$ if and only if $(X, Y) = \left({P_{2n}}^2, P_{2n}\, P_{2n-1}\right)$ where $n \geq 1$.

Proof:

Assume that $X^2 - 2XY - Y^2 + X = 0$ where $X, Y \in \mathbb{Z}^+$

Then by theorem II, $X = A^2$ and $Y = AB$ for some $A, B \in \mathbb{Z}^+$

Subsequently $A^2 - 2AB - B^2 + 1 = 0$

Again, by corollary 1.2 it is noticed that $(A, B) = (P_{2n}, P_{2n-1}) \Rightarrow (X, Y) = \left({P_{2n}}^2, P_{2n}\, P_{2n-1}\right)$ where $n \geq 1$.

Conversely, if $(X, Y) = \left({P_{2n}}^2, P_{2n}\, P_{2n-1}\right)$ with $n \geq 1$, then

$$X^2 - 2XY - Y^2 + X = {P_{2n}}^4 - 2{P_{2n}}^3\, P_{2n-1} - {P_{2n}}^2\, {P_{2n-1}}^2 + {P_{2n}}^2$$
$$= {P_{2n}}^2 \left\{{P_{2n}}^2 - 2P_{2n}\, P_{2n-1} - {P_{2n-1}}^2 + 1\right\}$$

By corollary 1.2 it is pursued that

$X^2 - 2XY - Y^2 + X = 0$.

Theorem: 7

The sequence of several positive integer solutions for the equation $X^2 - 6XY + Y^2 + 4X = 0$ are epitomized by $(X, Y) = \left({P_{2n-1}}^2,\ P_{2n-1}P_{2n+1}\right)$ where $n \geq 1$.

Proof:

Let $X, Y \in \mathbb{Z}^+$ be such that $X^2 - 6XY + Y^2 + 4X = 0$.

Then $X | Y^2$ and hence $Y^2 = XZ$ for some $Z \in \mathbb{Z}^+$.

If $p | X$ and $p | Z$ for some prime number p, then $p | Y$ and also the relation $X - 6Y + Z + 4 = 0$ is true for all $X, Y \in \mathbb{Z}^+$.

Thus, p divides 4.

It is evident that, the possibility of such p is $= 2$.

This condition offers that $X = 2X_1, Y = 2Y_1$ for some $X_1, Y_1 \in \mathbb{Z}^+$ and obviously the projected quadratic equation becomes ${X_1}^2 - 6X_1Y_1 + {Y_1}^2 + 2X_1 = 0$. Then $X_1 | {Y_1}^2$ and hence ${Y_1}^2 = X_1Z_1$ for some $Z_1 \in \mathbb{Z}^+$.

Again, if $p | X_1$ and $p | Z_1$ for some prime number p, then $p | Y_1$ and the relation $X_1 - 6Y_1 + Z_1 + 2 = 0$ is true for all $X_1, Y_1 \in \mathbb{Z}^+$.

Clearly, the chance of such p is $p = 2$. This diagnosis provides that $X_1 = 2X_2, Y = 2Y_2$ implying that $X = 4X_1, Y = 4Y_1$. These contributions state that ${X_2}^2 - 6X_2Y_2 + {Y_2}^2 + X_2 = 0$, which has no positive integer solution [8]. Hence, our assumption that X and Z have common divisors is wrong. This proposes that $gcd(X, Z) = 1$. Thus, by the fact that the product two coprime numbers should be a perfect square if and only if each of them is a perfect square, $X = R^2$ and $Z = S^2$ for some positive integers R and S and $\gcd(R, S) = 1$. These choices of X and Z affords $Y = RS$ and

afterward the desired equation can be modified into $R^2 - 6RS + S^2 + 4 = 0$.

By theorem 3, $(R, S) = (P_{2n+1}, P_{2n-1})$ and hence $(X, Y) = \left({P_{2n+1}}^2, P_{2n+1}\, P_{2n-1}\right)$, $n \geq 1$.

Theorem: 8

Every solution in Pell-Lucas numbers for two dissimilar quadratic equations $X^2 - 2XY - Y^2 = 8$ and $X^2 - 2XY - Y^2 = -8$ are offered by $(X, Y) = (Q_{2n},\ Q_{2n-1})$, $n \geq 1$ and $(X, Y) = (Q_{2n+1},\ Q_{2n})$, $n \geq 0$ respectively.

Theorem: 9

Let $X, Y \in \mathbb{N}$, the set of natural numbers.

(i) If $X^2 - 6XY + Y^2 = 32$, then $(X, Y) = (Q_{2n+1},\ Q_{2n-1})$, $n \geq 1$.

(ii) If $X^2 - 6XY + Y^2 = -32$, then $(X, Y) = (Q_{2n+2},\ Q_{2n})$, $n \geq 0$.

Theorem: 10

If X, Y be any two positive integers such that $X^2 - 2XY - Y^2 + 8X = 0$, then either $(X, Y) = \left(8{P_{2n}}^2,\ 8P_{2n}P_{2n-1}\right)$, $n \geq 1$ or $(X, Y) = \left({Q_{2n+1}}^2,\ Q_{2n+1}Q_{2n}\right)$, $n \geq 0$.

Proof:

Consider that $X^2 - 2XY - Y^2 + 8X = 0$ for some positive integers X and Y.

If $8|X$, then $8|Y \Rightarrow X = 8A$ and $Y = 8B$ for some A and B belong to the set of all positive integers

Therefore, the original equation in two variables X and Y is converted into $A^2 - 2AB - B^2 + A = 0$.

By the theorem 5, $(A, B) = \left({P_{2n}}^2,\ P_{2n}P_{2n-1}\right) \Rightarrow (X, Y) = \left(8{P_{2n}}^2,\ 8P_{2n}P_{2n-1}\right)$.

If $8 \nmid X$, then by theorem [II], $X = A^2$ and $Y = AB$.

Then, $A^2 - 2AB - B^2 + 8 = 0$. By applying the theorem 8, it is concluded that

$(A, B) = (Q_{2n+1},\ Q_{2n}) \Longrightarrow (X, Y) = \left({Q_{2n+1}}^2,\ Q_{2n+1}Q_{2n}\right)$ with $n \geq 0$.

Conversely, if $(X, Y) = \left(8{P_{2n}}^2,\ 8P_{2n}P_{2n-1}\right)$, then

$$X^2 - 2XY - Y^2 + 8X = \left(8{P_{2n}}^2\right)^2 - 2\left(8{P_{2n}}^2\right)\left(8P_{2n}P_{2n-1}\right) - \left(8P_{2n}P_{2n-1}\right)^2 + 8\left(8{P_{2n}}^2\right)$$
$$= 64{P_{2n}}^2\left\{{P_{2n}}^2 - 2P_{2n}P_{2n-1} - {P_{2n-1}}^2 + 1\right\} = 0,$$

by the implementation of corollary 1.2.

Similarly, the same equation could be satisfied for $(X, Y) = \left({Q_{2n+1}}^2,\ Q_{2n+1}Q_{2n}\right)$.

Theorem: 11

Let X, Y be any two positive integers.

(i) If $X^2 - 2XY - Y^2 - 8X = 0$, then either $(X, Y) = \left(8{P_{2n+1}}^2,\ 8P_{2n+1}P_{2n}\right), n \geq 0$ or $(X, Y) = \left({Q_{2n}}^2,\ Q_{2n}Q_{2n-1}\right)$, $n \geq 1$.

(ii) If $X^2 - 2XY - Y^2 + 8Y = 0$, then either $(X, Y) = \left(8P_{2n}P_{2n-1}, 8{P_{2n-1}}^2\right)$, $n \geq 1$ or $(X, Y) = \left(Q_{2n}Q_{2n+1}, {Q_{2n+1}}^2\right)$, $n \geq 0$.

(iii) If $X^2 - 2XY - Y^2 - 8Y = 0$, then either $(X, Y) = \left(8P_{2n+1}P_{2n}, 8{P_{2n}}^2\right)$, $n \geq 0$ or $(X, Y) = \left(Q_{2n}Q_{2n-1}, {Q_{2n-1}}^2\right)$, $n \geq 1$.

Theorem: 12

Let $X, Y \in \mathbb{Z}^+$, the set of all positive integer. Then

(i) The positive integer solutions to the equation $X^2 - 6XY + Y^2 + 32X = 0$ are either $(X,Y) = \left(8{P_{2n+1}}^2,\ 8P_{2n+1}P_{2n-1}\right)$, $n \geq 1$ or $(X,Y) = \left({Q_{2n+2}}^2,\ Q_{2n+2}Q_{2n}\right)$, $n \geq 0$

(ii) The complete solutions in Pell numbers and Pell-Lucas numbers of the equation $X^2 - 6XY + Y^2 - 32X = 0$ are either $(X,Y) = \left(8{P_{2n}}^2,\ 8P_{2n}P_{2n+2}\right)$, $n \geq 0$ or $(X,Y) = \left({Q_{2n+1}}^2,\ Q_{2n+1}Q_{2n-1}\right)$, $n \geq 1$

CONCLUSION:

In this paper, the general solutions in terms of Pell and Pell-Lucas numbers for limited number of definite quadratic equations in two unknowns are discovered. In this way, one can gander for different quadratic or any higher degree Diophantine equations and pursuit solutions in any other renewed sequences of numbers.

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MORDELL'S EQUATION WHICH HAS NO SOLUTION FOR CERTAIN SELECTION OF K

P. SANDHYA AND V. PANDICHELVI

ABSTRACT
This article examines an incomparable Diophantine equation $B^2 = A^3 + C$, $C \in Z$, the set of all integers and demonstrates for which values of C, no solution in integer has been provided in the suggested equation

Keywords: *Diophantine equation, integer solutions, Mordell's equation, Bacht's equation.*

I. INTRODUCTION

The equation $y^2 = x^3 + k$, for $k \in Z$, is alluded to as Mordell's equation due to Mordell's deep passion in it. Mordell [4] established in 1920 that the equation $y^2 = x^3 + k$ has an infinite number of integral solutions for any $k \in \mathcal{Z}$. Michael A. Bennett and Amir Ghadermarzi [2] cast-off the traditional link between Mordell and cubic Thue equations to solve the Diophantine problem $y^2 = x^3 + k$ for all non-zero integers$k$ with $|k| \leq 10^7$.

In this artefact,an unrivalled Diophantine equation $B^2 = A^3 + C, C \in Z$, the set of all integers is studied and it is exposed that, for which values of C in theprofessed equation, no integer solution was supplied by using some classical congruence relations and Legendre symbols.

II. MAIN RESULTS

Congruence relations and Legendre symbolsare exploited to demonstrate that $B^2 = A^3 + C$does not have an integer solution for certain values of$C$.

Theorem: 2.1
Let U & V be integers such that $U \equiv 2 \pmod 4$, $V \equiv 3 \pmod 4$and Let$p \mid V$ and $p \equiv 1 \pmod 4$, where p is a prime number.Then the equation $B^2 = A^3 + C$, where $C = U^3 - V^2$ has no integer solution (A, B).

Proof:
Suppose that there exists a solution (A, B)in integers.

As$C = U^3 - V^2 \equiv -1 \pmod 4$& we have$B^2 \equiv A^3 - 1 \pmod 4$

Hence, $A \not\equiv 0 \pmod 2$and $A \not\equiv 3 \pmod 4$ and so $A \equiv 1 \pmod 4$.

Now,$B^2 + V^2 = A^3 + U^3 = (A + U)(A^2 - AU + U^2)$ (1)

As$A \equiv 1 \pmod 4$ and $U \equiv 2 \pmod 4$, it should be

$(A^2 - AU + U^2) \equiv 3 \pmod 4$.

Hence,$(X^2 - XU + U^2)$is odd and by (1) it has a prime factorp_1, $p_1 \equiv 3 \pmod 4$. Thus,$B^2 \equiv -V^2 \pmod{p_1}$.

By our assertion, $p_1 \nmid V$. Hence,

$\left(-\frac{1}{p_1}\right) = \left(\frac{-N^2}{p_1}\right) = \left(\frac{Y^2}{p_1}\right) = 1$,

denying to $p_1 \equiv 3 \pmod 4$.

This proves that the Diophantine equation $Y^2 = X^3 + K$ has no solution.

Theorem: 2.2
Let U and V be integers satisfying$U \equiv 3 \pmod 4$, $V \equiv 0, 2 \pmod 4$. If a prime number p divides $V/2$ and $p \equiv 1 \pmod 4$, then the Diophantine equation $B^2 = A^3 + C$ where $C = U^3 - V^2$ has no solution (A, B) in integers.

Proof:
Suppose that $(A, B)$is an integer solution of the equation $B^2 = A^3 + C, C = U^3 - V^2$

Since,$U^3 - V^2 \equiv 3 \pmod 4$, it is attained by$B^2 \equiv A^3 + 3 \pmod 4$.

Hence,$A \equiv 1 \pmod 4$.

Now, the original equation is converted for the prescribe value of C as

$B^2 + V^2 = A^3 + U^3 = (A + U)(A^2 - AU + U^2)$ (2)

Since $A \equiv 1\ (mod\ 4)$& $U \equiv 3\ (mod\ 4)$,

$$A^2 - AU + U^2 \equiv 3\ (mod\ 4)$$

Hence,$A^2 - AU + U^2$is odd and by (2) it has a prime factorp_1, $p_1 \equiv 3\ (mod\ 4)$. Thus,$B^2 \equiv -V^2\ (mod\ p_1)$.

By our assertion, $p_1 \nmid V/2$. Hence,$p_1 \nmid V$.

$\left(\frac{-1}{p_1}\right) = \left(\frac{-V^2}{p_1}\right) = \left(\frac{B^2}{p_1}\right) = 1$,

denying$p_1 \equiv 3\ (mod\ 4)$.

Hence, the Diophantine equation $B^2 = A^3 + C$ has no solution.

Theorem: 2.3
Let U and V be integers nourishing with the conditions$U \equiv 2\ (mod\ 8), V \equiv 1\ (mod\ 2)$. If$p$is a prime such that$p \equiv 1, 3\ (mod\ 8)$ and p divides V, then the equation $B^2 = A^3 + C$, where $C = U^3 - 2V^2$has no integral solution.

Proof:
Since,$C = U^3 - 2V^2 \equiv 2\ (mod\ 4)$, it must be

$$B^2 \equiv A^3 + 2\ (mod\ 4)$$

Therefore, $A \not\equiv 0\ (mod\ 2)$, $A \not\equiv 1\ (mod\ 4)$and consequently$A \equiv 3\ (mod\ 4)$

Hence,$A \equiv 3$ or $7\ (mod\ 8)$

Moreover,$C \equiv -2\ (mod\ 8)$.

So that $A \not\equiv 7\ (mod\ 8) \Rightarrow A \equiv 3\ (mod\ 8)$

Now,$B^2 + 2V^2 = A^3 + U^3 = (A + U)(A^2 - UA + U^2)$

As $A \equiv 3\ (mod\ 8)$ and $U \equiv 2\ (mod\ 8)$, it is seen that$A^2 - UA + U^2 \equiv 7\ (mod\ 8)$ and $A + U \equiv 5\ (mod\ 8)$

$\therefore B^2 + 2V^2$has a prime factorp_1 such that$p_1 \equiv 5\ or\ 7\ (mod\ 8)$

By our assumption,$p_1 \nmid V$ and $B^2 \equiv -2V^2\ (mod\ p_1)$

$$\left(-\frac{2}{p_1}\right) = \left(-\frac{2V^2}{p_1}\right) = \left(\frac{B^2}{p_1}\right) = 1$$

contrasting$p_1 \equiv 5\ or\ 7\ (mod\ p)$

Hence no solution for the Diophantine equation $B^2 = A^3 + C$, if $C = U^3 - 2V^2$.

Theorem: 2.4
Assume $U, V \in \mathcal{Z}$ and$U \equiv 6\ (mod\ 8)$, $V \equiv 1\ (mod 4)$. Let$p$ be a prime number such that $p \mid V$ and $p \equiv \pm 1\ (mod\ 8)$. Then the equation $B^2 = A^3 + C$, where $C = 2V^2 + U^3$ does not embrace any integer solution.

Proof:
Since, $C = 2V^2 + U^3 \equiv 2\ (\ mod\ 8)$, then $B^2 \equiv A^3 + 2\ (mod\ 8)$

$\Rightarrow A \not\equiv 0\ (mod\ 2)$, $A \not\equiv 1\ (mod\ 4)$and hence$A \equiv 3\ (mod\ 4)$

Hence $A \equiv 3\ or\ 7\ (mod\ 8)$

If $\equiv 3\ (mod\ 8)$, then $B^2 \equiv 5\ (mod\ 8)$which is not possible.

Thus,$A \equiv 7\ (mod\ 8)$.

Now, $B^2 - 2V^2 = (A + U)(A^2 - UA + U^2)$

$$\Rightarrow A^2 - UA + U^2 \equiv 3\ (mod\ 8.)$$

Therefore$A^2 - UA + U^2$ is odd and is divisible by an odd prime p_1 with $p_1 \equiv 3\ (mod\ 8)$

$\Rightarrow B^2 \equiv 2V^2 \ (mod\ p_1)$.

By our postulation, $p_1 \nmid V$

$$\therefore \left(\frac{2}{p_1}\right) = \left(\frac{2V^2}{p_1}\right) = \left(\frac{B^2}{p_1}\right) = 1$$

disagreeing$p_1 \equiv 3\ (mod\ 8)$

Hence the proof.

III. CONCLUSION

As a result, using certain traditional mathematical tools like as congruence and Legendre symbols, it is sensed that the hypothesized equation does not have a solution in integers for some specific integer values of C. Thus, one may ascertain the values of C for which the Mordell's equation has no integer solution by following the steps described above.

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Research Article

Exploration of Solutions for an Exponential Diophantine Equation $p^x + (p+1)^y = z^2$

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Abstract: In this text, the exclusive exponential Diophantine equation $p^x + (p+1)^y = z^2$such that the sum of integer powers x and y of two consecutive prime numbers engrosses a square is examined or estimating enormous integer solutions by exploiting the fundamental notion of Mathematics and the speculation of divisibility or all possibilities of x + y = 1, 2, 3, 4..

Keywords: exponential Diophantine equation; integer solutions

1. INTRODUCTION

The study of an exponential Diophantine equations has stimulated the curiosity of plentiful Mathematicians since ancient times as can be seen from [2-6, 9].BanyatSroysang [7] showed that $7^x + 8^y = z^2$ has a unique non-negative integer solution (x, y, z) as (0,1,3) in 2013 and he proposed an open problem where x, y and z are non-negative integers and p is a positive odd prime number. In 2014, Suvarnamani. A [8] proved that$p^x + (p+1)^y = z^2$has a unique solution (p, x, y, z) = (3, 1, 0, 2) and was disproved by Nechemia Burshtein [1] by few examples. In this text, the list of infinite numbers of integer solutions of the equation$p^x + (p+1)^y = z^2$ where p is a prime number by using the basic concept of Mathematics and the theory of divisibility.

2. APPROACHOF RECEIVINGINTEGERSOLUTIONS

The approach of search out an integer solution to the equation under contemplation is proved by the following theorem.

Theorem:

If p is any prime and x, y and z are integers persuading the condition that x + y = 1, 2, 3, 4,then all feasible integer solutions to the exponential Diophantine equation $p^x + (p+1)^y = z^2$ are given by (p, x, y, z) = {(2,0,1,2), (3,1,0,2), (3,2,2,5)}when p = 2, 3and (p, x, y, z) = $(4n^2 + 4n - 1, 0, 1, 2n + 1)$ where $n \in N$ for p > 3.

Proof:

The equation for performing solutions in integer is taken as

$$p^x + (p+1)^y = z^2 \quad (1)$$

All doable predilection ofthe supposition$x + y = 1,2,3,4$ arecarried out by eight cases for assessing solutions in integers.

Case 1: x = 0, y = 1

Equation (1)toexplore solutionsinintegertrimsdown by

$$p + 2 = z^2 \quad (2)$$

If p = 2, then z = 2. Hence, theoneandonly oneinteger solution iscommunicated as (p, x, y, z) = (2,0,1,2).
If p is an odd prime, then p + 2 is an odd number.
Thismeansthat $z^2$is anoddnumber and consequently z is also an odd number.
Ifz = 1, then p + 2 =1which isimpossible.
Asaresult, $z \geq 3$.

$$\text{Describe } z = 2n + 1,\ n \in N \quad (3)$$

The square of the selection of z in (3) can be characterized by $z^2 = 4n^2 + 4n + 1$, $n \in N$. In sight of (2), the promising value of anodd prime complied with the specified equation is distinguished by $p = 4n^2 + 4n - 1$, $n \in N$
Hence, the enormous solutions to (1) is (p, x, y, z) = $(4n^2 + 4n - 1, 0, 1, 2n + 1)$ where $n \in N$

Case 2: x = 1, y = 0

The inventive equation (1) is diminished as

$$p + 1 = z^2 \quad (4)$$

If p =2, then z^2= 3 which is not possible for integer value of z.
If p is an odd prime, then p + 1 is an even number which can be articulated by
$p + 1 = 2n, n \in N$
Match up the above equation with (4), 2n is a perfect square only if $n = m^2$ where $m \in N$.
Thus, $p = (2m)^2 - 1$.
If m = 1, then p = 3. Therefore, the solution belongs to the set Z of integers is (p, x, y, z) = (3,1,0,2).
If $m \neq 1$, then $p = (2m - 1)(2m + 1)$
If p divides $(2m - 1)$, then $2m - 1 = ap$ and as a consequence $2m + 1 = ap + 2$.
Thus, $p = ap(ap + 2)$ and leads to the ensuing equation $1 = a(a + 2)$.
But the above equation is not true for any integer value of a.
Ifp divides$(2m + 1)$, then $2m + 1 = bp$ and from now$2m - 1 = bp - 2$.
Therefore, $p = bp(bp - 2)$ andconsequently $1 = b(b + 2)$ which is not factual for any integer options for b.
Hence, in this case there exists a unique solution to (1) given by (p, x, y, z) = (3,1,0,2)

Case 3: x = 1, y = 1
The creative equation (1) is adjust by
$2p + 1 = z^2$
Since $z^2$is an odd number for all selections of*p*, it follows that
$z^2 \equiv 1 \pmod 4$
$\Rightarrow 2p + 1 \equiv 1 \pmod 4$
$\Rightarrow 2p \equiv 0 \pmod 4$
Capture that 2p = 4kwhich means that p = 2k for some positive integer k.
This declaration is possible only when k = 1.
Then p = 2, and 2p + 1 =5 which is not a perfect square of an integer.
Hence, in this case there is no integer solution to the presupposed equation.

Case 4 : x = 1, y = 2
The resourceful equation (1) is reconstructed as
$P^2 + 3p = z^2 - 1$
$\Rightarrow p(p + 3) = (z - 1)(z + 1)$ (5)
If $p \mid (z - 1)$, then $z - 1 = kp$, and $z + 1 = kp + 2$.Executions of these two equations in (5) go along with the subsequent quadratic equation in k
$pk^2 + 2k - (p + 3) = p$,
which consent the value of $k = (-1 \pm \sqrt{(p(p + 3) + 1)}) / p$. It is deeply monitored that no prime number p provides an integer value for k.
An alternative vision of $p|(z + 1)$ reveals that $z + 1 = lp$ and $z - 1 = lp - 2$
By make use of these two equations in (5) espouse the second degree equation in l as
$pl^2 - 2l - (p + 3) = 0$
which yields $l = (1 \pm \sqrt{(1 - p(p + 3) 1)}) / p$.
The above value ofl is a complex number or any prime p.
Hence, the ultimate result is no integer solutions to the most wanted equation (1).

Case 5 :x = 2, y = 1
The quick-witted equation (1) is restructured as$p^2 + p = z^2 - 1$
$\Rightarrow p(p + 1) = (z - 1)(z + 1)$ (6)
It follows from equation (6) that p must divide any one of the values $(z - 1)$ or $(z + 1)$
If $p|(z - 1)$, then $z - 1 = mp$ and $z + 1 = mp + 2$for some integer m.
Then, (6) make available with the value ofp as
$p = (1 - 2m) / (m^2 - 1)$ (7)

Accordingly, one can easily notice that the right-hand side of (7) can never be a prime number for $m \in Z$.

This circumstance offers that p does not divide $(z-1)$.
If $p|(z+1)$, then $z+1 = np$ and $z-1 = np-2$ for some integer n. Then, (6) endow with the value ofpas
$p = (1+2n)/(n^2-1)$ (8)
None of the value of $n \in Z$ in the right-hand side of (8) supplies the prime number establish that p does not divide$(z+1)$
Hence, this case does not grant an integer solution for (1).

Case 6 :x = 1, y = 3
For these choices ofx and y, the well-groomed equation (1) be converted into
$(p+1)^3 + p = z^2$ (9)
If p = 2, $z^2 = 29$ which make sure that z cannot be an integer.If p is any odd prime, then p takes any one of the forms 4N + 1 or 4 N + 3.
If p = 4N + 1 and the perception thatz^2must be odd reduces (9) to
$64N^3 + 96N^2 + 52N + 9 = (2T-1)^2$
$\Rightarrow 16N^3 + 24N^2 + 13N + 8 = T(T-1)$ (10)
It is perceived that none of the values ofN ensure that the left hand side of (10) as the product of two consecutive integers.
Similarly, the chance of p = 4N + 3, and the discernment thatz^2 is an odd integer reduces (9) to
$64N^3 + 192N^2 + 196N + 67 = (2T-1)^2$
$\Rightarrow 2(32N^3 + 96N^2 + 94N + 33) = 4T(T-1)$
The above equality does not hold since the left hand side is a twice an odd number and the right hand side is a multiple of 4.
Hence, in this case there does not exist an integer solution.

Case 7: x = 2, y = 2
These preferences of x and y altered the well-designed equation (1) into
$p^2 + (p+1)^2 = z^2$
$\Rightarrow 2p(p+1) + 1 = z^2$ (11)
Sincez^2 is an odd number, $z^2 \equiv 1 \pmod 4$
Then, $2p(p+1) \equiv 0 \pmod 4$
Hence, either p or p + 1 is a multiple of 2.
If p is a multiple of 2, then p must be 2.
Implementation of this value of p in (7) furnishes $z^2 = 13$ which does not enable as an integer for *z*.
If p + 1 is a multiple of 2, thenp + 1 = 2A, for some $A \in Z$.
The only odd prime satisfying all the above conditions is 3 and the corresponding value of z = 3
Consequently, the only integer solution to (1) is
(p, x, y, z) = (3, 2, 2, 5)

Case 8: x = 3, y = 1
The original equation (1) can be written as
$p^3 + p + 1 = z^2$
If p = 2, then $z^2 = 11$ which cannot acquiesce an integer for z.
Also, $z^2 \equiv 1 \pmod 4$ and $p(p^2+1) \equiv 0 \pmod 4$ which implies that either $4|p$ or $4|(p^2+1)$
For the reason that p is an odd prime, 4 does not divide pand so $p^2 + 1 = 4n$
This is not possible since p can take either of the form4N + 1, $N \geq 1$ or 4N + 3, $N \geq 0$.
Hence, the conclusion of this case is there cannot discover an integer solution to (1).

3. CONCLUSION

In this text, the special exponential Diophantine equation $p^x + (p + 1)^y = z^2$ where p is a prime number and x, y and z are integers is studied by developing the fundamental concept of Mathematics and the conjecture of divisibility for all possibilities of x + y = 1, 2, 3, 4. In this manner, one can find an integersolutions by using the property ofcongruence and other thoughts of Number theory.

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RESEARCH ARTICLE OPEN ACCESS

INVESTIGATION OF SOLUTIONS TO AN EXPONENTIAL DIOPHANTINE EQUATION $p_1^x + p_2^y + p_3^z = M^2$ FOR PRIME TRIPLETS (p_1, p_2, p_3)

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Abstract:

In this article, the solutions to the Diophantine equation $p_1^x + p_2^y + p_3^z = M^2$ where (p_1, p_2, p_3) is a prime triplet of the forms $(p, p+2, p+6)$ and $(p, p+4, p+6)$ for x, y, z are integers takes the values of1 or 2 is investigated by applying the basic concepts of Mathematics. Also, few choices of x, y, z are not possible solutions of the equation is confirmed by MATLAB Program.

Keywords **-- Diophantine equation, integer solutions**

I. INTRODUCTION

Primeval and wide-ranging, the discipline of Diophantine equations lacks a standardised approach for determining if an equation has any solutions or how many solutions. Numerous variations of the well-known general equation $p^x + q^y = z^2$emerge. It is possible to locate a substantial quantity of literature on non-linear equations using specific primes and powers of numerous kinds.With varied degrees of success, numerous writers [1,2,5,6,7] have revisited the above problem and tried many primes, such as the Mersenne prime, in an effort to solve it.In [3,4], Nechemia Burshteinprotracted the above equation as $p^x + (p+1)^y + (p+2)^z = M^2$ and $p^x + (p+1)^y + (p+2)^z = M^3$, for all primes p and for specific powers ofx, y, z and the findings were supported by simple mathematical tools

If there are three prime numbers in the set and the smallest and biggest vary by six places, then the set is called a prime triplet. To be precise, the sets must be $(p, p+2, p+6)$ or $(p, p+4, p+6)$ in form. This is the closest conceivable collection of three prime numbers, with the exception of (2, 3, 5) and (3, 5, 7).

In this article, the Diophantine equation ${p_1}^x + {p_2}^y + {p_3}^z = M^2$ is established and results are analyzed for the prime triplets where p is of the form $4n+1$or $4n+3$and the powers of primes are either 1 or 2.p

II. METHODOF EXTRACTING INTEGER SOLUTIONS

The approach of existence ofinteger solutions to an equation${p_1}^x + {p_2}^y + {p_3}^Z = M^2$where (p_1, p_2, p_3) is a prime 3-tuples is analyzed in the following theorems.

Theorem 2.1

If $x, y, z \in \{1,2\}$ and $(p, p+2, p+6)$ is a prime triplet of the form $(4n+1, 4n+3, 4n+7)$, $n \in \mathcal{N}$ then an equation $p^x + (p+2)^y + (p+6)^Z = M^2$ has no solution.

Proof:

The theorem is proved by considering the following eight cases.

Case 1: $x = 1, y = 1, z = 1$

Then,$p^x + (p+2)^y + (p+6)^Z = M^2$

$$\Rightarrow 4n + 1 + 4n + 3 + 4n + 7 = M^2$$

$$\Rightarrow 12n + 11 = M^2$$

It is scrutinized that the expression $12n+11$ is not a perfect square for any $n \in \mathcal{N}$.

The following MATLAB Programdemonstrates the statement given above.

```
clc; clear all;
n = input('Enter a natural number n');
for i = 1:n
    p1 = 4 * i + 1; p2 = 4 * i + 3; p3 = 4 * i + 7;
    if (isprime(p1) == 1 & isprime(p2) == 1 & isprime(p3) == 1)
        MS = 12 * n + 11;
        M = sqrt(MS);
        if (rem(M,1) == 0)
            fprintf('p1 = %d,p2 = %d,p3 = %d,M = %d',p1,p2,p3,M)
        end
```


end

end

Case2: $x = 2, y = 1, z = 1$

The equation to analyze solutions in integers can be written as

$$(4n+1)^2 + (4n+3) + (4n+7) = M^2$$
$$\Rightarrow 16n^2 + 16n + 11 = M^2$$
$$\Rightarrow (4n+2)^2 + 7 = M^2$$
$$\Rightarrow M^2 - (4n+2)^2 = 7$$

This is possible only when $M = 4$ and $4n + 2 = 3$. But no such $n \in \mathcal{N}$ satisfying the equation $4n + 2 = 3$.

Hence, there exists no integer solution.

Case3: $x = 1, y = 2, z = 1$

For these choices of x, y, z, the original equation is reduced into

$$(4n+1) + (4n+3)^2 + (4n+7) = M^2$$
$$\Rightarrow 16n^2 + 32n + 17 = M^2$$
$$\Rightarrow (4n+4)^2 + 1 = M^2$$
$$\Rightarrow M^2 - (4n+4)^2 = 1$$

It is well-known that, difference of two square number cannot be 1.

Therefore, this case does not yield a solution.

Case 4 : $x = 1, y = 1, z = 2$

The selected values of the variables convert the given equation as follows

$$(4n+1) + (4n+3) + (4n+7)^2 = M^2$$
$$\Rightarrow 16n^2 + 64n + 53 = M^2$$
$$\Rightarrow (4n+8)^2 - 11 = M^2$$
$$\Rightarrow (4n+8)^2 - M^2 = 11$$

This is true only if $M = 5$ and $4n + 8 = 6$.But for any $n \in N$, $4n + 8 = 6$ is not valid.

Consequently, there is no solution to an equation.

Case 5 : $x = 2, y = 2, z = 1$

The desired equation becomes

$$(4n+1)^2 + (4n+3)^2 + (4n+7) = M^2$$

$$\Rightarrow 32n^2 + 36n + 17 = M^2$$

Case 6 : $x = 2, y = 1, z = 2$

The considered equation becomes

$$(4n + 1)^2 + (4n + 3) + (4n + 7)^2 = M^2$$

$$\Rightarrow 32n^2 + 68n + 53 = M^2$$

Case 7: $x = 1, y = 2, z = 2$

Then, $p^x + (p + 2)^y + (p + 6)^Z = M^2$

$$\Rightarrow (4n + 1) + (4n + 3)^2 + (4n + 7)^2 = M^2$$

$$\Rightarrow 32n^2 + 84n + 59 = M^2$$

Case 8: $x = 2, y = 2, z = 2$

Then $p^x + (p + 2)^y + (p + 6)^Z = M^2$

$$\Rightarrow (4n + 1)^2 + (4n + 3)^2 + (4n + 7)^2 = M^2$$

$$\Rightarrow 48n^2 + 88n + 59 = M^2$$

It is a well-known fact that if $b^2 = 4ac$, the quadratic polynomial $ax^2 + bx + c$ is a perfect square.

But, the quadratic equation in $n$mentioned above from case 5 to case 8does not meet this criterion. As a conclusion, none of these choices of x, y, z considered from case 5 to case 8 provides solutions to an equation.

Theorem 2.2

A solution to the equation$p^x + (p + 2)^y + (p + 6)^Z = M^2$ is inconceivable if $x, y, z \in \{1,2\}$ and $(p, p + 2, p + 6)$ is a prime triplet of the form $(4n + 3, 4n + 5, 4n + 9)$.

Proof:

This theorem is showed by the succeeding eight cases as in theorem 2.1

Case 1: $x = 1, y = 1, z = 1$

Then, $p^x + (p + 2)^y + (p + 6)^Z = M^2$

$$\Rightarrow 4n + 3 + 4n + 5 + 4n + 9 = M^2$$

$$\Rightarrow 12n + 17 = M^2$$

This is not true for any $n \in \mathcal{N}$. This statement is confirmed by the succeeding MATLAB Program.

$$clc;\ clear\ all;$$

$$n = input('Enter\ a\ natural\ number\ n');$$

$for\ i = 1{:}n$

$p1 = 4 * i + 3;\ p2 = 4 * i + 5;\ p3 = 4 * i + 9;$

$if\,(isprime(p1) == 1\ \&\ isprime(p2) == 1\ \&\ isprime(p3) == 1)$

$MS = 12 * n + 11;$

$M = sqrt(MS);$

$if\,(rem(M, 1) == 0)$

$fprintf(\text{'}p1 = \%d, p2 = \%d, p3 = \%d, M = \%d\text{'}, p1, p2, p3, M)$

end

end

end

Case 2: $x = 2, y = 1, z = 1$

The required equation to be solved becomes

$$(4n + 3)^2 + (4n + 5) + (4n + 9) = M^2$$
$$\Rightarrow 16n^2 + 32n + 23 = M^2$$
$$\Rightarrow (4n + 4)^2 + 7 = M^2$$
$$\Rightarrow M^2 - (4n + 4)^2 = 7$$

This declaration is true only when $M = 4$ and $4n + 4 = 3$. But there is no $n \in \mathcal{N}$ sustaining the condition$4n + 4 = 3$.

Case 3: $x = 1, y = 2, z = 1$

The developed equation can be modified into

$$(4n + 3) + (4n + 5)^2 + (4n + 9) = M^2$$
$$\Rightarrow 16n^2 + 48n + 37 = M^2$$
$$\Rightarrow (4n + 6)^2 + 1 = M^2$$
$$\Rightarrow M^2 - (4n + 6)^2 = 1$$

As is case 2 of theorem 2.1, this is impossible.

Case 4 : $x = 1, y = 1, z = 2$

The given equation can be rewritten as

$$(4n + 3) + (4n + 5) + (4n + 9)^2 = M^2$$
$$\Rightarrow 16n^2 + 80n + 89 = M^2$$
$$\Rightarrow (4n + 10)^2 - 11 = M^2$$
$$\Rightarrow (4n + 10)^2 - M^2 = 11$$

$\Rightarrow M = 5$ and $4n + 10 = 6$

But for any $n \in N$, $4n + 10 = 6$ is not possible.

Case 5 : $x = 2, y = 2, z = 1$

The stated equation becomes

$$(4n + 3)^2 + (4n + 5)^2 + (4n + 9) = M^2$$

$$\Rightarrow 32n^2 + 68n + 43 = M^2$$

Case 6 : $x = 2, y = 1, z = 2$

The considered equation is

$$(4n + 3)^2 + (4n + 5) + (4n + 9)^2 = M^2$$

$$\Rightarrow 32n^2 + 100n + 95 = M^2$$

Case 7: $x = 1, y = 2, z = 2$

These options of the variables reduce the scrutinized equation into

$$(4n + 3) + (4n + 5)^2 + (4n + 9)^2 = M^2$$

$$\Rightarrow 32n^2 + 116n + 109 = M^2$$

Case 8: $x = 2, y = 2, z = 2$

The equation in which solutions to be discovered becomes

$$(4n + 3)^2 + (4n + 5)^2 + (4n + 9)^2 = M^2$$

$$\Rightarrow 48n^2 + 136n + 115 = M^2$$

As in theorem 2.1, in this theorem also case 5 to case 8 does not yield the solution to an equation. Hence, there exists no solution in integer to the given equation.

Theorem 2.3

There are infinitely many solutions to the equation $p^x + (p + 4)^y + (p + 6)^Z = M^2$ if $(p, p + 4, p + 6)$ is a prime triplet the form $(4n + 1, 4n + 5, 4n + 7), n \in \mathcal{N}$ and if x, y, z are either of 1 or 2.

Proof:

The theorem is proved as in previous two theorems.

Case 1: $x = 1, y = 1, z = 1$

Then, $p^x + (p + 4)^y + (p + 6)^Z = M^2$

$$\Rightarrow 4n + 1 + 4n + 5 + 4n + 7 == M^2$$

$$\Rightarrow 12n + 13 = M^2$$

It is observedfrom the following MATLAB Program,there are enormous prime triplets can be extracted as a solution. For instance, if$n = 3, 9, 69, 153$provides the prime triplets $(13, 17, 19), (31, 41, 43), (277, 281, 283), (613, 617, 619)$ as solutions to the designated equation.

$$clc;\ clear\ all;$$

$$n = input('Enter\ a\ natural\ number\ n');$$

$for\ i = 1{:}n$

$p1 = 4 * i + 1;\ p2 = 4 * i + 5;\ p3 = 4 * i + 7;$

$if(isprime(p1) == 1\ \&\ isprime(p2) == 1\ \&\ isprime(p3) == 1)$

$MS = 12 * n + 13;$

$M = sqrt(MS);$

$if(rem(M, 1) == 0)$

$fprintf('p1 = \%d, p2 = \%d, p3 = \%d, M = \%d', p1, p2, p3, M)$

end

end

end

Case 2:$x = 2, y = 1, z = 1$

The given equation becomes

$$p^x + (p+4)^y + (p+6)^z = M^2$$

$$\Rightarrow (4n+1)^2 + (4n+5) + (4n+7) = M^2$$

$$\Rightarrow 16n^2 + 16n + 13 = M^2$$

$$\Rightarrow (4n+2)^2 + 9 = M^2$$

$$\Rightarrow M^2 - (4n+2)^2 = 9$$

This is achievable only when $M = 5$ and $4n + 2 = 4$. However, for every $n \in \mathcal{N}$, the equation$4n + 2 = 4$ is not feasible.

Case 3:$x = 1, y = 2, z = 1$

The elected choices of x, y, z minimizes the given equation as

$$(4n+1) + (4n+5)^2 + (4n+7) = M^2$$

$$\Rightarrow 6n^2 + 48n + 33 = M^2$$

$$\Rightarrow (4n+6)^2 - 3 = M^2$$

$$\Rightarrow (4n+6)^2 - M^2 = 3$$

$\Rightarrow 4n+6=2$ and $M=1$are the only values that enable the above equation to be accomplished. But $4n+6=2$ is not conceivable for any $n \in \mathcal{N}$.

Case 4 : $x=1, y=1, z=2$

For these options of x, y, z, the equation to be resolved is

$$(4n+1)+(4n+5)+(4n+7)^2=M^2$$
$$\Rightarrow 16n^2+64n+55=M^2$$
$$\Rightarrow (4n+8)^2-9=M^2$$
$$\Rightarrow (4n+8)^2-M^2=9$$

The only values which attain the above condition are $4n+8=5$ and $M=4$.

But for any $n \in \mathcal{N}$, $4n+8=5$ is invalid.

Case 5 :$x=2, y=2, z=1$

Therefore, the original equation is converted into the quadratic equation as follows

$$32n^2+52n+33=M^2$$

Case 6 :$x=2, y=1, z=2$

Then, the original equation is altered into the quadratic equation in n as given below.

$$32n^2+68n+55=M^2$$

Case 7: $x=1, y=2, z=2$

The similar form of the given equation is

$$32n^2+100n+75=M^2$$

Case 8: $x=2, y=2, z=2$

The identical from of the considered equation is

$$48n^2+104n+75=M^2$$

As the explanation given in theorem 2.1, there is no solution in integers for the cases listed above from 5 to 8.

Theorem 2.4:

For any $n \in \mathcal{N}$, if $p=4n+3$ and $(p, p+4, p+6)$ is a prime triplet, then$p^x+(p+4)^y+(p+6)^Z=M^2$ has no solution when x, y, z are either 1 or 2.

Proof:

The proof is analogous to theorem 2.1

Exceptional Prime Triplets

1. If $(p_1, p_2, p_3) = (2, 3, 5)$, then the possible solution of $2^x + 3^y + 5^z = M^2$ are $(x, y, z, M) = (1, 2, 1, 4)$ and $(1, 2, 2, 6)$.
2. If $(p_1, p_2, p_3) = (3, 5, 7)$, then there is no solution to the proposed equation $3^x + 5^y + 7^z = M^2$

III. CONCLUSION

This text investigates the spectacular exponential Diophantine equation ${p_1}^x + {p_2}^y + {p_3}^z = M^2$ where (p_1, p_2, p_3) is a prime triplet either of the form $(p, p + 2, p + 6)$ or $(p, p + 4, p + 6)$ and x, y, z are either 1 or 2.One may derive integer solutions by considering the sum of the variables or the product of the variables is either 1 or 2.

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Usage of Linear Diophantine Equation in the Resolution of Molecular Formulae for Various Chemical Substances

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Abstract

In this article, it is exhibited few examples for how to use the linear Diophantine equation to contract the molecular formulae of organic or inorganic chemical compounds in order to determine their structure.

Keywords: *linear Diophantine equation, integer solutions, molar mass, molecular formula.*

I. Introduction

There is a widespread belief that Number theory is the purest field of pure Mathematics, and that it has few meaningful applicability to real-world issues as a result of this belief. The significance of Number theory is derived from its prominent position in Mathematics; its ideas and problems have played a crucial role in the development of many areas of Mathematics throughout history. A Diophantine equation is a multivariable equation that concedes only integer solutions. One of its particular instances is the linear Diophantine equation which has the general form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where $a_1, a_2, \ldots., a_n, b \in Z, n \geq 2$ and whose solutions must be integers. For instance, [5] proposes techniques based on Euclidean algorithm arguments.
Furthermore, practical issues involving the splicing of telephone lines have been resolved by using techniques of basic number theory [9]. Many more fascinating applications may be found in the book Number Theory and the Periodicity of Matter [1], which has a large number of additional examples. Additionally, the reader is recommended to refer [2-4,7, 10-13] in this respect.
It is common to perceive linear Diophantine equations in many disciplines, but they are particularly prevalent in chemistry [6,8,14], In this article, it is revealed the application of linear Diophantine equation in Chemistry specially how to find the molecular formulae of organic or inorganic chemical compounds in order to determine their structure with few specimens.

Needed Theorem [1]

The linear Diophantine equation $ax + by = c$ has a solution if and only if d divides c where $d = gcd(a, b)$. Furthermore, if (x_0, y_0) is a solution of this equation, then the set of solution of the equation consists of all pairs (x, y) where $x = x_0 + t\,{}^{b}/_{d}$ and $y = y_0 - t\,{}^{a}/_{d}$, $t \in \mathcal{Z}$

II. Determination of chemical molecular formula

Enabling the possibility that a chemical substance with a molecular weight W encompasses elements A_1, A_2 and A_3 with atomic weights a_1, a_2 and a_3 respectively and that the numbers X, Y and Z represent the number of atoms of elements A_1, A_2 and A_3 visible in each of the elements' molecules. Then, it is obtained that

$$a_1X + a_2Y + a_3Z = W \quad (1)$$

Let α_1, α_2 and α_3 constitutes the integers closest to the values a_1, a_2 and a_3 and let w signify the integer closest to the value W.

Then the similar form of linear Diophantine equation (1) to be solved is represented by

$$\alpha_1X + \alpha_2Y + \alpha_3Z = w \quad (2)$$

If a limit is imposed on the integers X, Y and Z in (2), then it can be solved under a restriction

$$|(a_1 - \alpha_1)X + (a_2 - \alpha_2)Y + (a_3 - \alpha_3)Z| < |W - w| \quad (3)$$

If more solutions of (2) are retrieved, then the genuine values can be found by substituting them in (1) and assessing which of satisfies (2) with the least significant deviation from W.

The process of finding molecular formulae for three chemical substances using the linear Diophantine equation is enlightened as follows.

2.1. Molecular formula for substance 1

Consider the substance 1 as the chemical compound comprising Carbon, Hydrogen, and Oxygen having a molecular weight of 342.2965 g/mol.

Let X, Y and Z stand for the number of atoms of Carbon, Hydrogen and Oxygen respectively. Consider the first-degree Diophantine equation as

$$12.0107X + 1.00784Y + 15.999Z = 342.2965 \quad (4)$$

where $12.0107u, 1.00784u$ and $15.999u$ are the atomic weights of Carbon, Hydrogen and Oxygen respectively.

Next, it is clear that $\alpha_1 = 12, \alpha_2 = 1, \alpha_3 = 16$ and $w = 342$.

Furthermore, the corresponding linear form of (4) to discover molecular formula is converted into

$$12X + Y + 16Z = 342 \quad (5)$$

subject to the constraint

$$|0.0107X + 0.00784Y - 0.001Z| < 0.2965$$

which provides that $X \leq 12, Y \leq 23, Z \leq 13$

Modify (5) as in the following form

$12X + Y = 342 + 16T$ where $Z = -T$

Then, its common solution is given by

$$X = 28 + K$$
$$Y = 6 + 16T - 12K,\ K, T \in Z.$$

It is assured that the values of X, Y, Z must be greater than 0. Hence, it is enabled to discover the ranges for T and K.

Now,

$$X > 0 \Rightarrow 28 + K > 0 \Rightarrow K > -28$$
$$Y > 0 \Rightarrow 6 + 16T - 12K > 0 \Rightarrow 8T - 6K > -3$$
$$Z > 0 \Rightarrow -T > 0 \Rightarrow T < 0$$

In particular, if $T = -1$, then $8T - 6K > -3$ leads to $K < 1$.

Thus, the range for K should be $-28 < K < 1$.

Continuing the process for $T = -2, T = -3, \ldots, T = -21$, it is received that $K < 1$. But for $T = -22$, $K < -29$ contradicting $-28 < K < 1$.

Thus, the range for T should be $-22 < T \leq -1$.

Therefore, there exists 588 solutions in combinations of T and K.

Eliminate the solutions which violating the conditions that $0 < X < 13, 0 < Y < 24$ and $0 < Z < 14$ by the succeeding MATLAB Program:

```
clear all; clc;
for t=-21:-1
   for k=-27:0
      x=28+k;
      y=6+16*t-12*k;
      z=-t;
      if (x<13 && y<24 && z<14)
         fprintf('x=%d,y=%d,z=%d\n',x,y,z)
      end
   end
end
```

The residual solutions which satisfy the necessary conditions are listed in the table below.

X	*Y*	*Z*
10	14	13
11	2	13
11	18	12
12	6	12
12	22	11

Note that the last two solutions represent the compounds $C_{12}H_6O_{12}$ (Mellitic acid) and $C_{12}H_{22}O_{11}$ (Sucrose or Table Sugar) with molar mass 342.16 g/mol and 342.2965 g/mol respectively.

Therefore, the exact solution is $C_{12}H_{22}O_{11}$.

2.2. Molecular formula for substance 2

Let us choose the substance 2 as the chemical compound with the molecular weight of 98.079 g/mol and a mixture of Hydrogen, Sulphur, and Oxygen.

Let X, Y and Z be the number of atoms of Hydrogen, Sulphur, and Oxygen with respective atomic mass $1.00784u, 32.065u$ and $15.999u$ respectively.

As in Section 2.1, choose the linear Diophantine equation in three variables as

$$1.00784X + 32.065Y + 15.999Z = 98.079$$

Clearly, $\alpha_1 = 1, \alpha_2 = 32, \alpha_3 = 16$ and $w = 98$.

Consequently, let us solve the ensuing linear Diophantine equation

$$X + 32Y + 16Z = 98 \quad (6)$$

subject to the restriction

$$|0.0078X + 0.065Y - 0.001Z| < 0.079$$

The upper limit for the choices of X, Y and Z are noted by

$$X \leq 3, Y \leq 1, Z \leq 4 \quad (7)$$

From (6), $X = 98 - 32Y - 16Z > 0, Y > 0$ and $Z > 0$. (8)

All the possibility of X, Y and Z supporting (8) are evaluated by $\{(50,1,1), (18,2,1), (34,1,2), (2,2,2), (18,1,3), (2,1,4)\}$

The only choice of (X, Y, Z) that satisfies (7) is $(2,1,4)$.

Hence, the component is H_2SO_4, That is Sulphuric Acid with molar mass 98.079 g/mol.

2.3. Molecular formula for substance 3

Consider Substance 3 is a combination of Zinc, Sulphur, and Oxygen having a molecular weight of 161.47 g/mol.

Let X, Y and Z be the number of atoms of Zinc, Sulphur, and Oxygen with respective atomic mass $65.38u, 32.065u$ and $15.999u$ respectively.

As in the previous two sections, the equation to be resolved is

$$65.38\,X + 32.065\,Y + 15.999\,Z = 161.47 \quad (9)$$

With the same notations as in section 2.1, $\alpha_1 = 65, \alpha_2 = 32, \alpha_3 = 16$ and $w = 161$.

Then, an equivalent form of (9) to be solved is taken as

$$65X + 32Y + 16Z = 161 \quad (10)$$

together with the condition that

$$|0.38X + 0.065Y - 0.001Z| < 0.47$$

The options of such X, Y and Z in (10) are viewed by

$$X \leq 1, Y \leq 1, Z \leq 444$$

Since X, Y and Z are positive, the only possibility of X and Y are pointed out by

$$X = 1, Y = 1$$

Now, rearrange (10) the form as given below

$$65X + 16U = 161 \quad (11)$$

where $U = 2Y + Z$ (12)

In (11), $gcd(65,16) = 1$ and 1 divide 161.

Also, the least solution to (11) is taken as $X_0 = 161$ and $U_0 = -644$

Hence by theorem [1], there exists infinitely many integer solutions to (11) which are represented by

$$X = 161 + 16T \quad (13)$$

$$U = -644 - 65T \quad (14)$$

where $T \in \mathcal{Z}$

Since $X = 1$, the chance of T is evaluated from (13) as

$$T = -10$$

Note that (12) is satisfied by $Y_0 = U$ and $Z_0 = -U$. Also, $gcd(2,1) = 1$

Again, by Theorem [1] the infinitely many solutions to (12) are received by

$$Y = U + K = -644 - 65T + K \quad (15)$$

$$Z = -U - 2K = 644 + 65T - 2K, K \in \mathcal{Z} \quad (16)$$

Since $Y = 1$, the value of K is calculated from (15) by

$$K = -5$$

Substituting the values of T and K in (16), it is determined that

$$Z = 4$$

The only solution that satisfies (9) is $(X, Y, Z) = (1,1,4)$.

Hence, the component is $ZnSO_4$, that is Zinc Sulphate with molar mass 161.47 g/mol.

III. Conclusion

With the help of a few instances, the application of the linear Diophantine equation in finding the chemical molecular formula for three different substances are evaluated in this editorial. Finally, new and interesting applications of Number theory include cryptography, coding theory, and random number generation, among other things. These areas are developing at a breakneck pace as a result of the widespread use of computers, and their significance is growing all the time.

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