# AN ANALYSIS OF VARIOUS MEASURES OF FAREY AND CANTOR SETS

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**DOCTOR OF PHILOSOPHY IN MATHEMATICS**

**By**

**Ms. S. SUDHA**

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This is to certify that the thesis entitled **"An Analysis of Various Measures of Farey and Cantor Sets"** is a record of original research work carried out in partial fulfilment of the requirements for the award of the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICS** by **Ms. S. SUDHA ( Ref.No: 16238 / Ph.D.K1 / Mathematics / Full-Time/ (3-5) Oct- 2017 / Confirm.Date: 20.09.2021)** a Full – Time research scholar in Government Arts College affiliated to Bharathidasan University, Tiruchirappalli, Tamilnadu, India, under my supervision and guidance. The thesis has not been submitted for the award of any other degree / Associateship / Fellowship or similar title of this to any other University.

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## ABSTRACT

Number theory, known as queen of Mathematics has got many interesting topics. Thought number theory deals with integers in the beginning, it has branched into the field of rational numbers. Some of the fractal sequences in number theory are Sterm – Brocot sequence and Farey sequences.

In Real Analysis Cantor sets are unique in nature. It is probably, the best known example of a perfect nowhere-dense set in the real line. The Cantor set plays a very important role in many branches of mathematics. It has many definitions and many different constructions. Although Cantor originally provided a purely abstract definition, the most accessible is the Cantor "middle-thirds" or ternaryset construction beginning with the closed real interval $[0,1]$. In general, The Cantor Middle $(2m-1)^{th}$ set $C_{2m-1}$, where $m \geq 2$ contains the endpoints of all $2^n$ intervals, each of length $\frac{(m-1)^n}{(2m-1)^n}$ .

The Farey sequence of order $n$ is the sequence of completely reduced fractions between 0 and 1 whose lowest terms have denominators less than or equal to $n$, arranged in order of increasing size. The Farey fractions are irreducible fractions, reduced fractions or fractions in lowest terms between 0 and 1 with denominators less than or equal to some given value. When these fractions are arranged in increasing order the result is the Farey sequence. The maximum denominator is called the order of the sequence $F_n$, where $n$ denotes its order. Depending on how the sequence is to be used, one or the other or both of the end points $\frac{0}{1}$ and $\frac{1}{1}$ may be excluded. Unless noted otherwise, we will always include both the end points.

The study of measures and their application to integration is called measure theory. Measure concept may be a stronger assumption of countable additivity. Measure concept involves $\sigma$-algebras, measures, measurable features and integrals. Integration inside the

context of measure theory entails analogous sums and is based on capabilities consistent on sets of a few $\sigma$-algebras.

Extraction of Cantor sets from Farey sequences are already studied.

This thesis entitled **"An Analysis of Various Measures of Farey and Cantor Sets"** consists of six chapters.

**CHAPTER I**

This chapter provides the historical background and necessary literature survey for Farey sequences and Cantor set of odd order. Also measures like Lebesgue measure and Probability measure have a brief introduction.

**CHAPTER II**

This chapter has four sections. In section 2.1 Cantor Hexnary Sets, in section 2.2 Cantor Deca Sets, in section 2.3 Cantor Octanary Sets and in section 2.4 Cantor Dodeca Sets are developed.

Various patterns of removal of intervals are analyzed here.

**CHAPTER III**

A subsequence of Farey sequence, $\widetilde{F_N}$ ,Farey $N$ – subsequence has been established as a topological space and a Hausdorff space by appropriately defining basis and open sets. Also the $T_1$axiom has been discussed with an illustration. Here sequences are considered as sets.

This chapter provides two sections. In section 3.1 Farey sequence has been proved as a Topological Space and in section 3.2 Farey sequence with a slight change in the basis has been developed as a Hausdorff space .

**CHAPTER IV**

A Non Reduced Farey $N$-subsequence, a subsequence of Farey sequence consists of rational numbers with same denominator in $[0, 1]$. By reconstructing the non reduced Farey $N$ – subsequence it can be established as a $\sigma$- algebra and its Lebegue Measure has been found. Non –Reduced Farey $N$ - subsequence of even order has been studied.

**CHAPTER V**

This chapter is divided into two sections.Section 5.1 gives Probability Measure of Generalized Non - Reduced Farey $N$- Subsequence and the other section 5.2 is on Invariant Measure of Generalized Non - Reduced Farey $N$- Subsequence. An attempt has been to provide a theorem on the Probability measure of the Generalized Non reduced Farey $N$ – subsequence.

**CHAPTER VI**

This chapter deals with the Box Measure of Modified even ordered Cantor sets.While calculating box measure, boxes without dimensions are used in general but here, boxes are replaced by isosceles triangles and their areas are considred as measures. This chapter is divided into four sections.

Section 6.1 deals with Measure of Cantor Hexnary Sets and Section 6.2 is on Measure of Cantor Deca Sets.Varying from the previous section, section 6.3 provides a measure for Cantor Octanary Sets and following the similar lines Section 6.4 gives a measure of Cantor Dodeca Sets.

# CONTENTS

## NOTATIONS

| | | |
|---|---|---|
| 1. | $C_{\left(\frac{1}{6}\right)}$ | Cantor Hexnary Set |
| 2. | $C_{\left(\frac{1}{10}\right)}$ | Cantor Deca Set |
| 3. | $C_{\left(\frac{1}{8}\right)}$ | Cantor Octanary Set |
| 4. | $C_{\left(\frac{1}{12}\right)}$ | Cantor Dodeca Set |
| 5. | $F_n$ | Farey Sequence of order $n$ |
| 6. | $\langle F'_N \rangle$ | Farey $N$- Subsequence |
| 7. | $\tilde{F}_N$ | Non - Reducible Farey $N$- Subsequence |
| 8. | $C\tilde{F}_N$ | Non - Reducible Farey $N$- Subsequence as Union of closed set |
| 9. | $HC\tilde{F}_N$ | Non - Reducible Farey $N$- Subsequence as Union of half closed set |
| 10. | $TMC_{\left(\frac{1}{6}\right)}$ | Triangular Measure of Cantor Hexnary Set |
| 11. | $TMC_{\left(\frac{1}{10}\right)}$ | Triangular Measure of Cantor Deca Set |
| 12. | $TMC_{\left(\frac{1}{8}\right)}$ | Triangular Measure of Cantor Octanary Set |
| 13. | $TMC_{\left(\frac{1}{12}\right)}$ | Triangular Measure of Cantor Dodeca Set |

# CHAPTER - I

# INTRODUCTION

Number theory is one of the oldest branches of Mathematics. Numbers were used for keeping records and for commercial transactions for over 5000 years before anyone thought of studying numbers themselves in a systematic way. Dickson always said "Mathematics is the queen of science and that the theory of numbers is the Queen of Mathematics" [30].

The theory of numbers is concerned with properties of integers and more particularly with positive integers $1, 2, 3, 4, ...$The number theory is nothing but study of the whole numbers and integer-valued functions. The origin of this misnomer harks back to the early Greeks when the word "number" meant positive integer and nothing else. The number theory has always occupied unique position within the planet of Mathematics. The more established term for number theory is arithmetic. By the early twentieth century, it had been superseded by "number theory" [40].This is often because of the unquestioned historical importance of the subject. Like Dickson German Mathematician Carl Friedrich Gauss also said, "Mathematics is the queen of the sciences and number theory is the queen of Mathematics" [31].

Number theory analyses the properties of integer systems in spite of their apparent complexity. Integers are often considered either in themselves or as solutions to equations. The natural numbers have been known to us for so long that the mathematician Kronecker once remarked, "God created the natural numbers, and all the rest is the work of man". Far from being a gift from Heaven, number theory has a long and sometimes painful evolution [12,13,20]. In number theory concepts are often best understood through study of analytical objects that encode properties of the whole numbers, primes or other number-theoretic objects in some fashion. Real numbers may be studied in reference to rational numbers.

Number theory has many subdivisions such as Algebraic number theory, Analytic number theory, Probabilistic number theory, Computational number theory, etc.

The theory of numbers has always occupied unique position in the world of mathematics. It is especially entitled to a separate history on account of the greatness attached to it, continuously through the centuries, from the time of Pythagoras [4, 5]. Regarding the true origin of the theory of numbers: It seems probable that the Greeks were largely indebted to the Babylonians and ancient Egyptians for a core of information about the properties of the natural numbers, the first rudiments of an actual theory are generally credited by Pythagoras and his disciples. The Pythagoreans believed that the key to an explanation of the universe lay in number and form their general thesis that “Everything is number”. [6,7,11,15,23].

As the mathematician Sierpinski once said, “The progress of our knowledge of numbers is advanced not only by what we already know about them, but also by realizing what we yet to not know about them”. It is a fact that the natural numbers $1,2,3,4,5, \ldots$ are closed under addition and multiplication, and that the integers $\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$ are closed under addition, multiplication and subtraction but neiither of these sets is closed under division. The entire collection of such fractions constitutes the rational numbers. Thus a rational number a number which can be put in the form $\frac{a}{b}$, where $'a'$ and $'b'$ are integers and $'b'$ is not zero [9,19,21,22].

The Egyptians worked only with unit fractions, fractions with numerator equal to one, known also as Egyptian fractions .Their problem was to write any given common fraction as a sum of different unit fractions.

In algebraic number theory an irrational number is any complex quantity that is an answer to some polynomial equation $f(x) = 0$ with rational co-efficients. Algebraic number theory studies irrational number fields [35]. Number fields are often studied as extensions of smaller number fields, a field $L$ is claimed to be an extension of a field $K$ if $L$ contains$K$.

In analytic number theory, the number theory problem has brought light and elegance to this field, especially to the problem of the distribution of prime numbers. Through the centuries, an outsized sort of tools has been developed to understand a better understanding of this particular problem [8, 41].

In Probabilistic number theory, the study of variables that are almost mutually independent is often seen as a crucial special case [45].

Computational number theory, also referred to as algorithmic number theory, is the study of computational methods to analyze and solve problems in number theory. The theory of Computational number has cryptography applications and is used in number theory to research conjectures and open problems, including the Riemann hypothesis [45].

Number theory is not a systematic study of mathematics but also a popular diversion, it is part of recreational mathematics, including numerical curiosities and puzzle solving [22]. The dimension of number theory is not emphasized, unless the questions are related to general propositions. A comprehensive analysis of the theory is definitely beneficial for someone looking into recreational mathematics problems [22, 10].

Proofs within the principle of numbers rely on many exclusive thoughts and methods. Of these, special attention may be given to principle of mathematical induction [44]. If the important numbers are necessary for the study of rational numbers from the standpoint of their size, the $p$-adic numbers play a totally analogous role in question connected with divisibility by powers of the prime $p$ [44]. The analogy between real and $p$-adic numbers is often developed in other ways. The $p$-adic numbers are often constructed ranging from the rational numbers, in just an equivalent way that the important numbers are constructed by adjoining the bounds of Cauchy sequences [44].

The most rudimentary class of polygonal numbers described by the early Pythagoreans was that of the oblong numbers. The $n^{th}$ oblong number, denoted by $O_n$, is given by $n\ (n+1)$ and represents the number of focuses in a rectangular array having $n+1$ row and $n$ columns [29].

Number theory has long been a favourite subject for students and teachers of mathematics. It is a classical subject and has a reputation for being the "purest" part of Mathematics, yet recent developments in cryptology and software engineering depend on basic elementary number theory. Number theory, from general perspective, is the investigation of numbers and their properties [29]. The fundamental theorem of arithmetic is that each positive integer can be written uniquely as the product of primes. One of the most significant uses of number theory to computer science is in the zone of cryptography. Congruences can be utilized to create various types of ciphers. Recently, another new type of cipher system, called a public-key cipher system, has been devised. When a public-key cipher is utilized, every individual has a public enciphering key and a private deciphering key [29]. Messages are enciphered utilizing the public key of the receiver. Moreover, only the receiver can decipher the message, since a mind-boggling measure of computer time is required to decipher when just the enciphering key is known. The most widely utilized public-key cipher system relies on the difference in computer time required to discover large primes and to factor huge integers [29].

We describe two applications of Number theory in cryptography to computer science. The Chinese remainder theorem is utilized in two applications. The first application includes the enciphering of a data base [29]. A database is a collection of computer documents or records. In [29] it is shown how to encipher an entire database so that individual files may be deciphered without jeopardizing the security of other files in the database [29].

### 1.1 Significance of Number Theory:

Number theory was classified as a discipline without direct application and it only demonstrated the basic properties of Mathematics. With the great and profound scientific and technological transformation brought by the emergence and development of computer, number theory has been widely used, and is no longer just a pure mathematics, but a mathematical discipline with practical application value. At present, number theory is widely and fully applied in many fields, such as computing, cryptography, physics, chemistry, biology, acoustics, electronics, communication, graphics and even musicology.

This also proves the significance of number theory that it can be widely and fully applied to many other fields involving mathematics, and has developed into a new applied mathematics discipline - applied number theory. Therefore, number theory is no longer just a pure discipline, but a veritable applied discipline. Judged from the current development trend and applications of number theory, this ancient discipline is bound to be vigorous [43].

### 1.1.1 Development of Number Theory:

Many questions in number theory have been proposed and then solved, which attracts more and more people to focus on number theory. In the long history, techniques and methods to solve problems have emerged, and some theories have been formed. Algebraic number theory has been advanced with the expansion of number field and practical applications. Bacon, the famous philosopher, said that history makes people smart, so it is necessary to explore the development of early algebraic number theory. Domestic researches on algebraic number theory are mainly comprehensive discussions on the progress of algebraic number theory [26, 42].

a) The discovery of irrational numbers: The followers of Pythagorean school discovered the first irrational number, shocking the leaders of the school at that time. It was proposed that all numbers could be expressed as ratios of integers that later led to the first mathematical crisis. b) Creation of arithmetic operators and solution to irrational equations: In India, the mathematician Brahmagupta introduced a group of symbols used to express concepts and describe operations in the $7^{th}$ century, and Posgallo later put forward the concept of negative square root, the solution to irrational equations and the algorithm of irrational numbers in the $12^{th}$ century, which fostered the study of algebra to a new stage. c) Establishment of imaginary number theory: Cardano was the first mathematician to formulate the square root of a negative integer. Now it has been developed to find negative Jacobsthal numbers.

### 1.1.2 Other Basic Fields

Number theory also plays a surprising role in other theories. In quantum theory, Hermite operator is one of the most basic concepts. Apart from that, number theory is also widely used in non-mathematical disciplines, such as information science, theoretical physics, quantum chemistry and so on [29].

## 1.2 Applications of Number Theory:

### 1.2.1 Cryptography:

With the development of network encryption technology, number theory has found its own place in cryptography. Professor Wang Xiaoyun who cracked the $MD5$ code a few years ago is from the number theory school of Shandong University. Because of the irregular appearance of prime factors in composite numbers, it is very difficult to decompose composite numbers into product of prime numbers. At the same time, it is this difficulty that enlightens people to use it to design difficult codes. When studying number theory, especially cryptography, we pursue deterministic algorithm rather than probabilistic algorithm, and we will only lower our requirements and apply probabilistic algorithm if there is no deterministic algorithm [29].

## 1.3 Cantor Middle Set:

George Cantor (1845-1918) was the originator of quite a bit of a modern set theory. Among his commitments to mathematics was the thought of the Cantor set, which comprises points along a line segment, and has various interesting properties. The Cantor middle $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$ sets, in general, the Cantor middle $\left\{\frac{1}{2m-1}, \quad 2 \leq m < \infty\right\}$ set is called generalized Cantor sets and it is denoted by $C_{1/(2m-1)}$ [16,33].

In 2008, Gerald Edgar introduced different properties of Cantor sets using an iterated function system. Moreover, many other general Cantor sets were built by eliminating various parts of various lengths from the initiator and likewise introduced a few properties utilizing an iterated function system [32].

The Cantor set has numerous definitions and a wide range of developments. Despite the fact that Cantor initially gave an absolutely dynamic definition, the most available is Cantor's "middle - third" or ternary set in which the development starts with the closed real interval [0,1] and partitions it into three equivalent open subintervals. The methodology is to remove the central open interval.

$$I_1 = \left(\frac{1}{3}, \frac{2}{3}\right) \text{ such that } K_1 = [0,1] - I_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

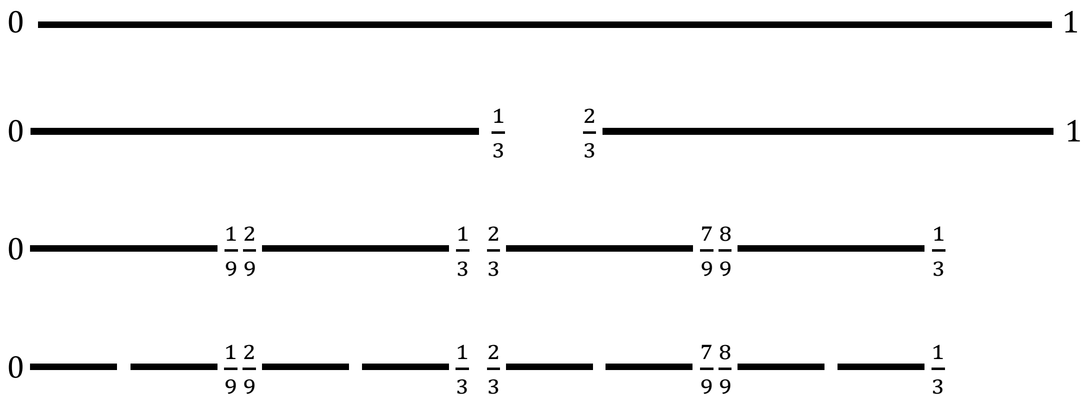
Next, subdivide each one of these two remaining intervals into three equivalent open subintervals, and from each remove the central third. Let $I_2$ the removed set, at that point is

$$I_2 = \left(\frac{1}{3^2}, \frac{2}{3^2}\right) \cup \left(\frac{7}{3^2}, \frac{8}{3^2}\right) \text{ and } K_2 = [0,1] - (I_1 \cup I_2) = \left[0, \frac{1}{3^2}\right] \cup \left[\frac{2}{3^2}, \frac{3}{3^2}\right] \cup \left[\frac{6}{3^2}, \frac{7}{3^2}\right] \cup \left[\frac{8}{3^2}, 1\right]$$

We can subdivide every one of the intervals that involve $[0, 1] - (I_1 \cup I_2)$ into three subintervals, eliminating their middle thirds, and proceed in a past way. The sequence of open sets is then disjoint, and we traditionally define the Cantor set $C_3$ as the closed interval with the union of these $I_n{}'$s subtracted out [11, 14, 36].

$$C_3 = [0,1] - \cup I_n$$

The graphical representation is given as




In general, The Cantor Middle $(2m-1)^{th}$ Cantor's set $C_{(2\mathrm{m}-1)}$, where $m \geq 2$ contains the endpoints of every $2^{\mathrm{n}}$ intervals, every one of length $\frac{(m-1)^n}{(2m-1)^n}$ [18, 33].

### 1.3.1 Self – Similarity:

Generally, a self-similar object is one that is comparable, or approximately comparable, to part of itself. That will be, that the object can be scaled and translated on top of a portion of the original object. In spite of the fact that self-similarity is difficult to define generally, much study has been done on self-similar objects.

A fractal is defined freely as a geometric shape that is exceptionally self-similar. A traditional illustration of a fractal is the Cantor middle thirds set. The Cantor set is developed by starting with the unit interval $[0, 1]$ and at each progression removing the middle third of the remaining intervals. The following are the few iterations of the process used to make the Cantor middle third set. We study the Cantor sets all the more for the most generally later as developments outside of Euclidean space [25].

### 1.3.2 Fractal Dimension and the Cantor Set:

An altogether different definition is shown up when we consider covering the object with copies of itself at a smaller scale and count the number of such copies needed. Righteousness of this definition is that it permits us to construct objects with a 'fractional dimension'. Such items are called fractals, and the Cantor set is one of the earliest illustrations of such an object [38].

The Cantor set is the model of a fractal. It is an object that seems self-similar under fluctuating levels of amplification. One of the typical highlights of the fractal is its fractal dimension, which is basically a measure of self-similarity. It is sometimes alluded to as the similarity dimension. There are various non-identical methods of characterizing fractal dimensions. Quite possibly the most well-known method of computing the dimension of a set is to find the box-counting dimension of it [17].

## 1.4 Farey Sequence:

Aside from including $\frac{0}{1}$ and $\frac{1}{1}$, Charles Haros had formulated $F_{99}$. To do this he used the mediant property to find the fractions with higher denominators and even provided a sketch proof that it worked. Also noted that if two numbers $\frac{a}{b}$ and $\frac{c}{d}$ are neighbours in the table then $|bc - ad| = 1$. From here the history of the Farey sequence travels to Britain, and to a man called Henry Goodwyn. Henry Goodwyn ran and owned a brewery and made mathematical tables in his spare time. In his retirement he set out to create a table of fractions and decimal equivalents. However, Goodwyn's tables were to contain every irreducible fraction with denominators between 1 and $1024$. The First Centenary of a Series of concise and useful Tables of all the complete decimal Quotients which can arise from dividing a Unit or any whole Number less than each Divisor by all Integers from 1 to $1024$" [27].

Farey sequence is widely applied both in engineering and combinatorics, some of which are presented below:

Employing the Farey sequence of Fibonacci numbers for a circuit constructed from $n$ equal resistors combined in series and in parallel, strict upper and lower bounds for the order of the set of equivalent resistances have been established in [37].

Generalized Fibonacci word is used in the study of combinatorics of sturmian words, while Farey codes and Languages associated with Farey sequence are discussed in [Arturo Carpi and Aldo de Luca, Farey codes and languages, European Journal of Combinatorics 2007]. The Mandelbrot set is one of the most elegant, interms of aesthetic appeal and complicated structure, yet complex objects in all of mathematics. This set has been the area of intense research since the first pictures of it were drawn in 1978, due to the hard work of Benoit Mandelbrot and others who created awareness about this branch of mathematics. The Farey sequence is used in determining the rotation number of any bulb in the Mandelbrot set [27].

When the process of constructing Farey sequence is extended to the whole real line we get the Stern- Brocot tree is arrived, which is used in the construction of clocks [28].

The link between Riemann-Hypothesis and Farey sequence has been established in [28].

An algorithm for boundary based shape decomposition has been proposed in [37]. This algorithm uses Farey sequence for determining several measures such as slopes of edges and turn types at vertices of the polygonal cover corresponding to the concerned shape.

The study of the topological properties of the Julia sets of rational maps is a central problem in complex dynamics. A family of rational maps whose Julia's sets is a Cantor set of circles must topologically conjugate to one map in this family on their corresponding Julia's set [12].

### 1.5 Measure:

One of the most fundamental concepts in Euclidean geometry is that of the measure $m(E)$ of a solid body $E$ in one or more dimensions. In one, two, and three dimensions, we refer to this measure as the length, area, or volume of $E$ respectively. In the classical approach to geometry, the measure of a body was often computed by partitioning that body into finitely many components, moving around each component by a rigid motion and then reassembling those components to form a simpler body which presumably has the same area. One could also obtain lower and upper bounds on the measure of a body by computing the measure of some inscribed or circumscribed body; this ancient idea goes all the way back to the work of Archimedes at least. Such arguments can be justified by an appeal to geometric intuition, or simply by postulating the existence of a measure $m(E)$ that can be assigned to all solid bodies $E$, and which obeys a collection of geometrically reasonable axioms. One can also justify the concept of measure on "physical" or "reductionistic" grounds, viewing the measure of a macroscopic body as the sum of the measures of its microscopic components [18, 50].

The physical intuition of defining the measure of a body $E$ to be the sum of the measure of its component "atoms" runs into an immediate problem: a typical solid body would consist of an infinite (and uncountable) number of points, each of which has a measure of zero; and the product $\infty \cdot 0$ is indeterminate. To make matters worse, two bodies that have exactly the same number of points need not have the same measure. For instance, in one dimension, the intervals $A = [0,1]$ and $B = [0,2]$ are in one-to-one correspondence using the bijection $x \rightarrow 2x$ from $A$ to $B$, but of course $B$ is twice as long as $A$. So $A$ can be rearranged into a set of uncountable number of points and reassemble them to form a set of twice the length.

Here, the problem is that the pieces used in this decomposition are highly pathological in nature; among other things, their construction requires use of the axiom of choice. Such pathological sets almost never come up in practical applications of mathematics. Because of this, the standard solution to the problem of measure has been to abandon the goal of measuring every subset $E$ of $R^d$ , and instead to settle for only measuring a certain subclass of "non-pathological" subsets of $R^d$ , which are then referred to as the measurable sets

The class of measurable sets can be expanded at the expense of losing one or more nice properties of measure namely, finite or countable additivity, translation invariance, or rotation invariance in the process. However, there are two basic concepts that are sufficient for most applications. The first is the concept of Jordan measure of a Jordan measurable set, which is a concept closely related to that of the Riemann integral [46].

This concept is elementary enough to be systematically studied in an undergraduate analysis course, and suffices for measuring most of the "ordinary" sets in many branches of mathematics. But , when the type of sets that arise in analysis are considered , and in particular those sets that arise as limits of other sets, the Jordan concept of measurability is not quite adequate, and must be extended to the more general notion of Lebesgue measurability, with the corresponding notion of Lebesgue measure that extends Jordan measure. With the Lebesgue theory , one keeps almost all of the desirable properties of Jordan measure, but with the crucial additional property that many features of the Lebesgue theory are preserved under limits (as exemplified in the fundamental convergence theorems of the Lebesgue theory, such as the monotone convergence theorem and the dominated convergence theorem which do not hold in the Jordan-Darboux-Riemann setting). As such, they are particularly well suited for applications in analysis, where limits of functions or sets arise all the time [46].

### 1.5.1 Elementary measure:

Simpler notion of elementary measure, which allows one to measure a very simple class of sets, namely the elementary sets

### 1.5.2 Jordan measure:

We now have a satisfactory notion of measure for elementary sets. But of course, the elementary sets are a very restrictive class of sets, far too small for most applications. For instance, a solid triangle or disk in the plane will not be elementary, or even a rotated box. On the other hand, as essentially observed long ago by Archimedes, such sets $E$ can be approximated from within and without by elementary sets $A \subset E \subset B$, and the inscribing elementary set $A$ and the circumscribing elementary set $B$ can be used to give lower and upper bounds on the putative measure of $E$. As one makes the approximating sets $A, B$ increasingly fine, one can hope that these two bounds eventually match.

### 1.5.3 Lebesgue measure:

The classical theory of Jordan measure on Euclidean spaces $R^d$. This theory proceeded in the following stages:

First, one defined the notion of a box B and its volume $|B|$.Using this, one defined the notion of an elementary set $E$, a finite union of boxes, and defines the elementary measure $m(E)$ of such sets. Even exist bounded open sets, or compact sets, which are not Jordan measurable, so the Jordan theory does not cover many classes of sets of interest. Another class that it fails to cover is countable unions or intersections of sets that are already known to be measurable. Lebesgue outer measure zero, in contrast to Jordan outer measure. Lebesgue outer measure is a special case of a more general concept known as an outer measure.

To define a concept of “Lebesgue inner measure” to complement that of outer measure. Here, there is an asymmetry which ultimately arises from the fact that elementary measure is subadditive rather than superadditive. one does not gain any increase in power in the Jordan inner measure by replacing finite unions of boxes with countable ones. But one can get a sort of Lebesgue inner measure by taking complements. This leads to one possible definition for Lebesgue measurability, namely the Caratheodory criterion for Lebesgue measurability, However, this is not the most intuitive formulation of this concept to work with, and we will instead use a different but logically equivalent definition of Lebesgue measurability. The starting point is the observation that Jordan measurable sets can be efficiently contained in elementary sets, with an error that has small Jordan outer measure. In a similar way, we will define Lebesgue measurable sets to be sets that can be efficiently contained in open sets, with an error that has small Lebesgue outer measure.

The Lebesgue integral and Lebesgue measure can be viewed as completions of the Riemann integral and Jordan measure respectively. This means three things. Firstly, the Lebesgue theory extends the Riemann theory: every Jordan measurable set is Lebesgue measurable, and every Riemann integrable function is Lebesgue measurable, with the measures and integrals from the two theories being compatible. Conversely, the Lebesgue theory can be approximated by the Riemann theory, every Lebesgue measurable set can be approximated by simpler sets, such as open sets or elementary sets, and in a similar fashion, Lebesgue measurable functions can be approximated by nicer functions, such as Riemann integrable or continuous functions.

### 1.6 Measurable functions:

Constant integral can be completed to the Riemann integral, the unsigned simple integral can be completed to the unsigned Lebesgue integral, by extending the class of unsigned simple functions to the larger class of unsigned Lebesgue measurable functions [47, 48].

#### 1.6.1 Outer measures, pre-measures, and product measures:

One specific example of a countable additive measure is Lebesgue measure. This measure was constructed from a more primitive concept of Lebesgue outer measure, which in turn was constructed from the even more primitive concept of elementary measure. This generalizes the construction of Lebesgue measure from Lebesgue outer measure. One can in turn construct outer measures from another concept known as a pre-measure. One can start constructing many more measures, such as Lebesgue-Stieltjes measures, product measures, and Hausdorff measures.

To construct a variety of measures on infinite-dimensional spaces, and is of particular importance in the foundations of probability theory, as it allows one to set up probability spaces associated to both discrete and continuous random processes, even if they have infinite length. The most important result about product measure, beyond the fact that it exists, is that one can use it to evaluate iterated integrals, and to interchange their order, provided that the integrand is either unsigned or absolutely integrable. This fact is known as the Fubini-Tonelli theorem, and is an absolutely indispensable tool for computing integrals, and for deducing higher-dimensional results from lower-dimensional ones. In this section we will however omit a very important way to construct measures, namely the Riesz representation theorem

### 1.6.2 Outer measures:

Lebesgue outer measure $m^*$ is an outer measure. On the other hand, Jordan outer measure $m_{*,(J)}$ is only finitely subadditive rather than countably subadditive and thus is not, strictly speaking, an outer measure, for this reason this concept is often referred to as Jordan outer content rather than Jordan outer measure. Outer measures are weaker than measures in that they are merely countably subadditive, rather than countably additive. On the other hand, they are able to measure all subsets of $X$, whereas measures can only measure a $\sigma$ - algebra of measurable sets [46].

## 1.7 Objectives and Scope of research work:

The Problem discussed in the thesis is finding various measures of fractal sequences. Here the fractal sequences considered are Farey sequences and Cantor sets. The Cantor ternary set is a good example of an elementary fractal set [51]. The extraction of Cantor sets from Farey sequence of points is done elaborately in [1]. So Farey sequence is slightly modified into Farey sets and various measures are calculated for both the sets.

## 1.8 Organization of the thesis:

The thesis consists of six chapters.

**CHAPTER I**

This chapter provides the historical background and necessary literature survey for Farey sequences and Cantor set of odd order. Also measures like Lebesgue measure and Probability measure have a brief introduction.

**CHAPTER II**

This chapter has four sections. In section 2.1 Cantor Hexnary Sets, in section 2.2 Cantor Deca Sets, in section 2.3 Cantor Octanary Sets and in section 2.4 Cantor Dodeca Sets are developed. Various patterns of removal of intervals are analyzed here.

**CHAPTER III**

A subsequence of Farey sequence, $\widetilde{F_N}$ ,Farey $N$ – subsequence has been established as a topological space and a Hausdorff space by appropriately defining basis and open sets. Also the $T_1$axiom has been discussed with an illustration. Here sequences are considered as sets.

This chapter provides two sections. In section 3.1 Farey sequence has been proved as a Topological Space and in section 3.2 Farey sequence with a slight change in the basis has been developed as a Hausdorff space .

**CHAPTER IV**

A Non Reduced Farey $N$-subsequence, a subsequence of Farey sequence consists of rational numbers with same denominator in $[0, 1]$. By reconstructing the non reduced Farey

$N$ – subsequence it can be established as a $\sigma$- algebra and its Lebegue Measure has been found. Non –Reduced Farey $N$ - subsequence of even order has been studied.

**CHAPTER V**

This chapter is divided into two sections.Section 5.1 gives Probability Measure of Generalized Non - Reduced Farey $N$- Subsequence and the other section 5.2 is on Invariant Measure of Generalized Non - Reduced Farey $N$- Subsequence. An attempt has been to provide a theorem on the Probability measure of the Generalized Non reduced Farey $N$ – subsequence.

**CHAPTER VI**

This chapter deals with the Box Measure of Modified even ordered Cantor sets.While calculating box measure , boxes without dimensions are used in general but here, boxes are replaced by isosceles triangles and their areas are considered as measures. This chapter is divided into four sections.

Section 6.1 deals with Measure of Cantor Hexnary Sets and Section 6.2 is on Measure of Cantor Deca Sets. Varying from the previous sections, section 6.3 provides a measure for Cantor Octanary Sets and following the similar lines Section 6.4 gives a measure of Cantor Dodeca Sets.

# CHAPTER – II

# MODIFIED EVEN ORDERED CANTOR SETS

In this chapter throughout we study the modified Cantor even ordered sets. In Cantor ternary sets middle third is removed and the pattern of removal continues indefinitely. Taking the number of divisions as order here, even ordered Cantor sets are considered. Unlike normal Cantor sets, here lengths of unequal intervals are removed. In this pattern of removal middle interval in successive iteration follows a geometric sequence of powers of two. The intervals equally spaced from the middle to the left and right follows different nature as the iteration increases. Its characteristics are studied. Also, the diagrammatic representation of modified even ordered Cantor sets has been exhibited.

Unlike Cantor ternary sets, in even ordered sets the intervals of lengths one, two, four, eight etc., are removed successively. Again, if intervals of lengths two are taken away the formulas for retaining terms are given. In this chapter Cantor sets of even numbers are considered. Contrary to the procedure followed by Cantor, intervals of various lengths are removed in different patterns. These various patterns of removals of intervals are also analyzed here.

This chapter deals with four sections. In section 2.1 Cantor Hexnary Sets, in section 2.2 Cantor Deca Sets, in section 2.3 Cantor Octanary Sets and in section 2.4 Cantor Dodeca Sets are analyzed.

## 2.1 Cantor Hexnary Sets:

**Definition 2.1.1: Cantor Hexnary Set**

Divide the closed interval $[0,1]$ into six equal intervals. Remove the second and fifth intervals $\left(\frac{1}{6},\frac{2}{6}\right)$ and $\left(\frac{4}{6},\frac{5}{6}\right)$**.** The intervals retained are $\left[\frac{0}{6}=0,\frac{1}{6}\right],\left[\frac{2}{6},\frac{4}{6}\right]$ and $\left[\frac{5}{6},\frac{6}{6}=1\right]$. Now for these intervals continue the procedure indefinitely. The set obtained is known as **Cantor Hexnary Set** and is denoted by $C_{\left(\frac{1}{6}\right)}$.

**Theorem 2.1.1:**

In the Cantor Hexnary set the middle most interval is given by $\left[\frac{k}{6^n},\frac{k+2^n}{6^n}\right]$, $n = 1,2,3,\ldots$ where k is represented by the series $2\sum_{i=1}^{n}6^{n-1}\,2^{i-1}$ .

**Proof:**

Proof follows by induction.

The closed interval $[0,1]$ is divided into six equal parts. Following the theory of Cantor Hexnary set, the open intervals $\left(\frac{1}{6},\frac{2}{6}\right)$ and $\left(\frac{4}{6},\frac{5}{6}\right)$ are removed. The remaining parts $\left[\frac{0}{6}=0,\frac{1}{6}\right],\left[\frac{2}{6},\frac{4}{6}\right]$ and $\left[\frac{5}{6},\frac{6}{6}=1\right]$ are again subdivided as follows. The length of the middlemost part is $2/6$. For the $2^{\text{nd}}$ iteration, the parts $\left[\frac{0}{6}=0,\frac{1}{6}\right]$ and $\left[\frac{5}{6},\frac{6}{6}\right]$ are each divided into six equal parts thereby giving six parts

$\left[\frac{0}{36}=0,\frac{1}{36}\right],\left[\frac{1}{36},\frac{2}{36}\right]\left[\frac{2}{36},\frac{3}{36}\right],\left[\frac{3}{36},\frac{4}{36}\right],\left[\frac{4}{36},\frac{5}{36}\right],\left[\frac{5}{36},\frac{6}{36}\right]$ and

$\left[\frac{30}{36},\frac{31}{36}\right],\left[\frac{31}{36},\frac{32}{36}\right],\left[\frac{32}{36},\frac{33}{36}\right],\left[\frac{33}{36},\frac{34}{36}\right],\left[\frac{34}{36},\frac{35}{36}\right]\ \left[\frac{35}{36},\frac{36}{36}\right]$ respectively.

The open intervals $\left(\frac{1}{36},\frac{2}{36}\right)$ and $\left(\frac{4}{36},\frac{5}{36}\right)$ are removed for the left of $\frac{1}{2}$ . The same procedure continues to the right of $\frac{1}{2}$. Applying the removal pattern for the middle part $\left[\frac{\mathbf{2}}{\mathbf{6}},\frac{\mathbf{4}}{\mathbf{6}}\right]$

again give rise to $\left[\frac{12}{36},\frac{14}{36}\right]$, $\left[\frac{14}{36},\frac{16}{36}\right]$,$\left[\mathbf{\frac{16}{36}},\mathbf{\frac{20}{36}}\right]$, $\left[\frac{20}{36},\frac{22}{36}\right]$, $\left[\frac{22}{36},\frac{24}{36}\right]$. The open intervals $\left(\frac{14}{36},\frac{16}{36}\right)$ and $\left(\frac{20}{36},\frac{22}{36}\right)$ are removed. The length of the middlemost part is 4/36. Continue the process indefinitely.

When $n = 1$,

$$k = 2.6^0.2^0 = 2$$

$$k + 2^1 = 2.6^0.2^0 + 2 = 4$$

Therefore the middlemost interval in the first iteration is $\left[\frac{2}{6},\frac{4}{6}\right]$

When $n = 2$,

$$k = 2.6^{2-1}.2^0 + 2.6^{2-2}.2^{2-1} = 16$$

$$k + 2^n = k + 2^2 = 16 + 4 = 20$$

Therefore the middlemost interval in the second iteration is $\left[\frac{16}{6^2},\frac{20}{6^2}\right]$

Assume for $n = r$

$$k = 2.6^{r-1}.2^0 + 2.6^{r-2}.2^{2-1} + 2.6^{r-3}.2^{3-1} + \cdots\cdots\cdots + 2.6^{r-r}.2^{r-1}$$

$$k + 2^r = (2.6^{r-1}.2^0 + 2.6^{r-2}.2^{2-1} + 2.6^{r-3}.2^{3-1} + \cdots\cdots\cdots + 2.6^{r-r}.2^{r-1}) + 2^r$$

Therefore the middlemost parts are $\left[\frac{k}{6^r},\frac{k+2^r}{6^r}\right]$

We prove the result for $n = r + 1$

The above interval can be divided into 6 equal parts as

$$\left[\frac{6k}{6^{r+1}},\frac{6k+2^r}{6^{r+1}}\right],\left[\frac{6k+2^r}{6^{r+1}},\frac{6k+2.2^r}{6^{r+1}}\right],\left[\frac{6k+2.2^r}{6^{r+1}},\frac{6k+3.2^r}{6^{r+1}}\right],\left[\frac{6k+3.2^r}{6^{r+1}},\frac{6k+4.2^r}{6^{r+1}}\right],\left[\frac{6k+4.2^r}{6^{r+1}},\frac{6k+5.2^r}{6^{r+1}}\right],\left[\frac{6k+5.2^r}{6^{r+1}},\frac{6k+6.2^r}{6^{r+1}}\right]$$

The middlemost part is $\left[\frac{6k+2.2^r}{6^{r+1}},\frac{6k+4.2^r}{6^{r+1}}\right]$.

$$6k + 2.2^r = 6\,[2.6^{r-1}.2^0 + 2.6^{r-2}.2^{2-1} + 2.6^{r-3}.2^{3-1} + \cdots\cdots\cdots + 2.6^{r-r}.2^{r-1}] + 2.2^r$$

$$= 2.6^r.2^0 + 2.6^{r-1}.2^1 + 2.6^{r-2}.2^2 + \cdots\cdots\cdots + 2.6^1.2^{r-1} + 2.6^0.2^r$$

Let $6k + 2.2^r = l$ (say)

$$6k + 4.2^r = (6k + 2.2^r) + 2.2^r$$

$$= l + 2.2^r = l + 2^{r+1}$$

$\left[\frac{l}{6^{r+1}}, \frac{l+2^{r+1}}{6^{r+1}}\right]$ result is true for $n = r + 1$ and hence for all positive integers .

Hence the theorem is proved by induction method.

The Following **Figure 2.1.1** shows the graphical representation of Cantor Hexnary sets.

**First iteration:**

The closed interval $[0,1]$ is subdivided into 6 equal sub- intervals

$$\left[\frac{0}{6} = 0, \frac{1}{6}\right], \left[\frac{1}{6}, \frac{2}{6}\right], \left[\frac{2}{6}, \frac{3}{6}\right], \left[\frac{3}{6}, \frac{4}{6}\right], \left[\frac{4}{6}, \frac{5}{6}\right], \left[\frac{5}{6}, \frac{6}{6} = 1\right]$$

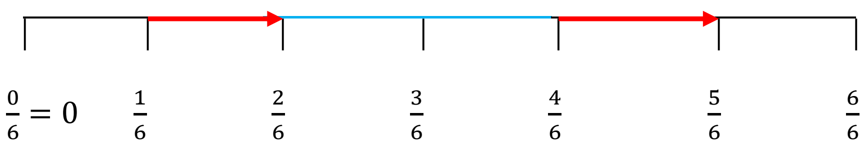



**Figure 2.1.1:** Cantor Hexnary sets

The intervals removed are $\left(\frac{1}{6}, \frac{2}{6}\right), \left(\frac{4}{6}, \frac{5}{6}\right)$ .

The remaining intervals are $\left[\frac{0}{6} = 0, \frac{1}{6}\right], \left[\frac{2}{6}, \frac{4}{6}\right], \left[\frac{5}{6}, \frac{6}{6} = 1\right]$

Therefore $$C_{\left(\frac{1}{6}\right)^1} = \left[\frac{0}{6} = 0, \frac{1}{6}\right] \cup \left[\frac{2}{6}, \frac{4}{6}\right] \cup \left[\frac{5}{6}, \frac{6}{6} = 1\right] \qquad (2.1.1)$$

**Second iteration:**

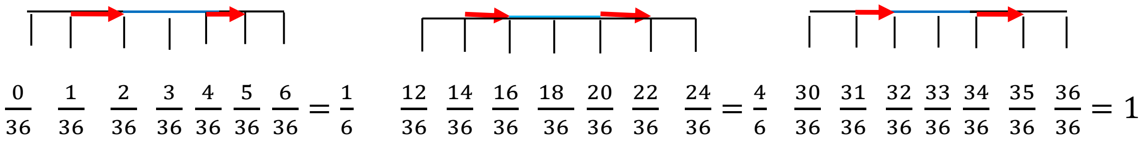



**Figure 2.1.2:** Second iteration – Cantor Hexnary sets

The intervals removed are $\left(\frac{1}{36},\frac{2}{36}\right),\left(\frac{4}{36},\frac{5}{36}\right),\left(\frac{14}{36},\frac{16}{36}\right),\left(\frac{20}{36},\frac{22}{36}\right),\left(\frac{31}{36},\frac{32}{36}\right),\left(\frac{34}{36},\frac{35}{36}\right)$ from each of the subintervals will result in modified cantor set.

Therefore

$$C_{\left(\frac{1}{6}\right)^2} = \left[\frac{0}{36} = 0,\frac{1}{36}\right] \cup \left[\frac{2}{36},\frac{4}{36}\right] \cup \left[\frac{5}{36},\frac{6}{36}\right] \cup \left[\frac{12}{36},\frac{14}{36}\right] \cup \left[\frac{16}{36},\frac{20}{36}\right] \cup \left[\frac{22}{36},\frac{24}{36}\right] \cup \left[\frac{30}{36},\frac{31}{36}\right] \cup \left[\frac{32}{36},\frac{34}{36}\right] \cup \left[\frac{35}{36},\frac{36}{36} = 1\right] \quad (2.1.2)$$

**Third Iteration:**

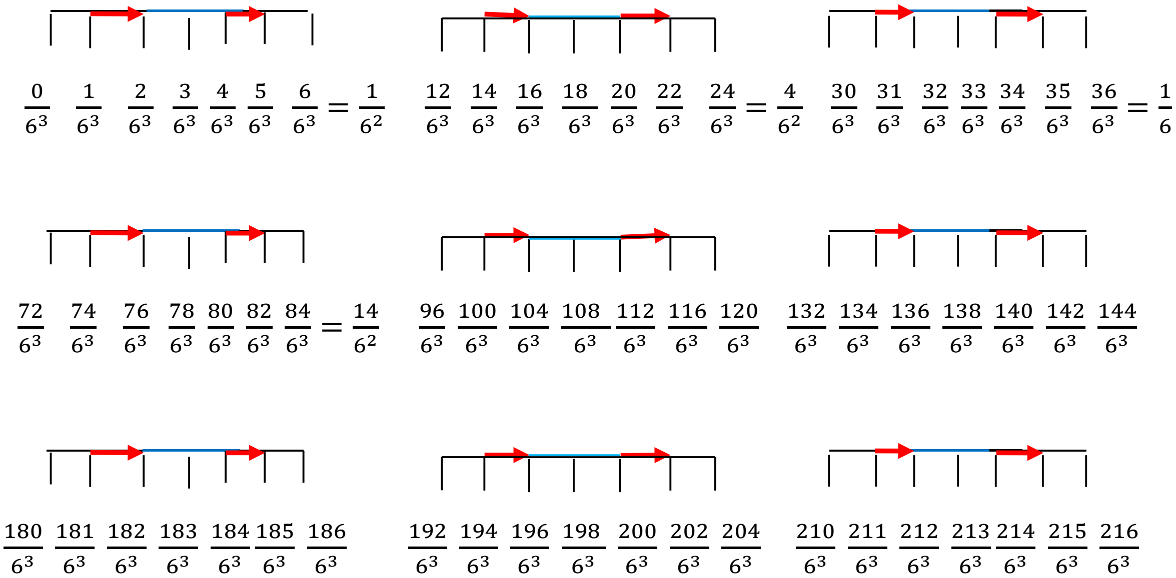



**Figure 2.1.3:** Third iteration – Cantor Hexnary sets

The intervals removed are

$\left(\frac{1}{216},\frac{2}{216}\right),\left(\frac{4}{216},\frac{5}{216}\right),\left(\frac{14}{216},\frac{16}{216}\right),\left(\frac{20}{216},\frac{22}{216}\right),\left(\frac{31}{216},\frac{32}{216}\right),\left(\frac{34}{216},\frac{35}{216}\right),\left(\frac{74}{216},\frac{76}{216}\right),\left(\frac{80}{216},\frac{82}{216}\right),\left(\frac{100}{216},\frac{104}{216}\right),$

$\left(\frac{112}{216},\frac{116}{216}\right),\left(\frac{134}{216},\frac{136}{216}\right),\left(\frac{140}{216},\frac{142}{216}\right),\left(\frac{181}{216},\frac{182}{216}\right),\left(\frac{184}{216},\frac{185}{216}\right),\left(\frac{194}{216},\frac{196}{216}\right),\left(\frac{200}{216},\frac{204}{216}\right),\left(\frac{211}{216},\frac{212}{216}\right),\left(\frac{214}{216},\frac{215}{216}\right)$

from each of the subintervals will result in modified cantor set.

Therefore

$$C_{\left(\frac{1}{6}\right)^3} = \left[\frac{0}{216}=0,\frac{1}{216}\right] \cup \left[\frac{2}{216},\frac{4}{216}\right] \cup \left[\frac{5}{216},\frac{6}{216}\right] \cup \left[\frac{12}{216},\frac{14}{216}\right] \cup \left[\frac{16}{216},\frac{20}{216}\right] \cup \left[\frac{22}{216},\frac{24}{216}\right] \cup \left[\frac{30}{216},\frac{31}{216}\right] \cup$$
$$\left[\frac{32}{216},\frac{34}{216}\right] \cup \left[\frac{35}{216},\frac{36}{216}\right] \cup \left[\frac{72}{216},\frac{74}{216}\right] \cup \left[\frac{76}{216},\frac{80}{216}\right] \cup \left[\frac{82}{216},\frac{84}{216}\right] \cup \left[\frac{96}{216},\frac{100}{216}\right] \cup \left[\frac{104}{216},\frac{112}{216}\right] \cup$$
$$\left[\frac{116}{216},\frac{120}{216}\right] \cup \left[\frac{132}{216},\frac{134}{216}\right] \cup \left[\frac{136}{216},\frac{140}{216}\right] \cup \left[\frac{142}{216},\frac{144}{216}\right] \cup \left[\frac{180}{216},\frac{181}{216}\right] \cup \left[\frac{182}{216},\frac{184}{216}\right] \cup \left[\frac{185}{216},\frac{186}{216}\right] \cup$$
$$\left[\frac{192}{216},\frac{194}{216}\right] \cup \left[\frac{196}{216},\frac{200}{216}\right] \cup \left[\frac{202}{216},\frac{204}{216}\right] \cup \left[\frac{210}{216},\frac{211}{216}\right] \cup \left[\frac{212}{216},\frac{214}{216}\right] \cup \left[\frac{215}{216},\frac{216}{216}\right] \qquad (2.1.3)$$

This procedure proceeds in every iteration to get the entire Cantor Hexnary set.

The iteration procedure for the middlemost part and the general procedure are shown in the following tree diagrams Figure 2.1.4 and Figure 2.1.5.

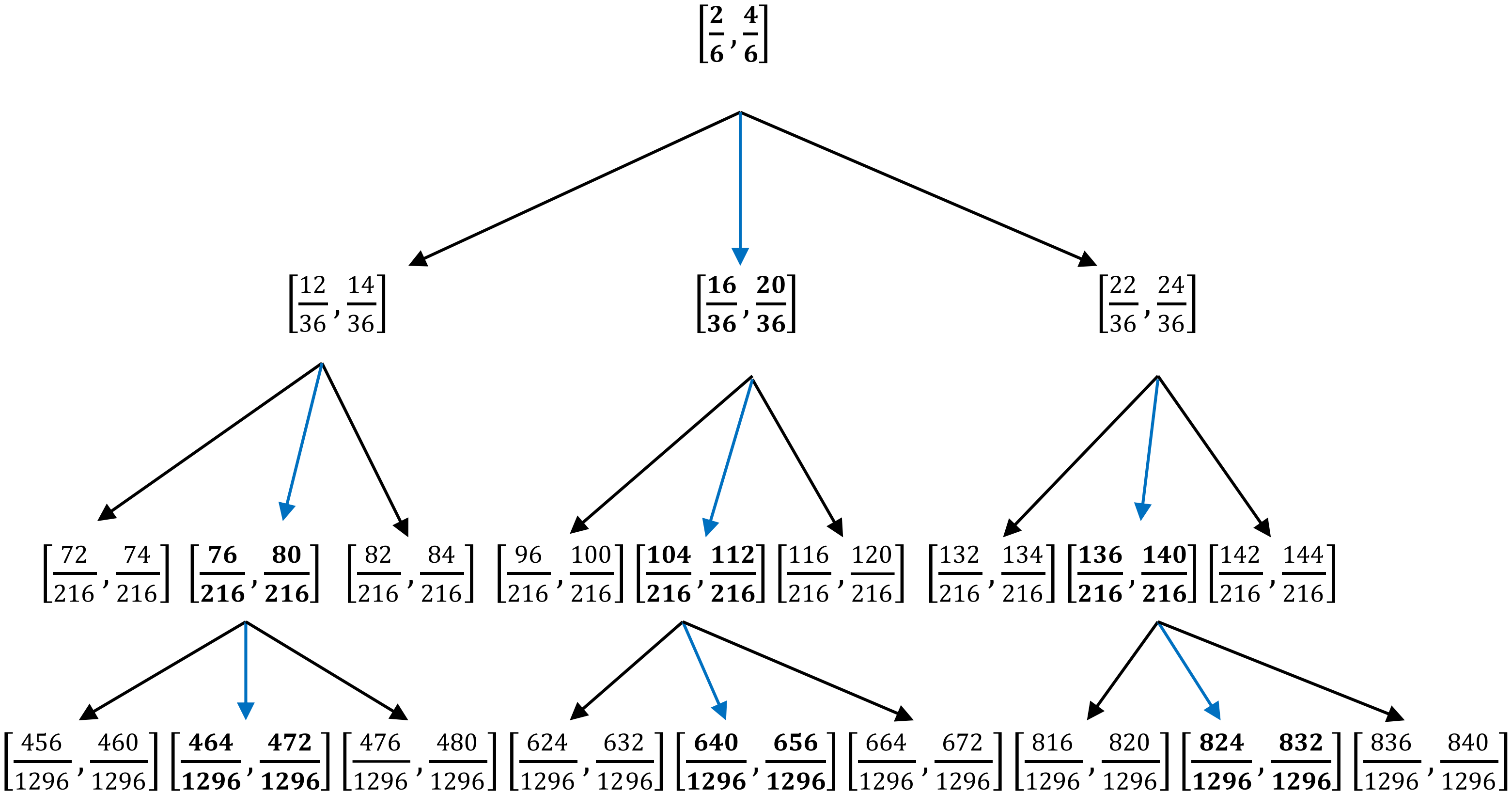



**Figure 2.1.4:** Middlemost part of Cantor Hexnary sets

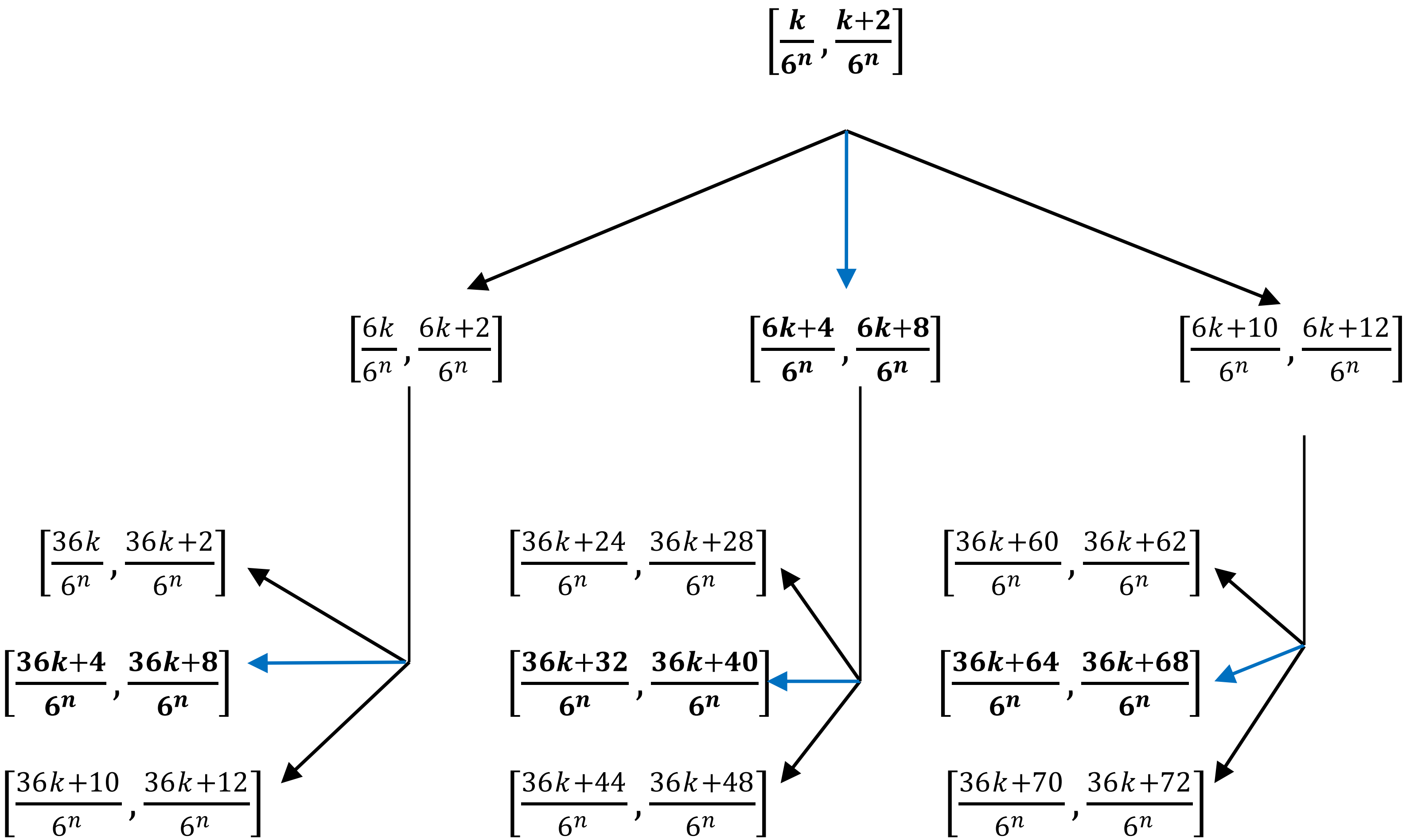



**Figure 2.1.5:** General form of middlemost part of Cantor Hexnary sets

## 2.2 Cantor Deca Sets:

**Definition 2.2.1: Cantor Deca Set**

Divide the closed interval [0,1] into ten equal intervals. Remove the second, fourth, seventh and ninth intervals $\left(\frac{1}{10}, \frac{2}{10}\right)$, $\left(\frac{3}{10}, \frac{4}{10}\right)$, $\left(\frac{6}{10}, \frac{7}{10}\right)$ and $\left(\frac{8}{10}, \frac{9}{10}\right)$. The intervals retained are $\left[\frac{0}{10}=0, \frac{1}{10}\right], \left[\frac{2}{10}, \frac{3}{10}\right], \left[\mathbf{\frac{4}{10}}, \mathbf{\frac{6}{10}}\right], \left[\frac{7}{10}, \frac{8}{10}\right]$ and $\left[\frac{9}{10}, \frac{10}{10}=1\right]$. Now for these intervals continue the procedure indefinitely. The set obtained is known as **Cantor Deca Set** and is denoted by $C_{\left(\frac{1}{10}\right)}$.

The Cantor Hexnary set pattern followed in Cantor Deca sets.

**First iteration:**

The closed interval [0,1] is subdivided into 10 equal sub- intervals

$$\left[\frac{0}{10}=0, \frac{1}{10}\right], \left[\frac{1}{10}, \frac{2}{10}\right], \left[\frac{2}{10}, \frac{3}{10}\right], \left[\frac{3}{10}, \frac{4}{10}\right], \left[\frac{4}{10}, \frac{5}{10}\right], \left[\frac{5}{10}, \frac{6}{10}\right], \left[\frac{7}{10}, \frac{8}{10}\right], \left[\frac{9}{10}, \frac{10}{10}\right]$$

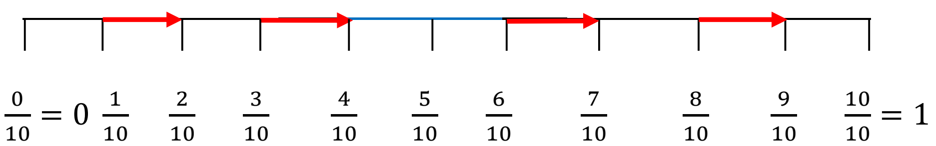



**Figure 2.2.1:** Cantor Deca sets

The intervals removed are $\left(\frac{1}{10}, \frac{2}{10}\right), \left(\frac{3}{10}, \frac{4}{10}\right), \left(\frac{6}{10}, \frac{7}{10}\right), \left(\frac{8}{10}, \frac{9}{10}\right)$ .

The remaining intervals are $\left[\frac{0}{10}=0, \frac{1}{10}\right], \left[\frac{2}{10}, \frac{3}{10}\right], \left[\mathbf{\frac{4}{10}}, \mathbf{\frac{6}{10}}\right], \left[\frac{7}{10}, \frac{8}{10}\right], \left[\frac{9}{10}, \frac{10}{10}=1\right]$

Therefore $C_{\left(\frac{1}{10}\right)^1} = \left[\frac{0}{10}=0, \frac{1}{10}\right] \cup \left[\frac{2}{10}, \frac{3}{10}\right] \cup \left[\mathbf{\frac{4}{10}}, \mathbf{\frac{6}{10}}\right] \cup \left[\frac{7}{10}, \frac{8}{10}\right] \cup \left[\frac{9}{10}, \frac{10}{10}=1\right]$ (2.2.1)

**Second iteration:**

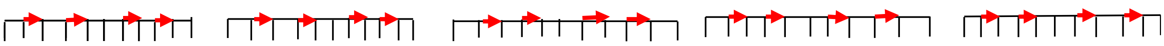


**Figure 2.2.2:** Second iteration- Cantor Deca sets

The intervals removed are

$$\left(\frac{1}{100},\frac{2}{100}\right),\left(\frac{3}{100},\frac{4}{100}\right),\left(\frac{6}{100},\frac{7}{100}\right),\left(\frac{8}{100},\frac{9}{100}\right),\left(\frac{21}{100},\frac{22}{100}\right),\left(\frac{23}{100},\frac{24}{100}\right),$$

$$\left(\frac{26}{100},\frac{27}{100}\right),\left(\frac{28}{100},\frac{29}{100}\right),\left(\frac{42}{100},\ \frac{44}{100}\right),\left(\frac{46}{100},\ \frac{48}{100}\right),\left(\frac{52}{100},\ \frac{54}{100}\right),\left(\frac{56}{100},\ \frac{58}{100}\right),\ \left(\frac{71}{100},\ \frac{72}{100}\right),$$

$$\left(\frac{73}{100},\ \frac{74}{100}\right),\ \left(\frac{76}{100},\frac{77}{100}\right),\left(\frac{78}{100},\ \frac{79}{100}\right),\left(\frac{91}{100},\ \frac{92}{100}\right),\left(\frac{93}{100},\ \frac{94}{100}\right),\left(\frac{96}{100},\ \frac{97}{100}\right),\ \left(\frac{98}{100},\ \frac{99}{100}\right)\ .$$

from each of the subintervals will result in modified cantor set.

Therefore

$$\begin{aligned} C_{\left(\frac{1}{10}\right)^2} = {} & \left[\frac{0}{100},\frac{1}{100}\right] \cup \left[\frac{2}{100},\frac{3}{100}\right] \cup \left[\frac{4}{100},\frac{6}{100}\right] \cup \left[\frac{7}{100},\frac{8}{100}\right] \cup \left[\frac{9}{100},\frac{10}{100}\right] \cup \left[\frac{20}{100},\frac{21}{100}\right] \cup \left[\frac{22}{100},\frac{23}{100}\right] \cup \\ & \left[\frac{24}{100},\frac{26}{100}\right] \cup \left[\frac{27}{100},\frac{28}{100}\right] \cup \left[\frac{29}{100},\frac{30}{100}\right] \cup \left[\frac{40}{100},\frac{42}{100}\right] \cup \left[\frac{44}{100},\frac{46}{100}\right] \cup \left[\frac{48}{100},\frac{52}{100}\right] \cup \left[\frac{54}{100},\frac{56}{100}\right] \cup \\ & \left[\frac{58}{100},\frac{60}{100}\right] \cup \left[\frac{70}{100},\frac{71}{100}\right] \cup \left[\frac{72}{100},\frac{73}{100}\right] \cup \left[\frac{74}{100},\frac{76}{100}\right] \cup \left[\frac{77}{100},\frac{78}{100}\right] \cup \left[\frac{79}{100},\frac{80}{100}\right] \cup \left[\frac{90}{100},\frac{91}{100}\right] \cup \\ & \left[\frac{92}{100},\frac{93}{100}\right] \cup \left[\frac{94}{100},\frac{96}{100}\right] \cup \left[\frac{97}{100},\frac{98}{100}\right] \cup \left[\frac{99}{100},\frac{100}{100}\right] \end{aligned} \tag{2.2.2}$$

This procedure proceeds in every iteration to get the entire Cantor Deca set.

## 2.3 Cantor Octanary Sets:

**Definition 2.3.1: Cantor Octanary Set**

The closed interval $[0,1]$ is divided into eight equal parts. By removing the second part, last but one part and middlemost part, the open intervals $\left(\frac{1}{8},\frac{2}{8}\right)$, $\left(\frac{3}{8},\frac{5}{8}\right)$ and $\left(\frac{6}{8},\frac{7}{8}\right)$ are removed. The middlemost removable interval is of length $\left(\frac{2}{8}\right)$. Other retained intervals are of length $\left(\frac{1}{8}\right)$. Continue the process indefinitely and the set obtained is known as the **Cantor Octanary Set** and is denoted by $C_{\left(\frac{1}{8}\right)}$.

**Graphical Representation:**

The Following **Figure 2.3.1** shows the graphical representation of Cantor Octanary sets.

**First iteration:**

The closed interval $[0,1]$ is subdivided into 8 equal sub- intervals

$$\left[\frac{0}{8}=0,\frac{1}{8}\right],\left[\frac{1}{8},\frac{2}{8}\right],\left[\frac{2}{8},\frac{3}{8}\right],\left[\frac{3}{8},\frac{4}{8}\right],\left[\frac{4}{8},\frac{5}{8}\right],\left[\frac{5}{8},\frac{6}{8}\right],\left[\frac{6}{8},\frac{7}{8}\right],\left[\frac{7}{8},\frac{8}{8}=1\right]$$

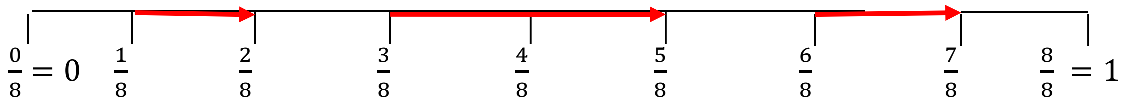



**Figure 2.3.1:** Cantor Octanary sets

The intervals removed are $\left(\frac{1}{8},\frac{2}{8}\right),\left(\frac{3}{8},\frac{5}{8}\right),\left(\frac{6}{8},\frac{7}{8}\right)$.

The remaining intervals are $\left[\frac{0}{8}=0,\frac{1}{8}\right],\left[\frac{2}{8},\frac{3}{8}\right],\left[\frac{5}{8},\frac{6}{8}\right]$ and $\left[\frac{7}{8},\frac{8}{8}=1\right]$

Therefore

$$C_{\left(\frac{1}{8}\right)^1} = \left[\frac{0}{8}=0,\frac{1}{8}\right]\cup\left[\frac{2}{8},\frac{3}{8}\right]\cup\left[\frac{5}{8},\frac{6}{8}\right]\cup\left[\frac{7}{8},\frac{8}{8}=1\right] \qquad (2.3.1)$$

**Second iteration:**

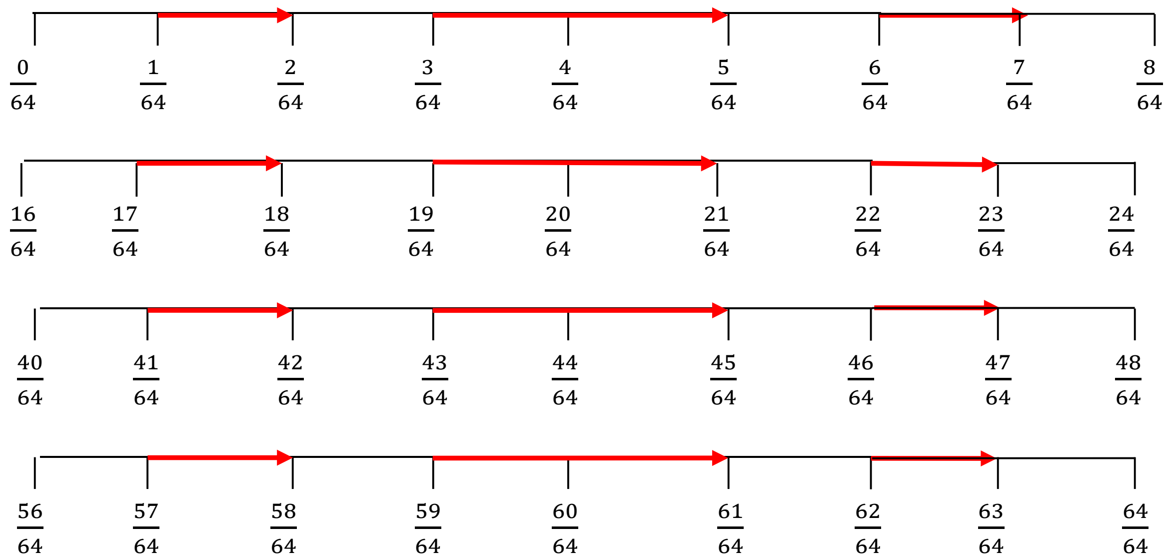



**Figure 2.3.2:** Second iteration – Cantor Octanary sets

The intervals removed are $\left(\frac{1}{64},\frac{2}{64}\right)$, $\left(\frac{3}{64},\frac{5}{64}\right)$, $\left(\frac{6}{64},\frac{7}{64}\right)$, $\left(\frac{17}{64},\frac{18}{64}\right)$, $\left(\frac{19}{64},\frac{21}{64}\right)$, $\left(\frac{22}{64},\frac{23}{64}\right)$, $\left(\frac{41}{64},\frac{42}{64}\right)$, $\left(\frac{46}{64},\frac{47}{64}\right)$, $\left(\frac{57}{64},\frac{58}{64}\right)$, $\left(\frac{59}{64},\frac{61}{64}\right)$, $\left(\frac{62}{64},\frac{63}{64}\right)$

For each of the remaining subintervals apply the same above procedure and the resultant set is will result in Cantor Octanary set.

Therefore

$$C_{\left(\frac{1}{8}\right)^2} = \left[\frac{0}{64},\frac{1}{64}\right] \cup \left[\frac{2}{64},\frac{3}{64}\right] \cup \left[\frac{5}{64},\frac{6}{64}\right] \cup \left[\frac{7}{64},\frac{8}{64}\right] \cup \left[\frac{16}{64},\frac{17}{64}\right] \cup \left[\frac{18}{64},\frac{19}{64}\right] \cup \left[\frac{21}{64},\frac{22}{64}\right] \cup \left[\frac{23}{64},\frac{24}{64}\right] \cup \left[\frac{40}{64},\frac{41}{64}\right] \cup \left[\frac{42}{64},\frac{43}{64}\right] \cup \left[\frac{45}{64},\frac{46}{64}\right] \cup \left[\frac{47}{64},\frac{48}{64}\right] \qquad (2.3.2)$$

This procedure proceeds in every iteration to get the entire Cantor Octanary set

**Theorem 2.3.1:**

In Cantor Octanary set each retained interval is represented as $\left[\frac{k}{8^n},\frac{k+1}{8^n}\right]$, $n = 1,2,3,\ldots$ where $k = 8^{n-1}\alpha + 8^{n-2}\gamma + \cdots\cdots\cdots\cdots + 8\gamma + \gamma$ ; $n = 1, 2, 3, \ldots$; $\alpha,\gamma = 0,2,5,7$

**Proof:**

The proof of the theorem follows by induction

The closed interval $[0,1]$ is divided into eight equal parts.

The open intervals $\left(\frac{1}{8},\frac{2}{8}\right)$, $\left(\frac{3}{8},\frac{5}{8}\right)$ and $\left(\frac{6}{8},\frac{7}{8}\right)$ are removed.

The remaining parts $\left[\frac{0}{8}=0,\frac{1}{8}\right]$, $\left[\frac{2}{8},\frac{3}{8}\right]$, $\left[\frac{5}{8},\frac{6}{8}\right]$ and $\left[\frac{7}{8},\frac{8}{8}=1\right]$ are again subdivided as follows for the 2nd iteration. The part $\left[\frac{0}{8}=0,\frac{1}{8}\right]$ is subdivided into eight equal parts thereby giving eight parts $\left[\frac{0}{64}=0,\frac{1}{64}\right]$, $\left[\frac{1}{64},\frac{2}{64}\right]$, $\left[\frac{2}{64},\frac{3}{64}\right]$ $\left[\frac{3}{64},\frac{4}{64}\right]$, $\left[\frac{4}{64},\frac{5}{64}\right]$, $\left[\frac{5}{64},\frac{6}{64}\right]$, $\left[\frac{6}{64},\frac{7}{64}\right]$ $\left[\frac{7}{64},\frac{8}{64}\right]$.

The open intervals $\left(\frac{1}{64},\frac{2}{64}\right)$, $\left(\frac{3}{64},\frac{5}{64}\right)$ and $\left(\frac{6}{64},\frac{7}{64}\right)$ are removed.

Now the part $\left[\frac{2}{8},\frac{3}{8}\right]$ is subdivided into eight equal parts thereby giving eight parts $\left[\mathbf{\frac{16}{64}},\mathbf{\frac{17}{64}}\right]$, $\left[\frac{17}{64},\frac{18}{64}\right]$, $\left[\mathbf{\frac{18}{64}},\mathbf{\frac{19}{64}}\right]$, $\left[\frac{19}{64},\frac{20}{64}\right]$, $\left[\mathbf{\frac{20}{64}},\mathbf{\frac{21}{64}}\right]$, $\left[\frac{21}{64},\frac{22}{64}\right]$, $\left[\frac{22}{64},\frac{23}{64}\right]$, $\left[\mathbf{\frac{23}{64}},\mathbf{\frac{24}{64}}\right]$ and the open intervals $\left(\frac{17}{64},\frac{18}{64}\right)$, $\left(\frac{19}{64},\frac{21}{64}\right)$ and $\left(\frac{22}{64},\frac{23}{64}\right)$ are removed.

Next $\left[\frac{5}{8},\frac{6}{8}\right]$ is divided into eight equal parts namely $\left[\mathbf{\frac{40}{64}},\mathbf{\frac{41}{64}}\right]$, $\left[\frac{41}{64},\frac{42}{64}\right]$, $\left[\mathbf{\frac{42}{64}},\mathbf{\frac{43}{64}}\right]$, $\left[\frac{43}{64},\frac{44}{64}\right]$, $\left[\frac{44}{64},\frac{45}{64}\right]$, $\left[\mathbf{\frac{45}{64}},\mathbf{\frac{46}{64}}\right]$, $\left[\frac{46}{64},\frac{47}{64}\right]$, $\left[\mathbf{\frac{47}{64}},\mathbf{\frac{48}{64}}\right]$ and the open intervals $\left(\frac{41}{64},\frac{42}{64}\right)$, $\left(\frac{43}{64},\frac{45}{64}\right)$, $\left(\frac{46}{64},\frac{47}{64}\right)$ are removed.

Last part $\left[\frac{7}{8},\frac{8}{8}=1\right]$ is again subdivided into eight equal parts namely $\left[\frac{56}{64},\frac{57}{64}\right]$, $\left[\frac{57}{64},\frac{58}{64}\right]$, $\left[\frac{58}{64},\frac{59}{64}\right]$, $\left[\frac{59}{64},\frac{60}{64}\right]$, $\left[\frac{60}{64},\frac{61}{64}\right]$, $\left[\frac{61}{64},\frac{62}{64}\right]$, $\left[\frac{62}{64},\frac{63}{64}\right]$, $\left[\frac{63}{64},\frac{64}{64}\right]$, the open intervals $\left(\frac{57}{64},\frac{58}{64}\right)$, $\left(\frac{59}{64},\frac{61}{64}\right)$ and $\left(\frac{62}{64},\frac{63}{64}\right)$ are removed.

Here it is noted that in the successive iterations the lengths of each of the retained intervals follow a Geometric progression $\frac{1}{8^1},\frac{1}{8^2},\frac{1}{8^3},\frac{1}{8^4},\cdots\cdots$.

When $n=1$; $\quad k=\alpha+\gamma$

Therefore the result is true for $n=1$

When $n=2$; $\quad k=8\alpha+\gamma$

Therefore the result is true for $n=2$

Assume the result for $n=p$;

$$k=\ 8^{p-1}\alpha+8^{p-2}\gamma+\cdots\cdots\cdots\cdots\cdots+8\gamma+\gamma$$

We prove the result for $n=p+1$

The retained intervals are of the form $\left[\frac{k}{8^n},\frac{k+1}{8^n}\right]$ where $k$ is given by

$8^{p-1}\alpha+8^{p-2}\gamma+\cdots\cdots\cdots\cdots\cdots+8\gamma+\gamma$ ; $\alpha,\gamma=0,2,5,7$

After subdivision $k$ is multiplied by 8 and $\gamma=0,2,5,7$ is added and the resulting intervals are retained.

Here when $n=p+1$

if retained intervals of the form

$$\left[\frac{l}{8^n},\frac{l+1}{8^n}\right],$$

where $l=8k+\gamma=8(\,8^{p-1}\alpha+8^{p-2}\gamma+\cdots\cdots\cdots\cdots\cdots+8\gamma+\gamma)+\gamma$

$$=\ 8^{p}\alpha+8^{p-1}\gamma+\cdots\cdots\cdots\cdots\cdots+8^2\gamma+8\gamma+\gamma$$

Hence the theorem is proved by induction.

The retained part and its general procedure are shown in the following tree diagrams Figure 2.3.3 and Figure 2.3.4.

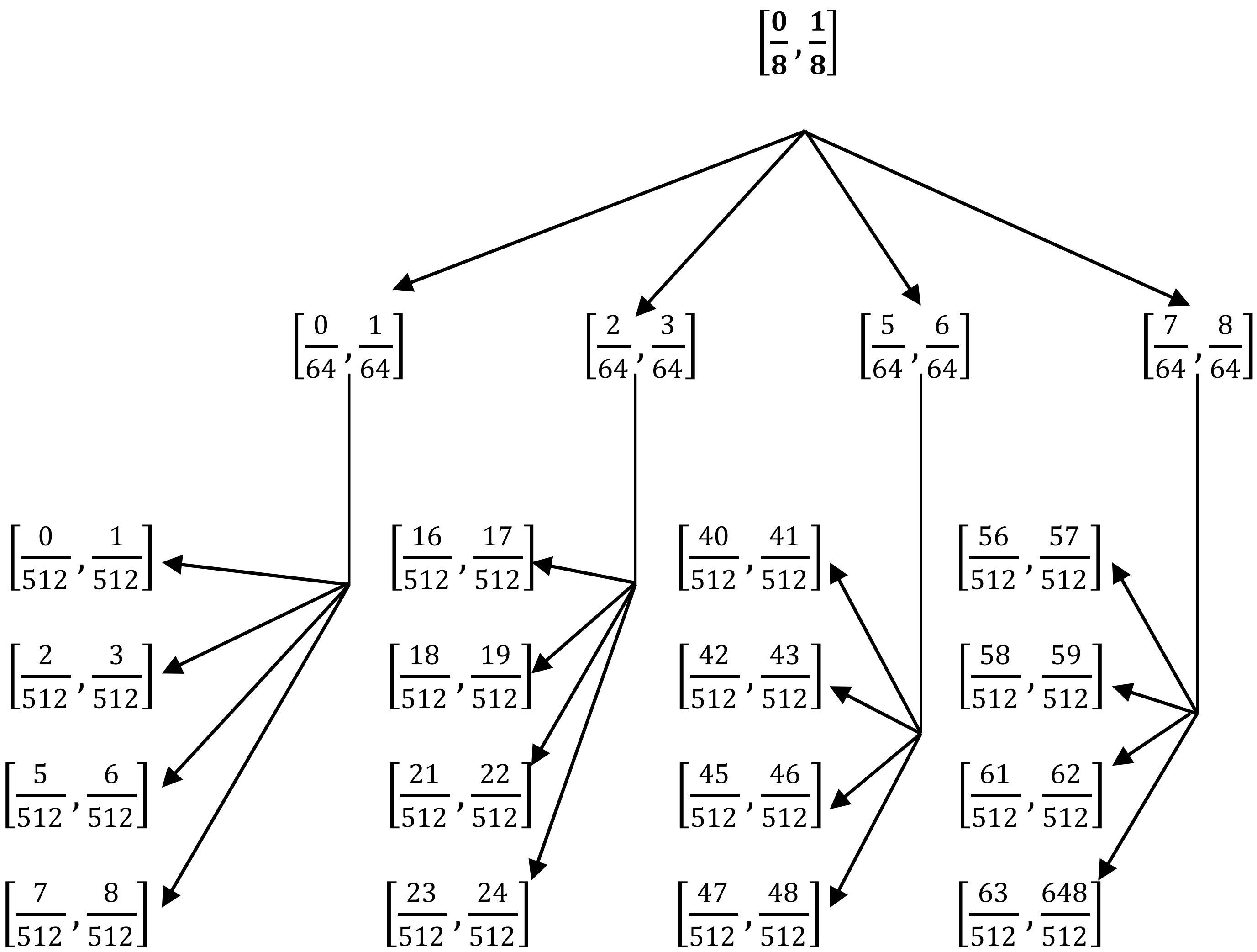



**Figure 2.3.3:** Retained part of Cantor Octanary sets.

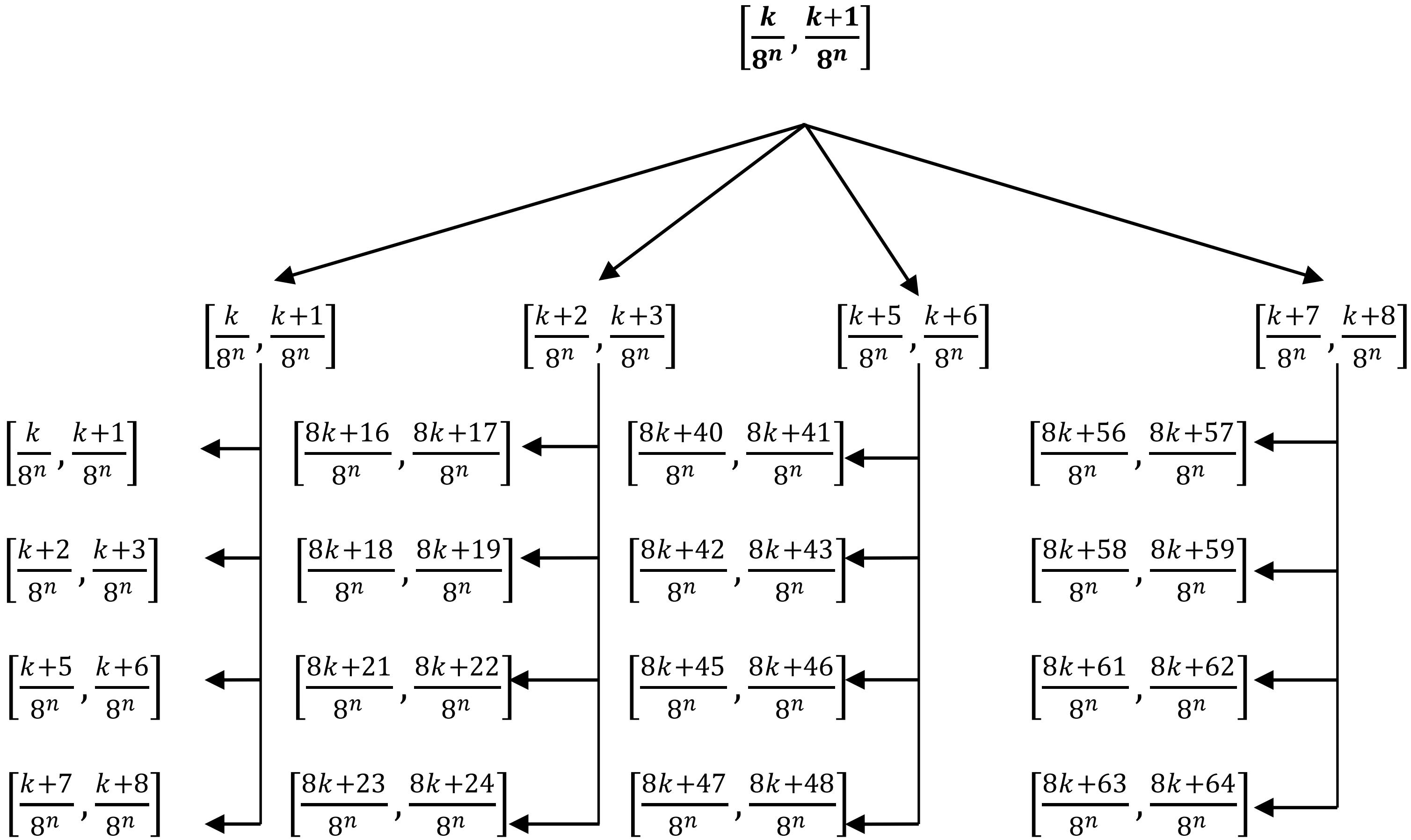



**Figure 2.3.4:** General form of Retained part of Cantor Octanary sets.

## 2.4 Cantor Dodeca Set:

**Definition 2.4.1: Cantor Dodeca Set**

The closed interval $[0, 1]$ is divided into twelve equal parts. By removing the second part, fourth part, ninth part, last but one part and middlemost part, the open intervals $\left(\frac{1}{12}, \frac{2}{12}\right)$, $\left(\frac{3}{12}, \frac{4}{12}\right)$, $\left(\frac{5}{12}, \frac{7}{12}\right)$, $\left(\frac{8}{12}, \frac{9}{12}\right)$ and $\left(\frac{10}{12}, \frac{11}{12}\right)$ are removed. The middlemost removable interval is of length $\left(\frac{2}{12}\right)$. Other retained intervals are of length $\left(\frac{1}{12}\right)$. Continue the process indefinitely and the set obtained is known as the **Cantor Dodeca Set** and is denoted by $C_{\left(\frac{1}{12}\right)}$.

The Cantor Octanary set pattern is followed by Cantor Dodeca sets.

**First iteration:**

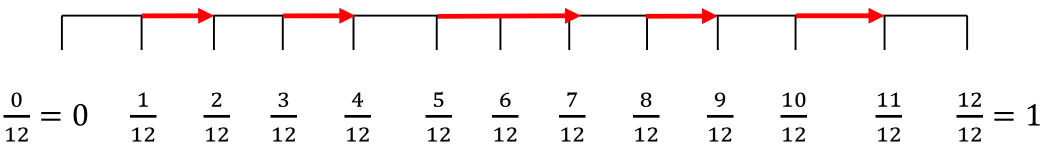



**Figure 2.4.1:** Cantor Dodeca sets

The intervals removed are $\left(\frac{1}{12}, \frac{2}{12}\right), \left(\frac{3}{12}, \frac{4}{12}\right), \left(\frac{5}{12}, \frac{7}{12}\right), \left(\frac{8}{12}, \frac{9}{12}\right), \left(\frac{10}{12}, \frac{11}{12}\right)$ .

The remaining intervals are $\left[\frac{0}{12} = 0, \frac{1}{12}\right], \left[\frac{2}{12}, \frac{3}{12}\right], \left[\frac{4}{12}, \frac{5}{12}\right], \left[\frac{7}{12}, \frac{8}{12}\right], \left[\frac{9}{12}, \frac{10}{12}\right], \left[\frac{11}{12}, \frac{12}{12} = 1\right]$

Therefore $C_{\left(\frac{1}{12}\right)^1} = \left[\frac{0}{12} = 0, \frac{1}{12}\right] \cup \left[\frac{2}{12}, \frac{3}{12}\right] \cup \left[\frac{4}{12}, \frac{5}{12}\right] \cup \left[\frac{7}{12}, \frac{8}{12}\right] \cup \left[\frac{9}{12}, \frac{10}{12}\right] \cup \left[\frac{11}{12}, \frac{12}{12} = 1\right]$ (2.4.1)

**Second iteration:**

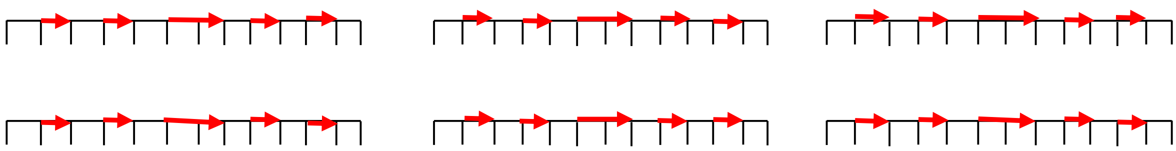


**Figure 2.4.2:** Second iteration - Cantor Dodeca sets

The intervals removed are $\left(\frac{1}{144}, \frac{2}{144}\right), \left(\frac{3}{144}, \frac{4}{144}\right), \left(\frac{5}{144}, \frac{7}{144}\right), \left(\frac{8}{144}, \frac{9}{144}\right), \left(\frac{10}{144}, \frac{11}{144}\right), \left(\frac{25}{144}, \frac{26}{144}\right),$

$\left(\frac{27}{144}, \frac{28}{144}\right), \left(\frac{29}{144}, \frac{31}{144}\right), \left(\frac{32}{144}, \frac{33}{144}\right), \left(\frac{34}{144}, \frac{35}{144}\right), \left(\frac{49}{144}, \frac{50}{144}\right), \left(\frac{51}{144}, \frac{52}{144}\right),$

$\left(\frac{53}{144}, \frac{55}{144}\right), \left(\frac{56}{144}, \frac{57}{144}\right), \left(\frac{58}{144}, \frac{59}{144}\right), \left(\frac{85}{144}, \frac{86}{144}\right), \left(\frac{87}{144}, \frac{88}{144}\right), \left(\frac{89}{144}, \frac{91}{144}\right),$

$\left(\frac{92}{144}, \frac{93}{144}\right), \left(\frac{94}{144}, \frac{95}{144}\right), \left(\frac{109}{144}, \frac{110}{144}\right), \left(\frac{111}{144}, \frac{112}{144}\right), \left(\frac{113}{144}, \frac{115}{144}\right), \left(\frac{116}{144}, \frac{117}{144}\right),$

$\left(\frac{118}{144}, \frac{119}{144}\right), \left(\frac{133}{144}, \frac{134}{144}\right), \left(\frac{135}{144}, \frac{136}{144}\right), \left(\frac{137}{144}, \frac{139}{144}\right), \left(\frac{140}{144}, \frac{141}{144}\right), \left(\frac{142}{144}, \frac{143}{144}\right)$

Therefore $C_{\left(\frac{1}{12}\right)^2} = \left[\frac{0}{144} = 0, \frac{1}{144}\right] \cup \left[\frac{2}{144}, \frac{3}{144}\right] \cup \left[\frac{4}{144}, \frac{5}{144}\right] \cup \left[\frac{7}{144}, \frac{8}{144}\right] \cup \left[\frac{9}{144}, \frac{10}{144}\right] \cup \left[\frac{11}{144}, \frac{12}{144}\right] \cup$

$\left[\frac{24}{144}, \frac{25}{144}\right] \cup \left[\frac{26}{144}, \frac{27}{144}\right] \cup \left[\frac{28}{144}, \frac{29}{144}\right] \cup \left[\frac{31}{144}, \frac{32}{144}\right] \cup \left[\frac{33}{144}, \frac{34}{144}\right] \cup \left[\frac{35}{144}, \frac{36}{144}\right] \cup$

$\left[\frac{48}{144}, \frac{49}{144}\right] \cup \left[\frac{50}{144}, \frac{51}{144}\right] \cup \left[\frac{52}{144}, \frac{53}{144}\right] \cup \left[\frac{55}{144}, \frac{56}{144}\right] \cup \left[\frac{57}{144}, \frac{58}{144}\right] \cup \left[\frac{59}{144}, \frac{60}{144}\right] \cup$

$\left[\frac{84}{144}, \frac{85}{144}\right] \cup \left[\frac{86}{144}, \frac{87}{144}\right] \cup \left[\frac{88}{144}, \frac{89}{144}\right] \cup \left[\frac{91}{144}, \frac{92}{144}\right] \cup \left[\frac{93}{144}, \frac{94}{144}\right] \cup \left[\frac{95}{144}, \frac{96}{144}\right] \cup$

$\left[\frac{108}{144}, \frac{109}{144}\right] \cup \left[\frac{110}{144}, \frac{111}{144}\right] \cup \left[\frac{112}{144}, \frac{113}{144}\right] \cup \left[\frac{115}{144}, \frac{116}{144}\right] \cup \left[\frac{117}{144}, \frac{118}{144}\right] \cup \left[\frac{119}{144}, \frac{120}{144}\right] \cup$

$\left[\frac{132}{144}, \frac{133}{144}\right] \cup \left[\frac{134}{144}, \frac{135}{144}\right] \cup \left[\frac{136}{144}, \frac{137}{144}\right] \cup \left[\frac{139}{144}, \frac{140}{144}\right] \cup \left[\frac{141}{144}, \frac{142}{144}\right] \cup \left[\frac{143}{144}, \frac{144}{144}\right]$

(2.4.2)

This procedure proceeds in every iteration to get the entire Cantor Dodeca set.

**Remark:**

Another way of forming Cantor modified set may be given as follows for Cantor Octanary Set.

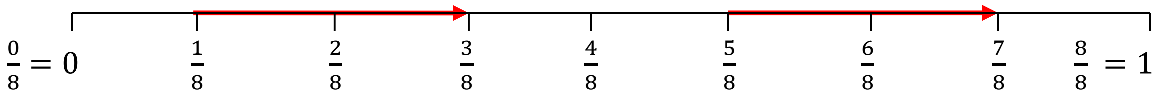



**Figure 2.4.2:** Another way of Cantor Octanary sets.

The closed interval $[0, 1]$ is divided into eight equal parts. Here in this pattern the intervals of lengths $\frac{2}{8}$ are removed and retained intervals are of two types. In the successive iterations the same pattern is followed.

**It is to be noted that in one pattern in the middlemost point $\frac{1}{2}$ is retained and in the other $\frac{1}{2}$ is removed.**

# CHAPTER –III

# A SUBSEQUENCE OF FAREY SEQUENCE – TOPOLOGICAL SPACE

In this chapter a subsequence of Farey sequence, $\widetilde{F_N}$, Farey $N$ – subsequence has been developed as a topological space and a Hausdorff space by appropriately defining basis and open sets. Also the $T_1$axiom has been discussed with an illustration.

Having identified the terms of $F_N$, $F_{N+1}$ is written by writing the mediant of all the successive terms of $F_N$.With slight modification, sequence whose terms are Farey sequences has been established as various spaces namely topological space, Hausdorff space and $T_1$ space. A Hausdorff space is basically a topological space. To form a topology a nonempty set with basis elements should be defined clearly.

This chapter provides two sections. In section 3.1 Topological Space and section 3.2 Hausdorff space has been analyzed. Here each sequence is treated as an element.

## 3.1 Topological Space:

**Definition3.1.1: Farey sequence [1,49]**

A Sequence of rational numbers $\frac{p}{q}$ with $(p,q) = 1$ in $[0,1]$ and $q \leq n$ is called a Farey Sequence of order $n$, denoted by $F_n$.

**Example 3.1.1:**

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$

$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}\frac{1}{1}\right\}$$

$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

**Definition 3.1.2: Farey $N$ – subsequence [1,2]**

In a Farey sequence $F_N$ the elements with denominators precisely $N$ are classified as Farey $N$ – subsequence and denoted by $\langle F_N' \rangle$ .

$$\langle F_N' \rangle = \left\{ \frac{u_i}{N} / \, 0 \le u_i \le N, 0 \le i \le N \right\}$$

**Example 3.1.2:**

The Farey $N$ – Sequence of order 4 is

$$\langle F_4' \rangle = \left\{ \frac{0}{1} = \frac{0}{4} < \frac{1}{4} < \frac{3}{4} < \frac{4}{4} = \frac{1}{1} \right\}$$

**Definition 3.1.3: Non - Reducible Farey Sequence [1]**

A subsequence of the Farey sequence $F_N$ whose denominators not exceeding $N$, listed in order of their size, is taken as Non – Reducible Farey Sequence. It is denoted by $\widetilde{F_N}$.

**Example 3.1.3:**

The quaternary Non - Reducible Farey Sequence of order 4 is

$$\tilde{F}_4 = \left\{ \frac{0}{1} = \frac{0}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} = \frac{2}{4}, \frac{2}{3}, \frac{3}{4}, \frac{4}{4} = \frac{1}{1} \right\}$$

**Definition 3.1.4: Non - Reducible Farey $N$ – Subsequence [1, 3]**

For $F_N$, the element of the sequence with denominator $N$ is taken as Non –Reducible Farey $N$ - subsequence. It is denoted by $\tilde{F}_N$.

**Example 3.1.4:**

The Non- Reducible Farey $N$ - Subsequence of order 4 is

$$\tilde{F}_4 = \left\{ \frac{0}{4} = 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = \frac{1}{1} \right\}$$

**Definition 3.1.5: Topological Space [24]**

A topology on a set $X$ is a collection $\tau$ of subsets of $X$ having the following properties.

(i) $\varphi$ and $X$ belong to $\tau$.

(ii) The union of the elements of $\tau$ is included in $\tau$.

(iii) The intersection of the elements is at $\tau$.

A set $X$ with the given topology $\tau$ is called a **topological space**.

**Definition 3.1.6: Basis [24]**

If $X$ is a set, a basis for a topology on $X$is a collection Ɓ of subsets of $X$ **called basis elements** such that

(i) For each $x \in X$, there ia at least one basis element $B$ containing $x$.

(ii) If $x$ belongs to the intersection of two basis element $B_1$ and $B_2$, then there is a basis element $B_3$ containing $x$ such that $B_3 \subset B_1 \cap B_2$.

**Example 3.1.5:**

Let Ɓ be the collection of all circular regions (interiors of circles) in the plane. Then Ɓ satisfies both conditions for a basis.

**Theorem 3.1.1:**

For any integer $N \geq 3$ a subsequence of Farey sequence, Farey $N$ – subsequence denoted by $\tilde{F}_{N^k}$ is a topological space.

**Proof:**

To define a topology on a set first a basis and hence open sets should be described clearly. Here the basis is defined as follows.

Consider $X = \left\{ \frac{i}{N^k} \; / \; \substack{i=0,1,2,3,\ldots.\\ k=1,2,3,4\ldots.. \\ N=3,4,5,\ldots\ldots} \right\}$

Take $B = \{ \tilde{F}_{N^k}, k = 1,2,3, \ldots \}$

Here every element of $B$ is a sequence of real numbers.

**Claim:**

$B$ constitute a basis for $X$

Take $F = \bigcap \tilde{F}_{N^k}, k = 1,2,3, \ldots \infty$

**Case (i)**:

If $x \in F$ then choose basis $B$ element as any one of $\tilde{F}_{N^k}, k \geq 2$

**Case (ii)**:

Suppose that $x$ is not in $F$. $x$ may be any one of the following forms.

$$x = \frac{i}{N^{k+1}} \; ; \substack{0 \leq i \leq N^k - 1 \\ N^k + 1 \leq i \leq j * N^k - 1} \; / \; j = 1,2,3, \ldots N \;\; for \; k = 1,2,3, \ldots \infty$$

Choose basis elements as any one of $\tilde{F}_{N^k}, k = 2,3, \ldots \infty$

Then clearly $B_i \cap B_j$contains a basis element in which $x$ is a member.

The open sets $U$ may be taken as a sequence of union of members of $B$.

Then for every element $x \in F$ there exists a member in $B$ and a set $U$ such that $x \in B \subseteq U$

**Illustration 3.1.1 :**

Consider $X = \left\{\frac{i}{N^k}, \begin{smallmatrix} i=0,1,2,3,.... \\ k=1,2,3,4..... \\ N=3,4,5,...... \end{smallmatrix}\right\}$

$B = \left\{\tilde{F}_{N^k}, k = 1,2,3, \dots\right\}$

Take $N = 4\,; k = 1,2,3, \dots$

$B = \{\tilde{F}_{4^1}, \tilde{F}_{4^2}, \tilde{F}_{4^3}, \dots.\}$

Consider higher ordered Farey sequences

$$\tilde{F}_{4^1} = \left\{\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = 1\right\}$$

$$\tilde{F}_{4^2} = \left\{\frac{0}{16}, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \frac{4}{16}, \frac{5}{16}, \frac{6}{16}, \frac{7}{16}, \frac{8}{16}, \frac{9}{16}, \frac{10}{16}, \frac{11}{16}, \frac{12}{16}, \frac{13}{16}, \frac{14}{16}, \frac{15}{16}, \frac{16}{16} = 1\right\}$$

$$\tilde{F}_{4^3} = \left\{\frac{0}{64}, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \frac{4}{64}, \frac{5}{64}, \frac{6}{64}, \frac{7}{64}, \frac{8}{64}, \frac{9}{64}, \frac{10}{64}, \dots \dots \dots \dots \dots \dots \dots \dots., \frac{64}{64} = 1\right\}$$

$\vdots$

$\vdots$

$U = \tilde{F}_{4^2} \cup \tilde{F}_{4^3}$

Take the element $x = \frac{3}{4^2}$ Clearly $x \in \tilde{F}_{4^2} \subseteq U$

Let $\tau = \{\tilde{F}_{4^1}, \tilde{F}_{4^2}, \tilde{F}_{4^3}, \dots.\}$

Consider $\tau_1 = \tilde{F}_{4^1}$

$= \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$

and $\tau_2 = \tilde{F}_{4^2}$

$= \left\{0, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \dots \dots \frac{15}{16}, 1\right\}$

Then

$$\tau_1 \cup \tau_2 = \tilde{F}_{4^1} \cup \tilde{F}_{4^2}$$

$$= \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\} \cup \left\{0, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \dots \dots \frac{15}{16}, 1\right\}$$

$$= \left\{0, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \dots \dots \frac{15}{16}, 1\right\} \in \tau$$

Therefore the union of the elements of subsets of $\tau$ is in $\tau$.

Consider $\tau_1 = \tilde{F}_{4^3}$

$$= \left\{0, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \dots \dots \frac{63}{64}, 1\right\}$$

and $\tau_2 = \tilde{F}_{4^4}$

$$= \left\{0, \frac{1}{256}, \frac{2}{256}, \frac{3}{256}, \dots \dots \frac{255}{256}, 1\right\}$$

Then

$$\tau_1 \cap \tau_2 = \tilde{F}_{4^3} \cap \tilde{F}_{4^4}$$

$$= \left\{0, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \dots \dots \frac{63}{64}, 1\right\} \cap \left\{0, \frac{1}{256}, \frac{2}{256}, \frac{3}{256}, \dots \dots \frac{255}{256}, 1\right\}$$

$$= \left\{0, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \dots \dots \frac{63}{64}, 1\right\} \in \tau$$

Therefore the intersection of the elements of any finite sub-collection of $\tau$ is in $\tau$.

It is well known that in a Hausdorff space every pair of elements is separated by open sets. The following is the theorem of Farey $N$ – subsequence as Hausdorff space.

## 3.2 Hausdorff space:

**Definition 3.2.1: Hausdorff space [24]**

A topological space $S = [0, 1]$ is said to be a **Hausdorff space** if for each pair of elements $s_1$ and $s_2$ in $S$ their exist neighbours $W_1$ and $W_2$ respectively of points $s_1$ and $s_2$ that do not intersect.

**Theorem 3.2.1:**

For a set consisting of rational numbers of the form $\frac{i}{N^k}$ the basis defined in the above theorem forms a Hausdorff space.

**Proof:**

A Hausdroff space is in fact a topological space. To define a topology a basis should be described in the basis for the topology defined as above. Here an open set is taken in the form

$$\widetilde{W}_{N^k} = X - \bigcup_{j=1}^{k-1} \tilde{F}_{N^j} \text{ where } k = 1,2,3, \dots ; N = 3,4,5, \dots \dots$$

Consider the points $x_1 = \frac{i}{N^r}$ and $x_2 = \frac{i}{N^t}$ ; $r, t = 1,2,3, \dots \dots$ .Clearly $\frac{i}{N^r} \in \tilde{F}_{N^r}$and $\frac{i}{N^t} \in \tilde{F}_{N^t}$ Then the sets $\widetilde{W}_{N^r}$ $\widetilde{W}_{N^t}$ constitute disjoint disjoint neighbourhoods of the points $x_1$ and $x_2$ respectively. Hence $X$ is a Hausdorff space.

**Illustration 3.2.1:**

$\widetilde{W}_{N^k} = X - \bigcup_{j=1}^{k-1} \tilde{F}_{N^j}$ where $k = 1,2,3, \dots ; N = 3,4,5, \dots \dots$

Take $N = 4, k = 1,2,3, \dots$

Consider $S = \{\widetilde{W}_{4^1}, \widetilde{W}_{4^2}, \widetilde{W}_{4^3}, \dots\}$

Let $s_1 = \frac{61}{4^3}$ and $s_2 = \frac{251}{4^4}$ be two distinct points of $S$.

Then there exist neighbourhoods $D_1 = \widetilde{W}_{4^3}$ and $D_2 = \widetilde{W}_{4^4}$ of $s_1$ and $s_2$ that are also disjoint.

Therefore the topological space S is a Hausdorff space

**Definition 3.2.2: $T_1$ axiom [24]**

A topological space $(X,T)$ is said to be $T_1$ given two points $p_1$ , $p_2$ of $X$, there exist open sets, $O_1$and $O_2$ such that $p_1$ is an element of $O_1$ and $p_2$ is not an element of $O_1$ and $p_2$ is an element of $O_2$ and $p_1$ is not an element of $O_2$.

**Corollary 3.2.1:**

On the same construction above the topological space $X$ also satisfies $T_1$ axioms.

**Illustration 3.2.2:**

$\widetilde{W}_{N^k} = X - \cup_{j=1}^{k-1} \tilde{F}_{N^j}$ where $k = 1,2,3, \dots; N = 3,4,5, \dots \dots$

Take $N = 4, k = 1,2,3, \dots$

Consider $T = \widetilde{W}_{4^1}, \widetilde{W}_{4^2}, \widetilde{W}_{4^3}, \dots$

Given two points $q_1 = \frac{15}{4^2}$ and $q_2 = \frac{517}{4^5}$ of $T$

There exist an open set $I_1 = \widetilde{W}_{4^2}$ and $I_2 = \widetilde{W}_{4^5}$ of $T$ such that $q_1 \in \widetilde{W}_{4^2}$ and $q_1 \notin \widetilde{W}_{4^5}$

Again $q_2 \in \widetilde{W}_{4^5}$ and $q_2 \notin \widetilde{W}_{4^2}$.

**Theorem 3.2.2:**

$F = \cup F_N$ , $F_N$ is a Farey sequence bounded by 0 and 1. The subsequence $V = \cup \tilde{F}_{N^k}$ of $F$ has convergent subsequence.

**Proof:**

Consider the Farey sequence $F_N$, where $N = 1, 2, 3, \dots$for all positive integers $N, F_N$ is a bounded sequence and is bounded by 0 and 1.The set $F$ is defined above is also bounded by 0 and 1. Now the subsequence of $V$ namely $\left\{\frac{k}{N^k} / k = 1,2,3, \dots\right\}$ it is a convergence sequence and converges to $0$. This is because $\frac{1}{N^k} \to 0$ as $k \to \infty$ and for all positive integers $N$.

The Farey $N$ subsequence of order 4 can be depicted in the graph as follows :

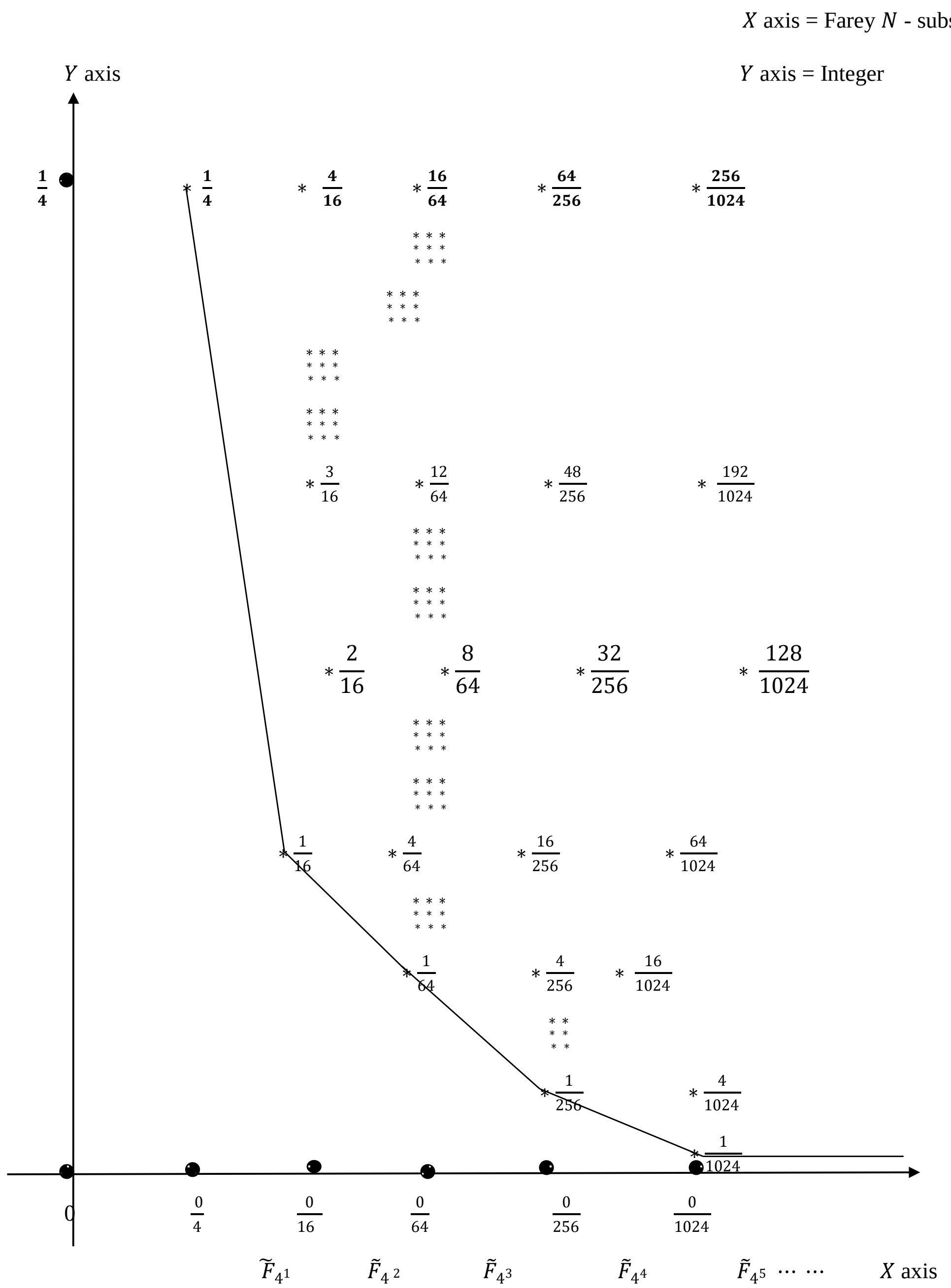



**Figure 3.2.1:** Farey N subsequence of order 4

From the above graph, it is clear that the curve resembles inverse exponential curve.

# CHAPTER – IV

# $\sigma$ ALGEBRA AND BOREL SET OF SUBSEQUENCE OF GENERALIZED FAREY SEQUENCE

In this chapter, Construction of measurable sets from Non reduced Farey $N$ – subsequence is discussed in detail. For the construction of measurable sets the Non reduced Farey $N$ - subsequence has been considered as union of intervals, half- open, closed (or) open sets as the case may be with a sequential points as end points. Also we have analysed a few points on $\sigma$ algebra and Borel Set of Subsequence of Generalized Farey Sequence. We derive theorem on the Lebesgue measure of the Generalized Non reduced Farey $N$ - subsequence .

**Definition 4.1: $\sigma$ – algebra [34]**

Let $X$ be a set and $\mathcal{A} \subseteq P(X)$ is called a $\sigma$ – algebra if

(i) $\varphi, X \in \mathcal{A}$

(ii) $A \subset \mathcal{A} \Rightarrow A^{\complement} = X/A \in \mathcal{A}$

(iii) $A_i \in \mathcal{A}$ , $i \in N \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

**Example 4.1:**

If $X = \{a, b, c, d\}$, one possible $\sigma$ – algebra on $X$ is $\Sigma = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$, where $\emptyset$ is the empty set. In general, a finite algebra is always a $\sigma$ – algebra.

**Definition 4.2: Borel Set [34]**

A Borel set is any set in topological space that can be formed from open sets (or) equivalently from closed sets through the operations of countable union, countable intersection and relative complement.

**Definition 4.3: Lebesgue Measure [34]**

A set $A \subset E$ is Lebesgue measurable or measurable if $\lambda^*(A) = \lambda_*(A)$. The measure of $A$ is denoted by $\lambda(A)$ and is given by $\lambda(A) = \lambda^*(A) = \lambda_*(A)$

**Theorem 4.1:**

The Lebesgue measure of the Generalized Non reduced Farey $N$ – subsequence of order $m$ is zero.

**Proof:**

Construction of measurable sets from Non reduced Farey $N$ - subsequence.

By definition, a Non reduced Farey $N$ - subsequence is given by

$$\tilde{F}_{(m)^1} = \left\{\frac{0}{(m)^1}, \frac{1}{(m)^1}, \frac{2}{(m)^1}, \cdots\cdots, \frac{(m)^1-3}{(m)^1}, \frac{(m)^1-2}{(m)^1}, \frac{(m)^1-1}{(m)^1}, \frac{(m)^1}{(m)^1}\right\} \text{ ; where } 1 \leq m \leq N$$

From this sequence construct sets as follows:

$$C\,\tilde{F}_{(m)^1} = \left\{\frac{0}{(m)^1}, \frac{1}{(m)^1}, \frac{2}{(m)^1}, \cdots\cdots, \frac{(m)^1-3}{(m)^1}, \frac{(m)^1-2}{(m)^1}, \frac{(m)^1-1}{(m)^1}, \frac{(m)^1}{(m)^1}\right\} \text{; where } 1 \leq m \leq N$$

$$= \left[\frac{0}{(m)^1}, \frac{1}{(m)^1}\right] \cup \left[\frac{1}{(m)^1}, \frac{2}{(m)^1}\right] \cup \cdots\cdots \cup \left[\frac{(m)^1-3}{(m)^1}, \frac{(m)^1-2}{(m)^1}\right] \cup \left[\frac{(m)^1-2}{(m)^1}, \frac{(m)^1-1}{(m)^1}\right] \cup \left[\frac{(m)^1-1}{(m)^1}, \frac{(m)^1}{(m)^1}\right]$$

$$= D_{(m)^1\,1} \cup D_{(m)^1\,2} \cup D_{(m)^1\,3} \cup \cdots\cdots \cup D_{(m)^1\,r}$$

In the next iteration the sequence is given by

$$C\tilde{F}_{(m)^2} = \left\{\frac{0}{(m)^2}, \frac{1}{(m)^2}, \frac{2}{(m)^2}, \cdots\cdots, \frac{(m)^2-3}{(m)^2}, \frac{(m)^2-2}{(m)^2}, \frac{(m)^2-1}{(m)^2}, \frac{(m)^2}{(m)^2}\right\}; \text{ where } 1 \le m \le N$$

Again writing in a set format we have

$$C\tilde{F}_{(m)^2} = \left[\frac{0}{(m)^2}, \frac{1}{(m)^2}\right] \cup \left[\frac{1}{(m)^2}, \frac{2}{(m)^2}\right] \cup \cdots\cdots \cup \left[\frac{(m)^2-3}{(m)^2}, \frac{(m)^2-2}{(m)^2}\right] \cup \left[\frac{(m)^2-2}{(m)^2}, \frac{(m)^2-1}{(m)^2}\right] \cup \left[\frac{(m)^2-1}{(m)^2}, \frac{(m)^2}{(m)^2}\right]$$

$$= D_{(m)^2\,1} \cup D_{(m)^2\,2} \cup D_{(m)^2\,3} \cup \cdots\cdots \cup D_{(m)^2\,r}$$

the $n^{th}$ term is

$$C\tilde{F}_{(m)^n} = \left\{\frac{0}{(m)^n}, \frac{1}{(m)^n}, \frac{2}{(m)^n}, \cdots\cdots, \frac{(m)^n-3}{(m)^n}, \frac{(m)^n-2}{(m)^n}, \frac{(m)^n-1}{(m)^n}, \frac{(m)^n}{(m)^n}\right\}; \text{ where } 1 \le m \le N$$

$$= \left[\frac{0}{(m)^n}, \frac{1}{(m)^n}\right] \cup \left[\frac{1}{(m)^n}, \frac{2}{(m)^n}\right] \cup \cdots\cdots \cup \left[\frac{(m)^n-3}{(m)^n}, \frac{(m)^n-2}{(m)^n}\right] \cup \left[\frac{(m)^n-2}{(m)^n}, \frac{(m)^n-1}{(m)^n}\right] \cup \left[\frac{(m)^n-1}{(m)^n}, \frac{(m)^n}{(m)^n}\right]$$

$$= D_{(m)^n1} \cup D_{(m)^n2} \cup D_{(m)^n3} \cup \cdots\cdots \cup D_{(m)^nr}$$

Let

$E_{(m)^n1}$ = Set of all possible union of two elements.

$E_{(m)^n2}$ = Set of all possible union of three elements.

And so on , the r$^{th}$ term is

$E_{(m)^n\,r}$ = Set of all possible union of $(r + 1)$ elements.

Take $X = \left\{C\,\tilde{F}_{(m)^1}, C\tilde{F}_{(m)^2}, \cdots\cdots\cdots\cdots\cdots\cdots\cdots C\tilde{F}_{(m)^n}\right\}$

$$P(X) = \{D_{(m)^1\,1}, D_{(m)^1\,2}, D_{(m)^1\,3}, \cdots\cdots, D_{(m)^1\,r}, E_{(m)^1\,1}, E_{(m)^1\,2},$$

$$E_{(m)^1\,3}, \cdots\cdots, E_{(m)^1\,r} \cdots\cdots\cdots D_{(m)^n1}, D_{(m)^n2}, D_{(m)^n3}$$

$$, \cdots\cdots, D_{(m)^nr}, E_{(m)^n\,1}, E_{(m)^n\,2}, \cdots\cdots E_{(m)^n\,r}\}$$

**Claim 4.1:**

The Set $P(X)$ is a $\sigma$ – algebra

(i) Empty set $\varphi, P(X) \in \mathcal{A}$.

(ii) Now $\left(D_{(m)^1 2}\right)^{C} = P(X)/D_{(m)^1 2}$

$= E_{(m)^1\,1} =$ Set of all possible union of two elements

(iii) Consider the elements $D_{(m)^n 1}, D_{(m)^n 2}, D_{(m)^n 3}, \cdots\cdots, D_{(m)^n r} \in \mathcal{A}$

Then $D_{(m)^n 1} \cup D_{(m)^n 2} \cup \cdots \cup D_{(m)^n r} = E_{(m)^n r}$

$=$ Set of all possible union of $(\,m)^n$ of elements $\in \mathcal{A}$

Therefore

$P(X)$ is a $\sigma$ – algebra.

**Claim 4.2:**

$P(X)$ is a Borel set

Consider the elements $D_{(m)^n 1}, D_{(m)^n 2}, D_{(m)^n 3}, \cdots D_{(m)^n r}, E_{(m)^n\,1}, E_{(m)^n 2}, E_{(m)^n 3} \cdots\cdots E_{(m)^n r} \in \mathcal{A}$

Then $D_{(m)^n 1} \cap D_{(m)^n 2} \cap \cdots \cap D_{(m)^n r} \cap E_{(m)^n\,1} \cap E_{(m)^n 2} \cdots \cap E_{(m)^n r} = D_{(m)^n 1} \in \mathcal{A}$

The Set $P(X)$ satisfies all the conditions together with claim 4.1

So the set $P(X)$ is $\sigma$ – algebra and Borel set and hence a measurable set.

Now, the Lebesgue Measure of $P(X)$ is calculated.

Since measure of an interval is its length, we have

$$\lambda\left( C\tilde{F}_{(m)^n} \right) = \lim_{n \to \infty} \lambda( C\tilde{F}_{(m)^n} )$$

$$= \lim_{n\to\infty} \lambda \left\{ \begin{matrix} \left[\frac{0}{(m)^n}, \frac{1}{(m)^n}\right] \cup \left[\frac{1}{(m)^n}, \frac{2}{(m)^n}\right] \cup \cdots\cdots \cup \left[\frac{(m)^n-3}{(m)^n}, \frac{(m)^n-2}{(m)^n}\right] \cup \left[\frac{(m)^n-2}{(m)^n}, \frac{(m)^n-1}{(m)^n}\right] \cup \\ \left[\frac{(m)^n-1}{(m)^n}, \frac{(m)^n}{(m)^n}\right] \end{matrix} \right\}$$

$$= \lim_{n\to\infty} \left\{ \lambda\left[\frac{0}{(m)^n}, \frac{1}{(m)^n}\right] + \lambda\left[\frac{1}{(m)^n}, \frac{2}{(m)^n}\right] + \cdots + \lambda\left[\frac{(m)^n-3}{(m)^n}, \frac{(m)^n-2}{(m)^n}\right] + \lambda\left[\frac{(m)^n-2}{(m)^n}, \frac{(m)^n-1}{(m)^n}\right] + \lambda\left[\frac{(m)^n-1}{(m)^n}, \frac{(m)^n}{(m)^n}\right] \right\}$$

$$= \lim_{n\to\infty} \left\{ \begin{matrix} \left(\frac{1}{(m)^n} - \frac{0}{(m)^n}\right) + \left(\frac{2}{(m)^n} - \frac{1}{(m)^n}\right) + \cdots + \left(\frac{(m)^n-2}{(m)^n} - \frac{(m)^n-3}{(m)^n}\right) + \\ \left(\frac{(m)^n-1}{(m)^n} - \frac{(m)^n-2}{(m)^n}\right) + \left(\frac{(m)^n}{(m)^n} - \frac{(m)^n-1}{(m)^n}\right) \end{matrix} \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{1}{(m)^n} + \frac{1}{(m)^n} + \frac{1}{(m)^n} + \cdots\cdots + \frac{1}{(m)^n} + \frac{1}{(m)^n} + \frac{1}{(m)^n} \right\}$$

$$= 0$$

Therefore $\lambda\left( C\tilde{F}_{(m)^n} \right) = 0$. Hence $C\, \tilde{F}_{(m)^n}$ has Lebesgue measure zero. Hence the Lebesgue measure of the Non reduced Farey $N$ – subsequence of order $m$ is zero.

**Illustration 4.1:**

$C\tilde{F}_{\ (m)^n} = \left\{\frac{0}{(m)^n}, \frac{1}{(m)^n}, \frac{2}{(m)^n}, \cdots, \frac{(m)^n-3}{(m)^n}, \frac{(m)^n-2}{(m)^n}, \frac{(m)^n-1}{(m)^n}, \frac{(m)^n}{(m)^n}\right\}$; where $1 \leq m \leq N$

Put $m = 4$

Non – Reduce to Farey $N$ – Subsequence of order 4 is

$$C\tilde{F}_{4^1} = \left\{\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}\right\} = \left[\frac{0}{4}, \frac{1}{4}\right] \cup \left[\frac{1}{4}, \frac{2}{4}\right] \cup \left[\frac{2}{4}, \frac{3}{4}\right] \cup \left[\frac{3}{4}, \frac{4}{4}\right]$$

$$= D_{4^1 1} \cup D_{4^1 2} \cup D_{4^1 3} \cup D_{4^1 4}$$

$$C\tilde{F}_{4^2} = \left\{\frac{0}{16}, \frac{1}{16}, \frac{2}{16}, \dots\dots \frac{16}{16}\right\} = \left[\frac{0}{16}, \frac{1}{16}\right] \cup \left[\frac{1}{16}, \frac{2}{16}\right] \cup \left[\frac{2}{16}, \frac{3}{16}\right] \cup \left[\frac{3}{16}, \frac{4}{16}\right] \cup \dots\dots \cup \left[\frac{15}{16}, \frac{16}{16}\right]$$

$$= D_{4^2 1} \cup D_{4^2 2} \cup D_{4^2 3} \cup \cdots\cdots\cdots\cdots\cdots \cup D_{4^2 (15)} \cup D_{4^2 (16)}$$

For $n^{\text{th}}$ term

$$C\tilde{F}_{4^n} = \left\{\frac{0}{4^n}, \frac{1}{4^n}, \frac{2}{4^n}, \frac{3}{4^n}, \cdots\cdots \frac{4^n-3}{4^n}, \frac{4^n-2}{4^n}, \frac{4^n-1}{4^n}, \frac{4^n}{4^n}\right\}$$

$$= \left[\frac{0}{4^n}, \frac{1}{4^n}\right] \cup \left[\frac{1}{4^n}, \frac{2}{4^n}\right] \cup \left[\frac{2}{4^n}, \frac{3}{4^n}\right] \cup \cdots\cdots \cup \left[\frac{4^n-3}{4^n}, \frac{4^n-2}{4^n}\right] \cup \left[\frac{4^n-2}{4^n}, \frac{4^n-1}{4^n}\right] \cup \left[\frac{4^n-1}{4^n}, \frac{4^n}{4^n}\right]$$

$$= D_{4^n 1} \cup D_{4^n 2} \cup D_{4^n 3} \cup \cdots\cdots\cdots \cup D_{4^n m}$$

Let

$E_{4^n 1}$ = Set of all possible union of two elements.

$E_{4^n 2}$ = Set of all possible union of three elements.

$lll^{ly}$ for $m^{\text{th}}$ term

$E_{4^n m}$ = Set of all possible union of $(m+1)$ elements.

$E_{4^n m} = \bigcup_{m=1}^{4^n-1}(\mathrm{m}+1)$

Take $X = \{\tilde{F}_{4^1}, \tilde{F}_{4^2}, \cdots\cdots\cdots\cdots\cdots\cdots\cdots \tilde{F}_{4^n}\}$

$$P(X) = \{D_{4^1 1}, D_{4^1 2}, D_{4^1 3}, D_{4^1 4}, E_{4^1 1}, E_{4^1 2}, E_{4^1 3}, \cdots D_{4^n 1}, D_{4^n 2}, D_{4^n 3}, \cdots, D_{4^n m}, E_{4^n 1}, E_{4^n 2} \cdots E_{4^n m}\}$$

The Set $P(X)$ is a $\sigma$ – algebra

(i) Empty set $\varphi, P(X) \in A$.

(ii) Take $(D_{4^1 2})^{C} = P(X)/D_{4^1 2}$

$= E_{4^1 1}$ = Set of all possible union of two elements

(iii) Consider the elements $D_{4^n 1}, D_{4^n 2}, D_{4^n 3}, \cdots, D_{4^n m} \in \mathcal{A}$

Then $D_{4^n 1} \cup D_{4^n 2} \cup \cdots \cup D_{4^n m} = E_{4^n m}$ (Set of all possible union of $4^n$ elements) $\in \mathcal{A}$

Therefore $P(X)$ is a $\sigma$ – algebra.

The Set $P(X)$ is Borel set

(iv) The Set $P(X)$ satisfies the countable intersection

Consider the elements $D_{4^1 1}, D_{4^1 2}, \cdots\cdots, D_{4^1 n}, E_{4^1 1} E_{4^1 2} \cdots\cdots E_{4^1 n} \in \mathcal{A}$

Then $D_{4^1 1} \cap D_{4^1 2} \cap \cdots \cap D_{4^1 n} \cap E_{4^1 1} \cap E_{4^1 2} \cap \cdots\cdots \cap E_{4^1 n} = D_{4^1 1} \in \mathcal{A}$

$P(X)$ is a Borel set.

Therefore $P(X)$ is a $\sigma$ Algebra as well as Borel Set.

**Lebesgue Measure:**

$$\lambda(C\tilde{F}_{4^n}) = \lim_{n\to\infty} \lambda(C\tilde{F}_{4^n})$$

$$= \lim_{n\to\infty} \lambda\left\{\left[\frac{0}{4^n}, \frac{1}{4^n}\right] \cup \left[\frac{1}{4^n}, \frac{2}{4^n}\right] \cup \left[\frac{2}{4^n}, \frac{3}{4^n}\right] \cup \cdots\cdots \cup \left[\frac{4^n-3}{4^n}, \frac{4^n-2}{4^n}\right] \cup \left[\frac{4^n-2}{4^n}, \frac{4^n-1}{4^n}\right] \cup \left[\frac{4^n-1}{4^n}, \frac{4^n}{4^n}\right]\right\}$$

$$= \lim_{n\to\infty} \left\{\lambda\left[\frac{0}{4^n}, \frac{1}{4^n}\right] + \lambda\left[\frac{1}{4^n}, \frac{2}{4^n}\right] + \lambda\left[\frac{2}{4^n}, \frac{3}{4^n}\right] + \cdots + \lambda\left[\frac{4^n-3}{4^n}, \frac{4^n-2}{4^n}\right] + \lambda\left[\frac{4^n-2}{4^n}, \frac{4^n-1}{4^n}\right] + \lambda\left[\frac{4^n-1}{4^n}, \frac{4^n}{4^n}\right]\right\}$$

$$= \lim_{n\to\infty} \left\{\left(\frac{1}{4^n} - \frac{0}{4^n}\right) + \left(\frac{2}{4^n} - \frac{1}{4^n}\right) + \left(\frac{3}{4^n} - \frac{2}{4^n}\right) + \cdots + \left(\frac{4^n-2}{4^n} - \frac{4^n-3}{4^n}\right) + \left(\frac{4^n-1}{4^n} - \frac{4^n-2}{4^n}\right) + \left(\frac{4^n}{4^n} - \frac{4^n-1}{4^n}\right)\right\}$$

$$= 0$$

Therefore $\lambda(C\tilde{F}_{4^n}) = 0$. Hence $C\tilde{F}_{4^n}$ has Lebesgue measure zero.

Since Cantor sets may be extracted from Farey sequence consider as sets. Here theorem on measures of even ordered Cantor sets discussed in chapter I are presented.

**Theorem 4.2:**

The Lebesgue measure of Cantor Hexnary set and Cantor Deca set are zero.

**Proof:**

Considering Cantor Hexnary set and Cantor Decaset defined in Chapter I.

The lengths of intervals retained in Cantor Hexnary sets in each of the iterations are given below:

First iteration: $2\left(\frac{1}{6}\right) + \left(\frac{2}{6}\right) = \left(\frac{1}{6}\right)[2+2] = \left(\frac{1}{6}\right)[2^1 + 2^1]$

Second iteration: $4\left(\frac{1}{6^2}\right) + 4\left(\frac{2}{6^2}\right) + 1\left(\frac{4}{6^2}\right) = \left(\frac{1}{6^2}\right)(2^2 + 2(2^2) + 2^2)$

Third iteration: $8\left(\frac{1}{6^3}\right) + 12\left(\frac{2}{6^3}\right) + 6\left(\frac{4}{6^3}\right) + 1\left(\frac{8}{6^3}\right) = \left(\frac{1}{6^3}\right)(2^3 + 3(2^3) + 3(2^3) + 2^3)$

Therefore

The $n^{th}$ iteration: $\left(\frac{1}{6^n}\right)(2^n + nc_1 2^n + nc_2 2^n + \cdots\cdots + nc_{n-1}2^n + 2^n)$

$$\text{Lebesgue measure} = \lambda\left(C_{\left(\frac{1}{6}\right)^n}\right)$$

$$= \lim_{n\to\infty}\left\{\frac{1}{6^n}(2^n + nc_1 2^n + nc_2 2^n + \cdots\cdots\cdots + nc_{n-1}2^n + 2^n)\right\}$$

$$= \lim_{n\to\infty}\left(\frac{2}{3}\right)^n \to 0$$

**Cantor Deca set**

First iteration: $1\left(\frac{1}{10}\right) + \left(\frac{1}{10}\right) + \left(\frac{2}{10}\right) + \left(\frac{1}{10}\right) + \left(\frac{1}{10}\right) = \left(\frac{1}{10}\right)[4+2] = \left(\frac{1}{10}\right)2[2^1 + 2^0]$

Second iteration: $16\left(\frac{1}{10^2}\right) + 8\left(\frac{2}{10^2}\right) + \left(\frac{4}{10^2}\right) = \left(\frac{1}{10^2}\right)(2^4 + 2(2^3) + 2^2)$

$$= \left(\frac{1}{10^2}\right)2^2(2^2 + 2(2^1) + 2^0)$$

Third iteration: $64\left(\frac{1}{10^3}\right) + 48\left(\frac{2}{10^3}\right) + 12\left(\frac{4}{10^3}\right) + 1\left(\frac{8}{10^3}\right) = \left(\frac{1}{10^3}\right)(2^6 + 3(2^5) + 3(2^4) + 2^3)$

$$= \left(\frac{1}{10^3}\right)2^3(2^3 + 3(2^2) + 3(2^1) + 2^0)$$

Therefore

The $n^{th}$ iteration: $\left(\frac{1}{10^n}\right) 2^n(2^n + nc_1 2^{n-1} + nc_2 2^{n-2} + \cdots\cdots + nc_{n-1} 2^1 + 2^0)$

Lebesgue measure $= \lambda\left(C_{\left(\frac{1}{10}\right)^n}\right)$

$$= \lim_{n\to\infty}\left\{\frac{1}{10^n} 2^n(2^n + nc_1 2^{n-1} + nc_2 2^{n-2} + \cdots\cdots + nc_{n-1} 2^1 + 2^0)\right\}$$

$$= \lim_{n\to\infty}\left(\frac{2}{5}\right)^n \to 0$$

**Theorem 4.3:**

If $[0,1]$ is subdivided into $8 + 4k, k = 0,1,2$ ... parts Cantor sets of order $8,12,16,$ ... are developed and the Lebesgue measure of each of the sets is zero

**Proof**

Referring to the diagrams of Cantor Octanary set and Cantor Dodeca sets we have calculated the Lebesgue measures:

For Cantor Octanary sets

Lengths of retained intervals of first iteration: $\left(\frac{1}{8}\right) + \left(\frac{1}{8}\right) + \left(\frac{1}{8}\right) + \left(\frac{1}{8}\right) = \left(\frac{1}{8}\right)\ 4$

Lengths of retained intervals of second iteration: $\left(\frac{1}{8^2}\right)(16) = \left(\frac{1}{8^2}\right) 4^2$

Lengths of retained intervals of third iteration: $\left(\frac{1}{8^3}\right)(64) = \left(\frac{1}{8^3}\right) 4^3$

Therefore

Lengths of retained intervals of $n^{th}$ iteration: $\left(\frac{1}{8^n}\right) 4^n$

Lebesgue measure $= \lambda\left(C_{\left(\frac{1}{8}\right)^n}\right)$

$$= \lim_{n\to\infty}\left\{\left(\frac{1}{8^n}\right) 4^n\right\}$$

$$= \lim_{n\to\infty}\left(\frac{1}{2}\right)^n \to 0$$

For Cantor Dodeca Sets

Lengths of retained intervals of first iteration: $\left(\frac{1}{12}\right)+\left(\frac{1}{12}\right)+\left(\frac{1}{12}\right)+\left(\frac{1}{12}\right)+\left(\frac{1}{12}\right)+\left(\frac{1}{12}\right)=\left(\frac{1}{12}\right)\ 6$

Lengths of retained intervals of second iteration: $\left(\frac{1}{12^2}\right)(36)=\left(\frac{1}{8^2}\right)6^2$

Lengths of retained intervals of third iteration: $\left(\frac{1}{12^3}\right)(216)=\left(\frac{1}{12^3}\right)6^3$

Therefore

Lengths of retained intervals of $n^{th}$ iteration: $\left(\frac{1}{12^n}\right)6^n$

$$\text{Lebesgue measure } =\lambda\left(C_{\left(\frac{1}{12}\right)^n}\right)$$

$$=\lim_{n\to\infty}\left\{\left(\frac{1}{12^n}\right)6^n\right\}$$

$$=\lim_{n\to\infty}\left(\frac{1}{2}\right)^n\to 0$$

It is observed that in the $n^{th}$ iteration sum of the length of the retained intervals is $\left(\frac{1}{2}\right)^n$.

# CHAPTER -V

# PROBABILITY MEASURE OF GENERALIZED NON - REDUCED FAREY $N$- SUBSEQUENCE

This chapter is divided into two sections. Section 5.1 is on Probability Measure of Generalized Non - Reduced Farey $N$- Subsequence and section 5.2 is on Invariant Measure of Generalized Non - Reduced Farey $N$- Subsequence. We derive the Probability measure of the Generalized Non reduced Farey $N$ – subsequence.

For the construction of measurable sets the Farey sequence has been considered as union of intervals, half- open, closed (or) open sets as the case may be with a sequential points as end points. For the same above construction Probability measure has also been calculated.

## 5.1 Probability Measure:

By a slight modification, writing Non reduced Farey N – subsequence as a measurable set the sequence is written as union of closed and semi – open intervals.

**Definition 5.1.1: Probability Measures [34]**

A Probability measure on $\Omega$ is a function $P$ from subsets of $\Omega$ to the real numbers that satisfies the following axioms

(i) $P(\Omega) = 1$

(ii) If $A \subset \Omega$, then $P(A) \geq 0$

(iii) If $A_1$ and $A_2$ are disjoint, then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ more generally, If $A_1, A_2, \cdots\cdots, A_n$ are mutually disjoint, then $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

**Theorem 5.1.1:**

The Probability measure of the Non reduced Farey $N$ – subsequence of even order is one.

**Proof:**

Construction of measurable sets from Farey $N$ – subsequence even order

$$\tilde{F}_{(2m)^n} = \left\{\frac{0}{(2m)^n}, \frac{1}{(2m)^n}, \frac{2}{(2m)^n}, \cdots\cdots, \frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right\}$$

$$= \left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right] \cup \left[\frac{1}{(2m)^n}, \frac{2}{(2m)^n}\right] \cup \cdots \cup \left[\frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}\right] \cup \left[\frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}\right] \cup \left[\frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right]$$

$$HC\ \tilde{F}_{(2m)^n} = \left\{\frac{0}{(2m)^n}, \frac{1}{(2m)^n}, \frac{2}{(2m)^n}, \cdots\cdots, \frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right\}$$

$$= \left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right), \left[\frac{1}{(2m)^n}, \frac{2}{(2m)^n}\right), \cdots, \left[\frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}\right), \left[\frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}\right), \left[\frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right]$$

Here Probability of an event in $HC\ \tilde{F}_{(2m)^n}$ is taken as follows

$$P\left(\left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right)\right) = \text{Length of the interval } \left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right)$$

$$\cong \frac{1}{(2m)^n}$$

$$P\left(\left[\frac{1}{(2m)^n}, \frac{2}{(2m)^n}\right)\right) = \text{Length of the interval } \left[\frac{1}{(2m)^n}, \frac{2}{(2m)^n}\right)$$

$$\cong \frac{1}{(2m)^n}$$

$$P\left(\left[\frac{2}{(2m)^n}, \frac{3}{(2m)^n}\right)\right) = \text{Length of the interval } \left[\frac{2}{(2m)^n}, \frac{3}{(2m)^n}\right)$$

$$\cong \frac{1}{(2m)^n}$$

For the $n^{th}$ term

$$P\left(\left[\frac{(2m)^n-1}{(2m)^n},\frac{(2m)^n}{(2m)^n}\right]\right) = \text{Length of the intervals } \left[\frac{(2m)^n-1}{(2m)^n},\frac{(2m)^n}{(2m)^n}\right]$$

$$\cong \frac{1}{(2m)^n}$$

For union of intervals consider only consecutive points in

$$\left[\frac{0}{(2m)^n},\frac{1}{(2m)^n}\right),\left[\frac{1}{(2m)^n},\frac{2}{(2m)^n}\right),\cdots,\left[\frac{(2m)^n-3}{(2m)^n},\frac{(2m)^n-2}{(2m)^n}\right),\left[\frac{(2m)^n-2}{(2m)^n},\frac{(2m)^n-1}{(2m)^n}\right),\left[\frac{(2m)^n-1}{(2m)^n},\frac{(2m)^n}{(2m)^n}\right]$$

Take any two elements say, $A_1 = \left[\frac{3}{(2m)^n},\frac{4}{(2m)^n}\right)$ and $A_2 = \left[\frac{4}{(2m)^n},\frac{5}{(2m)^n}\right)$

$$P(A_1 \cup A_2) = P\left(\left[\frac{3}{(2m)^n},\frac{4}{(2m)^n}\right) \cup \left[\frac{4}{(2m)^n},\frac{5}{(2m)^n}\right)\right)$$

$$= P\left(\left[\frac{3}{(2m)^n},\frac{5}{(2m)^n}\right)\right)$$

$$= \frac{2}{(2m)^n}$$

$$P(A_1) + P(A_2) = P\left(\left[\frac{3}{(2m)^n},\frac{4}{(2m)^n}\right)\right) + P\left(\left[\frac{4}{(2m)^n},\frac{5}{(2m)^n}\right)\right)$$

$$= \frac{2}{(2m)^n}$$

Therefore $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

Now we prove $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

$$P(\bigcup_{i=1}^{n} A_i) = P(A_1 \cup A_2 \cup \cdots\cdots \cup A_n)$$

$$= P\left(\left[\frac{0}{(2m)^n},\frac{1}{(2m)^n}\right) \cup \left[\frac{1}{(2m)^n},\frac{2}{(2m)^n}\right) \cup \cdots\cdots \cup \left[\frac{(2m)^n-1}{(2m)^n},\frac{(2m)^n}{(2m)^n}\right]\right)$$

$$= P\left(\left[\frac{0}{(2m)^n},\frac{(2m)^n}{(2m)^n}\right]\right) = \frac{(2m)^n}{(2m)^n} = 1$$

$$\sum_{i=1}^{n} P(A_i) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

$$= P\left(\left[\frac{0}{(2m)^n},\frac{1}{(2m)^n}\right)\right) + P\left(\left[\frac{1}{(2m)^n},\frac{2}{(2m)^n}\right)\right) + \cdots + P\left(\left[\frac{(2m)^n-1}{(2m)^n},\frac{(2m)^n}{(2m)^n}\right]\right)$$

$$\cong \frac{1}{(2m)^n} + \frac{1}{(2m)^n} + \frac{1}{(2m)^n} + \cdots\cdots + \frac{1}{(2m)^n}$$

$$\cong \frac{1}{(2m)^n}\ (1 + 1 + 1 + 1 + \cdots\cdots\cdots + (2m)^n\ times)$$

$$\cong \frac{(2m)^n}{(2m)^n} = 1$$

Therefore the Probability Measure of Non - Reduced Farey $N$- Subsequence of order $(2m)$ is **one,** where $m$ is a positive integer**.**

**Illustration 5.1.1: $\boldsymbol{m}$ is even**

$$HC\ \tilde{F}_{(2m)^n} \quad = \left\{\frac{0}{(2m)^n}, \frac{1}{(2m)^n}, \frac{2}{(2m)^n}, \cdots\cdots, \frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right\}$$

$$= \left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right) \cup \left[\frac{1}{(2m)^n}, \frac{2}{(2m)^n}\right) \cup \cdots\cdots \cup \left[\frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}\right) \cup$$

$$\left[\frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}\right) \cup \left[\frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right]$$

Put $m = 2, n = 1, l = 4$

$$HC\ \tilde{F}_4 = \left\{\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}\right\}$$

$$= \left[\frac{0}{4}, \frac{1}{4}\right) \cup \left[\frac{1}{4}, \frac{2}{4}\right) \cup \left[\frac{2}{4}, \frac{3}{4}\right) \cup \left[\frac{3}{4}, \frac{4}{4}\right]$$

Now

$$P\left(\bigcup_{i=1}^{l} A_i\right) = \sum_{i=1}^{l} P(A_i)$$

$$P\left(\bigcup_{i=1}^{4} A_i\right) = \sum_{i=1}^{4} P(A_i)$$

$$P\left(\bigcup_{i=1}^{4} A_i\right) = P(A_1 \cup A_2 \cup A_3 \cup A_4)$$

$$= P\left(\left[\frac{0}{4}, \frac{1}{4}\right) \cup \left[\frac{1}{4}, \frac{2}{4}\right) \cup \left[\frac{2}{4}, \frac{3}{4}\right) \cup \left[\frac{3}{4}, \frac{4}{4}\right]\right)$$

$$= P\left(\left[\frac{0}{4}, \frac{4}{4}\right]\right) = 1$$

$$\sum_{i=1}^{4} P(A_i) = P(A_1) + P(A_2) + P(A_3) + P(A_4)$$

$$= P\left(\left[\frac{0}{4}, \frac{1}{4}\right)\right) + P\left(\left[\frac{1}{4}, \frac{2}{4}\right)\right) + P\left(\left[\frac{2}{4}, \frac{3}{4}\right)\right) + P\left(\left[\frac{3}{4}, \frac{4}{4}\right]\right)$$

$$\cong \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

Probability Measure of $HC\ \tilde{F}_4 \ = 1$

Therefore

The probability measure of Non - Reduced Farey $N$- Subsequence of order 4 is **one.**

**Theorem 5.1.2:**

The Probability measure of the Generalized Non reduced Farey N – subsequence of odd order is one.

**Proof:**

Construction of measurable sets from Non reduced Farey $N$ – subsequence odd order is

$$\tilde{F}_{(2m-1)^n} = \left\{\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}, \cdots\cdots, \frac{(2m-1)^n-3}{(2m-1)^n}, \frac{(2m-1)^n-2}{(2m-1)^n}, \frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right\}$$

$$= \left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right] \cup \left[\frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}\right] \cup \cdots\cdots \cup \left[\frac{(2m-1)^n-3}{(2m-1)^n}, \frac{(2m-1)^n-2}{(2m-1)^n}\right] \cup \left[\frac{(2m-1)^n-2}{(2m-1)^n}, \frac{(2m-1)^n-1}{(2m-1)^n}\right] \cup \left[\frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right]$$

$$HC\ \tilde{F}_{(2m-1)^n} = \left\{\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}, \cdots, \frac{(2m-1)^n-2}{(2m-1)^n}, \frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right\}$$

$$= \left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right) \cup \left[\frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}\right) \cup \cdots\cdots \cup \left[\frac{(2m-1)^n-3}{(2m-1)^n}, \frac{(2m-1)^n-2}{(2m-1)^n}\right) \cup$$

$$\left[\frac{(2m-1)^n-2}{(2m-1)^n}, \frac{(2m-1)^n-1}{(2m-1)^n}\right) \cup \left[\frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right]$$

Here Probability of an event in $HC\ \tilde{F}_{(2m-1)^n}$ is taken as follows

$$P\left(\left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right)\right) = \text{Length of the interval } \left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right)$$

$$\cong \frac{1}{(2m-1)^n}$$

$$P\left(\left[\frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}\right)\right) = \text{Length of the interval } \left[\frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}\right)$$

$$\cong \frac{1}{(2m-1)^n}$$

$$P\left(\left[\frac{2}{(2m-1)^n}, \frac{3}{(2m-1)^n}\right)\right) = \text{Length of the interval } \left[\frac{2}{(2m-1)^n}, \frac{3}{(2m-1)^n}\right)$$

$$\cong \frac{1}{(2m-1)^n}$$

For the $n^{th}$ term

$$P\left(\left[\frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right]\right) = \text{Length of the intervals } \left[\frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right]$$

$$\cong \frac{1}{(2m-1)^n}$$

For union of intervals consider only consecutive points in

$$\left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right), \left[\frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}\right), \cdots, \left[\frac{(2m-1)^n-3}{(2m-1)^n}, \frac{(2m-1)^n-2}{(2m-1)^n}\right), \left[\frac{(2m-1)^n-2}{(2m-1)^n}, \frac{(2m-1)^n-1}{(2m-1)^n}\right),$$

$$\left[\frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right]$$

Take any two elements say, $A_1 = \left[\frac{6}{(2m-1)^n}, \frac{7}{(2m-1)^n}\right)$ and $A_2 = \left[\frac{7}{(2m-1)^n}, \frac{8}{(2m-1)^n}\right)$

$$P(A_1 \cup A_2) = P\left(\left[\frac{6}{(2m-1)^n}, \frac{7}{(2m-1)^n}\right) \cup \left[\frac{7}{(2m-1)^n}, \frac{8}{(2m-1)^n}\right)\right)$$

$$= P\left(\left[\frac{6}{(2m-1)^n}, \frac{8}{(2m-1)^n}\right)\right)$$

$$\cong \frac{2}{(2m-1)^n}$$

$$P(A_1) + P(A_2) = P\left(\left[\frac{6}{(2m-1)^n}, \frac{7}{(2m-1)^n}\right)\right) + P\left(\left[\frac{7}{(2m-1)^n}, \frac{8}{(2m-1)^n}\right)\right)$$

$$\cong \frac{2}{(2m-1)^n}$$

Therefore $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

Now

Probability Measure $HC\ \tilde{F}_{(2m)^n}$

$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

$$P(\bigcup_{i=1}^{n} A_i) = P(A_1 \cup A_2 \cup \cdots\cdots \cup A_n)$$

$$= P\left(\left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right) \cup \left[\frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}\right) \cup \cdots \cup \left[\frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right]\right)$$

$$= P\left(\left[\frac{0}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right]\right) = 1$$

$$\sum_{i=1}^{n} P(A_i) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

$$= P\left[\left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right)\right] + P\left(\left[\frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}\right)\right) + \cdots + P\left(\left[\frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right]\right)$$

$$\cong \frac{1}{(2m-1)^n} + \frac{1}{(2m-1)^n} + \frac{1}{(2m-1)^n} + \cdots\cdots + \frac{1}{(2m-1)^n}$$

$$\cong \frac{1}{(2m-1)^n}\ (1 + 1 + 1 + \cdots\cdots + (2m-1)^n\ times)$$

$$\cong \frac{(2m-1)^n}{(2m-1)^n} = 1$$

Therefore

Probability Measure of Non - Reduced Farey $N$- Subsequence of order $(2m-1)$ is **one.**

**Illustration 5.1.2: $m$ is odd**

$$\tilde{F}_{(2m-1)^n} = \left\{\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}, \cdots, \frac{(2m-1)^n-2}{(2m-1)^n}, \frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right\}$$

$$HC\ \tilde{F}_{(2m-1)^n} = \left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right) \cup \left[\frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}\right) \cup \cdots\cdots \cup \left[\frac{(2m-1)^n-3}{(2m-1)^n}, \frac{(2m-1)^n-2}{(2m-1)^n}\right) \cup \left[\frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right]$$

Put $m = 2, n = 1, l = 3$

$$HC\ \tilde{F}_3 = \left\{\frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}\right\}$$

$$= \left[\frac{0}{3}, \frac{1}{3}\right) \cup \left[\frac{1}{3}, \frac{2}{3}\right) \cup \left[\frac{2}{3}, \frac{3}{3}\right]$$

Now

$$P\left(\cup_{i=1}^{l} A_i\right) = \sum_{i=1}^{l} P(A_i)$$

$$P(\cup_{i=1}^{3} A_i) = \sum_{i=1}^{3} P(A_i)$$

$$P(\cup_{i=1}^{3} A_i) = P(A_1 \cup A_2 \cup A_3)$$

$$= P\left(\left[\frac{0}{3}, \frac{1}{3}\right) \cup \left[\frac{1}{3}, \frac{2}{3}\right) \cup \left[\frac{2}{3}, \frac{3}{3}\right]\right)$$

$$= P\left(\left[\frac{0}{3}, \frac{3}{3}\right]\right) = 1$$

$$\sum_{i=1}^{l} P(A_i) = P(A_1) + P(A_2) + P(A_3)$$

$$= P\left(\left[\frac{0}{3}, \frac{1}{3}\right)\right) + P\left(\left[\frac{1}{3}, \frac{2}{3}\right)\right) + P\left(\left[\frac{2}{3}, \frac{3}{3}\right]\right) \cong \frac{3}{3} = 1$$

Probability Measure $HC\ \tilde{F}_3 = 1$

Therefore the probability measure of Non - Reduced Farey $N$- Subsequence of order 3 is **one.**

## 5.2 Invariant Measure of Generalized Non - Reduced Farey $N$- Subsequence:

**Definition 5.2.1: Invariant Measure [48]**

Let ( $[0,1]$ , $\tilde{F}_{m^n}$ ) be a measurable space and let $g : [0,1] \rightarrow [0,1]$ be a measurable function from [0,1] to itself. A measure $\mu$ on ($[0,1]$ , $\tilde{F}_{m^n}$) is said to be invariant under $g$, if for every measurable set $A$ in $\tilde{F}_{m^n}$.

$$\mu\left(g^{-1}(A)\right) = \mu(A)$$

**Theorem 5.2.1:**

The invariant measure of the Generalized Non reduced Farey $N$ – subsequence of even order is one.

**Proof:**

Define the function $g$ as $g\ (\tilde{F}_{(2m)^n}) = C\tilde{F}_{(2m)^n}$

where $C\tilde{F}_{(2m)^n}$ is nothing but Non – Reduced Farey $N$ – Subsequence taken as union of closed sets.

Construction of measurable sets from Non reduced Farey $N$ – subsequence even order Non – Reduce Farey $N$ – Subsequence of even order is

$$\tilde{F}_{(2m)^n} = \left\{\frac{0}{(2m)^n}, \frac{1}{(2m)^n}, \frac{2}{(2m)^n}, \cdots\cdots, \frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right\} ; \text{where } 1 \leq m \leq N$$

$$C\tilde{F}_{(2m)^n} = \left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right] \cup \left[\frac{1}{(2m)^n}, \frac{2}{(2m)^n}\right] \cup \cdots\cdots \cup \left[\frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}\right] \cup$$
$$\left[\frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}\right] \cup \left[\frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right]$$

Let $A$ be taken as union of any two intervals

$$A = \left[\frac{0}{(2\mathrm{m})^{\mathrm{n}+1}}, \frac{1}{(2\mathrm{m})^{\mathrm{n}+1}}\right] \cup \left[\frac{2}{(2\mathrm{m})^{\mathrm{n}+1}}, \frac{3}{(2\mathrm{m})^{\mathrm{n}+1}}\right]$$

Then,

$$\mu(A) = \mu\left(\left[\frac{0}{(2\mathrm{m})^{\mathrm{n}+1}}, \frac{1}{(2\mathrm{m})^{\mathrm{n}+1}}\right]\right) + \mu\left(\left[\frac{2}{(2\mathrm{m})^{\mathrm{n}+1}}, \frac{3}{(2\mathrm{m})^{\mathrm{n}+1}}\right]\right)$$

$$= \int_{\frac{0}{(2\mathrm{m})^{\mathrm{n}+1}}}^{\frac{1}{(2\mathrm{m})^{\mathrm{n}+1}}} \frac{(2\mathrm{m})^{\mathrm{n}+1}}{2}\, d\mu + \int_{\frac{2}{(2\mathrm{m})^{\mathrm{n}+1}}}^{\frac{3}{(2\mathrm{m})^{\mathrm{n}+1}}} \frac{(2\mathrm{m})^{\mathrm{n}+1}}{2}\, d\mu$$

$$= \frac{(2\mathrm{m})^{\mathrm{n}+1}}{2}\left[\frac{1}{(2\mathrm{m})^{\mathrm{n}+1}} - \frac{0}{(2\mathrm{m})^{\mathrm{n}+1}}\right] + \frac{(2\mathrm{m})^{\mathrm{n}+1}}{2}\left[\frac{3}{(2\mathrm{m})^{\mathrm{n}+1}} - \frac{2}{(2\mathrm{m})^{\mathrm{n}+1}}\right]$$

$$= \frac{(2\mathrm{m})^{\mathrm{n}+1}}{2}\left[\frac{1}{(2\mathrm{m})^{\mathrm{n}+1}}\right] + \frac{(2\mathrm{m})^{\mathrm{n}+1}}{2}\left[\frac{1}{(2\mathrm{m})^{\mathrm{n}+1}}\right] = 1$$

Let $A = \left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right]$

$$\mu\left(g^{-1}(A)\right) = \mu\left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right]$$

$$= \int_{\frac{0}{(2m)^n}}^{\frac{1}{(2m)^n}} \frac{(2\mathrm{m})^{\mathrm{n}}}{1}\, d\mu$$

$$= (2\mathrm{m})^{\mathrm{n}}\left[\frac{1}{(2m)^n} - \frac{0}{(2m)^n}\right]$$

$$= (2\mathrm{m})^{\mathrm{n}}\left[\frac{1}{(2m)^n}\right] = 1$$

Therefore

$$\mu(A) = \mu\left(g^{-1}(A)\right)$$

Where $\mu$ is an invariant Measure for Non – Reduce to Farey $N$ – Sub sequence of even order $m$ with respect to $g$.

Invariant Measure for Non – Reduce to Farey $N$ – Sub sequence of even order is **one.**

**Illustration 5.2.1:**

Non – Reduce to Farey $N$ – Subsequence of even order is

$$\tilde{F}_{(2m)^n} = \left\{\frac{0}{(2m)^n}, \frac{1}{(2m)^n}, \frac{2}{(2m)^n}, \cdots\cdots, \frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right\}; \text{ where } 1 \le m \le N$$

$$C\tilde{F}_{(2m)^n} = \left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right] \cup \left[\frac{1}{(2m)^n}, \frac{2}{(2m)^n}\right] \cup \cdots\cdots \cup \left[\frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}\right]$$

$$\cup \left[\frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}\right] \cup \left[\frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right]$$

Let $A = \left[\frac{0}{(2\mathrm{m})^{\mathrm{n}+1}}, \frac{1}{(2\mathrm{m})^{\mathrm{n}+1}}\right] \cup \left[\frac{2}{(2\mathrm{m})^{\mathrm{n}+1}}, \frac{3}{(2\mathrm{m})^{\mathrm{n}+1}}\right]$

Take $m = 2, n = 1$;

$$A = \left[\frac{0}{(4)^{1+1}}, \frac{1}{(4)^{1+1}}\right] \cup \left[\frac{2}{(4)^{1+1}}, \frac{3}{(4)^{1+1}}\right]$$

$$= \left[\frac{0}{(4)^2}, \frac{1}{(4)^2}\right] \cup \left[\frac{2}{(4)^2}, \frac{3}{(4)^2}\right]$$

Then,

$$\mu(A) = \mu\left(\left[\frac{0}{(4)^2}, \frac{1}{(4)^2}\right]\right) + \mu\left(\left[\frac{2}{(4)^2}, \frac{3}{(4)^2}\right]\right)$$

$$= \int_{\frac{0}{(4)^2}}^{\frac{1}{(4)^2}} \frac{(4)^2}{2}\, d\mu + \int_{\frac{2}{(4)^2}}^{\frac{3}{(4)^2}} \frac{(4)^2}{2}\, d\mu$$

$$= \frac{(4)^2}{2}\left[\frac{1}{(4)^2} - \frac{0}{(4)^2}\right] + \frac{(4)^2}{2}\left[\frac{3}{(4)^2} - \frac{2}{(4)^2}\right]$$

$$= \frac{(4)^2}{2}\left[\frac{1}{(4)^2}\right] + \frac{(4)^2}{2}\left[\frac{1}{(4)^2}\right] = 1$$

Let $A = \left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right]$

Take $m = 2, n = 1$

$$\mu(g^{-1}(A)) = \mu\left[\frac{0}{(4)^1}, \frac{1}{(4)^1}\right]$$

$$= \int_{\frac{0}{(4)^1}}^{\frac{1}{(4)^1}} \frac{(4)^1}{1} d\mu = 1$$

Therefore

$$\mu(A) = \mu(g^{-1}(A))$$

where $\mu$ is an invariant Measure for Non – Reduce to Farey $N$ – Sub sequence of even order $m$ with respect to $g$.

Invariant Measure for Non – Reduce to Farey $N$ – Sub sequence of order 4 is **one.**

**Theorem 5. 2.2:**

The invariant measure of the Generalized Non reduced Farey $N$ – subsequence of odd order is one.

**Proof:**

Define a function $g\ (\tilde{F}_{(2m-1)^n}) = C\tilde{F}_{(2m-1)^n}$

where $C\tilde{F}_{(2m-1)^n}$ – Non – Reduced Farey $N$ – Subsequence as union of closed set.

Construction of measurable sets from Non reduced Farey $N$ – subsequence odd order

Non – Reduce to Farey $N$ – Subsequence of odd order is

$$\tilde{F}_{(2m-1)^n} = \left\{\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}, \cdots, \frac{(2m-1)^n-3}{(2m-1)^n}, \frac{(2m-1)^n-2}{(2m-1)^n}, \frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right\};$$

where $1 \leq m \leq N$

$$C\tilde{F}_{(2m-1)^n} = \left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right] \cup \left[\frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}\right] \cup \cdots\cdots \cup$$

$$\left[\frac{(2m-1)^n-3}{(2m-1)^n}, \frac{(2m-1)^n-2}{(2m-1)^n}\right] \cup \left[\frac{(2m-1)^n-2}{(2m-1)^n}, \frac{(2m-1)^n-1}{(2m-1)^n}\right] \cup \left[\frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right]$$

Let $A = \left[\frac{0}{(2m-1)^{n+1}}, \frac{1}{(2m-1)^{n+1}}\right] \cup \left[\frac{2}{(2m-1)^{n+1}}, \frac{3}{(2m-1)^{n+1}}\right]$

Then,

$$\mu(A) = \mu\left(\left[\frac{0}{(2m-1)^{n+1}}, \frac{1}{(2m-1)^{n+1}}\right]\right) + \mu\left(\left[\frac{2}{(2m-1)^{n+1}}, \frac{3}{(2m-1)^{n+1}}\right]\right)$$

$$= \int_{\frac{0}{(2m-1)^{n+1}}}^{\frac{1}{(2m-1)^{n+1}}} \frac{(2m-1)^{n+1}}{2}\, d\mu + \int_{\frac{2}{(2m-1)^{n+1}}}^{\frac{3}{(2m-1)^{n+1}}} \frac{(2m-1)^{n+1}}{2}\, d\mu$$

$$= \frac{(2m-1)^{n+1}}{2}\left[\frac{1}{(2m-1)^{n+1}} - \frac{0}{(2m-1)^{n+1}}\right] + \frac{(2m-1)^{n+1}}{2}\left[\frac{3}{(2m-1)^{n+1}} - \frac{2}{(2m-1)^{n+1}}\right]$$

$$= \frac{(2m-1)^{n+1}}{2}\left[\frac{1}{(2m-1)^{n+1}}\right] + \frac{(2m-1)^{n+1}}{2}\left[\frac{1}{(2m-1)^{n+1}}\right] = 1$$

Let $A = \left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right]$

$$\mu\left(g^{-1}(A)\right) = \mu\left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right]$$

$$= \int_{\frac{0}{(2m-1)^n}}^{\frac{1}{(2m-1)^n}} \frac{(2m-1)^n}{1}\, d\mu$$

$$= (2m-1)^n \left[\frac{1}{(2m-1)^n} - \frac{0}{(2m-1)^n}\right]$$

$$= (2m-1)^n \left[\frac{1}{(2m-1)^n}\right] = 1$$

Therefore

$$\mu(A) = \mu\left(g^{-1}(A)\right)$$

where $\mu$ is an invariant Measure for Non – Reduce to Farey $N$ – Sub sequence of odd order $m$ with respect to $g$.

Invariant Measure for Non – Reduce to Farey $N$ – Sub sequence of odd order is **one.**

**Illustration 5.2.2:**

Non – Reduce to Farey $N$ – Subsequence of odd order is

$$\tilde{F}_{(2m-1)^n} = \left\{\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}, \cdots, \frac{(2m-1)^n-3}{(2m-1)^n}, \frac{(2m-1)^n-2}{(2m-1)^n}, \frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right\} \quad ;$$

where $1 \leq m \leq N$

$$C\tilde{F}_{(2m-1)^n} = \left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right] \cup \left[\frac{1}{(2m-1)^n}, \frac{2}{(2m-1)^n}\right] \cup \cdots \cdots \cup \left[\frac{(2m-1)^n-3}{(2m-1)^n}, \frac{(2m-1)^n-2}{(2m-1)^n}\right] \cup$$

$$\left[\frac{(2m-1)^n-2}{(2m-1)^n}, \frac{(2m-1)^n-1}{(2m-1)^n}\right] \cup \left[\frac{(2m-1)^n-1}{(2m-1)^n}, \frac{(2m-1)^n}{(2m-1)^n}\right]$$

Let $A = \left[\frac{0}{(2\mathrm{m}-1)^{\mathrm{n}+1}}, \frac{1}{(2\mathrm{m}-1)^{\mathrm{n}+1}}\right] \cup \left[\frac{2}{(2\mathrm{m}-1)^{\mathrm{n}+1}}, \frac{3}{(2\mathrm{m}-1)^{\mathrm{n}+1}}\right]$

Take $m = 2, n = 1$; By the equation $\mu\big(g^{-1}(A)\big) = \mu(A)$

$$A = \left[\frac{0}{(3)^{1+1}}, \frac{1}{(3)^{1+1}}\right] \cup \left[\frac{2}{(3)^{1+1}}, \frac{3}{(3)^{1+1}}\right]$$

$$= \left[\frac{0}{(3)^2}, \frac{1}{(3)^2}\right] \cup \left[\frac{2}{(3)^2}, \frac{3}{(3)^2}\right]$$

Then,

$$\mu(A) = \mu\left(\left[\frac{0}{(3)^2}, \frac{1}{(3)^2}\right]\right) + \mu\left(\left[\frac{2}{(3)^2}, \frac{3}{(3)^2}\right]\right)$$

$$= \int_{\frac{0}{(3)^2}}^{\frac{1}{(3)^2}} \frac{(3)^2}{2}\, d\mu + \int_{\frac{2}{(3)^2}}^{\frac{3}{(3)^2}} \frac{(3)^2}{2}\, d\mu$$

$$= \frac{(3)^2}{2}\left[\frac{1}{(3)^2} - \frac{0}{(3)^2}\right] + \frac{(3)^2}{2}\left[\frac{3}{(3)^2} - \frac{2}{(3)^2}\right]$$

$$= \frac{(3)^2}{2}\left[\frac{1}{(3)^2}\right] + \frac{(3)^2}{2}\left[\frac{1}{(3)^2}\right] = 1$$

Let $A = \left[\frac{0}{(2m-1)^n}, \frac{1}{(2m-1)^n}\right]$

Take $m = 2, n = 1$

$$\mu\left(g^{-1}(A)\right) = \mu\left[\frac{0}{(3)^1}, \frac{1}{(3)^1}\right]$$

$$= \int_{\frac{0}{(3)^1}}^{\frac{1}{(3)^1}} \frac{(3)^1}{1} d\mu = 1$$

Therefore

$$\mu(A) = \mu\left(g^{-1}(A)\right)$$

where $\mu$ is an invariant Measure for Non – Reduce to Farey $N$ – Sub sequence of odd order $m$ with respect to $g$.

Hence, invariant Measure for Non – Reduce to Farey $N$ – Sub sequence of order 3 is **one.**

# CHAPTER –VI

# MEASURE OF MODIFIED EVEN ORDERED CANTOR SETS

This chapter deals with the Box Measure of Modified even ordered Cantor sets. While calculating box measure, boxes without dimensions are used in general but here, boxes are replaced by isosceles triangles and their areas are considred as measures. This chapter is divided into four sections.

Section 6.1 deals with Measure of Cantor Hexnary Sets and Section 6.2 is on Measure of Cantor Deca Sets.Varying from the previous sections, section 6.3 provides a measure for Cantor Octanary Sets and following the similar lines Section 6.4 gives a measure of Cantor Dodeca Sets.

## 6.1 Measure of Cantor Hexnary Sets:

**Definition 6.1.1: Triangular Measure of Cantor Hexnary Sets**

The closed interval $[0, 1]$ is divided into six equal parts ,second and fifth intervals are removed. The left and right intervals of length $\left(\frac{1}{6}\right)$ and middle interval of length $\left(\frac{2}{6}\right)$ is retained. Draw a smaller triangle of retained intervals and calculate he area of triangle. This is known as **Triangular Measure of Cantor Hexnary Sets** and it is denoted by $TMC_{\left(\frac{1}{6}\right)}$ .

**Theorem 6.1.1:**

The Box measure of Cantor sets can be converted into triangular measure and it is given by

$$TMC_{\left(\frac{1}{6}\right)^n} = \left(\frac{1}{2}\right)\left(\frac{1}{6^n}\right)^2 [2^n + nc_1 2^{n-1} + nc_2 2^{n-2} + \cdots\cdots + nc_r 2^{n-r} + \cdots\cdots + nc_{n-1} 2^{2n-1} + 2^{2n}] .$$

The areas follows Geometric Progression with common ratio $\left(\frac{1}{6}\right)$.

**Proof:**

Proof of the theorem follows by induction method.

The closed interval $[0,1]$ is divided into six equal intervals. Following the theory of Cantor Hexnary set, the open intervals $\left(\frac{1}{6},\frac{2}{6}\right)$ and $\left(\frac{4}{6},\frac{5}{6}\right)$ are removed. The remaining parts $\left[\frac{0}{6}=0,\frac{1}{6}\right], \left[\frac{2}{6},\frac{4}{6}\right]$ and $\left[\frac{5}{6},\frac{6}{6}=1\right]$ are again subdivided as follows. The length of the middlemost part is $2/6$. For the 2nd iteration, the parts $\left[\frac{0}{6}=0,\frac{1}{6}\right]$ and $\left[\frac{5}{6},\frac{6}{6}\right]$ are each divided into six equal intervals there by giving six intervals

$$\left[\frac{0}{36}=0,\frac{1}{36}\right], \left[\frac{1}{36},\frac{2}{36}\right], \left[\frac{2}{36},\frac{3}{36}\right], \left[\frac{3}{36},\frac{4}{36}\right], \left[\frac{4}{36},\frac{5}{36}\right], \left[\frac{5}{36},\frac{6}{36}\right]$$

and $$\left[\frac{30}{36},\frac{31}{36}\right], \left[\frac{31}{36},\frac{32}{36}\right], \left[\frac{32}{36},\frac{33}{36}\right], \left[\frac{33}{36},\frac{34}{36}\right], \left[\frac{34}{36},\frac{35}{36}\right], \left[\frac{35}{36},\frac{36}{36}\right]$$

respectively. The open intervals $\left(\frac{1}{36},\frac{2}{36}\right)$ and $\left(\frac{4}{36},\frac{5}{36}\right)$ are removed. Draw a smaller triangular of retained intervals and calculate area of triangle.

When $n = 1$,

$$TMC_{\left(\frac{1}{6}\right)^n} = \left(\frac{1}{2}\right)\left(\frac{1}{6^n}\right)^2 [2^n + nc_1 2^{n-1} + nc_2 2^{n-2} + \cdots\cdots + nc_r 2^{n-r} + \cdots\cdots + nc_{n-1} 2^{2n-1} + 2^{2n}]$$

$$TMC_{\left(\frac{1}{6}\right)^1} = \left(\frac{1}{2}\right)\left(\frac{1}{6^1}\right)^2 [2^1 + 2^{2(1)}] = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right)$$

Therefore the area of triangle is $\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)$

When $n = 2$,

$$TMC_{\left(\frac{1}{6}\right)^n} = \left(\frac{1}{2}\right)\left(\frac{1}{6^n}\right)^2 [2^n + nc_1 2^{n-1} + nc_2 2^{n-2} + \cdots\cdots + nc_r 2^{n-r} + \cdots\cdots + nc_{n-1} 2^{2n-1} + 2^{2n}]$$

$$TMC_{\left(\frac{1}{6}\right)^2} = \left(\frac{1}{2}\right)\left(\frac{1}{6^2}\right)^2 \left[2^2 + 2c_1 2^{4-1} + 2^{2(2)}\right]$$

$$= [2^2 + 2.2^3 + 2^4] = \left(\frac{1}{2}\right)\left(\frac{1}{6^2}\right)$$

The area of triangle is $\left(\frac{1}{2}\right)\left(\frac{1}{6^2}\right)$

Assume the result for $n = e$

$$TMC_{\left(\frac{1}{6}\right)^n} = \left(\frac{1}{2}\right)\left(\frac{1}{6^n}\right)^2 [2^n + nc_1 2^{n-1} + nc_2 2^{n-2} + \cdots\cdots + nc_r 2^{n-r} + \cdots\cdots + nc_{n-1} 2^{2n-1} + 2^{2n}]$$

$$TMC_{\left(\frac{1}{6}\right)^e} = \left(\frac{1}{2}\right)\left(\frac{1}{6^e}\right)^2 \left[2^e + ec_{e-1} 2^{2e-1} + 2^{2(e)}\right]$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{6^e}\right)^2 [6^e] = \left(\frac{1}{2}\right)\left(\frac{1}{6^e}\right)$$

We prove the result for $n = e + 1$

Now

$$TMC_{\left(\frac{1}{6}\right)^n} = \left(\frac{1}{2}\right)\left(\frac{1}{6^n}\right)^2 [2^n + nc_1 2^{n-1} + nc_2 2^{n-2} + \cdots\cdots + nc_r 2^{n-r} + \cdots\cdots + nc_{n-1} 2^{2n-1} + 2^{2n}]$$

$$TMC_{\left(\frac{1}{6}\right)^{e+1}} = \left(\frac{1}{2}\right)\left(\frac{1}{6^{e+1}}\right)^2 \left[2^{(e+1)} + (e+1)c_{e+1-1} 2^{2(e+1)-1} + 2^{2(e+1)}\right]$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{6^{e+1}}\right)^2 [6^{e+1}] = \left(\frac{1}{2}\right)\left(\frac{1}{6^{e+1}}\right)$$

Hence the result is true for all positive integers

**Graphical Representation:**

The Following **Figure 6.1.1** shows the graphical representation of Triangular Measure of Cantor Hexnary Sets.

**First iteration:**

The closed interval $[0,1]$ is subdivided into 6 equal sub- intervals $\left[\frac{0}{6}=0,\frac{1}{6}\right]$, $\left[\frac{1}{6},\frac{2}{6}\right]$, $\left[\frac{2}{6},\frac{3}{6}\right]$, $\left[\frac{3}{6},\frac{4}{6}\right]$, $\left[\frac{4}{6},\frac{5}{6}\right]$,

$\left[\frac{5}{6},\frac{6}{6}=1\right]$

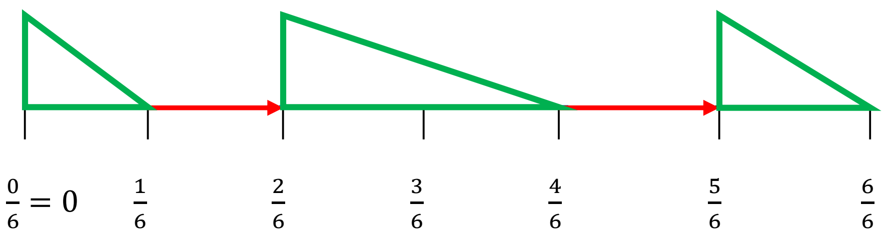



**Figure 6.1.1:** Triangular Measure of Cantor Hexnary Sets

The intervals removed are $\left(\frac{1}{6},\ \frac{2}{6}\right)$, $\left(\frac{4}{6},\ \frac{5}{6}\right)$ .

| **Length** | **No.of Triangle** |
|---|---|
| $\left(\frac{1}{6}\right)$ | 2 |
| $\left(\frac{2}{6}\right)$ | 1 |

**Table 6.1.1**

The above table 6.1.1 shows the number of triangle appear in the first iteration.

**Second iteration:**

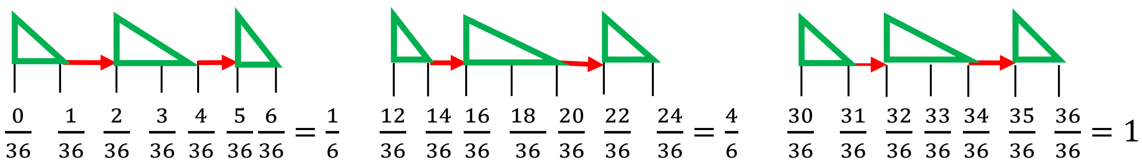



**Figure 6.1.2:** Second iteration – Triangular Measure of Cantor Hexnary Sets

The intervals removed are $\left(\frac{1}{36}, \frac{2}{36}\right), \left(\frac{5}{36}, \frac{6}{36}\right), \left(\frac{14}{36}, \frac{16}{36}\right), \left(\frac{20}{36}, \frac{22}{36}\right), \left(\frac{31}{36}, \frac{32}{36}\right), \left(\frac{34}{36}, \frac{35}{36}\right)$

| **Length** | **No.of Triangle** |
|---|---|
| $\left(\frac{1}{36}\right)$ | 4 |
| $\left(\frac{2}{36}\right)$ | 4 |
| $\left(\frac{4}{36}\right)$ | 1 |

**Table 6.1.2**

The above table 6.1.2 shows the number of triangle appear in the second iteration.

## 6.2 Measure of Cantor Deca Sets:

**Definition 6.2.1: Triangular Measure of Cantor Deca Sets:**

The closed interval $[0, 1]$ is divided into ten equal parts. Second, fourth, seventh and ninth intervals are removed. The left and right intervals of length $\left(\frac{1}{10}\right)$ and middle interval of length $\left(\frac{2}{10}\right)$ is retained. Draw a smaller triangle of retained intervals and calculating the area of triangle. This is known as **Triangular Measure of Cantor Deca Sets** and it is denoted by $TMC_{\left(\frac{1}{10}\right)}$.

**Graphical Representation:**

The Following **Figure 6.2.1** shows the graphical representation of Triangular Measure of Cantor Deca Sets.

**First iteration:**

The closed interval $[0,1]$ is subdivided into 10 equal sub- intervals

$$\left[\frac{0}{10}=0,\frac{1}{10}\right],\left[\frac{1}{10},\frac{2}{10}\right],\left[\frac{2}{10},\frac{3}{10}\right],\left[\frac{3}{10},\frac{4}{10}\right],\left[\frac{4}{10},\frac{5}{10}\right],\left[\frac{5}{10},\frac{6}{10}\right],\left[\frac{7}{10},\frac{8}{10}\right],\left[\frac{9}{10},\frac{10}{10}\right]$$

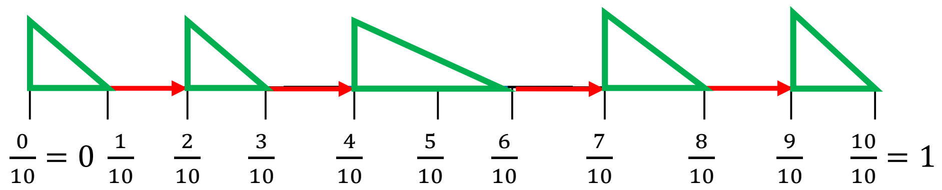



**Figure 6.2.1:** Triangular Measure of Cantor Deca Sets

The intervals removed are $\left(\frac{1}{10},\ \frac{2}{10}\right),\left(\frac{3}{10},\ \frac{4}{10}\right),\left(\frac{6}{10},\ \frac{7}{10}\right),\left(\frac{8}{10},\ \frac{9}{10}\right)$ .

The remaining intervals are $\left[\frac{0}{10}=0,\frac{1}{10}\right]\cup\left[\frac{2}{10},\frac{3}{10}\right]\cup\left[\frac{\mathbf{4}}{\mathbf{10}},\frac{\mathbf{6}}{\mathbf{10}}\right]\cup\left[\frac{7}{10},\frac{8}{10}\right]\cup\left[\frac{9}{10},\frac{10}{10}=1\right]$

| **Length** | **No.of Triangle** |
|---|---|
| $\left(\frac{1}{10}\right)$ | 4 |
| $\left(\frac{2}{10}\right)$ | 1 |

**Table 6.2.1**

The above table 6.2.1 shows the number of triangle appear in the first iteration.

**Second iteration:**

The retained intervals are $\left[\frac{0}{10}=0,\frac{1}{10}\right],\left[\frac{2}{10},\frac{3}{10}\right],\left[\frac{\mathbf{4}}{\mathbf{10}},\frac{\mathbf{6}}{\mathbf{10}}\right],\left[\frac{7}{10},\frac{8}{10}\right]\left[\frac{9}{10},\frac{10}{10}=1\right]$ each closed interval is subdivided into 10 equal sub- intervals

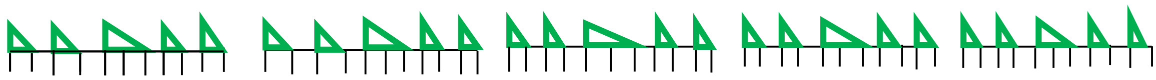


**Figure 6.2.2:** Second iteration -Triangular Measure of Cantor Deca Sets

The intervals removed are $\left(\frac{1}{100},\frac{2}{100}\right),\left(\frac{3}{100},\frac{4}{100}\right),\left(\frac{6}{100},\frac{7}{100}\right),\left(\frac{8}{100},\frac{9}{100}\right),\left(\frac{21}{100},\frac{22}{100}\right),$ $\left(\frac{23}{100},\frac{24}{100}\right),\left(\frac{26}{100},\frac{27}{100}\right),\left(\frac{28}{100},\frac{29}{100}\right),\left(\frac{42}{100},\frac{44}{100}\right),\left(\frac{46}{100},\frac{48}{100}\right),\left(\frac{52}{100},\frac{54}{100}\right),\left(\frac{56}{100},\frac{58}{100}\right),$ $\left(\frac{71}{100},\frac{72}{100}\right),\left(\frac{73}{100},\frac{74}{100}\right),\left(\frac{76}{100},\frac{77}{100}\right),\left(\frac{78}{100},\frac{79}{100}\right),\left(\frac{91}{100},\frac{92}{100}\right),\left(\frac{93}{100},\frac{94}{100}\right),\left(\frac{96}{100},\frac{97}{100}\right),$ $\left(\frac{98}{100},\frac{99}{100}\right)$ .

| **Length** | **No.of Triangle** |
|---|---|
| $\left(\frac{1}{100}\right)$ | 16 |
| $\left(\frac{2}{100}\right)$ | 8 |
| $\left(\frac{4}{100}\right)$ | 1 |

**Table 6.2.2**

The above table 6.2.2 shows the number of triangle appear in the second iteration.

**Theorem 6.2.1:**

The Triangular Measure of Cantor Deca Sets it is given by

$$TMC_{\left(\frac{1}{10}\right)^n} = \left(\frac{1}{2}\right)\left(\frac{1}{10^n}\right)^2 [2^{2n} + nc_1 2^{2n} + nc_2 2^{2n} + \cdots + nc_r 2^{2n} + \cdots\cdots + nc_{n-1} 2^{2n} + 2^{2n}] .$$

The areas follows Geometric Progression with common ratio $\left(\frac{1}{10}\right)$.

**Proof:**

The theorem is proved by induction method.

When $n = 1$,

$$TMC_{\left(\frac{1}{10}\right)^n} = \left(\frac{1}{2}\right)\left(\frac{1}{10^n}\right)^2 [2^{2n} + nc_1 2^{2n} + nc_2 2^{2n} + \cdots + nc_r 2^{2n} + \cdots\cdots + nc_{n-1} 2^{2n} + 2^{2n}]$$

$$TMC_{\left(\frac{1}{10}\right)^1} = \left(\frac{1}{2}\right)\left(\frac{1}{10^1}\right)^2 \left[2^2 + 2^{2(1)}\right] = \left(\frac{1}{10}\right)^2 2^2$$

Therefore

The area of triangle is $\left(\frac{1}{10}\right)^2 2^2$

$n = 1$ is true.

When $n = 2$,

$$TMC_{\left(\frac{1}{10}\right)^n} = \left(\frac{1}{2}\right)\left(\frac{1}{10^n}\right)^2 [2^{2n} + nc_1 2^{2n} + nc_2 2^{2n} + \cdots + nc_r 2^{2n} + \cdots\cdots + nc_{n-1} 2^{2n} + 2^{2n}]$$

$$TMC_{\left(\frac{1}{10}\right)^2} = \left(\frac{1}{2}\right)\left(\frac{1}{10^2}\right)^2 \left[2^4 + 2c_1 2^{4-1} + 2^{2(2)}\right]$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{10^2}\right)^2 [2^4 + 2.\,2^3 + 2^4] = \left(\frac{1}{10^2}\right)^2\ 2^5$$

Therefore

The area of triangle is $\left(\frac{1}{10^2}\right)^2 2^5$

Assume the result for $n = e$

$$TMC_{\left(\frac{1}{10}\right)^n} = \left(\frac{1}{2}\right)\left(\frac{1}{10^n}\right)^2 [2^{2n} + nc_1 2^{2n} + nc_2 2^{2n} + \cdots + nc_r 2^{2n} + \cdots\cdots + nc_{n-1} 2^{2n} + 2^{2n}]$$

$$TMC_{\left(\frac{1}{10}\right)^e} = \left(\frac{1}{2}\right)\left(\frac{1}{10^e}\right)^2 \left[2^{2e} + ec_{e-1} 2^{2e} + 2^{2(e)}\right] = \left(\frac{1}{10^e}\right)^2 2^{3e-1}$$

Therefore

The area of triangle is $\left(\frac{1}{10^e}\right)^2 2^{3e-1}$

We prove the result for $n = e + 1$

$$TMC_{\left(\frac{1}{10}\right)^n} = \left(\frac{1}{2}\right)\left(\frac{1}{10^n}\right)^2 [2^{2n} + nc_1 2^{2n} + nc_2 2^{2n} + \cdots + nc_r 2^{2n} + \cdots\cdots + nc_{n-1} 2^{2n} + 2^{2n}]$$

$$TMC_{\left(\frac{1}{10}\right)^{e+1}} = \left(\frac{1}{2}\right)\left(\frac{1}{10^{e+1}}\right)^2 \left[2^{2(e+1)} + (e+1)c_e 2^{2(e+1)} + 2^{2(e+1)}\right] = \left(\frac{1}{10^{e+1}}\right)^2 2^{3e+2}$$

Therefore the area of triangle is $\left(\frac{1}{10^{e+1}}\right)^2 2^{3e+2}$

Therefore the triangular measure of $TMC_{\left(\frac{1}{10}\right)^n} = \left(\frac{1}{10^n}\right)^2 2^{3n-1}$

## 6.3 Measure of Cantor Octanary Sets:

**Definition 6.3.1: Triangular Measure of Cantor Octanary Sets:**

The closed interval $[0, 1]$ is divided into eight equal parts. By removing the second part, last but one part and middlemost part, the open intervals $\left(\frac{1}{8}, \frac{2}{8}\right)$, $\left(\frac{3}{8}, \frac{5}{8}\right)$ and $\left(\frac{6}{8}, \frac{7}{8}\right)$ are removed. The middlemost removable interval is of length $\left(\frac{2}{8}\right)$. Each retained intervals are of length $\left(\frac{1}{8}\right)$. Continue the process indefinitely. Draw a smaller triangle of retained intervals and calculating the area of triangle the set obtained is known as the **Triangular Measure of Cantor Octanary Sets** and is denoted by $TMC_{\left(\frac{1}{8}\right)}$.

**Graphical Representation:**

The Following **Figure 6.3.1** shows the graphical representation of Triangular Measure of Cantor Octanary sets.

**First iteration:**

The closed interval $[0,1]$ is subdivided into 8 equal sub- intervals

$$\left[\frac{0}{8}=0,\frac{1}{8}\right],\left[\frac{1}{8},\frac{2}{8}\right],\left[\frac{2}{8},\frac{3}{8}\right],\left[\frac{3}{8},\frac{4}{8}\right],\left[\frac{4}{8},\frac{5}{8}\right],\left[\frac{5}{8},\frac{6}{8}\right],\left[\frac{6}{8},\frac{7}{8}\right],\left[\frac{7}{8},\frac{8}{8}=1\right]$$

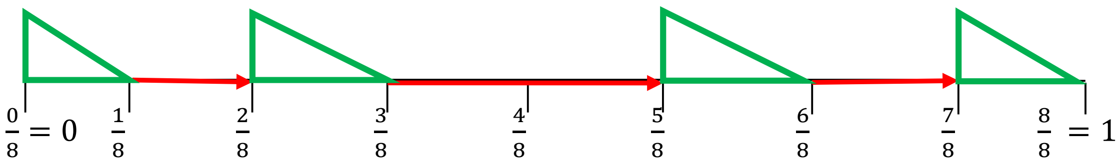



**Figure 6.3.1:** Triangular Measure of Cantor Octanary sets

The intervals removed are $\left(\frac{1}{8},\ \frac{2}{8}\right),\left(\frac{3}{8},\ \frac{5}{8}\right),\left(\frac{6}{8},\ \frac{7}{8}\right)$.

The remaining intervals are $\left[\frac{0}{8}=0,\frac{1}{8}\right],\left[\frac{2}{8},\frac{3}{8}\right],\left[\frac{5}{8},\frac{6}{8}\right]$ and $\left[\frac{7}{8},\frac{8}{8}=1\right]$

| **Length** | **No.of Triangle** |
|---|---|
| $\left(\frac{1}{8}\right)$ | 4 |

**Table 6.3.1**

The above table 6.3.1 shows the number of triangle appear in the first iteration.

**Second iteration:**

The remaining intervals are $\left[\frac{0}{8}=0, \frac{1}{8}\right], \left[\frac{2}{8}, \frac{3}{8}\right], \left[\frac{5}{8}, \frac{6}{8}\right]$ and $\left[\frac{7}{8}, \frac{8}{8}=1\right]$ subdivided into 8 equal sub- intervals.

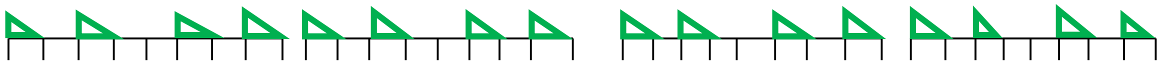


**Figure 6.3.2:** Second iteration - Triangular Measure of Cantor Octanary sets

| Length | No.of Triangle |
|---|---|
| $\left(\frac{1}{64}\right)$ | 16 |

**Table 6.3.2**

The above table 6.3.2 shows the number of triangle appear in the second iteration.

**First iterations:**

The area of triangle is $TMC_{\left(\frac{1}{8}\right)^1} = \left(\frac{1}{2}\right)\left(\frac{1}{8^1}\right)^2 (4)^1$

**Second iteration:**

The area of triangle is $TMC_{\left(\frac{1}{8}\right)^2} = \left(\frac{1}{2}\right)\left(\frac{1}{8^2}\right)^2 (4)^2$

Therefore the triangular measure of $TMC_{\left(\frac{1}{8}\right)^n} = \left(\frac{1}{2}\right)\left(\frac{1}{8^n}\right)^2 (4)^n$

## 6.4 Measure of Cantor Dodeca Sets:

The same procedure follows by the Triangular Measure of Cantor Dodeca Set.

**Definition 6.4.1: Triangular Measure of Cantor Dodeca Sets:**

The closed interval $[0, 1]$ is divided into twelve equal parts. By removing the second part, fourth part, ninth part, last but one part and middlemost part, the open intervals

$\left(\frac{1}{12}, \frac{2}{12}\right)$, $\left(\frac{3}{12}, \frac{4}{12}\right)$, $\left(\frac{5}{12}, \frac{7}{12}\right)$, $\left(\frac{8}{12}, \frac{9}{12}\right)$ and $\left(\frac{10}{12}, \frac{11}{12}\right)$ are removed. The middlemost removable interval is of length $\left(\frac{2}{12}\right)$. Other retained intervals are of length $\left(\frac{1}{12}\right)$. Continue the process indefinitely. Draw a smaller triangle of retained intervals and calculating the area of triangle the set obtained is known as the **Triangular Measure of Cantor Dodeca Sets** and is denoted by $TMC_{\left(\frac{1}{12}\right)}$.

**Graphical Representation:**

The Following **Figure 6.4.1** shows the graphical representation of Triangular Measure of Cantor Dodeca sets.

**First iteration:**

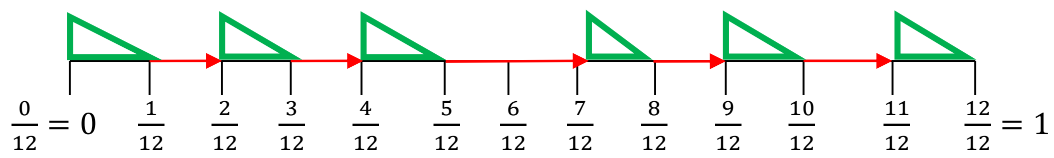



**Figure 6.4.1:** Triangular Measure Cantor Dodeca sets

The intervals removed are $\left(\frac{1}{12}, \frac{2}{12}\right), \left(\frac{3}{12}, \frac{4}{12}\right), \left(\frac{5}{12}, \frac{7}{12}\right), \left(\frac{8}{12}, \frac{9}{12}\right), \left(\frac{10}{12}, \frac{11}{12}\right)$ .

The remaining intervals are $\left[\frac{0}{12} = 0, \frac{1}{12}\right] \cup \left[\frac{2}{12}, \frac{3}{12}\right] \cup \left[\frac{4}{12}, \frac{5}{12}\right] \cup \left[\frac{7}{12}, \frac{8}{12}\right], \left[\frac{9}{12}, \frac{10}{12}\right] \cup \left[\frac{11}{12}, \frac{12}{12} = 1\right]$.

| **Length** | **No.of Triangle** |
|---|---|
| $\left(\frac{1}{12}\right)$ | 6 |

**Table 6.4.1**

The above table 6.4.1 shows the number of triangle appear in the first iteration.

**Second iteration:**

The remaining intervals are

$\left[\frac{0}{12}=0,\frac{1}{12}\right],\left[\frac{2}{12},\frac{3}{12}\right],\left[\frac{4}{12},\frac{5}{12}\right],\left[\frac{7}{12},\frac{8}{12}\right],\left[\frac{9}{12},\frac{10}{12}\right],\left[\frac{11}{12},\frac{12}{12}=1\right]$ subdivided into 12 equal sub- intervals.

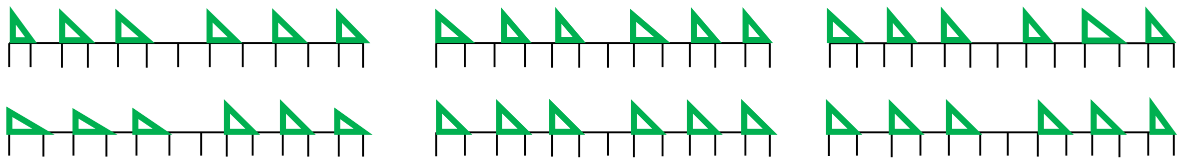


**Figure 6.4.2:** Second iteration- Triangular Measure Cantor Dodeca sets

| Length | No.of Triangle |
|---|---|
| $\left(\frac{1}{144}\right)$ | 36 |

**Table 6.4.2**

The above table 6.4.2 shows the number of triangle appear in the second iteration.

**First iteration:**

The area of triangle is $TMC_{\left(\frac{1}{12}\right)^1} = \left(\frac{1}{2}\right)\left(\frac{1}{12^1}\right)^2 (6)^1$

**Second iteration:**

The area of triangle is $TMC_{\left(\frac{1}{12}\right)^2} = \left(\frac{1}{2}\right)\left(\frac{1}{12^2}\right)^2 (6)^2$

Therefore the triangular measure of $TMC_{\left(\frac{1}{12}\right)^n} = \left(\frac{1}{2}\right)\left(\frac{1}{12^n}\right)^2 (6)^n$

# CONCLUSION

We have established modified Cantor $\left(\frac{1}{6}\right)$ and $\left(\frac{1}{8}\right)$ sets and corresponding sequential sets namely Cantor sets of order $6 + 4k$ and $8 + 4k$ , $k = 0,1,2,3, \cdots$ . Unlike Cantor ternary sets here removal of sets in various patterns are considered. One is usual away of removing intervals of lengths $1/2n$ and the middle most intervals of lengths $2/2n$. The process is continued successively, so that the general portion of removable intervals can be identified.

In this pattern of removal middle most interval in successive iteration follows a geometric sequence of powers of two. The intervals equally spaced from the middle to the left and right follows different nature as the iteration increases. Its characteristics are studied. Also, the diagrammatic representation of modified even ordered Cantor sets has been exhibited.

In analyzing Cantor sets of even order it is obtained that more than one pattern of removal of intervals can be considered. Also it is noted that starting with six and eight every increment of four gives the same mode of removal of sets. The general formulas for the existing intervals have been given for the middle most intervals.

Farey $N$ – subsequence has been developed as a topological space Hausdorff space and $T_1$ space.

The Non Reduced Farey $N$ – subsequence has been established as a $\sigma$- algebra and Borel set. By reconstructing the sequence of elements also the Lebegue Measure and Probability Measure have been calculated for this $\sigma$- algebra. Also the Probability measure of the Generalized Non reduced Farey $N$ – subsequence has been evaluated. Measure of Modified Even ordered Cantor Sets are analyzed.

In finding box measure rectangles without dimensions are used. Here isosceles triangles are constructed and triangular measures are calculated.

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6. S. Sudha and A. Gnanam, **"Iterated Function System of Generalized Non – Reducible Farey N – Subsequence of order (4, 6 ...) by using HB Operator"** IOSR Journal of Mathematics (IOSR-JM), e - ISSN: 2278-5728, P-ISSN: 2319-765X, Volume 16, Issue 4 Ser. I (Jul-Aug 2020), PP: 47 – 51.

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1. Extraction of Cantor - Middle ($\boldsymbol{\omega}$=$\mathbf{2/6}$) Set From Farey Sequence **"International Conference on Mathematical Computer Engineering",** Mathematics Division, School of Advanced Studies (SAS), Vellore Institute of Technology, Chennai, on 21st and 22nd February 2020.

2. Probability Measures for IFS of Generalized Cantor Middle $\left(\frac{1}{4}, \frac{1}{6}, \cdots\right)$ Set By Using HB Operator **"International e-Conference On Advanced Research in Science, Engineering, Technology & Management",** Shadan Women's College of Engineering & Technology, Hyderabad, on 24th-25th July 2020.

**Workshop Attended**

1. Attended a one day National Level Workshop on "Teaching Excellence in Higher Education"on 7th Dec 2017, Bishop Heber College, Trichy.

2. Attended a one week National Level Workshop on "Mathematical Sciences for CSIR-UGC NET Aspirants" on 9th - 13th May 2018, organized by VIT, Chennai.

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# MODIFIED EVEN ORDERED CANTOR SETS

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**Abstract:** In Cantor ternary sets middle third is removed and the pattern of removal continues indefinitely. Taking the number of divisions as order here, even ordered Cantor sets are considered. Unlike Cantor sets here lengths of unequal intervals are removed. In this pattern of removal middle interval in successive iteration follows a geometric sequence of powers of two. The intervals equally spaced from the middle to the left and right follows different nature as the iteration increases. Its characteristics are studied in this paper. Also, the diagrammatic representation of modified even ordered Cantor sets has been exhibited.

**2010 AMS Classification:** 26A30

**Keywords:** Cantor set

## 1. Introduction

The Cantor ternary set is a set of rational numbers in the closed interval [0, 1] obtained by dividing the interval into 3 parts successfully after removing the middle third. There are many publications describing various properties of Cantor middle sets. The Cantor middle sets are considered only for odd integers. The analysis is done only for $C_{2m-1}$ middle sets $(2 \leq m < \infty)$. Cantor sets for even integers are not so far studied in detail.

Unlike Cantor ternary sets, in even ordered sets the intervals of lengths one and two are removed successively. Again, if intervals of lengths two are taken away the formulas for retaining terms are given. In this paper Cantor sets of even numbers are considered. Contrary to the procedure followed by Cantor, intervals of various lengths are removed in different patterns. These various patterns of removal of intervals are analyzed here.

## 2. Preliminaries

Throughout this paper we study the modified Cantor even ordered sets.

**Definition 1: Cantor Hexnary Set**

Divide the closed interval $[0,1]$ into six equal intervals. Remove the second and last but one of the six intervals of length $\left(\frac{1}{6}\right)$**.** The middle interval of length $\left(\frac{2}{6}\right)$ is only retained. Now for the first, last and middle intervals continue the procedure indefinitely. The set obtained is known as **Cantor Hexnary Set.**

**Definition 2: Cantor Octanary Set**

The closed interval [0, 1] is divided into eight equal parts. By removing the second part, last but one part and middlemost part, the open intervals $\left(\frac{1}{8},\frac{2}{8}\right)$, $\left(\frac{3}{8},\frac{5}{8}\right)$ and $\left(\frac{6}{8},\frac{7}{8}\right)$ are removed. The middlemost removable interval is of length $\left(\frac{2}{8}\right)$. Each retained intervals are of length $\left(\frac{1}{8}\right)$.

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Continue the process indefinitely and the set obtained is known as the **Cantor Octanary Set.**

### 3. Modified Cantor $\left(\frac{1}{6}\right)$ Sets

**Theorem 3.1:**

In $C_6$ , if the middle interval of length ${}^{2}/_{6}$ is retained and subdivided successfully as in Cantor then in the successive iterations the middlemost interval is retained that follow a series of the form ${}^{2}/_{6}, ({}^{2}/_{6})^2, ({}^{2}/_{6})^3, \ldots.$ for all these intervals. The general term of the middlemost interval is given by $\left[\frac{k}{6^n}, \frac{k+2^n}{6^n}\right]$ were k can be represented by the series $2(6)^{n-1}(2)^0 + 2(6)^{n-2}(2)^1 + 2(6)^{n-3}(2)^2 + \cdots + 2(6)^0(2)^{n-1}$

**Proof:**

The closed interval $[0,1]$ is divided into six equal parts. Following the theory of Cantor set, the open intervals $\left(\frac{1}{6}, \frac{2}{6}\right)$ and $\left(\frac{4}{6}, \frac{5}{6}\right)$ are removed. In first iteration the number of parts removed is $2*3^0$.The remaining parts $\left[\frac{0}{6} = 0, \frac{1}{6}\right]$, $\left[\frac{2}{6}, \frac{4}{6}\right]$ and $\left[\frac{5}{6}, \frac{6}{6} = 1\right]$ are again subdivided as follows. The length of the middlemost part is 2/6. For the 2nd iteration, the parts $\left[\frac{0}{6} = 0, \frac{1}{6}\right]$ and $\left[\frac{5}{6}, \frac{6}{6}\right]$ are each divided into six equal parts thereby giving six parts $\left[\frac{0}{36} = 0, \frac{1}{36}\right]$, $\left[\frac{1}{36}, \frac{2}{36}\right]$,$\left[\frac{2}{36}, \frac{3}{36}\right]$, $\left[\frac{3}{36}, \frac{4}{36}\right]$, $\left[\frac{4}{36}, \frac{5}{36}\right]$, $\left[\frac{5}{36}, \frac{6}{36}\right]$. The open intervals $\left(\frac{1}{36}, \frac{2}{36}\right)$ and $\left(\frac{4}{36}, \frac{5}{36}\right)$ are removed. Applying the removal pattern of the middle part $\left[\mathbf{\frac{2}{6}, \frac{4}{6}}\right]$ again give rise to $\left[\frac{12}{36}, \frac{14}{36}\right]$, $\left[\frac{14}{36}, \frac{16}{36}\right]$,$\left[\mathbf{\frac{16}{36}, \frac{20}{36}}\right]$, $\left[\frac{20}{36}, \frac{22}{36}\right]$, $\left[\frac{22}{36}, \frac{24}{36}\right]$. The open intervals $\left(\frac{14}{36}, \frac{16}{36}\right)$ and $\left(\frac{20}{36}, \frac{22}{36}\right)$ are removed. The length of the middlemost part is 4/36. In second iteration the number of parts removed is $2*3^1$. The third iteration the middle part $\left[\mathbf{\frac{16}{36}, \frac{20}{36}}\right]$ is again subdivided into six equal parts are $\left[\frac{96}{216}, \frac{100}{216}\right]$,$\left[\frac{100}{216}, \frac{104}{216}\right]$,$\left[\frac{104}{216}, \frac{108}{216}\right]$, $\left[\frac{108}{216}, \frac{112}{216}\right]$, $\left[\frac{112}{216}, \frac{116}{216}\right]$,$\left[\frac{116}{216}, \frac{120}{36}\right]$. The intervals $\left(\frac{100}{216}, \frac{104}{216}\right)$ and $\left(\frac{112}{36}, \frac{116}{36}\right)$ are removed. The retained intervals are $\left[\frac{96}{216}, \frac{100}{216}\right]$,$\left[\frac{100}{216}, \frac{104}{216}\right]$, $\left[\mathbf{\frac{104}{216}, \frac{112}{216}}\right]$, $\left[\frac{112}{216}, \frac{116}{216}\right]$, $\left[\frac{116}{216}, \frac{120}{36}\right]$. The length of the middlemost part is 8/216. In third iteration the number of parts removed is $2*3^2$.

Here it is noted that in the successive iterations every interval which are equally spaced from the middle interval are subdivided into six equal parts whose length $\frac{1}{6^1}, \frac{1}{6^2}, \frac{1}{6^3}, \frac{1}{6^4}, \cdots\cdots$. The middlemost part when subdivided into six equal parts are of length is $\frac{2}{6^1}, \frac{2^2}{6^2}, \frac{2^3}{6^3}, \frac{2^4}{6^4}, \cdots\cdots$. In each iteration the number of parts removed are $(2*3^0)$, $(2*3^1)$, $(2*3^2)$, $(2*3^3)$ … $(2*3^n)$ successively.

Therefore in $C_6$ , if the middle interval of length ${}^{2}/_{6}$ is retained and subdivided successfully as in Cantor then in successive iterations the middlemost intervals retained follow a series of the form ${}^{2}/_{6}, ({}^{2}/_{6})^2, ({}^{2}/_{6})^3, \ldots$ for all these intervals. The general representation of the middlemost term at the nth iteration is given by $\left[\frac{k}{6^n}, \frac{k+2^n}{6^n}\right]$ where k = $2(6)^{n-1}(2)^0 + 2(6)^{n-2}(2)^1 + 2(6)^{n-3}(2)^2 + \cdots + 2(6)^0(2)^{n-1}$

The Following **Figure 1** shows the graphical representation of modified Cantor $\left(\frac{1}{6}\right)$ sets and **Figure 3** shows the tree representation of middlemost part of modified Cantor $\left(\frac{1}{6}\right)$ sets.

**First iteration:**

The closed interval $[0,1]$ is subdivided into 6 equal sub- intervals

$\left[\frac{0}{6}=0,\frac{1}{6}\right],\left[\frac{1}{6},\frac{2}{6}\right],\left[\frac{2}{6},\frac{3}{6}\right],\left[\frac{3}{6},\frac{4}{6}\right],\left[\frac{4}{6},\frac{5}{6}\right],\left[\frac{5}{6},\frac{6}{6}=1\right]$

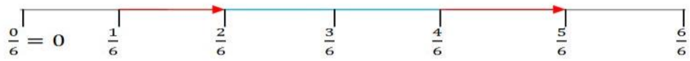


**Figure 1:** Modified Cantor $\left(\frac{1}{6}\right)$ sets

The Removed intervals are $\left(\frac{1}{6},\frac{2}{6}\right),\left(\frac{4}{6},\frac{5}{6}\right)$ .

The remaining intervals are $\left[\frac{0}{6}=0,\frac{1}{6}\right]\cup\left[\frac{2}{6},\frac{4}{6}\right]\cup\left[\frac{5}{6},\frac{6}{6}=1\right]$

$$\therefore\ C_{6^1}=\left\{\frac{0}{6}=0,\ \frac{1}{6},\frac{2}{6},\ \frac{4}{6},\ \frac{5}{6},\frac{6}{6}=1\right\} \qquad (3.1)$$

**Second iteration:**

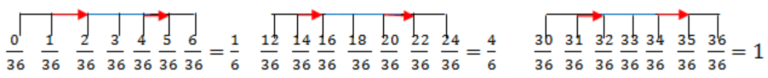


**Figure 2:** Second iteration - Modified Cantor $\left(\frac{1}{6}\right)$ sets

The Removable intervals are

$\left(\frac{1}{36},\frac{2}{36}\right),\left(\frac{5}{36},\frac{6}{36}\right),\left(\frac{14}{36},\frac{16}{36}\right),\left(\frac{20}{36},\frac{22}{36}\right),\left(\frac{31}{36},\frac{32}{36}\right),\left(\frac{34}{36},\frac{35}{36}\right)$ from each of the subintervals will result in modified cantor set.

$$\therefore\ C_{6^2}=\left\{\frac{0}{36},\frac{1}{36},\frac{2}{36},\frac{4}{36},\frac{5}{36},\frac{6}{36},\frac{12}{36},\frac{14}{36},\frac{16}{36},\frac{20}{36},\frac{22}{36},\frac{24}{36},\frac{30}{36},\frac{31}{36},\frac{32}{36},\frac{34}{36},\frac{35}{36},\frac{36}{36}\right\} \quad (3.2)$$

This procedure proceeds in every iteration to get the entire modified cantor set.

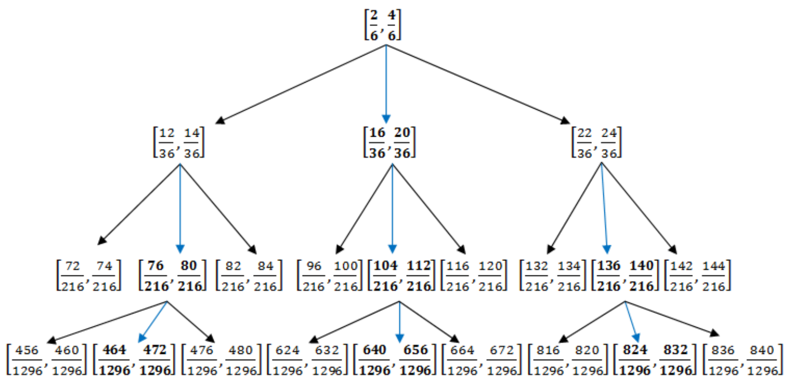


**Figure 3:** Middlemost part of modified Cantor $\left(\frac{1}{6}\right)$ sets

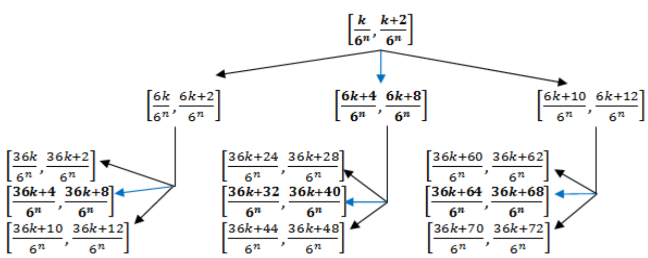


**Figure 4:** General form of middlemost part of modified Cantor $\left(\frac{1}{6}\right)$ sets

### 3.1 MODIFIED CANTOR $\left(\frac{1}{10}\right)$ SETS:

The above (Theorem 3.1) same pattern followed by modified Cantor $\left(\frac{1}{10}\right)$ sets.

**First iteration:**

The closed interval $[0,1]$ is subdivided into 10 equal sub- intervals $\left[\frac{0}{10}=0,\frac{1}{10}\right],\left[\frac{1}{10},\frac{2}{10}\right],\left[\frac{2}{10},\frac{3}{10}\right],\left[\frac{3}{10},\frac{4}{10}\right],\left[\frac{4}{10},\frac{5}{10}\right],\left[\frac{5}{10},\frac{6}{10}\right],\left[\frac{7}{10},\frac{8}{10}\right],\left[\frac{9}{10},\frac{10}{10}\right]$

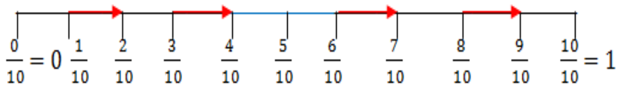


**Figure 5:** Modified Cantor $\left(\frac{1}{10}\right)$ sets

The Removed intervals are $\left(\frac{1}{10},\frac{2}{10}\right),\left(\frac{3}{10},\frac{4}{10}\right),\left(\frac{6}{10},\frac{7}{10}\right),\left(\frac{8}{10},\frac{9}{10}\right)$ .

The remaining intervals are $\left[\frac{0}{10}=0,\frac{1}{10}\right]\cup\left[\frac{2}{10},\frac{3}{10}\right]\cup\left[\frac{\mathbf{4}}{\mathbf{10}},\frac{\mathbf{6}}{\mathbf{10}}\right]\cup$ $\left[\frac{7}{10},\frac{8}{10}\right],\left[\frac{9}{10},\frac{10}{10}=1\right]$

$$\therefore\ C_{10^1}=\left\{\frac{0}{10}=0,\ \frac{1}{10},\frac{2}{10},\frac{3}{10},\frac{4}{10},\frac{6}{10}\frac{7}{10},\frac{8}{10},\frac{9}{10},\frac{10}{10}=1\right\} \qquad (3.3)$$

This procedure proceeds in every iteration to get the entire modified cantor set.

### 4. Modified Cantor $\left(\frac{1}{8}\right)$ Sets

**Theorem 4.1:**

In $C_8$ , the intervals of lengths $\frac{1}{8^n},\frac{2}{8^n},\frac{1}{8^n}$ are successively removed and only $4*4^{n-1}$ intervals of length each $1/8$ is retained. For each interval $\left[\frac{k}{6^n},\frac{k+1}{6^n}\right]=\left(\frac{1}{(2s)^n}[(k-1)(2s)^{n-1}+(8)^{n-1}]\right),\left(\frac{1}{(2s)^n}[(k-1)(2s)^{n-1}+(8)^{n-1}+1]\right)$.

**Proof:**

The closed interval $[0,1]$ is divided into eight equal parts. The open interval $\left(\frac{1}{8},\frac{2}{8}\right),\left(\frac{3}{8},\frac{5}{8}\right)$ and $\left(\frac{6}{8},\frac{7}{8}\right)$ are removed. In first iteration the number of parts removed is $4*4^0$.The length of the first iteration is 1/8. The remaining parts $\left[\frac{0}{8}=0,\frac{1}{8}\right],\left[\frac{2}{8},\frac{3}{8}\right],\left[\frac{5}{8},\frac{6}{8}\right]$ and $\left[\frac{7}{8},\frac{8}{8}=1\right]$ are again subdivided as follows for the 2nd iteration. The part $\left[\frac{0}{8}=0,\frac{1}{8}\right]$ are subdivided into eight equal parts thereby giving eight parts$\left[\frac{0}{64}=0,\frac{1}{64}\right],\left[\frac{1}{64},\frac{2}{64}\right],\left[\frac{2}{64},\frac{3}{64}\right]\left[\frac{3}{64},\frac{4}{64}\right],\left[\frac{4}{64},\frac{5}{64}\right],\left[\frac{5}{64},\frac{6}{64}\right],\left[\frac{6}{64},\frac{7}{64}\right]\left[\frac{7}{64},\frac{8}{64}\right]$, the open intervals $\left(\frac{1}{64},\frac{2}{64}\right),\left(\frac{3}{64},\frac{5}{64}\right)$ and$\left(\frac{6}{64},\frac{7}{64}\right)$ are removed. Now the part $\left[\frac{2}{8},\frac{3}{8}\right]$ are subdivided into eight equal parts thereby giving eight parts $\left[\frac{16}{64},\frac{17}{64}\right],\left[\frac{17}{64},\frac{18}{64}\right],\left[\frac{18}{64},\frac{19}{64}\right],\left[\frac{19}{64},\frac{20}{64}\right],\left[\frac{20}{64},\frac{21}{64}\right],\left[\frac{21}{64},\frac{22}{64}\right],\left[\frac{22}{64},\frac{23}{64}\right],\left[\frac{23}{64},\frac{24}{64}\right]$, the open intervals $\left(\frac{17}{64},\frac{18}{64}\right),\left(\frac{19}{64},\frac{21}{64}\right)$ and$\left(\frac{22}{64},\frac{23}{64}\right)$ are removed. Next parts$\left[\frac{5}{8},\frac{6}{8}\right]$ are divided into eight equal parts namely $\left[\frac{40}{64},\frac{41}{64}\right],\left[\frac{41}{64},\frac{42}{64}\right],\left[\frac{42}{64},\frac{43}{64}\right],\left[\frac{43}{64},\frac{44}{64}\right],\left[\frac{44}{64},\frac{45}{64}\right],\left[\frac{45}{64},\frac{46}{64}\right],\left[\frac{46}{64},\frac{47}{64}\right],\left[\frac{47}{64},\frac{48}{64}\right]$, the open intervals $\left(\frac{41}{64},\frac{42}{64}\right),\left(\frac{43}{64},\frac{45}{64}\right)$ and$\left(\frac{46}{64},\frac{47}{64}\right)$ are removed. Last part $\left[\frac{7}{8},\frac{8}{8}=1\right]$ is again subdivided into eight equal parts namely $\left[\frac{56}{64},\frac{57}{64}\right],\left[\frac{57}{64},\frac{58}{64}\right],\left[\frac{58}{64},\frac{59}{64}\right],\left[\frac{59}{64},\frac{60}{64}\right],\left[\frac{60}{64},\frac{61}{64}\right],\left[\frac{61}{64},\frac{62}{64}\right],\left[\frac{62}{64},\frac{63}{64}\right],\left[\frac{63}{64},\frac{64}{64}\right]$, the open intervals $\left(\frac{57}{64},\frac{58}{64}\right),\left(\frac{59}{64},\frac{61}{64}\right)$ and$\left(\frac{62}{64},\frac{63}{64}\right)$ are removed.

Here it is noted that in the successive iterations every interval which are equally spaced. Left out parts for partitioned into eight equal parts whose length $\frac{1}{8^1}, \frac{1}{8^2}, \frac{1}{8^3}, \frac{1}{8^4}, \cdots\cdots$. In every cycle the number of parts eliminated $(4{*}4^0)$, $(4{*}4^1)$, $(4{*}4^2)$, $(4{*}4^3)$ ... The n$^{\text{th}}$ iteration is $(4{*}4^n)$.

If any part is of the form $\left[\frac{K}{8^n}, \frac{K+1}{8^n}\right]$. When the n$^{\text{th}}$ iteration the end parts of the iteration are of the form $\left(\frac{1}{(2s)^n}[(k-1)(2s)^{n-1} + 8^{n-1}]\right), \left(\frac{1}{(2s)^n}[(k-1)(2s)^{n-1} + 8^{n-1}]\right) + 1$.

The Following **Figure 6** shows the graphical representation of modified Cantor $\left(\frac{1}{8}\right)$ sets and **Figure 8** shows the tree representation of interval of length is ${}^{1}/_{8}$ retained part of modified Cantor $\left(\frac{1}{8}\right)$ sets.

**First iteration:**

The closed interval $[0,1]$ is subdivided into 8 equal sub- intervals

$$\left[\frac{0}{8} = 0, \frac{1}{8}\right], \left[\frac{1}{8}, \frac{2}{8}\right], \left[\frac{2}{8}, \frac{3}{8}\right], \left[\frac{3}{8}, \frac{4}{8}\right], \left[\frac{4}{8}, \frac{5}{8}\right], \left[\frac{5}{8}, \frac{6}{8}\right], \left[\frac{6}{8}, \frac{7}{8}\right], \left[\frac{7}{8}, \frac{8}{8} = 1\right]$$

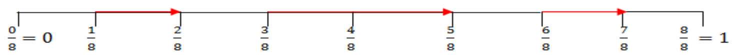


**Figure 6:** Modified Cantor $\left(\frac{1}{8}\right)$ sets

The Removed intervals are $\left(\frac{1}{8}, \frac{2}{8}\right), \left(\frac{3}{8}, \frac{5}{8}\right), \left(\frac{6}{8}, \frac{7}{8}\right)$.

The remaining intervals are $\left[\frac{0}{8} = 0, \frac{1}{8}\right], \left[\frac{2}{8}, \frac{3}{8}\right], \left[\frac{5}{8}, \frac{6}{8}\right]$ and $\left[\frac{7}{8}, \frac{8}{8} = 1\right]$

$$\therefore\ C_{8^1} = \left\{\frac{0}{8} = 0,\ \frac{1}{8}, \frac{2}{8},\ \frac{3}{8},\ \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \frac{8}{8} = 1\ \right\} \tag{4.1}$$

**Second iteration:**

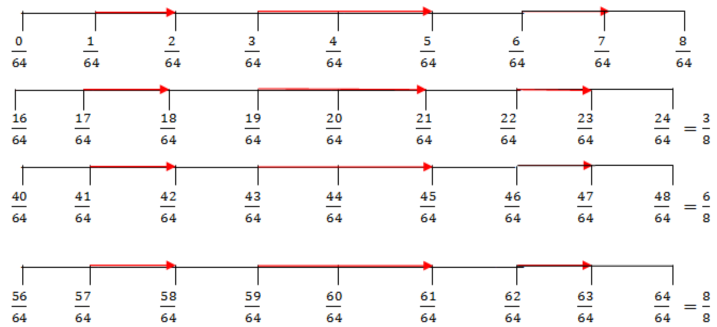


**Figure 7:** Second iteration - Modified Cantor $\left(\frac{1}{8}\right)$ sets

The Removable intervals are $\left(\frac{1}{64}, \frac{2}{64}\right), \left(\frac{3}{64}, \frac{5}{64}\right), \left(\frac{6}{64}, \frac{7}{64}\right), \left(\frac{17}{64}, \frac{18}{64}\right), \left(\frac{19}{64}, \frac{21}{64}\right), \left(\frac{22}{64}, \frac{23}{64}\right), \left(\frac{41}{64}, \frac{42}{64}\right)$ $\left(\frac{43}{64}, \frac{45}{64}\right), \left(\frac{46}{64}, \frac{47}{64}\right), \left(\frac{57}{64}, \frac{58}{64}\right), \left(\frac{59}{64}, \frac{61}{64}\right), \left(\frac{62}{64}, \frac{63}{64}\right)$ from each of the subintervals will result in modified cantor set.

Therefore

$$C_{8^2} = \left\{\begin{matrix} \frac{0}{64} = 0, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \frac{5}{64}, \frac{6}{64}, \frac{7}{64}, \frac{8}{64}, \frac{16}{64}, \frac{17}{64}, \frac{18}{64}, \frac{19}{64}, \frac{21}{64}, \frac{22}{64}, \frac{23}{64}, \frac{24}{64}, \\ \frac{40}{64}, \frac{41}{64}, \frac{42}{64}, \frac{43}{64}, \frac{45}{64}, \frac{46}{64}, \frac{47}{64}, \frac{48}{64} \end{matrix}\right\} \tag{4.2}$$

This procedure proceeds in every iteration to get the entire modified cantor set.

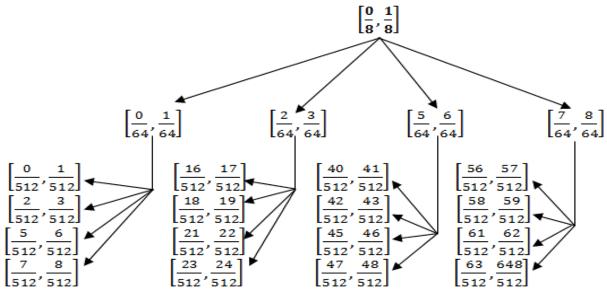


**Figure 8:** Retained part of modified Cantor $\left(\frac{1}{8}\right)$ sets.

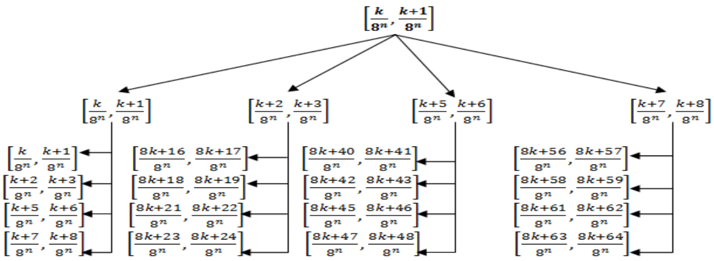


**Figure 9:** General form of Retained part of modified Cantor $\left(\frac{1}{8}\right)$ sets.

### 4.1 MODIFIED CANTOR $\left(\frac{1}{12}\right)$ SETS:

The above (Theorem 4.1) same pattern followed by modified Cantor $\left(\frac{1}{12}\right)$ sets.

**First iteration:**

The closed interval $[0,1]$ is subdivided into 12 equal sub- intervals $\left[\frac{0}{12}=0,\frac{1}{12}\right],\left[\frac{1}{12},\frac{2}{12}\right],\left[\frac{2}{12},\frac{3}{12}\right],\left[\frac{3}{12},\frac{4}{12}\right],\left[\frac{4}{12},\frac{5}{12}\right],\left[\frac{5}{12},\frac{6}{12}\right],\left[\frac{7}{12},\frac{8}{12}\right],$ $\left[\frac{9}{12},\frac{10}{12}\right],\left[\frac{11}{12},\frac{12}{12}=1\right]$

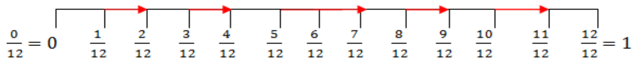


**Figure 10:** Modified Cantor $\left(\frac{1}{12}\right)$ sets

The Removed intervals are $\left(\frac{1}{12},\frac{2}{12}\right),\left(\frac{3}{12},\frac{4}{12}\right),\left(\frac{5}{12},\frac{7}{12}\right),\left(\frac{8}{12},\frac{9}{12}\right),\left(\frac{10}{12},\frac{11}{12}\right)$ .

The remaining intervals are $\left[\frac{0}{12}=0,\frac{1}{12}\right]\cup\left[\frac{2}{12},\frac{3}{12}\right]\cup\left[\frac{4}{12},\frac{5}{12}\right]\cup$ $\left[\frac{7}{12},\frac{8}{12}\right],\left[\frac{9}{12},\frac{10}{12}\right],\left[\frac{11}{12},\frac{12}{12}=1\right]$

$$\therefore\ C_{12^1}=\left\{\frac{0}{12}=0,\ \frac{1}{12},\frac{2}{12},\frac{3}{12},\frac{4}{12},\frac{5}{12},\frac{7}{12},\frac{8}{12},\frac{9}{12},\frac{10}{12},\frac{11}{12},\frac{12}{12}=1\ \right\} \qquad (4.3)$$

This procedure proceeds in every iteration to get the entire modified cantor set.

## 5. Conclusion

We have established modified cantor $\left(\frac{1}{6}\right)$ and $\left(\frac{1}{8}\right)$ sets. Unlike usual Cantor sets having removal sets of equal lengths here two cases are considered. One is usual away of removing intervals of lengths 1/2n and the middle most intervals of lengths 2/2n. The process is continued successively, so that the general portion of removable intervals can be identified.

In analyzing Cantor sets of even order it is obtained that more than one pattern of removal intervals can be considered. Also it is noted that starting with six and eight every increment of four gives the same mode of removal exists respectively. In this paper the general formula for the existing intervals have been given. The other modes may be considered for future work.

**Remark:**

Another way of forming Cantor modified set may be given as follows for Cantor Octanary Set.

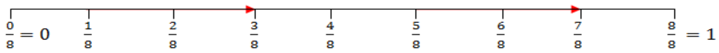


**Figure 11:** Another way of Cantor $\left(\frac{1}{8}\right)$ sets.

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**VARIOUS MEASURES OF NON - REDUCED FAREY N- SUBSEQUENCE**

*S.Sudha[1*], A.Gnanam[2] and P.Balamurugan[3]*

**Abstract**

A Non Reduced Farey N-subsequence, a subsequence of Farey sequence consists of rational numbers with same denominator in [0, 1]. By reconstructing the non reduced Farey N – subsequence it can be established as a $\sigma$- algebra. For this measurable space Lebegue Measure and Probability Measure have been calculated. Also it has been established as a Borel set. In this paper Farey subsequence of even order has been studied. Similarly the odd order Farey subsequence may be studied separately.

**2010 Mathematics Subject Classification:** : 11B57

**Keywords:-**

Farey N – Subsequence**,** $\sigma$- algebra, Borel set, Lebesgue measure, Probability Measure.

**Notations:**

1. $\tilde{F}_N$ - Non- Reduced Farey N – Subsequence.
2. $C\ \tilde{F}_N$ - Non- Reduced Farey N – Subsequence as union of closed set.
3. $HC\ \tilde{F}_N$ - Non- Reduced Farey N – Subsequence as union of half closed set.

## 1. INTRODUCTION

The Farey sequence of order n is the sequence of completely reduced fractions, either between o and 1, or without this restriction, which when in lowest terms have denominators less than or equal to n, arranged in order of increasing size [1,8]. Farey sequences are very useful to find rational approximations of irrational numbers.

The Farey sequence, sometimes called the Farey series, is a series of sequences in which each sequence consists of rational numbers ranging in value from 0 to 1. A non reduced Farey N – subsequence satisfies the conditions of Lebesgue measure [8].

For the construction of measurable sets the Farey sequence has been consider as union of intervals, half- open, closed (or) open sets as the case may be with a sequential points as a end points. For the same above construction Probability measure has also been calculated.

## 2. PRELIMINARIES

### Definition1: Farey sequence [1]

A Sequence of rational numbers $\frac{p}{q}$ with $(p, q) = 1$ in [0, 1] and $q \leq n$ is called a Farey Sequence of order n, denoted by $F_n$.

**Example1:**

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$
$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}\frac{1}{1}\right\}$$

$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

**Definition 2:Farey N – subsequence [1]**

In a Farey sequence $F_N$ the elements with denominators precisely N are classified as Farey N – subsequence and denoted by $\langle F'_N \rangle$ .

$$\langle F'_N \rangle = \left\{\frac{u_i}{N} / \, 0 \leq u_i \leq N, 0 \leq i \leq N\right\}$$

**Example 2:**

The FareyN – Sequence of order 4 is

$$\langle F'_4 \rangle = \left\{\frac{0}{1} = \frac{0}{4} < \frac{1}{4} < \frac{3}{4} < \frac{4}{4} = \frac{1}{1}\right\}$$

**Definition 3: Non - Reducible Farey N – Subsequence [1]**

For $F_N$, the element of the sequence with denominator N is taken as Non –Reducible Farey N - subsequence. It is denoted by $\tilde{F}_N$.

**Example 3:**

The Non- Reducible Farey N - Subsequence of order 2 is

$$\tilde{F}_2 = \left\{\frac{0}{2}, \frac{1}{2}, \frac{2}{2} = \frac{1}{1}\right\}$$

**Definition 4: $\sigma$ – algebra [8]**

$\mathcal{A} \subseteq P(X)$ is called a $\sigma$ – algebra

(i) $\varphi, X \in \mathcal{A}$

(ii) A$\subset \mathcal{A} \Rightarrow A^C = X/A \in \mathcal{A}$

(iii) $A_i \in \mathcal{A}$ , $i \in N \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

A$\in \mathcal{A}$ is called a $\mathcal{A}$ – measurable set.

**Definition 5: Borel Set [8]**

A Borel set is any set in topological space that can be formed from open sets (or) equivalently from closed sets through the operations of countable union, countable intersection and relative complement.

**Definition 6: Probability Measures [8]**

A Probability measure on $\Omega$ is a function P from subsets of $\Omega$ to the real numbers that satisfies the following axioms

(i) P($\Omega$) = 1

(ii) If A$\subset \Omega$, then P(A)$\geq$ 0

(iii) If $A_1$ and $A_2$ are disjoint, then P$(A_1) + P(A_2)$ more generally, If $A_1, A_2, \cdots\cdots, A_n$ are mutually disjoint, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

**Definition 7: Lebesgue Measure [8]**

A set A$\subset E$ is Lebesgue measurable or measurable if $\lambda^*(A) = \lambda_*(A)$. The measure of A is denoted by $\lambda(A)$ and is given by $\lambda(A) = \lambda^*(A) = \lambda_*(A)$

## 3. A SUBSEQUENCE OF FAREY SEQUENCE – $\sigma$ ALGEBRA AND BOREL SET

**Theorem 1:**

The Lebesgue measure of the Non reduced Farey N – subsequence of even order n is zero.

**Proof:**

**Construction of measurable sets from Farey N – subsequence.**

By definition, a Non reduced Farey N – subsequence of even order is given by

$\tilde{F}_{(2m)^1} = \left\{\frac{0}{(2m)^1}, \frac{1}{(2m)^1}, \frac{2}{(2m)^1}, \cdots\cdots, \frac{(2m)^1-3}{(2m)^1}, \frac{(2m)^1-2}{(2m)^1}, \frac{(2m)^1-1}{(2m)^1}, \frac{(2m)^1}{(2m)^1}\right\}$ ; Where $1 \leq m \leq N$

From this sequence construct set as follows:

$$C\,\tilde{F}_{(2m)^1} = \left\{\frac{0}{(2m)^1}, \frac{1}{(2m)^1}, \frac{2}{(2m)^1}, \cdots\cdots, \frac{(2m)^1-3}{(2m)^1}, \frac{(2m)^1-2}{(2m)^1}, \frac{(2m)^1-1}{(2m)^1}, \frac{(2m)^1}{(2m)^1}\right\} \text{ ; Where } 1 \leq m \leq N$$
$$= \left[\frac{0}{(2m)^1}, \frac{1}{(2m)^1}\right] \cup \left[\frac{1}{(2m)^1}, \frac{2}{(2m)^1}\right] \cup \cdots\cdots \cup \left[\frac{(2m)^1-3}{(2m)^1}, \frac{(2m)^1-2}{(2m)^1}\right] \cup \left[\frac{(2m)^1-2}{(2m)^1}, \frac{(2m)^1-1}{(2m)^1}\right] \cup \left[\frac{(2m)^1-1}{(2m)^1}, \frac{(2m)^1}{(2m)^1}\right]$$
$$= D_{(2m)^1\,1} \cup D_{(2m)^1\,2} \cup D_{(2m)^1\,3} \cup \cdots\cdots \cup D_{(2m)^1\,r}$$

In the next iteration the sequence is given by

$$C\tilde{F}_{(2m)^2} = \left\{\frac{0}{(2m)^2}, \frac{1}{(2m)^2}, \frac{2}{(2m)^2}, \cdots\cdots, \frac{(2m)^2-3}{(2m)^2}, \frac{(2m)^2-2}{(2m)^2}, \frac{(2m)^2-1}{(2m)^2}, \frac{(2m)^2}{(2m)^2}\right\} \text{ ; Where } 1 \leq m \leq N$$

Again writing in a set format we have

$$= \left[\frac{0}{(2m)^2}, \frac{1}{(2m)^2}\right] \cup \left[\frac{1}{(2m)^2}, \frac{2}{(2m)^2}\right] \cup \cdots\cdots \cup \left[\frac{(2m)^2-3}{(2m)^2}, \frac{(2m)^2-2}{(2m)^2}\right] \cup \left[\frac{(2m)^2-2}{(2m)^2}, \frac{(2m)^2-1}{(2m)^2}\right] \cup \left[\frac{(2m)^2-1}{(2m)^2}, \frac{(2m)^2}{(2m)^2}\right]$$

$$= D_{(2m)^2\,1} \cup D_{(2m)^2\,2} \cup D_{(2m)^2\,3} \cup \cdots\cdots \cup D_{(2m)^2\,r}$$

$lll^{ly}$ for the $n^{th}$ term is

$$C\tilde{F}_{(2m)^n} = \left\{\frac{0}{(2m)^n}, \frac{1}{(2m)^n}, \frac{2}{(2m)^n}, \cdots\cdots, \frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right\} \text{ ; Where } 1 \leq m \leq N$$
$$= \left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right] \cup \left[\frac{1}{(2m)^n}, \frac{2}{(2m)^n}\right] \cup \cdots\cdots \cup \left[\frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}\right] \cup \left[\frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}\right] \cup \left[\frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right]$$
$$= D_{(2m)^n 1} \cup D_{(2m)^n 2} \cup D_{(2m)^n 3} \cup \cdots\cdots \cup D_{(2m)^n r}$$

Let

$E_{(2m)^n 1} =$ Set of all possible union of two elements.

$E_{(2m)^n 2} =$ Set of all possible union of three elements.

for the $r^{th}$ term is

$E_{(2m)^n\,r} =$ Set of all possible union of (r +1) elements.

ie., $E_{(2m)^n\,r} = \cup_{r=1}^{(2m)^n-1}(r+1)$

Take $X = \{C\,\tilde{F}_{(2m)^1}, C\tilde{F}_{(2m)^2}, \cdots\cdots\cdots\cdots\cdots\cdots\cdots\; C\tilde{F}_{(2m)^n}\}$

*P(X)* = $\{D_{(2m)^1\,1}, D_{(2m)^1\,2}, D_{(2m)^1\,3}, \cdots\cdots, D_{(2m)^1\,r}, E_{(2m)^1\,1}, E_{(2m)^1\,2}, E_{(2m)^1\,3}, \cdots\cdots, E_{(2m)^1\,r}$
$\cdots\cdots\cdots D_{(2m)^n1}, D_{(2m)^n2}, D_{(2m)^n3}, \cdots\cdots, D_{(2m)^nr}, E_{(2m)^n\,1},$
$E_{(2m)^n\,2}, \cdots\cdots E_{(2m)^n\,r}\}$

**Claim 1:**

The Set *P(X)* is a $\sigma$ – algebra

By the definition of $\sigma$ – algebra.

(i) Empty set $\varphi$, *P(X)* $\in \mathcal{A}$.

(ii) Take $\left(D_{(2m)^12}\right)^C = P(X)/D_{(2m)^12}$

$= E_{(2m)^1\,1} =$ Set of all possible union of two elements

( iii) Consider the elements $D_{(2m)^n1}, D_{(2m)^n2}, D_{(2m)^n3}, \cdots\cdots, D_{(2m)^nr} \in \mathcal{A}$

Then

$D_{(2m)^n1} \cup D_{(2m)^n2} \cup \cdots \cup D_{(2m)^nr} = E_{(2m)^nr}$(Set of all possible union of $4^n$elements) $\in \mathcal{A}$

$\therefore$ P(X) is a $\sigma$ – algebra.

**Claim 2:**

The Set *P(X)* is a Borel set

By the definition of Borel set

Consider the elements $D_{(2m)^n1}, E_{(2m)^n\,1} \in \mathcal{A}$

$D_{(2m)^n1} \cap\; E_{(2m)^n\,1} = D_{(2m)^n1} \;\in \mathcal{A}$

The Set *P(X)* satisfies all the conditions.

Hence the set *P(X)* is $\sigma$ – algebra and Borel set. Also the set *P(X)* is measurable sets.

Now, the Lebesgue Measure of *P(X)* is calculated.

$$\lambda\left(C\tilde{F}_{(2m)^n}\right) = \lim_{n\to\infty} \lambda\left(C\tilde{F}_{(2m)^n}\right)$$

$$= \lim_{n\to\infty} \lambda\left\{\left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right] \cup \left[\frac{1}{(2m)^n}, \frac{2}{(2m)^n}\right] \cup \cdots\cdots \cup \left[\frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}\right] \cup \left[\frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}\right] \cup \left[\frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right]\right\}$$

$$= \lim_{n\to\infty} \left\{\lambda\left[\frac{0}{(2m)^n}, \frac{1}{(2m)^n}\right] + \lambda\left[\frac{1}{(2m)^n}, \frac{2}{(2m)^n}\right] + \cdots\cdots + \lambda\left[\frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}\right] + \lambda\left[\frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}\right] + \lambda\left[\frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right]\right\}$$

$$= \lim_{n\to\infty} \left\{\left(\frac{1}{(2m)^n} - \frac{0}{(2m)^n}\right) + \left(\frac{2}{(2m)^n} - \frac{1}{(2m)^n}\right) + \cdots\cdots + \left(\frac{(2m)^n-2}{(2m)^n} - \frac{(2m)^n-3}{(2m)^n}\right) + \left(\frac{(2m)^n-1}{(2m)^n} - \frac{(2m)^n-2}{(2m)^n}\right) + \left(\frac{(2m)^n}{(2m)^n} - \frac{(2m)^n-1}{(2m)^n}\right)\right\}$$

$$= \lim_{n\to\infty} \left\{\frac{1}{(2m)^n} + \frac{1}{(2m)^n} + \frac{1}{(2m)^n} + \cdots\cdots + \frac{1}{(2m)^n} + \frac{1}{(2m)^n} + \frac{1}{(2m)^n}\right\}$$

$$= 0$$

Therefore $\lambda\left(C\tilde{F}_{(2m)^n}\right) = 0$. Hence $C\,\tilde{F}_{(2m)^n}$ has Lebesgue measure zero.

Hence the Lebesgue measure of the Non reduced Farey N – subsequence of even order is zero.

**Illustration: 1**

$C\widetilde{F}_{(2m)^n} = \left\{\frac{0}{(2m)^n}, \frac{1}{(2m)^n}, \frac{2}{(2m)^n}, \cdots\cdots, \frac{(2m)^n-3}{(2m)^n}, \frac{(2m)^n-2}{(2m)^n}, \frac{(2m)^n-1}{(2m)^n}, \frac{(2m)^n}{(2m)^n}\right\}$ ; Where $1 \leq m \leq N$

Put m = 2

Non – Reduce to Farey N – Subsequence of order 4 is

$$C\widetilde{F}_{4^1} = \left\{\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}\right\} = \left[\frac{0}{4}, \frac{1}{4}\right] \cup \left[\frac{1}{4}, \frac{2}{4}\right] \cup \left[\frac{2}{4}, \frac{3}{4}\right] \cup \left[\frac{3}{4}, \frac{4}{4}\right]$$
$$= D_{41} \cup D_{42} \cup D_{43} \cup D_{44}$$

$$C\widetilde{F}_{4^2} = \left\{\frac{0}{16}, \frac{1}{16}, \frac{2}{16}, \ldots\ldots \frac{16}{16}\right\} = \left[\frac{0}{16}, \frac{1}{16}\right] \cup \left[\frac{1}{16}, \frac{2}{16}\right] \cup \left[\frac{2}{16}, \frac{3}{16}\right] \cup \left[\frac{3}{16}, \frac{4}{16}\right] \cup \ldots\ldots \cup \left[\frac{15}{16}, \frac{16}{16}\right]$$
$$= D_{4^2 1} \cup D_{4^2 2} \cup D_{4^2 3} \cup \cdots\cdots\cdots\cdots\cdots\cdots \cup D_{4^2(15)} \cup D_{4^2(16)}$$

For $n^{th}$ term is

$$C\widetilde{F}_{4^n} = \left\{\frac{0}{4^n}, \frac{1}{4^n}, \frac{2}{4^n}, \frac{3}{4^n}, \cdots\cdots \frac{4^n-3}{4^n}, \frac{4^n-2}{4^n}, \frac{4^n-1}{4^n}, \frac{4^n}{4^n}\right\}$$

$$= \left[\frac{0}{4^n}, \frac{1}{4^n}\right] \cup \left[\frac{1}{4^n}, \frac{2}{4^n}\right] \cup \left[\frac{2}{4^n}, \frac{3}{4^n}\right] \cup \cdots\cdots \cup \left[\frac{4^n-3}{4^n}, \frac{4^n-2}{4^n}\right] \cup \left[\frac{4^n-2}{4^n}, \frac{4^n-1}{4^n}\right] \cup \left[\frac{4^n-1}{4^n}, \frac{4^n}{4^n}\right]$$
$$= D_{4^n 1} \cup D_{4^n 2} \cup D_{4^n 3} \cup \cdots\cdots\cdots \cup D_{4^n m}$$

Let

$E_{4^n 1} =$ Set of all possible union of two elements.

$E_{4^n 2} =$ Set of all possible union of three elements.

lll$^{ly}$ for $m^{th}$ term is

$E_{4^n m} =$ Set of all possible union of (m+1) elements.

ie., $E_{4^n m} = \bigcup_{m=1}^{4^n-1} (\mathrm{m}+1)$

Take $X = \{\widetilde{F}_{4^1}, \widetilde{F}_{4^2}, \cdots\cdots\cdots\cdots\cdots\cdots\cdots \widetilde{F}_{4^n}\}$

*P(X)* $= \{D_{41}, D_{42}, D_{43}, D_{44}, E_{41}, E_{42}, E_{43}, \cdots D_{4^n 1}, D_{4^n 2}, D_{4^n 3}, \cdots, D_{4^n m}, E_{4^n 1}, E_{4^n 2} \cdots E_{4^n m}\}$

By the definition of $\sigma$ – algebra. The Set *P(X)* satisfies all the conditions.

Also the set *P(X)* is $\sigma$ – algebra and Borel set .

(i) Empty set $\varphi$, *P(X)* $\in \mathcal{A}$.

(ii) Take $(D_{42})^C = P(X)/D_{42}$

$= E_{41} =$ Set of all possible union of two elements

( iii) Consider the elements $D_{4^n 1}, D_{4^n 2}, D_{4^n 3}, \cdots, D_{4^n m} \in \mathcal{A}$

Then $D_{4^n 1} \cup D_{4^n 2} \cup D_{4^n 3} \cup \cdots \cup D_{4^n m} = E_{4^n m}$(Set of all possible union of $4^n$elements) $\in \mathcal{A}$

$\therefore$ *P(X)* is a $\sigma$ – algebra.

iv) The Set *P(X)* satisfies the countable intersection

Consider the elements $D_{43} \cap E_{43} = \varphi \in \mathcal{A}$

$\therefore$ *P(X)* is a Borel set.

Therefore *P( X)* is a $\sigma$ Algebra as well as Borel Set.

**Lebesgue Measure:**

$$\lambda(C\widetilde{F}_{4^n}) = \lim_{n\to\infty} \lambda(C\widetilde{F}_{4^n})$$
$$= \lim_{n\to\infty} \lambda\left\{\left[\frac{0}{4^n}, \frac{1}{4^n}\right] \cup \left[\frac{1}{4^n}, \frac{2}{4^n}\right] \cup \left[\frac{2}{4^n}, \frac{3}{4^n}\right] \cup \cdots\cdots \cup \left[\frac{4^n-3}{4^n}, \frac{4^n-2}{4^n}\right] \cup \left[\frac{4^n-2}{4^n}, \frac{4^n-1}{4^n}\right] \cup \left[\frac{4^n-1}{4^n}, \frac{4^n}{4^n}\right]\right\}$$

$$= \lim_{n\to\infty}\left\{\lambda\left[\frac{0}{4^n},\frac{1}{4^n}\right]+\lambda\left[\frac{1}{4^n},\frac{2}{4^n}\right]+\lambda\left[\frac{2}{4^n},\frac{3}{4^n}\right]+\cdots\cdots+\lambda\left[\frac{4^n-3}{4^n},\frac{4^n-2}{4^n}\right]+\lambda\left[\frac{4^n-2}{4^n},\frac{4^n-1}{4^n}\right]+\lambda\left[\frac{4^n-1}{4^n},\frac{4^n}{4^n}\right]\right\}$$

$$= \lim_{n\to\infty}\left\{\left(\frac{1}{4^n}-\frac{0}{4^n}\right)+\left(\frac{2}{4^n}-\frac{1}{4^n}\right)+\left(\frac{3}{4^n}-\frac{2}{4^n}\right)+\cdots+\left(\frac{4^n-2}{4^n}-\frac{4^n-3}{4^n}\right)+\left(\frac{4^n-1}{4^n}-\frac{4^n-2}{4^n}\right)+\left(\frac{4^n}{4^n}-\frac{4^n-1}{4^n}\right)\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1}{4^n}+\frac{1}{4^n}+\frac{1}{4^n}+\cdots\cdots+\frac{1}{4^n}+\frac{1}{4^n}+\frac{1}{4^n}\right\}$$

$$= 0$$

Therefore $\lambda\left(C\tilde{F}_{4^n}\right)=0$. Hence $C\tilde{F}_{4^n}$ has Lebesgue measure zero.

By a slight modification in writing non reduced Farey N – subsequence as a measurable set. The sequence is written union of closed and semi – open intervals of Probability.

## 4. PROBABILITY MEASURE OF GENERALIZED NON - REDUCE TO FAREY N- SUBSEQUENCE OF EVEN ORDER

**Theorem: 2**

The Probability measure of the Non reduced Farey N – subsequence of even order n is one.

**Proof:**

Construction of measurable sets from Farey N – subsequence

$$HC\ \tilde{F}_{(2m)} = \left\{\frac{0}{(2m)},\frac{1}{(2m)},\frac{2}{(2m)},\cdots\cdots,\frac{(2m)-3}{(2m)},\frac{(2m)-2}{(2m)},\frac{(2m)-1}{(2m)},\frac{(2m)}{(2m)}\right\}$$

$$= \left[\frac{0}{(2m)},\frac{1}{(2m)}\right]\cup\left[\frac{1}{(2m)},\frac{2}{(2m)}\right]\cup\cdots\cdots\cup\left[\frac{(2m)-3}{(2m)},\frac{(2m)-2}{(2m)}\right]\cup\left[\frac{(2m)-2}{(2m)},\frac{(2m)-1}{(2m)}\right]\cup\left[\frac{(2m)-1}{(2m)},\frac{(2m)}{(2m)}\right]$$

ie., HC $\tilde{F}_{(2m)} = \left\{\frac{0}{(2m)},\frac{1}{(2m)},\frac{2}{(2m)},\cdots\cdots,\frac{(2m)-3}{(2m)},\frac{(2m)-2}{(2m)},\frac{(2m)-1}{(2m)},\frac{(2m)}{(2m)}\right\}$

$$= \left[\frac{0}{(2m)},\frac{1}{(2m)}\right)\cup\left[\frac{1}{(2m)},\frac{2}{(2m)}\right)\cup\cdots\cdots\cup\left[\frac{(2m)-3}{(2m)},\frac{(2m)-2}{(2m)}\right)\cup\left[\frac{(2m)-2}{(2m)},\frac{(2m)-1}{(2m)}\right)\cup\left[\frac{(2m)-1}{(2m)},\frac{(2m)}{(2m)}\right]$$

By the definition of Probability Measure P (A)$\geq 0$ and If $A_1$ and $A_2$ are disjoint, then P$(A_1)+P(A_2)$

More generally,if $A_1, A_2,\cdots\cdots, A_n$ are mutually disjoint, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Now

Probability Measure HC $\tilde{F}_{(2m)} = \left(\frac{1}{(2m)}-\frac{0}{(2m)}\right)+\left(\frac{2}{(2m)}-\frac{1}{(2m)}\right)+\cdots+\left(\frac{(2m)}{(2m)}-\frac{(2m)-1}{(2m)}\right)$

$$= \frac{1}{(2m)}+\frac{1}{(2m)}+\frac{1}{(2m)}+\cdots\cdots+\frac{1}{(2m)}$$

$$= \frac{1}{(2m)}\ (1+1+1+1+\cdots\cdots\cdots+(2m)\ times)$$

$$= \frac{2m}{2m}$$

$$= 1$$

$\therefore$ Probability Measure of Non - Reduced Farey N- Subsequence of order (2m) is **one.**

**Illustration: 2**

HC $\tilde{F}_{(2m)} = \left\{\frac{0}{(2m)}, \frac{1}{(2m)}, \frac{2}{(2m)}, \cdots\cdots, \frac{(2m)-3}{(2m)}, \frac{(2m)-2}{(2m)}, \frac{(2m)-1}{(2m)}, \frac{(2m)}{(2m)}\right\}$

$$= \left[\frac{0}{(2m)}, \frac{1}{(2m)}\right) \cup \left[\frac{1}{(2m)}, \frac{2}{(2m)}\right) \cup \cdots\cdots \cup \left[\frac{(2m)-3}{(2m)}, \frac{(2m)-2}{(2m)}\right) \cup \left[\frac{(2m)-2}{(2m)}, \frac{(2m)-1}{(2m)}\right) \cup \left[\frac{(2m)-1}{(2m)}, \frac{(2m)}{(2m)}\right]$$

Put m = 2

HC $\tilde{F}_4 = \left\{\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}\right\}$

$$= \left[\frac{0}{4}, \frac{1}{4}\right) \cup \left[\frac{1}{4}, \frac{2}{4}\right) \cup \left[\frac{2}{4}, \frac{3}{4}\right) \cup \left[\frac{3}{4}, \frac{4}{4}\right]$$

By the definition of Probability Measure P (A)$\geq 0$

Now

Probability Measure HC $\tilde{F}_4 = \left(\frac{1}{4} - \frac{0}{4}\right) + \left(\frac{2}{4} - \frac{1}{4}\right) + \left(\frac{3}{4} - \frac{2}{4}\right) + \left(\frac{4}{4} - \frac{3}{4}\right)$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$= 1$$

∴ Non - Reduce to Farey N- Subsequence of order 4 of Probability Measure is **one.**

In this paper m may be taken as a positive integer.

## 5. CONCLUSION

In this paper the Non reduced Farey N – subsequence has been established as a $\sigma$- algebra and Borel set. By reconstructing the sequence of elements also the Lebegue Measure and Probability Measure have been calculated for this $\sigma$- algebra. Here only Farey N- subsequence of even order has been studied. In future work, we have established the even order of Cantor sets also.

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# ESTABLISHMENT OF EVEN ORDERED A SUBSEQUENCE OF A FAREY SEQUENCE AS A TOPOLOGICAL SPACE

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## Abstract

Farey sequence in [0, 1] is a sequence of real numbers generated using median properties. A subsequence of Farey sequence, $\widetilde{F_N}$ Farey $N$-subsequence has been established as a topological space and a Hausdorff space by appropriately defining basis and open sets. Also the axiom has been discussed with an illustration. A Lebesgue measure of Farey sequence of both odd and even order has been presented in this paper.

## 1. Introduction

The Farey Sequence usually referred to as the Farey series, maybe a chain of sequences during which every series includes rational numbers that move in estimate from zero to one [2].

In "Extraction of Cantor Middle $\left(\omega = \frac{2}{5}, \frac{3}{7}\right)$ from Non-Reducible Farey

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Subsequence" -Cantor ternary sets and some higher ordered Cantor sets have been extracted from Farey sequences of like order. There are many properties observed on the Farey sequence in "Farey to Cantor" [1]. In "Farey Triangle Graphs and Farey Triangle Matrices" [4] the terms of a Farey sequence have been considered as ordered pairs and a pattern of matrices and graphs have been identified.

It is mentioned in "https://ncatlab.org/nlab" that Cantor sets may be developed into topological space and Hausdorff space. Having identified the terms of $F_N$, $F_{N+1}$ is written by writing the mediant of all the successive terms of $F_N$. With slight modification, sequence whose terms are Farey sequences has been establishedas various spaces namely topological space, Hausdorff space and $T_1$ space. A Hausdorff space is basically a topological space. To form a topology a nonempty set with basis elements should be defined clearly.

## 2. Preliminaries

**Definition 1. Farey sequence** [1] A Sequence of rational numbers $\frac{p}{q}$ with $(p, q) = 1$ in $[0, 1]$ and $q \leq n$ is called a Farey Sequence of order $n$, denoted by $F_n$.

**Example 1.**

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$

$$F_2 = \left\{\frac{0}{1}, \frac{1}{2} \frac{1}{1}\right\}$$

$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

**Definition 2. Farey *N*-subsequence** [1] In a Farey sequence $F_N$ the elements with denominators precisely $N$ are classified as Farey $N$-subsequence and denoted by $\langle F_N' \rangle$.

$$\langle F'_N \rangle = \left\{ \frac{u_i}{N} / 0 \le u_i \le N,\ 0 \le i \le N \right\}$$

**Example 2.** The Farey $N$-Sequence of order 4 is

$$\langle F'_N \rangle = \left\{ \frac{0}{1} = \frac{0}{4} < \frac{1}{4} < \frac{3}{4} < \frac{4}{4} = \frac{1}{1} \right\}$$

**Definition 3. Non - Reducible Farey Sequence** [1] A subset of the Farey sequence $F'_N$ whose denominator not exceeding $N$ is taken as Non – Reducible Farey Sequence. It is denoted by $\widetilde{F_N}$.

**Example 3.** The quaternary Non - Reducible Farey Sequence of order 4 is

$$\widetilde{F}_4 = \left\{ \frac{0}{1} = \frac{0}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} = \frac{2}{4}, \frac{2}{3}, \frac{3}{4}, \frac{4}{4} = \frac{1}{1} \right\}$$

**Definition 4. Non-Reducible Farey $N$-Subsequence** [1] For $F_N$, the element of the sequence with denominator $N$ is taken as Non-Reducible Farey $N$-subsequence. It is denoted by $\widetilde{F}_N$.

**Example 4.** The Non- Reducible Farey $N$-Subsequence of order 4 is

$$\widetilde{F}_4 = \left\{ \frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = \frac{1}{1} \right\}$$

Non-Reducible Farey N-subsequenceoforder $\widetilde{F}_4$ and its higher powers have been given below.

$$\widetilde{F}_{4^1} = \left\{ \frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = 1 \right\} \tag{2.1}$$

$\widetilde{F}_{4^2} =$

$$\left\{ \frac{0}{16}, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \frac{4}{16}, \frac{5}{16}, \frac{6}{16}, \frac{7}{16}, \frac{8}{16}, \frac{9}{16}, \frac{10}{16}, \frac{11}{16}, \frac{12}{16}, \frac{13}{16}, \frac{14}{16}, \frac{15}{16}, \frac{16}{16} = 1 \right\} \tag{2.2}$$

$$\widetilde{F}_{4^3} = \left\{ \frac{0}{64}, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \frac{4}{64}, \frac{5}{64}, \frac{6}{64}, \frac{7}{64}, \frac{8}{64}, \frac{9}{64}, \frac{10}{64}, \ldots, \frac{64}{64} = 1 \right\} \tag{2.3}$$

## 3. A Subsequence of Farey Sequence-Topological Space

**Theorem 1.** *For any integer* $N \geq 3$ *a subsequence of Farey sequence, Farey N-subsequence denoted by* $\widetilde{F}_{N^k}$ *is a topological space.*

**Proof.** To define a topology on a set first a basis and hence open sets should be described clearly. Here the basis is defined as follows.

Consider $X = \left\{ \frac{i}{N^k}, \quad \begin{matrix} i = 0, 1, 2, 3, \ldots \\ k = 1, 2, 3, 4, \\ N = 3, 4, 5, \ldots \end{matrix} \right\}$

Take $B = \{\widetilde{F}_{N^k}, k = 1, 2, 3, \ldots\}$

Here every element of $B$ is a sequence of real numbers.

**Claim.** B constitute a basis for X

Clearly $F = \cap \widetilde{F}_{N^k}, k = 1, 2, 3, \ldots \infty = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$

**Case (i).** If $x \in F$ then choose basis elements as any one of $\widetilde{F}_{n^k}, k \geq 2$

**Case (ii).** Suppose that $x$ is not in $F$. $X$ may be any one of the following forms.

$$X = \frac{i}{N^{k+1}}; \overset{0 \leq i \leq N^k - 1}{N^k + 1 \leq i \leq j * N^k - 1 / j} = 1, 2, 3, \ldots, N \text{ for } k = 1, 2, 3, \ldots, \infty.$$

Choose basis elements as any one of $\widetilde{F}_{N^k}, k = 2, 3, \ldots, \infty$

Then clearly $B_i \cap B_j$ contains a basis element in which $x$ is a members.

The open sets may taken as a sequence of union of members of $B$. Then for every elements in $U$ there exists a member in $B$ such that $x \in B \subseteq U$.

**Illustration 1.** Consider $X = \left\{ \frac{i}{N^k}, \quad \begin{matrix} i = 0, 1, 2, 3, \ldots \\ k = 1, 2, 3, 4, \\ N = 3, 4, 5, \ldots \end{matrix} \right\}$

$B = \{\widetilde{F}_{N^k}, k = 1, 2, 3, \ldots$

Take $N = 4$, $k = 1, 2, 3, \ldots$

$B = \{\widetilde{F}_{4^1}, \widetilde{F}_{4^2}, \widetilde{F}_{4^3}, \ldots\}$

Consider higher ordered Farey sequence

$$\widetilde{F}_{4^1} = \left\{\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = 1\right\}$$

$\widetilde{F}_{4^2} =$

$$\left\{\frac{0}{16}, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \frac{4}{16}, \frac{5}{16}, \frac{6}{16}, \frac{7}{16}, \frac{8}{16}, \frac{9}{16}, \frac{10}{16}, \frac{11}{16}, \frac{12}{16}, \frac{13}{16}, \frac{14}{16}, \frac{15}{16}, \frac{16}{16} = 1\right\}$$

$$\widetilde{F}_{4^3} = \left\{\frac{0}{64}, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \frac{4}{64}, \frac{5}{64}, \frac{6}{64}, \frac{7}{64}, \frac{8}{64}, \frac{9}{64}, \frac{10}{64}, \ldots, \frac{64}{64} = 1\right\}$$

$\vdots$

$\vdots$

$U = \widetilde{F}_{4^2} \cup \widetilde{F}_{4^3}$

Such that $x \in \frac{3}{4^2}$ and $x \in \widetilde{F}_{4^2} \subseteq U$

Let $\tau = \{\widetilde{F}_{4^1}, \widetilde{F}_{4^2}, \widetilde{F}_{4^3}, \ldots\}$

Consider $\tau_1 = \widetilde{F}_{4^1}$

$$= \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$$

and $\tau_1 = \widetilde{F}_{4^2}$

$$= \left\{0, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \ldots, \frac{15}{16}, 1\right\}$$

Then

$$\tau_1 \cup \tau_2 = \widetilde{F}_{4^1} \cup \widetilde{F}_{4^2}$$

$$= \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\} \cup \left\{0, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \dots, \frac{15}{16}, 1\right\}$$

$$= \left\{0, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \dots, \frac{15}{16}, 1\right\} \in \tau$$

Therefore the union of the elements of a subset of $\tau$ is in $\tau$.

Consider $\tau_1 = \widetilde{F}_{4^3}$

$$= \left\{0, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \dots, \frac{63}{64}, 1\right\}$$

and $\tau_1 = \widetilde{F}_{4^4}$

$$= \left\{0, \frac{1}{256}, \frac{2}{256}, \frac{3}{256}, \dots, \frac{255}{256}, 1\right\}$$

Then

$$\tau_1 \cup \tau_2 = \widetilde{F}_{4^3} \cup \widetilde{F}_{4^4}$$

$$= \left\{0, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \dots, \frac{63}{64}, 1\right\} \cup \left\{0, \frac{1}{256}, \frac{2}{256}, \frac{3}{256}, \dots, \frac{255}{256}, 1\right\}$$

$$= \left\{0, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \dots, \frac{63}{64}, 1\right\} \in \tau$$

Therefore the intersection of the elements of any finite subcollection of $\tau$ is in $\tau$. In well known that in a Hausdorff space every pair of elements is separated by open sets the following is the theorem of Farey N-subsequenceas Hausdorff space.

**Theorem 2.** *For set consisting of rational numbers of* $\frac{i}{N^k}$ *the form the basis defined in the above theorem forms a Hausdorff space.*

**Proof.** A Hausdroff space is in fact a topological space. To define a topology a basis should be describe in the basis for the topology is defined as above. Here an open set is taken in the form

$$\widetilde{W}_{N^k} = X - \bigcup_{j=1}^{k-1} \widetilde{F}_{N^j} \text{ where } \begin{matrix} k = 1, 2, 3, \dots \\ N = 3, 4, 5, \dots \end{matrix}$$

Consider the points $x_1 = \frac{i}{N^r}$ and $x_2 = \frac{i}{N^t}$, $r, t = 1, 2, 3, \dots$ clearly $\frac{i}{N^r} \in \widetilde{F}_{N^r}$ and $\frac{i}{N^t} \in \widetilde{F}_{N^t}$ which are disjoint by their construction. Therefore any two disjoint points of $X$ have disjoint neighbourhoods. Therefore $X$ is a Hausdorff space.

**Illustration 2.** $\widetilde{W}_{N^k} = X - \bigcup_{j=1}^{k-1} \widetilde{F}_{N^j}$ where $\begin{matrix} k = 1, 2, 3, \dots \\ N = 3, 4, 5, \dots \end{matrix}$

Take $N = 4$, $k = 1, 2, 3, \dots$

Consider $S = \widetilde{W}_{4^1}, \widetilde{W}_{4^2}, \widetilde{W}_{4^3}, \dots$

$s_1 = \frac{61}{4^3}$ and $s_2 = \frac{251}{4^4}$ are distinct points of $S$.

Then there exist neighborhoods

$D_1 = \widetilde{W}_{4^3}$ and $D_2 = \widetilde{W}_{4^4}$ of $s_1$ and $s_2$ that are also disjoint.

Therefore the topological space $S$ is called a Hausdorff space.

**Corollary 1.** *On the same construction above the topological space $X$ also satisfies $T_1$ axioms.*

**Illustration 3.** $\widetilde{W}_{N^k} = X - \bigcup_{j=1}^{k-1} \widetilde{F}_{N^j}$ where $\begin{matrix} k = 1, 2, 3, \dots \\ N = 3, 4, 5, \dots \end{matrix}$

Take $N = 4$, $k = 1, 2, 3, \dots$

Consider $T = \widetilde{W}_{4^1}, \widetilde{W}_{4^2}, \widetilde{W}_{4^3}, \dots$

Given two points $q_1 = \frac{15}{4^2}$ and $q_2 = \frac{517}{4^5}$ of $T$

There exist an open set $I_1 = \widetilde{W}_{4^2}$ and $I_2 = \widetilde{W}_{4^5}$ of $T$

Therefore $q_1 \in \widetilde{W}_{4^2}$ and $q_1 \notin \widetilde{W}_{4^5}$

$\therefore q_2 \in \widetilde{W}_{4^5}$ and $q_2 \notin \widetilde{W}_{4^2}$.

**Theorem 3.** $F = \cup F_N$, $F_N$ *is a Farey sequence is bounded by* 0 *and* 1. *The subsequence* $V = \cup F_{N^k}$ *of F it has convergent subsequence.*

**Proof.** Consider the Farey sequence $F_N$, where $N = 1, 2, 3, \ldots$ for all positive integers $N$, $F_N$ is a bounded sequence and is bounded by 0 and 1.

The set $F$ is defined above is also bounded by 0 and 1.

Now the subsequence of $V$ namely $\left\{\frac{k}{N^k} / k = 1, 2, 3, \ldots\right\}$ it is a convergence sequence and converges to 0. This is because $\frac{1}{N^k} \to 0$ as $k \to \infty$ and for all positive integers $N$.

The Farey $N$ subsequence of order 4 can be depicted in the graph as follows $X$ axis = Farey $N$-subsequence

$Y$ axis $Y$ axis = Integer

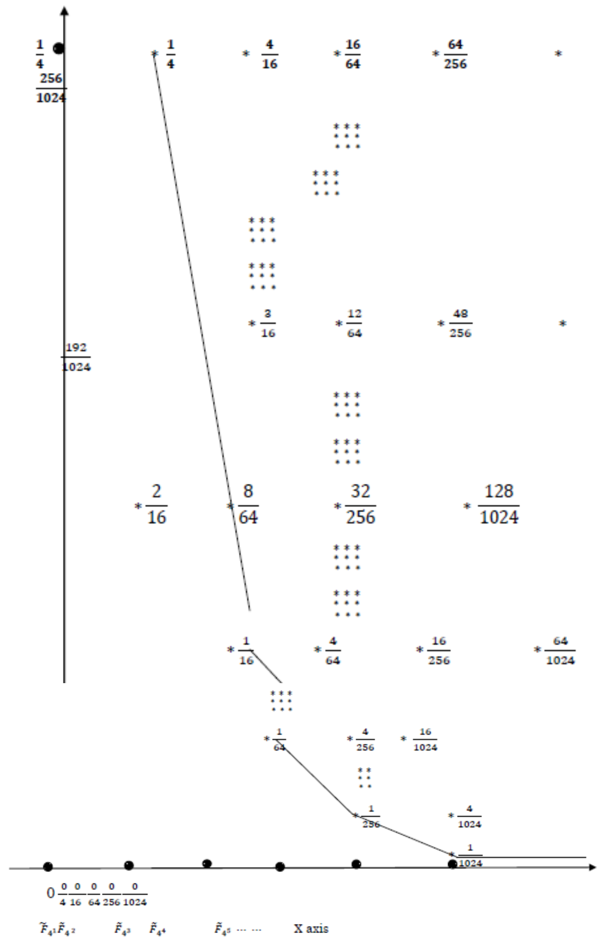


**Figure 1.**

From the above graph, it is clear that the curve resembles inverse exponential curve.

## 5. Conclusion

We have established the Farey $N$-Subsequence and also we have shown their topology space, Hausdorff space and space.

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# Complex Method on Octagonal Number

S. Sudha, A. Gnanam

***Abstract:** In Number theory Study of polygonal numbers is rich in varity. In this paper we establish a Complex Octagonal Number using Continued Fraction algorithm.*

***Keywords:** Algorithm, Continuedfraction, complex Number. Octagonal Number.*

## I. INTRODUCTION

A Simple continued fraction [1] is an expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\ddots}}}$$

Where the $a_i$ are a possibly infinite sequence of integers such that $a_1$ is non-negative and the rest of the sequence is positive. We write $\langle a_1; a_2, a_3 \dots\dots \rangle$ . The above fraction also calls them Regular continued fractions.

## II. CONTINUED FRACTION ALGORITHM

Suppose we wish to find continued fraction expansion[2] of $x \in R$.
Let $x_0 \in x$ and set $a_0 = [x_0]$,
Define $x_1 = \frac{1}{x_0 - [x_0]}$ and set $a_1 = [x_1]$ and $x_2 = \frac{1}{x_1 - [x_1]} \Rightarrow$ $a_2 = [x_2] \dots . x_k = \frac{1}{x_{k-1} - [x_{k-1}]} \Rightarrow a_k = [x_k] \dots$
This process is continued infinitely or to some finite stage till an $x_i \in N$ exists such that $a_i = [x_i]$.

## III. OCTAGONAL NUMBER

**A. Definition:** Centered Octagonal Number[3]
The Number 1,9,25,49,81,121,…….. are called centered octagonal numbers. The number that represents associate in nursing polygonal shape with a dot within the center and every one dots different dots encompassing the middle dot in associate in nursing polygonal shape lattice .

The $n^{th}$ centered octagonal number is given by the formula
$O_n = 4n(n-1) + 1$

**B. Theorem:**

For $n \geq 3$,

$$\frac{O_n}{O_{n+1}} + i\frac{O_{n+2}}{O_{n+3}} = \begin{cases} \langle 0; 1, \left[\frac{n}{2}\right], 8n\rangle + i\,\langle 0; 1, \left[\frac{n+1}{2}\right], 8(n+2)\rangle & \textit{when n is odd} \\ \langle 0; 1, \frac{n}{2} - 1,1,1,2n-1,2\,\rangle + i\,\langle 0; 1, \frac{(n+2)}{2} - 1,1,1,2(n+2)-1,2\,\rangle & \textit{when n is even} \end{cases}$$

**Proof:**
**Case (i):-** n is odd
Let n = 2k-1, Where 3≤ $k \leq n$
Then

$$\frac{O_{2k-1}}{O_{2k}} + i\frac{O_{2k+1}}{O_{2k+2}} = \langle 0; 1, \left[\frac{2k-1}{2}\right], 8(2k-1)\rangle + i\langle 0; 1, [k], 8(2k+1)\rangle$$

Next we have to prove that n = 2k+1
To find the continued fraction of

$$\frac{O_{2k+1}}{O_{2k+2}} + i\frac{O_{2k+3}}{O_{2k+4}}$$

**A. Real Part:-[3]**

$$\frac{O_{2k+1}}{O_{2k+2}} = \frac{4(2k+1)(2k+1-1)+1}{4(2k+2)(2k+2-1)+1}$$
$$= \frac{16k^2+8k+1}{16k^2+2k+9}$$

$$x_0 = \frac{16k^2+8k+1}{16k^2+2k+9}, a_0 = 0$$
$$x_1 \Rightarrow 1 + \frac{16k+8}{16k^2+8k+1} \Rightarrow a_1 = 1$$
$$x_2 \Rightarrow k + \frac{1}{16k+8} \Rightarrow a_2 = k$$
$$x_3 \Rightarrow 16k+8 \Rightarrow a_3 = 16k+8$$
=8(2k+1)

$$\therefore \frac{O_{2k+1}}{O_{2k+2}} = \langle 0; 1, k, 8(2k+1)\rangle$$

**B. Imaginary part:-**

$$\frac{O_{2k+3}}{O_{2k+4}} \Rightarrow \frac{4(2k+3)(2k+3-1)+1}{4(2k+4)(2k+4-1)+1}$$
$$= \frac{4[4k^2+6k-2k+6k+9-3]+1}{4[4k^2+8k-2k+8k+16-4]+1}$$
$$= \frac{4[4k^2+4k+6k+6]+1}{4[4k^2+14k+12]+1}$$
$$= \frac{4[4k^2+10k+6]+1}{4[4k^2+14k+12]+1}$$
$$= \frac{16k^2+40k+24+1}{16k^2+56k+48+1}$$
$$\frac{O_{2k+3}}{O_{2k+4}} \Rightarrow \frac{16k^2+40k+25}{16k^2+56k+49}$$
$$x_0 = \frac{16k^2+40k+25}{16k^2+56k+49};\ a_0 = 0$$

Then

$$x_1 \Rightarrow \frac{16k^2+56k+49}{16k^2+40k+25} = 1 + \frac{16k+24}{16k^2+40k+25} \Rightarrow a_1 = 1$$

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# Algebraic Approach onOctagonal Numbers

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***Abstract*--In number theory study of polygonal number. In this paper we have to find centered octagonal number of continued fractions. Next we have to prove monoidunder multiplication of centered octagonal number and square number**.

***Keywords*--**Continued fraction Algorithm, Centered Octagonal Number, Square Number,Monoid.

**Notations:-**

1. $\langle a_0; a_1, a_2, \ldots \ldots . a_n \rangle$ : Continued fraction Expansion
2. $\left[\frac{n}{2}\right]$ : Integer part of the rational number n/2.
3. $O_n = 4n(n-1)+1$ : n[th]centered octagonal number.

## I. Introduction

A continued fraction refers to all expressions of the form

$$a_0 + \frac{b_0}{a_1 + \frac{b_1}{a_2 + \frac{b_2}{\ddots}}}$$

Where $a_1, a_2, a_3 \ldots$ ....and $b_1, b_2, b_3 \ldots$ ....are either real or complex values. The number of terms can be either finite or infinite[7].

A Simple Continued fraction is a continued fraction[7] in which the value of $b_n = 1$ for all n.

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

The value of $a_n$is a positive integer for all n $\geq 1. a_1$can be any integer value, including 0. The above fraction is sometimes represented by $\langle a_1; a_2, a_3 \ldots \ldots \rangle$

## II. Continued Fraction Algorithm

To find continued fraction expansion of $x \in R$.

Let $x \in R$ and we write [2]

$$x = a_0 + t_0$$

With $a_0 \in Z$and $0 \leq t_0 < 1$

If$t_0 \neq 0$, then we write $\frac{1}{t_0} = a_1 + t_1$with $a_1 \in N$ and $0 \leq t_1 < 1$

Thus we can write

$$t_0 = \frac{1}{a_1 + t_1} = \langle 0; a_1 + t_1 \rangle$$

This is a continued fraction expansion of$t_0$. Continue in this process $t_n \neq 0$

We write

$$\frac{1}{t_n} = a_{n+1} + t_{n+1}$$

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# Iterated Function System of Generalized Non – Reducible Farey N – Subsequence of order (4, 6...) by using HB Operator

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***Abstract:-***
*This paper establishes Non - Reducible Farey N - Subsequence of order (4, 6,...). This research formulates iterated function systems by using HB Operator and additionally shows the generalized Non - Reducible Farey N - Subsequence of order (4, 6,...).*

***Keywords:-***
*Arithmetic Mediant, Farey 'N' subsequence, Hausdorff dimension, Invariant Measure, Iterated Function System, Markov Operator, Non - Reducible Farey Sequence.*

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## I. Introduction

The Farey sequence is an example that has its inception for all intents and common numbers. The Farey sequence was so named after the British born geologist, John Farey (1766-1826). Given a sequence $F_N$ where $b, d$ and $b+d$ are all less than $N$, what Farey noticed is that if two fractions $\frac{a}{b}$ and $\frac{c}{d}$ were combined in the way $\frac{a+c}{b+d}$, the resulting fraction was also in the series. Farey was not able to prove this but prolific French mathematician Augustin Cauchy (1789-1857) had the option to give a proof in 1816 and published in Exerices de mathematiques [1,3,5,6,8,9].

This paper is organized as follows: Section 2 Basic definitions with example. Section 3 iterated function system of Non – Reducible Farey N – subsequence of order 4. Section 4 Research formulates iterated function system of Non – Reducible Farey N – subsequence of order 6. Section 5 the researcher shows the iterated function system of Non – Reducible Farey N – subsequence of order (2m-2).

## II. Preliminaries

Throughout this paper we study the Non – Reducible Farey N – subsequence of order (4, 6,...)

**Definition 2.1: Farey sequence [2]**
A Farey sequence $F_n$ is the set of rational numbers $\frac{p}{q}$ with $p$ and q coprime, with $0 < p < q < n$, ordered by size.

**Example 2.1.1:**

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$
$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$
$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

**Definition 2.2: Farey N – subsequence [2]**
The subsequence of farey sequence of order N whose denominators is equal to N is named as Farey N – subsequence and denoted by $\langle F_N' \rangle$.

$$\langle F_N' \rangle = \left\{\frac{u_i}{N} / \, 0 \le u_i \le N, 0 \le i \le N\right\}$$

**Definition 2.3: Non - Reducible Farey Sequence [2]**
The Sequence of non-reduced fractions with denominators not exceeding N listed in order of their size is called Non - Reducible Farey Sequence of order N. It is denoted by $\widetilde{F_N}$.