ALGEBRAIC STUDIES ON CERTAIN CLASSES OF GK ALGEBRA

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By

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This thesis is dedicated to

The Gods who brought me to this world

Mr. R. Jayavel and Mrs. J. Thilagavathi

and

Late Mr. P. R. Radhakrishnan, my Father-in-law.

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CERTIFICATE

Certified that this thesis titled **“ALGEBRAIC STUDIES ON CERTAIN CLASSES OF GK ALGEBRA”** is the bonafide work of **Ms. J. KAVITHA** who carried out the research under my supervision. Certified further, that to the best of my knowledge the work reported herein does not form part of any other thesis or dissertation on the basis of which a degree or award was conferred on an earlier occasion on this or any other candidate.

Kumbakonam **Dr. R. GOWRI**

.06.2022 Research Advisor

CERTIFICATE FOR PLAGIARISM

It is Certified that Ph.D. Thesis Titled **“ALGEBRAIC STUDIES ON CERTAIN CLASSES OF GK ALGEBRA”** by J. KAVITHA, has been examined by me, I undertaken the follows:

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DECLARATION

I do hereby declare that this work has been originally carried out by me under the guidance and supervision of **Dr. R. GOWRI**, Assistant Professor, Department of Mathematics, Government College for Women (Autonomous), and this work has not been submitted elsewhere for the award of other degree or diploma.

Kumbakonam **J. KAVITHA**

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Contents

Notations

T	-	GK algebra
P	-	GK Sub algebra
$\circledast$	-	Binary operation on T
D	-	GK ideal
σ	-	GK homomorphism
Kerσ	-	Kernel of σ
Θ	-	Multiplier of GK algebra
$\curlywedge$	-	$y \circledast (y \circledast x)$
ξ	-	Derivation of GK algebra
$LD(\xi)$	-	Set of all GK left-right derivation
Ω	-	GK symmetric bi derivation
ρ_{gk}	-	Fuzzy GK sub algebra

Abstract

The construction of this research work has been mainly focused on evolving the new concept of algebraic structure, namely GK algebra. The newly constructed algebraic structure is named from the first letters of the authors' name. In this work, the characteristics of GK algebra are studied and showed that GK algebra is different from all other algebraic structures such as BCK / BCI / BM / C / BE / TM / BG / Z / AB / B / BRK / KUS / KU / Q / QS / PS / SP, etc. with suitable illustrations. The GK sub algebra, GK ideal, self-distributivity, commutativity, and associativity of GK algebra are defined and their respective properties are expounded. Homomorphism and anti-homomorphism of GK algebra are introduced and discussed the relationship between homomorphism, anti-homomorphism, ideal, and kernel of GK algebra in detail. The theory of multipliers and kernel of multipliers of GK algebra are introduced and investigated their properties. The concept of the direct product of GK algebra is studied. The theory of derivation in GK algebra is initiated and particularly (left-right) and (right-left) derivation respectively are investigated. The concept of the symmetric bi derivation of GK algebra is deliberated. The newly constructed algebraic structure, GK algebra is fuzzified and investigated their aspects. The fuzzy GK sub algebra, fuzzy GK ideal, fuzzy GK homomorphism, and fuzzy GK anti-homomorphism, fuzzy GK anti-ideal are developed and explored their properties. The Cartesian product along with fuzzy GK algebra is examined and in continuation, few interesting results relevant to these are obtained.

CHAPTER 1

INTRODUCTION

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INTRODUCTION

1.1 Algebraic structure

The term "*algebra*" has been coined only in the 19^{th} Century. The word algebra was used first time in the book entitled *Hisab al-JabrW'al-muqabala*" a Mathematician, who lived in the city of Baghdad, knowingly, *Jafar Muhammad I.M. Al-Khwaizmi*. His area of interest was solving the algebraic equation, particularly quadratic equations. His method of finding the solution for an equation was quite different. He applied a transformation to the given equation and substituted it in a standard form and attained the method of solution to that equation. In connection with this, through the 19^{th} Century the meaning of "*algebra*" was only solving the equations, on which, mainly focused on the 4^{th} degree or less than that. In this regard, the method of finding the solution to an equation is known as Classical algebra. It was first developed by Babylonian. They were distinctly known for efficient "algebraists".

In the early decennium of the 20^{th} Century, algebra had progressed in the axiomatic approach. This approach is known as modern or abstract algebra. At the end of the 19^{th} Century, the transformation from Classical algebra to modern algebra was developed. The axiomatic approach is used while learning abstract mathematics. This leads to taking a group of $\mathbb{N}$ objects and presuming some conditions about their structure. These conditions are named axioms. Using this, we shall derive other formats about $\mathbb{N}$ by using logical statements.

The algebraic structure is a structure that can arise in the problem of Mathematics. It is defined through distinct order of axioms. In the field of Mathematics, the algebraic structure takes a part in an indispensable role with comprehensive application in several areas namely Engineering, Physics, Information Technology and Computer Science etc.

1.2 Fuzzy Structure

In 1965, the fuzzy set theory was coined by Zadeh [74]. In the observation of fuzzy sets, the word fuzzy generally represents the word, formless or unclear. He has initiated an analytical technique of decision making with fuzzy elucidation of some kind of information became possible. The concept of fuzzy set theory offered a gradient to perceive and explore the relationship between the sets and the corresponding elements in the set. Fuzzy set theory was followed by the presumption that classical sets were not realistic, applicable, or useful concepts in exploring real-life problems because everything confronted in this world carries a degree of fuzziness. Further, the concept of grade of membership is not a probabilistic concept.

It is very difficult to describe the concept of vagueness. At that time, the grade of membership has introduced. A fuzzy set is characteristics by a membership function that assigns to each object a degree of membership ranging from zero and one. Basically, fuzzy logic is a multi-valued logic that allows intermediate values to be defined between conventional evaluations like true or false, yes or no, high or low etc. Hence fuzzy set has become a wide area of research in Engineering, Medical science, Social science, Graph theory etc.

The fuzzy subgroups are introduced in a seminal paper which was delivered by Rosenfeld, in 1971[64]. This paper is one of the initiatives of attaining the new content in abstract algebra and also in fuzzy mathematics. In continuation of that, many researchers have come forward to do their research work with fuzzy concepts. Fuzzy algebraic structures, fuzzy topological spaces, and fuzzy graphs are some fuzzy extensions of the vital theories in the field of mathematics such as Algebra, Topology, and Graph theory respectively.

1.3 Literature Review

A review of the literature provides the fundamental facts, a clear graph of the subject under investigation, the research gaps that remain, and the path forward. This section provides the Chronological facts of the researches state of the art.

BCK-algebras and BCI-algebras are abridged to two B-algebras. The BCK algebra has been coined in 1966 by the Japanese mathematicians, Y. Imai and K. Iseki [24]. Two B-algebras have been created from two different provenances. One of the instigations is gleaned from set theory. The necessary and rudimentary operations on set theory are the union, intersection, and set difference. In addition that, the Boolean algebra is attained from these three operations and their aspects, and also their generalization. Considering the union and intersection both together then as an algebra, the concept of distributive lattices is attained. In addition, the observation of any one of the operations such as either union or intersection alone, the concept of upper semi-lattices is attained, otherwise, lower semi-lattices has appeared. Despite this, the set difference along with its aspects have not been observed as methodically before Iseki [24].

Later, in 1980, another class of algebraic structure was initiated by Iseki [25] named as BCI algebra and analyzed some of its aspects. It is shown that the BCK algebra is a proper subclass of the BCI-algebra. These two algebras are the paramount classes of logical- based algebras. In 1983, a wide range of abstract algebra named as, BCH- algebras was initiated by Hu and Li [23] and which is shown that the class of BCI algebras is a proper subclass of the class of BCH-algebras.

In 1990, the concept of BCC algebra has explored by Dudek [17]. Meanwhile, in 1998, the new idea which is called as BH algebra introduced by Jun Y.B, Roh, E. H, Kim, H. S. [27] as a generalization of BCH / BCI/BCK- algebras. In 1999, the new algebraic structure, which is named as d algebra has initiated by Neggers and Kim [53], which is another generalization of BCK-algebras and studied the relation between d algebra and BCK algebra.

At the same period of time, in 1999, one more new algebraic structure namely QS algebra which is also a generalization of BCK/BCI algebras, was explored by Ahn and Kim [6] and derived various results in terms of subalgebras, ideals, implicative, etc. In 2001, Q- algebras which is a generalization of BCH / BCI / BCK-algebras, and analyzed several theorems on BCI algebras, by Neggers, J, Ahn, S.S and Kim, H.S [52]. Then after, in 2002, the new notion called B algebra was introduced by Neggers and Kim [50] which is another generalization of the class of BCK/BCI/BCH –algebras.

During the same period of time, in 2002, a new concept which is called as β-algebras established by Neggers and Kim [51], where two operations are combined in this way as to reflect the logical reasoning which exists between the typical group operation and its associated B-algebra which is interpreted by it.

In 2006, the notion of BM algebra was introduced by Kim C. B and Kim H. S [30] which is a special class of B-algebras. They established that the class of BM algebras is a proper subclass of B-algebras and also derived that 0 commutative B-algebra is equivalent to BM algebra.

In 2006, the concept of theory of BE algebra was introduced by H. S. Kim and Y. H. Kim [32] as a generalization of a BCK-algebra which is intensely analyzed by S. S. Ahn and Y. H. Kim and K. S. So[7] and A. Walendziak [71]. In 2007, the new algebraic structure which is said to be BF algebra, was explored by Andrze J Walendziak [70] which is a generalization of BCI/BCK/B-algebras. In 2008, the generalization of B algebra called as BG algebra was initiated by Kim and Kim [31].

In 2009, Meng [40] has initiated the CI algebra in which the generalization of BG algebra and dual BCK/BCH/BCI are investigated and also discussed the relation between CI algebra and BE algebras. It is proved that the transitive BE algebra, the notion of ideals is equivalent to one of the filters. In the same year of 2009, KU algebra was introduced by Prabayak and Leerawat [54] which also a generalization of BCK / BCI / BCC –algebras and analysed its characteristics by using ideals and congruences. In 2010, TM algebra was explored by Megalai and Tamilarasi [37,38], and also, they showed that the TM algebra is a generalization of Q / BCK / BCI / BCH algebras and they derived that the TM-algebra fulfilled the several conditions framed in the Q / BCH / BCI /BCK / BCC algebras.

In 2012, the new concept which is called as BRK algebra, which is a generalization of BCK/BCI/BCH/Q/QS/BM algebras, was explored by Bandaru [12]. In 2013, a new notion KUS algebra was introduced by Samy M Mostafa, Naby, M.A.A and Elgendy, O.R [46], which is a generalization of BCK / BCI / BCH / BCC / Q / KU-algebras and explored related aspects. In 2014, a new algebraic structure, namely PS algebra initiated by Priya and Ramachandran [63], as a generalization of BCK / BCI / d / Q / KU algebras and studied its characteristics. Recently, in 2017, the notion which is called as Z algebra established by M. Chandramouleeswaran, P. Muralikrishna, K.Sujatha and S. Sabarinathan [15] and studied its characteristics.

Fuzzification of abstract algebras, which are useful to the successful progress of this thesis are analysed and listed the review of published works.

In 1991, the fuzzification of BCK algebras was introduced by O.G.Xi [72] discussed its characteristics and its properties. In 1993, the concept of Fuzzy BCI algebra was introduced by B. Ahamed [3], in this study he explored the properties of Fuzzy BCI algebras. In 2004, Ahn and Hu.lee [5] have been classified the sub algebra by their family of level sets in BG algebra. In 2010, R. Muthuraj, M.Sridharan, M.S. Muthuraman, and P. M. SitharSelvam [49] have initiated the notion of the anti-Q fuzzy BG ideal of BG algebra. The lower-level cuts and upper-level cuts are introduced and proved some results. In 2014, the characterization of anti-fuzzy PS ideals was introduced and discussed along with ideal and homomorphism by Priya. T and Ramachandran. T [61].

Recently, many researchers have initiated the algebras with subalgebras, ideals, Filter, G-part, medial, implicative ideals, derivation, symmetric bi derivation, multipliers, hyper structure, homomorphism, and relations, etc. and also investigated the fuzzification of algebras. In this more enthrallingly, they studied pseudo, Intuitionistic, bipolar, Neutrosophic structure, Smarandache structure, and cubic structures of the algebras and also their aspects. These are all induced us to study the new algebraic structure which is different from all other algebraic structures.

1.4 Organisation of the Thesis

This section deals with the arrangement of the research work which is to be needed for this thesis.

Chapter I deals with the introduction of research, literature review, basic definitions of both algebraic structure and fuzzy structure that are used to motivate the study of this thesis.

Chapter II deals with the introduction of a new class of algebraic structure, named as GK algebra. It is studied about the properties of GK algebra through GK sub algebra, self-distributive, associative law, ideal, Homomorphism, Kernel of GK algebra, and also, anti-homomorphism and investigated some of its aspects. Some of the paramount results in this chapter are

- In GK algebra left cancellation law and right cancellation law holds.
  - (i) Right cancellation law: if $i \circledast j = k \circledast j$ $\ then\ $ $i = k$.
  - (ii) Left cancellation law: if $k \circledast i = k \circledast j$ $\ then\ i = j$.
- Let T be a GK algebra. A relation ≤ on T is defined as $i \leq j$ if

$i \circledast j = 1. Then\ (T, \leq)$is a partially ordered set.

- Let (T,$\circledast_T, 1)$ and (P,$\circledast_P, 1')$ be a GK algebra and the mapping $\sigma: T \to P$ is a GK homomorphism. Let M be a GK ideal of B then $\sigma^{-1}(M)$is a GK ideal of T.

Chapter III establishes the concept of multipliers of GK algebra and the direct product of GK algebra. The right and left multipliers of GK algebra are defined and attained some interesting results and also, studied about its direct product. Some of the obtained results are given here,

- Let T be a GK algebra and Θ be a regular multiplier. Then the self-mapping Θ is an identity mapping if it satisfies the multiplier (left) is equal to the multiplier (right) that is $\Theta(i) \circledast j = i \circledast \Theta(j)\ \forall\, i, j \in T.$
- Direct product of any two GK algebras is again a GK algebra.

Chapter IV expresses the concept of the derivation of newly defined GK algebra and expounded about GK-LR derivation, GK-RL derivation, and regular in the derivation of GK algebra. Some of the interesting outcomes are,

- Let $(T, \circledast, 1)\ be$ a GK-algebra and ξ be a (GK-LR) derivation of T. Then the following hold $\forall\, i, j \in T$

  (i) $\xi(i \circledast j) = \xi(i) \circledast j$

  (ii) If ξ is regular then $\xi(i) \leq i$
- Let ξ: T→T be a derivation of T. Then ξ is a regular derivation if ξ is either a (GK-LR) derivation or a (GK-RL) derivation.

Chapter V exhibits the concept of the symmetric bi derivation of GK algebra and some of the properties are studied and obtained enthralling results. Some emerged results are,

- Let $(T, \circledast, 1)$ be a GK algebra. Let Ω be a GK-RL symmetric bi derivation on T. Then the following holds

  (i) $\Omega(i,j) = \Omega(i,j) \wedge \big(i \circledast \Omega(1,j)\big)\ for\ all\ \ i,j \in T.$

  (ii) $\Omega(i,1) = \delta(i) \circledast i$ where δ is the trace of Ω.

  (iii) $\Omega(1,j) = \Omega(i,j) \circledast i\ \ \forall\ \ i,j\ \in T.$

  (iv) $\Omega(j,1) = \Omega(j,1) \wedge j\ \forall\ j\ in\ T\ if\ \ \Omega\ \ is\ \delta - regular\ .$

  (v) $\Omega(j,1) = 1\ \ \ \forall\ j\ in\ T\ if\ \ \Omega\ \ is\ component\ wise\ regular.$

- Let T be the GK algebra and δ be the trace of the GK-RL symmetric bi derivation on T. Then

  (i) $\delta(1) = \Omega(1,i) \circledast i.$

  (ii) $\delta(i) = \delta(i) \wedge (i \circledast \Omega(i,1))$

  (iii) $If\ \Omega(1,i) = \Omega(1,j)\ \ \ \forall\ \ i,j\ \in T\ \ then\ \ \delta\ is\ 1 - 1.$

  (iv) δ is regular iff $\Omega(1,i) = i$.

Chapter VI explicates the concept of the fuzzification of GK algebra and discussed the terms, fuzzy GK algebra, fuzzy GK sub algebra, fuzzy GK ideal, Fuzzy homomorphism, anti- fuzzy Homomorphism, anti-fuzzy GK ideals and attained enthralling results such as,

- Every fuzzy GK ideal of a GK-algebra T is order overturn.
- In GK-algebra, the intersection of family of sets on fuzzy GK-ideals is also a fuzzy GK-ideal.
- Let $\rho_{gk}\ and\ \ \sigma_{gk}$ be fuzzy GK ideals of GK algebra X. Then $\rho_{gk} \times \sigma_{gk}$ is a fuzzy GK ideal of $T \times T$.
- A fuzzy set ρ_{gk} in GK algebra is an anti-fuzzy sub algebra iff for every q in [0,1], $\Gamma\big(\rho_{gk}, q\big)$ is either $\emptyset$ or a sub algebra of T.

1.5 Preliminaries

This section deals with the basic and necessary definitions of this current work that have been expressed. Particularly, basic definitions of BCK, BCI, BCH, BM, BG, BRK, d, BI, BE CI, BF, BH, QS, Q, KU, KUS, TM, BRK, KUS, and PS algebras, etc. presented by the researchers are listed and described their works which are already done related to derivation, symmetric bi derivation, multipliers, Cartesian product, anti-homomorphism, fuzzy sub algebra, fuzzy ideals, fuzzy homomorphism, anti-fuzzy sub algebra, and Anti fuzzy ideals, which are to be needed to the progress of this current work has emerged from a survey of the literature.

Definition 1.5.1 [41]

An algebra is called BCK algebra $(\mathcal{M},*,0)$ of type (2,0) satisfying the following axioms

(i) $(p_1 * q_1) * (p_1 * r_1) \leq (r_1 * q_1)$

(ii) $p_1 * (p_1 * q_1) \leq q_1$

(iii) $p_1 \leq p_1$

(iv) $p_1 \leq q_1 \; and \; y \leq p_1 \Longrightarrow p_1 = q_1$

(v) $0 \leq p_1 \Longrightarrow p_1 = 0, where \; p_1 \leq q_1 \; is \; defined \; by \; p_1 * q_1 = 0, \forall \, p_1, q_1, r_1 \in \mathcal{M}.$

Definition 1.5.2 [41]

Let $(\mathcal{M},*,0)$ be a BCK algebra. A non-empty subset I of $\mathcal{M}$ is called an ideal of $\mathcal{M}$ if it satisfies the following conditions

(i) $0 \in I$

(ii) $p_1 * q_1 \in I \, and \, q_1 \in I \Longrightarrow p_1 \in I \, for \, all \, p_1, q_1 \in \mathcal{M}.$

Definition 1.5.3 [25]

A BCI-algebra is an algebra ($\mathcal{M}$, *, 0) of type (2, 0) satisfying the following conditions:

(i) $\big((p_1 * q_1) * (p_1 * r_1)\big) * (r_1 * q_1) = 0$

(ii) $\big(p_1 * (p_1 * q_1)\big) * q_1 = 0$

(iii) $p_1 * p_1 = 0$

(iv) $p_1 * q_1 = 0 \, and \, q_1 * p_1 \Longrightarrow p_1 = q_1 \forall \, p_1, q_1, r_1 \in \mathcal{M}.$

Definition 1.5.4 [1, 23]

A BCH - algebra is an algebra $(\mathcal{M}, *, 0)$ of type (2, 0) satisfying the following axioms

(i) $p_1 * p_1 = 0$

(ii) $p_1 * q_1 = 0 \, and \, q_1 * p_1 = 0 \Longrightarrow p_1 = q_1$

(iii) $(p_1 * q_1) * r_1 = (p_1 * z) * q_1 \, for \, all \, p_1, q_1, r_1 \in \mathcal{M}.$

Definition 1.5.5 [27]

A BH-algebra is an algebra$(\mathcal{M}, *, 0)$, where $\mathcal{M}$ is a nonempty set, $*$ is a binary operation and 0 is a constant, satisfying the following axioms

(i) $p_1 * p_1 = 0$

(ii) $p_1 * q_1 = 0 \, and \, q_1 * p_1 = 0 \Longrightarrow p_1 = q_1$

(iii) $p_1 * 0 = p_1, \forall p_1, q_1 \in \mathcal{M}.$

Definition 1.5.6 [53]

A d-algebra is an algebra $(\mathcal{M},*,0)$ of type (2, 0) satisfying the following conditions

(i) $p_1 * p_1 = 0$

(ii) $0 * p_1 = 0$

(iii) $p_1 * q_1 = 0 \, and \, q_1 * p_1 = 0 \Longrightarrow p_1 = q_1 \, , for \, all \, p_1, q_1 \in \mathcal{M}.$

Definition 1.5.7[53]

Let $(\mathcal{M},*,0)$ be a d – algebra. A non-empty subset I of $\mathcal{M}$ is called a d ideal of $\mathcal{M}$ if it satisfies the following conditions

(i) $0 \in I$

(ii) $p_1 \in I \, and \, q_1 \in \mathcal{M} \Longrightarrow p_1 * q_1 \in I \, , that \, is \, , I * \mathcal{M} \subseteq I.$

Definition 1.5.8 [52]

A Q - algebra is an algebra $(\mathcal{M},*,0)$ of type (2, 0) satisfying the following conditions

(i) $p_1 * p_1 = 0$

(ii) $p_1 * 0 = p_1$

(iii) $(p_1 * q_1) * r_1 = (p_1 * r_1) * q_1, where \, p_1 \leq q_1 \, is \, defined \, by$

$p_1 * q_1 = 0, for \, all \, p_1, q_1, r_1 \in \mathcal{M}.$

Definition 1.5.9 [52]

Let $(\mathcal{M},*,0)$ be a Q – algebra. A non-empty subset I of $\mathcal{M}$ is called a Q- ideal of $\mathcal{M}$ if it satisfies the following conditions:

(i) $0 \in I$

(ii) $(p_1 * q_1) * r_1 \in I \, and \, q_1 \in I \Longrightarrow p_1 * r_1 \in I \, , for \, all \, p_1 \, , q_1, r_1 \in \mathcal{M}.$

Definition 1.5.10 [50]

A B-algebra is a non-empty set $\mathcal{M}$ with a constant 0 and a binary operation $*$ satisfying the following axioms

(i) $p_1 * p_1 = 0$

(ii) $p_1 * 0 = p_1$

(iii) $(p_1 * q_1) * r_1 = p_1 * (r_1 * (0 * q_1)), for\ all\ p_1, q_1, r_1 \in \mathcal{M}$.

Definition 1.5.11 [51]

A β-algebra is a non-empty set $\mathcal{M}$ with constant 0 and two binary operations + and $-$ satisfying the following axioms

(i) $p_1 - 0 = p_1$

(ii) $(0 - p_1) + p_1 = 0$

(iii) $(p_1 - q_1) - r_1 = p_1 - (r_1 + q_1), for\ allp_1, q_1, r_1\ in\ \mathcal{M}$.

Definition 1.5.12 [51]

A non-empty set I of anβ algebra is called a β −ideal of $\mathcal{M}$, if it satisfies the following conditions

(i) $0 \in I$,

(ii) $p_1 + q_1 \in I, \forall p_1, q_1 \in I, and$

(iii) $if\ p_1 - q_1 and\ q_1 \in I\ then\ p_1 \in I\ \forall\ p_1, q_1 \in \mathcal{M}$.

Definition 1.5.13 [30]

An algebra is said to be a BM-algebra with a non-empty set $\mathcal{M}$ and a constant 0, a binary operation $*$ satisfying the following axioms

(i) $p_1 * 0 = p_1$

(ii) $(r_1 * p_1) * (r_1 * q_1) = q_1 * p_1, for\ any\ p_1, q_1, r_1 \in \mathcal{M}.$

Note 1.5.14 [30]

(i) Every BM-algebra is a B-algebra.

(ii) If $(\mathcal{M},*,0)$ is a BM-algebra, then it is a 0-commutative B-algebra.

Definition 1.5.15 [32]

An algebra $(\mathcal{M},*,1)$ of type (2, 0) is called a BE-algebra if it satisfiesthe following conditions:

(i) $p_1 * p_1 = 1\ for\ allp_1 \in \mathcal{M}$

(ii) $p_1 * 1 = 1\ for\ all\ p_1 \in \mathcal{M}$

(iii) $1 * p_1 = p_1\ for\ allp_1 \in \mathcal{M}$

(iv) $p_1 * (q_1 * r_1) = q_1 * (p_1 * r_1)\ for\ all\ p_1, q_1, r_1 \in \mathcal{M}$

Here the relation $\leq$ on $\mathcal{M}$ is defined by $p_1 \leq q_1$ if and only if $p_1 * q_1 = 1$.

Definition 1.5.16 [32]

Let $(\mathcal{M},*,1)$ be a BE-algebra and let F be a non-empty subset of $\mathcal{M}$. Then F is said to be a filter of $\mathcal{M}$ if

(i) $1 \in F$

(ii) $p_1 * q_1 \in F\ and\ p_1 \in F\ imply\ q_1 \in F.$

Definition 1.5.17 [70]

A non-empty set $\mathcal{M}$ with a constant 0 and a single binary operation $*$, is said to be a BF-algebra if it satisfies the following

(i) $p_1 * p_1 = 0$

(ii) $p_1 * 0 = p_1$

(iii) $0 * (p_1 * q_1) = q_1 * p_1, for\ all\ p_1, q_1 \in \mathcal{M}.$

Definition 1.5.18[31]

A BG-algebra is a non-empty set $\mathcal{M}$ with a constant 0 and a binary operation $*$ satisfying the following axioms:

(i) $p_1 * p_1 = 0$

(ii) $p_1 * 0 = p_1$

(iii) $(p_1 * q_1) * (0 * q_1) = p_1, for\ all\ p_1, q_1 \in \mathcal{M}.$

Definition 1.5.19 [29,40]

An algebraic system $(\mathcal{M}, *, 1)$ of type (2, 0) is called a CI -algebra if itsatisfies the following axioms.

(i) $p_1 * p_1 = 1$

(ii) $1 * p_1 = p_1$

(iii) $p_1 * (q_1 * r_1) = q_1 * (p_1 * r_1), for\ all\ p_1, q_1, r_1 \in \mathcal{M}.$

In $\mathcal{M}$, we define a binary operation $\leq$ by $p_1 \leq q_1\ if\ and\ only\ if$

$p_1 * q_1 = 1\ for\ all\ p_1, q_1 \in \mathcal{M}.$

Definition 1.5.20 [54]

A KU - algebra is an algebra $(\mathcal{M}, *, 0)$ of type (2,0) satisfying the following conditions

(i) $(p_1 * q_1) * ((q_1 * r_1) * (p_1 * r_1)) = 0$

(ii) $p_1 * 0 = 0$

(iii) $0 * p_1 = p_1$

(iv) $p_1 * q_1 = 0\, and\, q_1 * p_1 = 0\, imply\, p_1 = q_1, for\, all\, p_1, q_1, r_1 \in \mathcal{M}$.

Definition 1.5.21 [54]

Let $(\mathcal{M},*,0)$ be a KU-algebra. A non-empty subset I of $\mathcal{M}$ is called KU ideal of $\mathcal{M}$ if it satisfies the following conditions

(i) $0 \in I$.

(ii) $p_1 * (q_1 * r_1) \in I\, and\, q_1 \in I \implies p_1 * r_1 \in I\, for\, all\, p_1, q_1, r_1 \in \mathcal{M}$.

Definition 1.5.22 [37,38]

A non-empty set $\mathcal{M}$ with a constant 0 and a binary operation $*$ is calleda TM - algebra if it satisfies the following axioms.

(i) $p_1 * 0 = p_1$

(ii) $(p_1 * q_1) * (p_1 * r_1) = z * q_1, for\, all\, p_1, q_1, r_1 \in \mathcal{M}$

In $\mathcal{M}$ we can define a binary operation $\leq$ by $p_1 \leq q_1$ if and only if $p_1 * q_1 = 0$.

Definition 1.5.23 [12]

A BRK-algebra is a nonempty set $\mathcal{M}$ with a constant 0 and a binaryoperation $*$ satisfying axioms

(i) $p_1 * 0 = p_1$,

(ii) $(p_1 * q_1) * p_1 = 0 * q_1\, for\, any\, p_1, q_1 \in A$.

In $\mathcal{M}$, we can define a binary relation $\leq$ by $p_1 \leq q_1$ if and only if $p_1 * q_1 = 0$.

Definition 1.5.24 [46]

Let $(\mathcal{M},*,0)$ be an algebra of type (2,0) with a binary Operation $*$ is called KUS-algebra if it satisfies the following axioms

(i) $(r_1 * q_1) * (r_1 * p_1) = (q_1 * p_1)$

(ii) $0 * p_1 = p_1$

(iii) $p_1 * p_1 = 0$

(iv) $p_1 * (q_1 * r_1) = q_1 * (p_1 * r_1)\ , for\ any\ p_1\ , q_1\ , r_1 \in \mathcal{M}$,

In $\mathcal{M}$, we define a binary relation $\leq$ by $p_1 \leq q_1$ if and only if $q_1 * p_1 = 0$.

Definition 1.5.25 [46]

A nonempty subset I of a KUS-algebra $\mathcal{M}$ is called a KUS-ideal of $\mathcal{M}$ if it satisfies:

(i) $0 \in I$

(ii) $r_1 * q_1 \in I\ and\ q_1 * p_1 \in I\ imply\ r_1 * p_1 \in I, for\ p_1, q_1, r_1 \in \mathcal{M}$.

Definition 1.5.26 [63]

A non-empty set $\mathcal{M}$ with a constant 0 and a binary operation $*$ is called PS–algebra if it satisfies the following axioms

(i) $p_1 * p_1 = 0$

(ii) $p_1 * 0 = 0$

(iii) $p_1 * q_1 = 0\ and\ q_1 * p_1 = 0 \Longrightarrow p_1 = q_1\ , \forall\ p_1\ , q_1 \in \mathcal{M}$.

In $\mathcal{M}$, we define a binary relation $\leq$ by $p_1 \leq q_1$ if and only if $q_1 * p_1 = 0$.

Definition 1.5.27 [63]

Let $\mathcal{M}$ be a PS-algebra and I be a subset of $\mathcal{M}$, then I is called a PS -ideal of $\mathcal{M}$ if it satisfies the following conditions:

(i) $0 \in I$

(ii) $q_1 * p_1 \in I$ *and* $q_1 \in I \Longrightarrow p_1 \in I$, *for all* $p_1, q_1 \in \mathcal{M}$.

Definition 1.5.28 [1]

Let $\mathcal{M}$ be a BCH-algebra and I be an ideal of $\mathcal{M}$. Then I is called a closed ideal with respect to an element $\beta \in \mathcal{M}$ (denoted a-closed ideal) if $\beta * (0 * p_1) \in I \; \forall \, p_1 \in I$.

Definition 1.5.29 [5]

In QS-algebra $\mathcal{M}$, the set $\mathcal{B}(\mathcal{M}) = \{p_1 \in \mathcal{M} \text{ such that } 0 * p_1 = 0\}$ is called a p-radical of $\mathcal{M}$. A QS-algebra $\mathcal{M}$ is said to be p-semi simple if $\mathcal{B}(\mathcal{M}) = \{0\}$.

Definition 1.5.30 [5]

Let $\mathcal{M}$ be a QS-algebra. For any subset $\mathbb{Q}$ of $\mathcal{M}$, we define $G(\mathbb{Q}) = \{p_1 \in \mathbb{Q} / 0 * p_1 = p_1\}$. In particular, if $\mathbb{Q} = \mathcal{M}$ then we say that G($\mathcal{M}$) is the G-part of a QS -algebra $\mathcal{M}$. The following property is obvious. G($\mathcal{M}$)∩B($\mathcal{M}$) = {0}.

Definition 1.5.31 [5]

A QS-algebra X satisfying $(x * y) * (z * u) = (x * z) * (y * u)$ for any x, y, z, u ∈ X is called a medial QS-algebra.

Definition 1.5.32 [11]

Let X be a B-algebra. By a (l, r)derivation of X, we mean a self-map d of X satisfying the identity $d(x * y) = (d(x) * y) \wedge (x * d(y))$ for all $x, y \in X$. If X satisfies the identity $d(x * y) = (x * d(y)) \wedge (d(x) * y)$ for all $x, y \in X$ then we say that d is a (r, l) - derivation of X . Moreover, if d is both a (l, r) $and\ a$ (r, l) - derivation, we say that d is a derivation of X.

Definition 1.5.33 [47]

Let X be a BCI-algebra. Then for any $t \in X$, a self map $d_t : X \to X$ is called a left-right t-derivation or (l, r) t-derivation of X if it satisfies the identity $d_t(x * y) = (d_t(x) * y) \wedge (x * d_t(y))$ for all $x, y \in X$.

Definition 1.5.34 [65]

Let X be a BCI-algebra and $D(.,.) : X \times X \to X$ be a symmetric mapping. If D satisfies the identity $D(x * y, z) = (D(x, z) * y) \wedge (x * D(y, z))$ for all $x, y, z \in X$, then D is called left − right symmetric bi − derivation (briefly (l, r)−symmetric bi−derivation). If D satisfies the identity

$$D(x * y, z) = (x * D(y, z)) \wedge (D(x, z) * y) \text{ for all } x, y, z \in X,$$

then we say that D is right − left symmetric bi − derivation (briefly (r, l) − symmetric bi − derivation). Moreover if d is both an (r, l) − and a (l, r) − symmetric bi − derivation, it is said that D is symmetric bi − derivation.

Definition 1.5.35 [74,75]

Let D be a non-empty set (to be called the universe of discourse or domain or universal set). A classical set on D is a mapping whose co -domain is 0 and 1. (i.e.) f: D $\longrightarrow$ {0, 1}.

Definition 1.5.36 [74]

Let X be a non-empty set. A fuzzy set A in X is characterized by its membership function $\mu_A: X \to [0,1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$. It is clear that A is completely determined by the set of tuples $A = \{(x, \mu(x)) \,/\, x \in X\}$.

Definition 1.5.37 [74,75]

The membership function of the intersection of two fuzzy sets A and B is defined as $\mu_{A\cap B}(x) = min\,\{\mu_A(x), \mu_B(x)\}$.

Definition 1.5.38 [9]

A fuzzy set μ in d-algebra X is called a fuzzy subalgebra of X if it satisfies $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for all x, y ∈ X.

Definition 1.5.39 [74,75]

The membership function of union of two fuzzy sets A and B is defined as

$$\mu_{A\cup B}(x) = max\,\{\mu_A(x), \mu_B(x)\}.$$

Definition 1.5.40 [74,75]

Let μ be a fuzzy set in D. Then the complement of μ is the fuzzysubset of D, which is given by $\mu^c\ (x) = 1 - \mu(x)$.

Definition 1.5.41 [74,75]

Let A and B be two fuzzy sets in D. Then A is said to be equal to B, denoted by $A = B$ if $\mu_A\ (x) = \mu_B\ (x)$ for every x in D.

Definition 1.5.42 [74,75]

The set of elements that belong to the fuzzy set μ at least to the degree α is called the α - level set. It is represented by $\mu_\alpha = \{x \in X \,/\, \mu\ (x) \geq \alpha\}$.

Definition 1.5.43 [72]

A fuzzy set μ in a BCK-algebra X is called a fuzzy subalgebra of X if

$$\mu(x * y) \geq min\,\{\mu(x), \mu(y)\}, \text{ for all } x, y \in X.$$

Definition 1.5.44 [72]

Let X be a BCK-algebra. A fuzzy subset μ in X is called a fuzzy ideal of X if it satisfies the following conditions:

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}, for\ all\ x, y \in X.$

Definition 1.5.45 [74]

The set of elements that belong to the fuzzy set μ at least to the degree t is called the t-level set. It is represented by $\mu^t = \{ x \in X / \mu(x) \geq t \}$.

Definition 1.5.46 [38, 42]

A fuzzy set μ in a TM-algebra X is called an anti-fuzzy sub algebra of X if

$$\mu(x * y) \leq max\{\mu(x), \mu(y)\}, \text{ for all } x, y \in X.$$

Definition 1.5.47 [43]

A fuzzy subset μ of a TM-algebra X is called an anti-fuzzy ideal of X,If

(i) $\mu(0) \leq \mu(x)$

(ii) $\mu(x) \leq \max\{\mu(x * y), \mu(y)\}, for\ all\ x, y \in X.$

Definition 1.5.48 [43]

Let $(X, *, 0)$ and $(Y, \Delta, 0`)$ be TM-algebras. A mapping $f: X \longrightarrow Y$ is said to be an anti-homomorphism if $f(x * y) = f(y)\, \Delta\, f(x)$ for all $x, y \in X$.

Definition 1.5.49[6]

A fuzzy set μ in X is called a fuzzy BE-algebra of X if it satisfies for all x, y ∈ X.

$$\mu(x * y) \geq \min\{(\mu(x), \mu(y)\}.$$

A fuzzy set μ in X is a function μ : X → [0, 1]. We note that x ∗ x = 1 for all x ∈ X and so if μ is a fuzzy BE-algebra of X, then μ(1) ≥ μ(x) for all x ∈ X.

Definition 1.5.50[6]

A fuzzy BE-algebra μ of X is said to be normal if there exists x ∈ X such that μ(x) = 1.

Definition 1.5.51 [71]

A fuzzy set μ in X is called fuzzy BCK-ideal of X if it satisfies the following inequalities

(i) $\mu(0) \geq \mu(x)$.

(ii) $\mu(x) \geq \min\{ \mu(x * y), \mu(y)\}$ for all x, y ∈ X.

Definition 1.5.52 [18]

A fuzzy set μ in a BCC-algebra X is called a fuzzy sub algebra of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \ \forall x, y \in X.$$

Definition 1.5.53 [7]

A fuzzy set μ in $X$is called a fuzzy BE-algebra of X if it satisfies: for all $x, y \in X$.

$$\mu(x * y) \geq min\{\mu (x), \mu(y)\}.$$

A fuzzy set μ in X is a function $\mu : X \to [0,1]$, we note that $x * x = 1$ for all $x \in X$ and so if μ is a fuzzy BE-algebra of X, then $\mu(1) \geq \mu(x)$ for all $x\ in\ X$.

CHAPTER 2

THE STRUCTURE OF GK ALGEBRA†

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CHAPTER 2

THE STRUCTURE OF GK ALGEBRA

In this chapter, a new algebraic structure which is called as GK algebra is introduced and analyzed its properties. It is expressed that the newly introduced notion of GK algebra is fully different from the previously defined algebraic structure such as BCK / BCI / BE / CI / BG / K / QS / PMS / KUS / KU / TM / PS / Q / KUS / SP / Z algebras etc. The basic concepts in which commutativity, associativity and distributivity of GK algebra are defined and investigated their properties. GK sub algebra, GK ideal, homomorphism and anti-homomorphism of GK algebra are initiated and discussed its aspects.

2.1 The structure of GK algebra

In this section, the new algebraic structure, namely GK algebra is introduced and investigated its properties and also, discussed GK sub algebra and GK ideals with necessary illustrations.

Definition 2.1.1

A non-empty set T with fixed constant 1 and a binary operation $\circledast$ is called GK algebra if it satisfying the following axioms

(i) $i \circledast i = 1$

(ii) $i \circledast 1 = i$

(iii) $i \circledast j = 1 \; and \; j \circledast i = 1 \; implies \; i = j$

(iv) $(j \circledast k) \circledast (i \circledast k) = j \circledast i$

(v) $(i \circledast j) \circledast (1 \circledast j) = i \quad \forall \; i, j, k \in T.$

Example 2.1.2

Consider the set T= {1,2,3}. The binary operation ⊛ is defined as follows

⊛	1	2	3
1	1	3	2
2	2	1	3
3	3	2	1

Table 2.1

Hence (T, ⊛ ,1) is a GK algebra.

Note 2.1.3

1. A GK algebra need not be a BE algebra for $2 \circledast 1 = 2 \neq 1$ and $3 \circledast 1 = 3 \neq 1$.
2. GK algebra need not be a CI algebra for $1 \circledast 3 = 2 \neq 1,\ 1 \circledast 2 = 3 \neq 2$.

Example 2.1.4

Let us consider T = {1, l, m, n} be the set, then the following table,

⊛	1	l	m	n
1	1	l	m	n
l	l	1	n	m
m	m	n	1	l
n	n	m	l	1

Table 2.2

Hence (T, ⊛ ,1) is a GK algebra.

Example 2.1.5

Consider the set T= {1,2,3,4,5} be set with the following table

$\circledast$	1	2	3	4	5
1	1	5	4	3	2
2	2	1	5	4	3
3	3	2	1	5	4
4	4	3	2	1	5
5	5	4	3	2	1

Table 2.3

Hence (T, $\circledast$,1) is a GK algebra.

Example 2.1.6

Consider the set T= $\{1,2,0\}$. The binary operation $\circledast$ is defined as follows

$\circledast$	1	2	0
1	1	0	2
2	2	1	0
0	0	2	1

Table 2.4

Hence (T, $\circledast$,1) is a GK algebra.

Note 2.1.7

1. A GK algebra is said to be a CI algebra if

   $1 \circledast i = i \;\; and \;\; i \circledast (j \circledast k) = j \circledast (i \circledast k).$

2. A GK algebra is said to be a BE algebra if

   i $\circledast\, 1 = 1{,}1 \circledast i = i \;\; and \;\; i \circledast (j \circledast k) = j \circledast (i \circledast k).$

Definition 2.1.8

Let T be a GK algebra and D be a subset of T, then D is called a GK ideal of T if it satisfying the following conditions

(i) $1 \in D$

(ii) $j \circledast k \in D \ \ and \ \ i \circledast k \in D \Rightarrow j \circledast i \in D \ \forall \ i,j,k \in T.$

Example 2.1.9

Consider the Example 2.1.4 in that D = {1, l, m} is a GK ideal.

Proposition 2.1.10

In GK algebra (T, ⊛ ,1) with $i \leq j$, the following holds for all $i,j,k \in T.$

(i) $1 \circledast (1 \circledast i) = i$

(ii) $(i \circledast j) \circledast 1 = (i \circledast 1) \circledast (j \circledast 1)$

(iii) $j \circledast (1 \circledast (1 \circledast j) = 1$

Proof

We know that $(i \circledast j) \circledast (1 \circledast j) = i$ in axiom V of definition

Replacing $j \ \ by \ \ i$

(i) We have $(i \circledast i) \circledast (1 \circledast i) = i$

$1 \circledast (1 \circledast i) = i$ by axiom (I) of definition.

(ii) We know that $i \circledast 1 = i$ by axiom (I)

Similarly, $(i \circledast j) \circledast 1 = i \circledast j$

$= (i \circledast 1) \circledast (j \circledast 1)$

Therefore $(i \circledast j) \circledast 1 = (i \circledast 1) \circledast (j \circledast 1)$

(iii) By (i) of proposition 2.1.10

We have $1 \circledast (1 \circledast j) = j$

Now, $j \circledast (1 \circledast (1 \circledast j)) = j \circledast j$

$j \circledast \big(1 \circledast (1 \circledast j)\big) = 1$

$Hence\ the\ proof.$

Proposition 2.1.11

In GK algebra (T, $\circledast$,1), the following holds for all $i, j, k \in T$,

(i) If $1 \circledast i = 1 \circledast j$ $then$ $i = j$

(ii) $\big(i \circledast (1 \circledast i)\big) \circledast i = i$

(iii) $i \circledast (i \circledast j) = i = j = j \circledast (i \circledast i)$

(iv) $i \circledast (j \circledast i) = i = j = j \circledast (i \circledast i)$

Proof

(i) Let us consider $1 \circledast i = 1 \circledast j$

Now $i = 1 \circledast (1 \circledast i)$

$= 1 \circledast (1 \circledast j)$ $\quad \because 1 \circledast i = 1 \circledast j$

$= j$ $\quad$ by proposition 2.1.10

(ii) Now $\big(i \circledast (1 \circledast i)\big) \circledast i$

$= \big(i \circledast (1 \circledast i)\big) \circledast \big(1 \circledast (1 \circledast i)\big)$

$= i \circledast 1$ $\quad$ by axiom (IV)

$= i$

(iii) Consider $i \circledast (i \circledast j) = i \circledast 1$ $\quad$ by axiom (III)

$= i$ $\quad$ by axiom (III)

$= j$ $\quad$ by axiom (III)

$= j \circledast 1$ $\quad$ by axiom (I)

$= j \circledast (i \circledast i)$

(iv) The proof of (iv) is similar to the proof of (iii).

Proposition 2.1.12

In GK algebra (T, $\circledast$,1), the following holds

(i) $1 \circledast (i \circledast j) = j \circledast i$

(ii) If $1 \circledast i = i \circledast 1 = 1$ then $i = 1$ for any $i, j \in$T.

Proof

(i) We know that$(j \circledast k) \circledast (i \circledast k) = j \circledast i$

Replacing k by j

We have, $(j \circledast j) \circledast (i \circledast j) = j \circledast i$

$$1 \circledast (i \circledast j) = j \circledast i.$$

Hence the result.

(ii) Let $1 \circledast i = i \circledast 1 = 1$

when $i = 1,\ 1 \circledast i = i \circledast 1 = 1.$

Hence the result.

Theorem 2.1.13

If every GK algebra T satisfies $i \circledast (j \circledast i) = i \circledast j \ \forall\, i, j \in T$ is a trivial algebra.

Proof

Let $(T, \circledast, 1)$ be a GK algebra.

Put $j = i$ in $i \circledast (j \circledast i) = i \circledast j$

$$\Longrightarrow\ i \circledast (i \circledast i) = i \circledast i$$

$$\Longrightarrow\ i \circledast 1 = 1$$

$$\Longrightarrow\ i = 1$$

This shows that T is a trivial algebra.

Theorem 2.1.14

In GK algebra left cancellation law and right cancellation law holds.

(i) Right cancellation law: if $i \circledast j = k \circledast j$ $then$ $i = k.$

(ii) Left cancellation law: if $k \circledast i = k \circledast j$ $then$ $i = j.$

Proof

(i) Let us consider $i \circledast j = k \circledast j$

From the definition, we know that

$$i = (i \circledast j) \circledast (1 \circledast j)$$

$$i = (k \circledast j) \circledast (1 \circledast j)$$

$$i = k \circledast 1$$

$$i = k.$$

(ii) Assume that $k \circledast i = k \circledast j$

Now $k \circledast (k \circledast i) = i \circledast (k \circledast k)$

$= i \circledast 1 \; = i$

and $k \circledast (k \circledast j) \; = j \circledast (k \circledast k)$

$= j \circledast 1 = \mathrm{j}$

From this $k \circledast i = k \circledast j \Longrightarrow i = j$.

Hence the proof.

Definition 2.1.15

A GK algebra (T, $\circledast$, 1) is said to be associative if it satisfies

$$(i \circledast j) \circledast k = i \circledast (j \circledast k) \; \forall \; i, j, k \in T$$

Theorem 2.1.16

Every GK algebra (T, $\circledast$,1) satisfying the associative law

$(i \circledast j) \circledast k = i \circledast (j \circledast k)$ is a group under $\circledast$.

Proof

Let (T, $\circledast$, 1) be a GK-algebra

Put $i = j = k$ in associative law,

$$(i \circledast j) \circledast k = i \circledast (j \circledast k)$$

$$\Longrightarrow (i \circledast i) \circledast i = i \circledast (i \circledast i)$$

$\Longrightarrow 1 \circledast i = i \circledast 1 = i$

This exhibits that 1 is the identity element of T.

By the definition, we get $i \circledast i = 1$.

This shows that every element $i\ of\ T$ has its own inverse.

Now,

$$(j \circledast k) \circledast (i \circledast k) = j \circledast (k \circledast (i \circledast k))$$

$$= j \circledast (i \circledast (k \circledast k))$$

$$= j \circledast (i \circledast 1)$$

$$(j \circledast k) \circledast (i \circledast k) = j \circledast i.$$

and $\quad (i \circledast j) \circledast (1 \circledast j) = i \circledast (j \circledast (1 \circledast j))$

$$= i \circledast 1 \ = \ i$$

Therefore (T, $\circledast$,1) is a group.

Theorem 2.1.17

Let T be a GK algebra. A relation $\leq$ on T is defined as $i \leq j$ if $i \circledast j = 1. Then\ (T, \leq)$ is a partially ordered set.

Proof

Let T be a GK algebra and let $i, j, k \in T$.

By definition of GK algebra, we know that $i \circledast i = 1$

$\Longrightarrow i \leq i$

$\therefore\ \leq$ is reflexive.

Suppose if $i \leq j\ \ and\ \ j \leq i$, then $i \circledast j = 1$ and $j \circledast i = 1$.

By definition of GK algebra

$i \circledast j = 1$ and $j \circledast i = 1 \Longrightarrow i = j$

$\therefore \leq$ is anti-symmetric.

Suppose $i \leq j\ \ and\ \ j \leq k$, then $i \circledast j = 1$ and $j \circledast k = 1$.

Now, $i \circledast k = (i \circledast k) \circledast 1$

$= (i \circledast k) \circledast (j \circledast k)$

$= i \circledast j$

$i \circledast k = 1$

$\Longrightarrow \ i \leq k.$

$\therefore \leq$ is transitive.

Hence $(T, \leq)$ is a partially ordered set.

Theorem 2.1.18

Let (T, $\circledast$,1) be a GK algebra with identity

$$(i \circledast j) \circledast k = i \circledast \Big(1 \circledast \big((1 \circledast j) \circledast k\big)\Big) \forall\, i, j, k \in T.$$

Then (T, $\circledast$, 1) is a group derived.

Proof

Define a binary operation $\diamond$ on T by $i \diamond j = i \circledast (1 \circledast j)$

Then $i \diamond 1 = i \circledast (1 \circledast 1) = i \circledast 1 = i$

$1 \diamond i = 1 \circledast (1 \circledast i) = i$ by proposition 2.1.10

Therefore 1 acts as an identity.

Also $i \diamond (1 \circledast i) = i \circledast \big(1 \circledast (1 \circledast i)\big)$

$= i \circledast i = 1$

and $(1 \circledast i) \diamond i = (1 \circledast i) \circledast (1 \circledast i) = 1$

Here $1 \circledast i$ acts like a multiplicative inverse.

Therefore (T, $\diamond$) is a semi group.

Now, $i \diamond (j \diamond k) = i \circledast \Big(1 \circledast \big(j \circledast (1 \circledast k)\big)\Big)$

$= i \circledast (1 \circledast \big(1 \circledast (1 \circledast j)\big) \circledast (1 \circledast k))$

$= \big(i \circledast (1 \circledast j)\big) \circledast (1 \circledast k)$

$= (i \diamond j) \diamond k$

Therefore it satisfies the associative law.

Theorem 2.1.19

If $i \circledast j = 1\ and\ j \circledast i = 1\ then\ i = j.$

Proof

Let $i \circledast j = 1\ and\ j \circledast i = 1$

Now, $i \circledast 1 = i$

$\Longrightarrow i \circledast (j \circledast i) = i$

$\Longrightarrow j \circledast (i \circledast i) = i$

$\Longrightarrow j \circledast 1 = i$

$\Longrightarrow j = i$

Now, $j \circledast 1 = j$

$\Longrightarrow j \circledast (i \circledast j) = j$

$\Longrightarrow i \circledast (j \circledast j) = j$

$\Longrightarrow i \circledast 1 = j$

$\Longrightarrow i = j$

Hence the theorem.

Theorem 2.1.20

Let (T, $\circledast$,1) be a group with respect to $i \circledast j = ij^{-1}$, then T is a GK algebra.

Proof

Let us consider $i \circledast j = ij^{-1}$

Now, $i \circledast i = ii^{-1} = 1$and $i \circledast 1 = i1^{-1} = i$

For any $i, j \in T$, then we have $i \circledast j = ij^{-1}$

Put $i = j\ in\ RHS$

$i \circledast j \quad = ij^{-1}$

$= ii^{-1}$

$= 1$ ---------------------- (1)

$j \circledast i = ji^{-1}$

$= ji^{-1}$

$= 1$ --------------------------(2)

From (1) and (2) $i \circledast j = j \circledast i$

For any $i, j, k \in T$, then we have $i \circledast j = ij^{-1}$

$(j \circledast k) \circledast (i \circledast k) \quad = (jk^{-1})(ik^{-1})^{-1}$

$= (jk^{-1})(ki^{-1})$

$= j(k^{-1}k)i^{-1}$

$= ji^{-1}$

$= j \circledast i$

For any $i, j \in T$

$(i \circledast j) \circledast (1 \circledast j) \quad = (ij^{-1})(1j^{-1})^{-1}$

$= (ij^{-1})(j1^{-1})$

$= i(j^{-1}j)1^{-1}$

$= i1^{-1}$

$= i \circledast 1 = i.$

Hence (T, $\circledast$,1) is a GK algebra.

Definition 2.1.21

A GK algebra (T, $\circledast$,1) is a self-distributive if the operation $\circledast$ satisfies

(i) Right distributive law: $(i \circledast j) \circledast k = (i \circledast k) \circledast (j \circledast k)$ $for\ all\ i, j, k \in T.$

(ii) Left distributive law: $i \circledast (j \circledast k) = (i \circledast j) \circledast (i \circledast k)$ $for\ all\ i, j, k \in T.$

Definition 2.1.22

In a GK-algebra, an element $i \in T$ is said to commute if

$(i \circledast j) \circledast j = (j \circledast i) \circledast i$, $\forall\, i, j \in T$. If this condition is true for all $i, j, \in T$, then (T, $\circledast$,1) is said to be a commutative GK-algebra.

Theorem 2.1.23

In GK algebra, for any $i, j, k \in T$ if associativity holds then the following are equivalent

(i) $i \circledast (j \circledast k) = (i \circledast k) \circledast j$

(ii) $(j \circledast k) \circledast (i \circledast k) = j \circledast i$

Proof

$(i) \implies (ii)$

Let us assume $i \circledast (j \circledast k) = (i \circledast k) \circledast j$

Now, $(j \circledast k) \circledast (i \circledast k) = ((j \circledast k) \circledast k) \circledast i$

$= (j \circledast (k \circledast k)) \circledast i$

$= (j \circledast 1) \circledast i$

$= j \circledast i$

$(ii) \implies (i)$

Assume $(j \circledast k) \circledast (i \circledast k) = j \circledast i$

$i \circledast (j \circledast k) = (i \circledast k) \circledast ((j \circledast k) \circledast k)$

$= (i \circledast k) \circledast (j \circledast (k \circledast k))$

$= (i \circledast k) \circledast (j \circledast 1)$

$= (i \circledast k) \circledast j$

Proposition 2.1.24

Let T be a GK algebra. If $i \neq j$ *and* $i \circledast j = 1$ *then* $j \circledast i \neq 1$

Proposition 2.1.25

Let (T, $\circledast$,1) be a GK algebra. Then for any $i, j, k \in T$,

(i) $i \circledast \big(i \circledast (j \circledast i)\big) = 1$

(ii) $j \circledast \big(j \circledast (i \circledast j)\big) = 1$

(iii) $(i \circledast j) \circledast i = (j \circledast i) \circledast j$

(iv) $(i \circledast j) \circledast j = (j \circledast i) \circledast i$

(v) $(i \circledast j) \circledast i = (i \circledast i) \circledast j$

(vi) $(i \circledast j) \circledast j = (j \circledast j) \circledast i$

Proof

(i) Let us consider $i \circledast (i \circledast (j \circledast i))$

$= i \circledast (i \circledast 1)$ by axiom (iii) of GK algebra

$= i \circledast i$

$= 1$

(ii) This proof is similar as of (i)

(iii) Consider $(i \circledast j) \circledast i$

$= 1 \circledast i$

$= 1 \circledast j$

$= (i \circledast j) \circledast j$ (or) $(j \circledast i) \circledast j$

(iv) Consider $(i \circledast j) \circledast j$

$= 1 \circledast j$

$= 1 \circledast i$

$= (i \circledast j) \circledast i$ (or) $(j \circledast i) \circledast i$

This proof shows that the commutativity of GK algebra.

(v) Consider $(i \circledast j) \circledast i = 1 \circledast i$

$= 1 \circledast j$

$= (i \circledast i) \circledast j$

(vi) The proof of this is similar to (v).

Definition 2.1.26

Let (T, $\circledast$,1) be a GK algebra. A non-empty subset P of T is called a sub algebra of T if $i \circledast j \in P \;\; for\ any\ i, j \in P$.

Theorem 2.1.27

Let (T, $\circledast$,1) be a GK algebra and $P \neq \emptyset\,, P \subseteq T$ then the following are equivalent

(i) P is a sub algebra.

(ii) $i \circledast (1 \circledast j), 1 \circledast j \in P \;\; for\ any\ i, j \; \in P$.

Proof

$(i) \implies (ii)$

Let P be a sub algebra of T. Since P is a subset of T, ∃ an element

$i \in P \;\; such\ that\ i \circledast i = 1 \in P$.

Since T is closed under $\circledast, j \in P, 1 \circledast j \in P \;\Rightarrow i \circledast (1 \circledast j) \in P$.

$(ii) \implies (i)$

We know that, $i \circledast j = i \circledast \big(1 \circledast (1 \circledast j)\big)$ by proposition 2.1.9.

$\implies i \circledast j \in P \;\; for\ any\ i, j \in P$.

$\therefore P$ is a sub algebra of T.

Theorem 2.1.28

Every GK ideal of GK algebra (T, $\circledast$,1) is a GK sub algebra.

Proof

Let D be a GK ideal of GK algebra T such that,

$i, j \in T, then\ i \circledast 1 \in D, j \circledast 1 \in D\ \forall\ i, j, k \in T$.

By definition of GK algebra,

$(i \circledast 1) \circledast (j \circledast 1) = i \circledast j \in D$

Hence D is the GK sub algebra.

Theorem 2.1.29

Let T be a GK algebra and D be a non-empty subset of T containing 1. Then D is a GK ideal of T if and only if

$$j \circledast k \in D \ \ and \ j \circledast i \notin D, \Longrightarrow i \circledast k \notin \ D \ for \ every \ i, j, k \in T.$$

Proof

Let D be a GK ideal of T and $j \circledast k \in D \ \ and \ j \circledast i \notin D$.

Let us consider $i \circledast k \in D$

Since D is a GK algebra, $j \circledast i \in D$, which leads the contradiction.

Conversely,

Let us assume that

$j \circledast k \in D \ and \ j \circledast i \notin D \Rightarrow i \circledast k \notin D \ \forall \ i, j, k \in T.$

If $j \circledast k \in D, i \circledast k \in D,$ it is obvious that $j \circledast i \in D.$

Therefore, D is a GK ideal of T.

Theorem 2.1.30

The intersection of family of GK- ideals on GK algebra T is again GK ideal.

Proof

Let us assume that $\{D_n / n \in T\}$ be a family of GK ideals on GK algebra T.

Then,

(i) $1 \in D_n \ for \ all \ n \in T \Rightarrow 1 \in \bigcap D_n \forall \ n \in T.$

(ii) For any $i, j, k \in T, suppose \ j \circledast k \in D_n \ and \ i \circledast k \in D_n \ for \ all \ n \in T.$

Since D_n is a GK ideal of T $\forall \ n \in T \ then \ j \circledast i \in D_n \ for \ all \ n \in T.$

This implies that $j \circledast i \in \cap \ D_n \ for \ all \ n \in T.$

Therefore $\cap_{n \in T} D_n$ is a GK ideal.

Corollary 2.1.31

Let T be a GK algebra and D be a non-empty subset of T then ,

- D is a GK ideal of T if $j \circledast k \in D$ and $i \circledast k \notin D \Longrightarrow j \circledast i \notin D$ $\forall\, i, j, k \in T$.
- D is a GK ideal of T if $j \circledast k \in D$ and $j \circledast i \in D \Longrightarrow i \circledast k \in D$ $\forall\, i, j, k \in T$.
- D is a GK ideal of T if $i \circledast k \in D$ and $j \circledast i \in D \Longrightarrow j \circledast k \in D$ $\forall\, i, j, k \in T$.

2.2 Homomorphism and Anti-homomorphism of GK algebra

In this section, homomorphism and anti-homomorphism of GK algebra are discussed and important results of kernel of GK algebra are explored.

Definition 2.2.1

Let (T,$\circledast_T, 1$) and (P,$\circledast_P, 1'$) be a GK algebra. Then the mapping $\sigma: T \to P$ of GK algebra is called homomorphism if

$$\sigma(i \circledast_T\ j) = \sigma(i) \circledast_P \sigma(j)\ \forall\, i, j \in T.$$

Theorem 2.2.2

If $\sigma: T \to P$ is a GK homomorphism of GK algebra then

(i) $\sigma(1) = 1'$

(ii) $If\ i \leq j\ then\ \sigma(i) \leq \sigma(j)\ for\ any\ i, j \in T$

(or)

If $i \circledast j = 1\ \forall\, i, j\ \in T\ then\ \sigma(i) \circledast_P \sigma(j) = 1'$

Proof

(i) Let σ be GK homomorphism.

We know that $1 \circledast 1 = 1$.

This implies that $\sigma(1) = \sigma(1 \circledast_T 1) = \sigma(1) \circledast_P \sigma(1) = 1'$.

(ii) If $i \leq j$ then $i \circledast j = 1$.

$\sigma(i) \circledast_P \sigma(j) = \ \sigma(i \circledast_T j) = \sigma(1) = 1$

$\Longrightarrow\ \sigma(i) \circledast_P \sigma(j) = 1$.

Then, we can write it as $\sigma(i) \leq \sigma(j)$.

Theorem 2.2.3

Let (T,$\circledast_T, 1$) and (P,$\circledast_P, 1'$) be a GK algebra and the mapping $\sigma: T \to P$ is a GK homomorphism. Let M be a GK ideal of B then $\sigma^{-1}(M)$is a GK ideal of T.

Proof

The definition of $\sigma^{-1}(M)$ is $\sigma^{-1}(M) = \{i \in T /\ \sigma(i) = j\ for\ j \in M\}$

Since M is a GK ideal of P, then $1' \in M\ and\ \sigma(1) = 1' \Rightarrow 1 \in \sigma^{-1}(M)$.

Assume that $j \circledast k \in \sigma^{-1}(M)\ and\ i \circledast k \in \sigma^{-1}(M)\ then$

$$\sigma(j \circledast k) \in M\ and\ \sigma(\ i \circledast k) \in M$$

Since σ is a homomorphism, $\sigma(j \circledast_T k) = \sigma(j) \circledast_P \sigma(k) \in M$.

Since M is a GK ideal of P, $\sigma(\ j \circledast i) \in M$

$\Longrightarrow\ j \circledast i \in \sigma^{-1}(M)$.

$\therefore \sigma^{-1}(M)$ is a GK ideal of T.

Definition 2.2.4

Let (T,$\circledast_T, 1$) and (P,$\circledast_P, 1'$) be a GK algebra. Then the mapping $\sigma: T \to P$ of GK algebra is called anti- homomorphism if

$$\sigma(i \circledast_T\ j) = \sigma(j) \circledast_P \sigma(i)\ \forall\ i, j \in T.$$

Definition 2.2.5

Let (T,$\circledast_T, 1$) and (P,$\circledast_P, 1'$) be a GK algebra and the mapping $\omega: T \to P$ be a GK homomorphism. Then the subset $ker\omega = \{i \in T/\omega(i) = 1'\}$ of T is called the kernel of ω.

Theorem 2.2.6

If $\sigma: T \to P$ is an anti- homomorphism of GK algebra then $\ker \sigma$ is a GK ideal of T.

Proof

It is clear that $1 \in \ker \sigma$

$\Rightarrow \sigma(1) = 1'$

Let if $j \circledast k \in ker\, \sigma\ and\ i \circledast k \in ker\, \sigma$

This implies that $\sigma(j \circledast k) = 1'\ and\ \sigma(i \circledast k) \in 1'$

$\Rightarrow \sigma(j \circledast i) = \sigma\big((j \circledast k) \circledast_T (i \circledast k)\big)$ $by\ definition\ of\ GK\ algebra$

$\Rightarrow \big(\sigma(i \circledast k)\big) \circledast_P \big(\sigma(j \circledast k)\big)$

$\Rightarrow 1 \circledast_P 1$

$\Rightarrow 1$

$\Longrightarrow j \circledast i \in ker\, \sigma$

Hence $ker\, \sigma$ is a GK ideal of T.

2.3 Summary

This chapter has been elaborated the content of newly defined GK algebra. The GK algebra and GK subalgebra with their adequate illustrations are explored. It is shown that this algebraic structure, GK algebra is different from BE algebra and CI algebra. The GK algebra satisfied the associativity, self-distributivity law, Commutativity law and also its properties are investigated. GK ideal, the kernel of GK algebra, anti-homomorphism of GK algebra are defined and attained remarkable results such as every GK ideal is a GK sub algebra, the intersection of the family of GK- ideals on GK algebra T is again GK ideal.

CHAPTER 3

MUTIPLIERS AND THE DIRECT PRODUCT OF GK ALGEBRA†

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CHAPTER 3

MULTIPLIERS AND THE DIRECT PRODUCT OF GK ALGEBRA

This Chapter is separated into two sections. In the first section, the theory of Multipliers in GK algebra is introduced. In that, the left multipliers, the right multipliers, and regular multipliers of GK algebra are initiated, and also, some of the interesting properties of the regular multiplier and kernel of multipliers in GK algebra are discussed. In the second section, the concept of the direct product of GK algebra is explored and derived some paramount results.

3.1 Multipliers in GK algebra

This section deals with the concept of Multipliers of GK algebra. The right multipliers, the left multipliers are explained with the necessary illustrations, and also in this section, the results of composition of multipliers, regular multipliers and kernel of multipliers of GK algebra are obtained.

Definition 3.1.1

Let $(T,\circledast,1)$ be a GK algebra. A self-map Θ is said to be a multiplier (right) of T if $\Theta(i \circledast j) = i \circledast \Theta(j)$ for all $i, j \in T$.

Example 3.1.2

Consider $T = \{1, 2, 3\}$ in which '$\circledast$' is defined by

$\circledast$	1	2	3
1	1	3	2
2	2	1	3
3	3	2	1

Table 3.1

Then T is a GK algebra.

Define a mapping $\Theta: \mathrm{T} \to \mathrm{T}$ by

$$\Theta(\mathrm{i}) = \begin{cases} 1 & if\ i = 1 \\ 2 & if\ i = 2 \\ 3 & if\ i = 3 \end{cases}$$

It is clearly shown that Θ is a multiplier (right) of GK algebra.

Definition 3.1.3

Let $(\mathrm{T}, \circledast, 1)$ be a GK algebra. A self-map Θ is called a multiplier (left) of T if

$$\Theta(\mathrm{i} \circledast \mathrm{j}) = \Theta(\mathrm{i}) \circledast \mathrm{j} \ \text{ for all } \ \mathrm{i}, \mathrm{j} \in \mathrm{T}.$$

Note 3.1.4

The above said Example 3.1.2 is also an example of the multiplier (left) of GK algebra. In this example, the surjectiveness (onto) is exists and also the defined mapping is identity mapping (one-one and onto). If the surjectiveness (onto) does not exists in the self-mapping which is defined in the above example, then it will affect the regularity condition in GK algebra.

Definition 3.1.5

A map Θ of a GK algebra T is called regular if $\Theta(1) = 1$.

Proposition 3.1.6

Let Θ be a multiplier (left) of T, then

(i) For every i in T, $\Theta(1) = \Theta(i) \circledast i$.

(ii) Θ is 1-1.

Proof

(i) Let $i \in T$. Then $i \circledast i = 1$.

We have $\Theta(1) = \Theta(\mathrm{i} \circledast \mathrm{i}) = \Theta(i) \circledast i \ \ forall \ \ i \in T$.

(ii) Let $i, j \in T$ such that $\Theta(i) = \Theta(j)$

Then by (i), we have $\Theta(1) = \Theta(\mathrm{i} \circledast \mathrm{i})$

$= \Theta(i) \circledast i$ and

$\Theta(1) = \Theta(j \circledast j) = \Theta(j) \circledast j.$

Then $\Theta(i) \circledast i = \Theta(j) \circledast j.$

By cancellation law, $i = j.$

$\therefore \; \Theta \; is \; 1 - 1$

Proposition 3.1.7

Let Θ be a multiplier (right) of T, then

(i) For every i in T, $\Theta(1) = i \circledast \Theta(i)$.

(ii) Θ is 1-1.

Proof

(i) Let i $\in T$. Then $i \circledast i = 1.$

We have $\Theta(1) = \Theta(i \circledast i)$

$= i \circledast \Theta(i) \; for \; all \; i \in T.$

(ii) Let $i, j \in T$ such that $\Theta(i) = \Theta(j)$.

Then by (i),

we have $\Theta(1) = \Theta(i \circledast i)$

$= i \circledast \Theta(i)$ and

$\Theta(1) = \Theta(j \circledast j)$

$= j \circledast \Theta(j).$

Then $i \circledast \Theta(i) = j \circledast \Theta(j).$

By cancellation law, $i = j.$

$\therefore \; \Theta \; is \; 1 - 1.$

Example 3.1.8

Let us consider T = {1,2,3,4} in which $\circledast$ is defined by

$\circledast$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Table 3.2

It can be checked that T is a GK algebra.

Now define a mapping $\Theta: \mathrm{T} \to \mathrm{T}$ by

$$\Theta(\mathrm{i}) = \begin{cases} 2 & if\ i = 1 \\ 1 & if\ i = 2 \\ 4 & if\ i = 3 \\ 3 & if\ i = 4 \end{cases}$$

It is clear that Θ is a multiplier of GK algebra.

This example shows that Θ is one-to-one and onto. But it is not regular,

since $\Theta(1) \neq 1$. Even surjectiveness exists here, it is not regular.

Consider the another mapping $\Theta: \mathrm{T} \to \mathrm{T}$ by

$$\Theta(\mathrm{i}) = \begin{cases} 3 & if\ i = 1 \\ 1 & if\ i = 2 \\ 3 & if\ i = 3 \end{cases}$$

in GK algebra example 3.1.2. Here Θ is one-to-one but not surjective (onto). Hence Θ is not multiplier of GK algebra and also not regular.

Hence, in GK algebra Θ should be an identity mapping (which means one-to-one and surjective (onto)).Then only Θ is regular. If surjectiveness does not exists, the regularity condition of GK algebra may be affected.

Theorem 3.1.9

Let Θ be a multiplier (left) of T. Then $\Theta(i) = i$ if and only if Θ is regular.

Proof

Let Θ is regular. Since $\Theta(1) = 1$.

Then we have $\Theta(1) = \Theta(i \circledast i) = \Theta(i) \circledast i = 1$.

By definition of GK algebra, $\Theta(i) = i$.

Conversely, let $\Theta(i) = i \; for \; i \; in \; T$.

It is clear that $\Theta(1) = 1$.

Hence Θ is regular.

Proposition 3.1.10

Let T be GK algebra and Θ be a multiplier (left) of T.

If $\Theta(i) \circledast i = 1 \; for \; every \; T$, then Θ is regular.

Proof

Let $\Theta(i) \circledast i = 1$ and Θ be a multiplier (left) of T.

By definition of GK algebra,

We have $\Theta(1) = \Theta(i \circledast i) = \Theta(i) \circledast i = 1$.

Hence Θ is regular.

Proposition 3.1.11

Let Θ be a multiplier (left) of T. Then the following holds

(i) If $\exists$ an element $i \in T \; \ni : \Theta(i) = i, \; \Theta$ is the identity.

(ii) If $\exists$ an element $i \in T \; \ni : \Theta(j) \circledast i = 1 \; for \; every \; j \in T$ then $\Theta(j) = i$.

Proof

(i) Let $\Theta(i) = i$ for some $i \in T$.

Then $\Theta(i) \circledast i = i \circledast i$

$\Rightarrow \Theta(i) \circledast i = 1$.

$Hence \; \Theta(1) = 1$ by the proposition 3.1.10 which implies that Θ is regular.

(ii) By the definition of GK algebra,

$\Theta(i \circledast j) = \Theta(j \circledast i) = \Theta(1)$

$\Rightarrow \Theta(i) \circledast j = \Theta(j) \circledast i = \Theta(1)$

$\Rightarrow \Theta(i) \circledast j = 1$

$\Rightarrow \Theta(i) = j.$

Proposition 3.1.12

Let T be GK algebra and Θ be a multiplier (left) of T. Then $\Theta(\Theta(i) \circledast i) = 1 \; \forall \; i \in T$

Proof

Let $i \in T$. Then we have $\Theta(\Theta(i) \circledast i) = \Theta(i) \circledast \Theta(i) = 1.$

Proposition 3.1.13

Let T be a GK algebra and Θ be a regular multiplier. Then the self-mapping Θ is an identity mapping if it satisfies multiplier (left) is equal to the multiplier (right) that is $\Theta(i) \circledast j = i \circledast \Theta(j) \; \forall \; i, j \in T.$

Proof

Since Θ is regular, we have $\Theta(1) = 1.$

Let $\Theta(i) \circledast j = i \circledast \Theta(j) \; \forall \; i, j \in T$

Then $\Theta(i) = \Theta(i \circledast 1) = \Theta(i) \circledast 1 = i \circledast \Theta(1) = i \circledast 1 = i.$

Hence Θ is an identity map.

Theorem 3.1.14

Let $(T, \circledast, 1)$ be a GK-algebra and Θ be a multiplier. Then

(i) $i \leq \Theta(i)$, for all $i \in T$.

(ii) $i \leq j \Longrightarrow i \leq \Theta(j)$, for all $i, j \in T$

Proof

(i) We have $i \circledast i = 1$, for all $i \in T$. So, $\Theta(i \circledast i) = \Theta(1)$

$i \circledast \Theta(i) = \Theta(1)$

$\Longrightarrow i \circledast \Theta(i) = 1$

$$\Longrightarrow i \leq \Theta(i)$$

(ii) We have $i \leq j$, $i \circledast j = 1$

$$i \circledast \Theta(i) = \Theta(1)$$

$$\Longrightarrow i \circledast \Theta(j) = 1$$

$$\Longrightarrow i \leq \Theta(j)$$

Definition 3.1.15

Let Θ be a multiplier of GK algebra. A set $\mathcal{H}_\Theta(\mathrm{T})$, the set of all invariant points of T, is defined by

$$\mathcal{H}_\Theta(\mathrm{T}) = \{\mathrm{i} \in \mathrm{T}/\Theta(\mathrm{i}) = \mathrm{i} \ \forall \mathrm{i} \in \mathrm{T}.\}$$

Proposition 3.1.16

Let T be a GK algebra and Θ be a multiplier (left) on T.If $j \in \mathcal{H}_\Theta(T)$, we have

$$i \wedge j \in \mathcal{H}_\Theta(T)\ \forall\, i, j \in T.$$

Proof

Let Θ be a left multiplier on T and let $j \in \mathcal{H}_\Theta(T)$.

$$\begin{aligned} \text{Now } \Theta(i \wedge j) &= \Theta\big(j \circledast (j \circledast i)\big) \\ &= \Theta(j) \circledast (j \circledast i) \\ &= j \circledast (j \circledast i) \\ &= i \wedge j. \end{aligned}$$

Hence $i \wedge j \in \mathcal{H}_\Theta(T)$.

Proposition 3.1.17

Let T be a GK algebra and Θ be a multiplier (right) on T.If $j \in \mathcal{H}_\Theta(T)$, we have $i \wedge j \in \mathcal{H}_\Theta(T)\ \forall\, i, j \in T$.

Proof

Let Θ be a multiplier (right) on T and let $j \in \mathcal{H}_\Theta(T)$.

Now, $\Theta(i \wedge j) = \Theta\big(j \circledast (j \circledast i)\big)$

$= j \circledast \Theta(j \circledast i)$

$= j \circledast (j \circledast \Theta(i))$

$= j \circledast (j \circledast i)$

$= i \wedge j.$

Hence $i \wedge j \in \mathcal{H}_{\Theta}(T)$.

Definition 3.1.18

Let T be a GK algebra and Θ_1, Θ_2 two self maps. We define a mapping

$$\Theta_1 \circ \Theta_2 : \mathrm{T} \to \mathrm{T} \text{ by } (\Theta_1 \circ \Theta_2)(\mathrm{i}) = \Theta_1(\Theta_2(\mathrm{i})) \ \forall\, \mathrm{i} \in \mathrm{T}.$$

Proposition 3.1.19

Let T be a GK algebra and Θ_1, Θ_2 two multipliers [right (left)] of T. The $\Theta_1 \circ \Theta_2$ is also multiplier [right (left)] of T.

Proof

Let T be a GK algebra and Θ_1, Θ_2 two multipliers (right) of T. Then we have

$$\begin{aligned}(\Theta_1 \circ \Theta_2)(i \circledast j) &= \Theta_1(\Theta_2(i \circledast j))\\ &= \Theta_1(i \circledast \Theta_2(j))\\ &= i \circledast \Theta_1(\Theta_2(j))\\ &= i \circledast (\Theta_1 \circ \Theta_2)(j)\end{aligned}$$

Let T be an GK algebra and Θ_1, Θ_2 two multipliers (left) of T. Then we have

$$\begin{aligned}(\Theta_1 \circ \Theta_2)(i \circledast j) &= \Theta_1(\Theta_2(i \circledast j))\\ &= \Theta_1(\Theta_2(i)) \circledast j\\ &= (\Theta_1 \circ \Theta_2)(i) \circledast j.\end{aligned}$$

Definition 3.1.20

Let T be a GK algebra and Θ_1, Θ_2 two self maps. We define $(\Theta_1 \wedge \Theta_2): \mathrm{T} \to \mathrm{T}$ by $(\Theta_1 \wedge \Theta_2)(\mathrm{i}) = \Theta_1(\mathrm{i}) \wedge \Theta_2(\mathrm{i})$.

Proposition 3.1.21

Let T be a GK algebra and Θ_1, Θ_2 two multipliers (left) of T. Then $\Theta_1 \wedge \Theta_2$ is also multiplier (left) of T.

Proof

Let T be a GK algebra and Θ_1, Θ_2 two multipliers (left) of T.

$(\Theta_1 \wedge \Theta_2)(i \circledast j) = \Theta_1(i \circledast j) \wedge \Theta_2(i \circledast j)$

$= (\Theta_1(i) \circledast j) \wedge (\Theta_2(i) \circledast j)$

$= (\Theta_2(i) \circledast j) \circledast ((\Theta_2(i) \circledast j) \circledast (\Theta_1(i) \circledast j))$

$= \Theta_1(i) \circledast j$ ……………….(1)

$(\Theta_1 \wedge \Theta_2)(i) \circledast j = (\Theta_1(i) \wedge \Theta_2(i)) \circledast j$

$= (\Theta_2(i) \circledast \left(\Theta_2(i) \circledast \Theta_1(i)\right)) \circledast j$

$= \Theta_1(i) \circledast j$ ………………..(2)

From (1) and (2)

$$(\Theta_1 \wedge \Theta_2)(i \circledast j) = (\Theta_1 \wedge \Theta_2)(i) \circledast j.$$

Hence $\Theta_1 \wedge \Theta_2$ is a multiplier (left) .

Definition 3.1.22

Consider the set of multipliers $Q(T)$, for any $\omega \in Q(T)$, the Kernel of ω is as follows $\mathcal{K}_\omega = \{i \in T / \omega(i) = 1\}$.

Proposition 3.1.23

Let ω be a multiplier and $1-1$.Then $\mathcal{K}_\omega$ is $\{1\}$

Proof

Let ω be one-to-one.

Let $i \in \mathcal{K}_\omega$. So $\omega(i) = 1 = \omega(1)$. Thus $i = 1$.

So, ker $(\omega) = \{1\}$.

3.2 Direct product of GK algebra

This section deals with the concept of the direct product of GK algebra. Some important results in which direct product of two GK algebra is again GK algebra as a particular case are derived, and also, the general case of the same is explored then after investigated the direct product of kernel of GK algebra.

Definition 3.2.1

Let (M,$\circledast, 1_M$) and (N,$\circledast, 1_N$) be GK algebras. Direct product $M \times N$ is defined as a structure $M \times N$ = ($M \times N$;$\otimes$;(1_M; 1_N)), where $M \times N$ is the set $\{(m, n)/m \in M, n \in N\}$ and $\otimes$ is given by

$$(m_1, n_1) \otimes (m_2, n_2) = (m_1 \circledast m_2, n_1 \circledast n_2)$$

This shows that the direct product of two sets of GK algebra M and N is denoted by $M \times N$, which each (m, n) is an ordered pair.

Theorem 3.2.2

The direct product of any two GK algebras is again a GK algebra.

Proof

Let M and N be GK algebras, let $m_1, m_2 \in M\ and\ n_1, n_2 \in N$

We know that $M \times N$ = ($M \times N$;$\otimes$;(1_M; 1_N))

Since $1_M \in M, 1_N \in N$

This implies that $(1_M, 1_N) \in M \times N$

$\therefore\ M \times N\ \ is\ non - empty.$

Now let us prove it is GK algebra.

Let $m_1, m_2 \in M\ and\ n_1, n_2 \in N$

(i) $(m_1, n_1) \otimes (m_1, n_1) = (m_1 \circledast m_1, n_1 \circledast n_1)$

= $(1_M, 1_N)$by definition of GK algebra

(ii) $(m_1, n_1) \otimes (1_M, 1_N) = (m_1 \circledast 1_M, n_1 \circledast 1_N)$

$= (m_1, n_1)$by definition of GK algebra

(iii) If $(m_1, n_1) \otimes (m_2, n_2) = (1_M, 1_N)$ and

$(m_2, n_2) \otimes (m_1, n_1) = (1_M, 1_N)$

then $(m_1 \circledast m_2, n_1 \circledast n_2) = (1_M, 1_N)$

$\Longrightarrow\ m_1 \circledast m_2 = 1_M\ \ and\ \ n_1 \circledast n_2 = 1_N$

$\Longrightarrow\ m_1 = m_2\ \ \ and\ \ n_1 = n_2$ by definition GK algebra.

(iv) $[((m_2, n_2) \otimes (m_3, n_3)] \otimes [(m_1, n_1) \otimes (m_3, n_3)]$

$\Longrightarrow (m_2 \circledast m_3, n_2 \circledast n_3) \otimes (m_1 \circledast m_3, n_1 \circledast n_3)$

$\Longrightarrow \{[(m_2 \circledast m_3) \circledast (m_1 \circledast m_3)] \circledast [(n_2 \circledast n_3) \circledast (n_1 \circledast n_3)]\}$

$\Longrightarrow (m_2 \circledast m_1, n_2 \circledast n_1)$

$\Longrightarrow (m_2, n_2) \otimes (m_1, n_1)$.

(v) $[(m_1, n_1) \otimes (m_2, n_2)] \otimes [(1_M, 1_N) \otimes (m_2, n_2)]$

$\Longrightarrow [(m_1 \circledast m_2), (n_1 \circledast n_2)] \otimes [(1_M \circledast m_2), (1_N \circledast n_2)]$

$\Longrightarrow [(m_1 \circledast m_2) \circledast (1_M \circledast m_2)], [(n_1 \circledast n_2) \circledast (1_N \circledast n_2)]$

$\Longrightarrow\ \ (m_1 \circledast 1_M, n_1 \circledast 1_N)$

$\Longrightarrow\ \ (m_1, n_1)$

Hence $M \times N$ is a GK algebra.

Theorem 3.2.3

Let $\{M_i\ /(M_i; \circledast; 1) : i = 1,2,3 \ldots.n\}\ and\ \{N_i\ /(N_i; \circledast; 1) : i = 1,2,3 \ldots.n\}$ be the family of GK algebras and let $\zeta_i : M_i \longrightarrow N_i, i = 1,2,3 \ldots..n$ be the set of isomorphism. If $\zeta\ from\ \prod_1^n M_i \longrightarrow \prod_1^n N_i$ given by $(m_i) = \zeta_i(m_i), i = 1,2, \ldots..n$, then ζ is also an isomorphism.

Proof

Let $\{M_i / (M_i; \circledast; 1) : i = 1,2,3 \dots . n\}$ and $\{N_i / (N_i; \circledast; 1) : i = 1,2,3 \dots . n\}$ be the family of GK algebras and let $\zeta_i: M_i \longrightarrow N_i, i = 1,2,3 \dots . . n$ be the set of isomorphism.

Let $\zeta\ from\ \prod_1^n M_i \longrightarrow \prod_1^n N_i$ given by $\zeta(m_i), (i = 1,2,3 \dots n) = \zeta_i(m_i), i = 1,2,3 \dots n$.

We have to prove ζ is an isomorphism.

If $(m_i, n_i) \in \prod_1^n M_i$ then $\zeta[(m_1, m_2, \dots . . m_n) \otimes (n_1, n_2, \dots \dots n_n)]$

$$= \zeta[m_1 \circledast n_1, m_2 \circledast n_2 \dots \dots m_n \circledast n_n]$$

$$= (\zeta_1(m_1 \circledast n_1), \zeta_2(m_2 \circledast n_2) \dots \dots \zeta_n(m_n \circledast n_n))$$

$$= ((\zeta_1(m_1) \circledast \zeta_1(n_1)),\ (\zeta_2(m_2) \circledast \zeta_2(n_2))\ \dots \dots (\zeta_n(m_n) \circledast \zeta_n(n_n))$$

$$= [\zeta_1(m_1), \zeta_2(m_2), \dots \dots . \zeta_n(m_n)] \otimes\ [\zeta_1(n_1), \zeta_2(n_2), \dots \dots . \zeta_n(n_n)]$$

$$= \zeta(m_1, m_2, \dots . . m_n)\ \otimes\ \zeta(n_1, n_2, \dots \dots n_n)$$

This implies that ζ is a homomorphism.

We have to prove ζ is onto, we have ζ_i is onto, where i=1,2,3….n.

Let $(n_1, n_2, \dots \dots n_n) \in N_1 \times N_2 \times \dots . \times N_n$

$\Longrightarrow Since\ \zeta\ is\ onto, n_i \in N_i, there\ exists\ m_i \in M_i\ such\ that\ \zeta_i(m_i) = n_i\ for\ i = 1,2,3 \dots n$

$\Longrightarrow (n_1, n_2, \dots \dots n_n) = [\zeta_1(m_1), \zeta_2(m_2), \dots \dots . \zeta_n(m_n)] = \zeta(m_1, m_2, \dots . . m_n)$

$\Longrightarrow\ \zeta\ is\ onto.$

Now, to prove $\zeta\ is\ 1 - 1$.

$\zeta(m_1, m_2, \dots . . m_n) = \zeta(n_1, n_2, \dots \dots n_n)$

$[\zeta_1(m_1), \zeta_2(m_2), \dots \dots . \zeta_n(m_n)] = [\zeta_1(n_1), \zeta_2(n_2), \dots \dots . \zeta_n(n_n)]$

$\Longrightarrow\ \zeta_i(m_i) = \zeta_i(n_i)$

$\Longrightarrow\ m_i = n_i$, where i=1,2,3….n , since ζ_i is 1-1.

$\Longrightarrow (m_1, m_2, \dots . . m_n) = (n_1, n_2, \dots \dots n_n)$

$\Longrightarrow\ \zeta\ is\ 1-1.$

Hence ζ is an isomorphism.

Theorem 3.2.4

Let $M_i, N_i, i = 1,2$ be GK algebras. Consider the mapping $\zeta_1: M_1 \longrightarrow N_1\ and$ $\zeta_2: M_2 \longrightarrow N_2$where ζ_1, ζ_2 are homomorphisms. If the map $\zeta: M_1 \times M_2 \longrightarrow N_1 \times N_2$ given by $\zeta(m_1, m_2) = \zeta_1(m_1), \zeta_2(m_2)$, then

(i) $\zeta$is a homomorphism.

(ii) $\ker\zeta = \ker\zeta_1 \times \ker\zeta_2$.

Proof

Let us consider the mapping $\zeta_1: M_1 \longrightarrow N_1 and \zeta_2: M_2 \longrightarrow N_2$ where ζ_1, ζ_2 are homomorphisms.

If the map $\zeta: M_1 \times M_2 \longrightarrow N_1 \times N_2$ given by $\zeta(m_1, n_1) = (\zeta_1(m_1), \zeta_2(n_1))$,

for $m_1, m_2 \in M_1\ and\ n_1, n_2 \in M_2$ then

(i) $\zeta[(m_1, n_1)\otimes(m_2, n_2)] = \zeta(m_1 \circledast m_2, n_1 \circledast n_2)$

$$= (\zeta_1(m_1 \circledast m_2), \zeta_2(n_1 \circledast n_2))$$

$$= (\zeta_1(m_1) \circledast \zeta_1(m_2), \zeta_2(n_1) \circledast \zeta_2(n_2))$$

$$= \big(\zeta_1(m_1), \zeta_2(n_1)\big) \otimes\ (\zeta_1(m_2), \zeta_2(n_2))$$

$$= \zeta_1(m_1, n_1)\otimes\zeta_2(m_2, n_2)$$

Therefore ζ is a homomorphism.

(ii) Let $(m, n) \in ker\zeta \iff \zeta(m, n) = \big(1_{M_1}, 1_{M_2}\big)$

$$\iff (\ \zeta_1(m), \zeta_2(n)) = \big(1_{M_1}, 1_{M_2}\big)$$

$$\iff \zeta_1\big(m\big) = 1_{M_1}, \zeta_2\big(n\big) = 1_{M_2}$$

$$\iff m \in \ker\zeta_1\ ,\ n \in \ker\zeta_2$$

$$\iff\ (m, n)\ \in\ \ker\zeta_1 \times \ker\zeta_2.$$

Hence $\ker\zeta = \ker\zeta_1 \times \ker\zeta_2$.

3.3 Summary

The study of multipliers (left and right) on GK algebra had been explored. It is derived that the composition of two multipliers is again a multiplier of GK algebra. In the second part of this chapter, the direct of GK algebras is introduced. In that, some important theorem such as the direct product of any two GK algebras is again a GK algebra and $\ker \zeta = \ker \zeta_1 \times \ker \zeta_2$ where ζ_1, ζ_2 are homomorphism is obtained.

CHAPTER 4

DERIVATIONS IN GK ALGEBRA

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CHAPTER 4

DERIVATIONS IN GK ALGEBRA

The theory of derivations in GK algebra is initiated in this Chapter. This new concept deals with some interesting properties of derivations in GK algebra and also, investigated the (GK-LR) (left-right) and (GK-RL) (Right left) derivations respectively, regular of GK derivations in GK algebra with necessary examples. It is showed that the set all GK-LR derivations is associative and also brought that few adequate results.

4.1 Derivations in GK algebra

This section explained about the concept of derivation in GK algebra. The concept of GK derivations, GK-LR derivation, GK-RL derivation are explained with the examples. The properties of GK (LR and RL) are scrutinized and obtained interesting results.

Definition 4.1.1

Let (T,$\circledast$,1) be a GK algebra. A map $\xi: T \to T$ is called a left-right derivation (Simply (GK-LR) derivation) of T if

$\xi(i \circledast j) = (\xi(i) \circledast j) \wedge \left(i \circledast \xi(j)\right) \forall\, i, j \in T.$

Definition 4.1.2

Let $(T,\circledast,1)$ be a GK-algebra. A map $\xi: T \to T$ is called a right-left derivation (Simply (GK-RL) derivation) of T if

$\xi(i \circledast j) = (i \circledast \xi(j)) \wedge (\xi(i) \circledast j) \forall\, i, j \in T.$

Remark 4.1.3

A map $\xi: T \to T$ is said to be a derivation of T if ξ is both a (GK-LR) derivation and a (GK-RL) derivation of T.

Note 4.1.4

Let $(T, \circledast, 1)$ be a GK-algebra T, $i, j \in$T. We denote $i \wedge j = j \circledast (j \circledast i)$.

Example 4.1.5

Let $T = \{1,2,3\}$ be a GK-algebra. The operation $\circledast$ is defined as follows

$\circledast$	1	2	3
1	1	3	2
2	2	1	3
3	3	2	1

Table 4.1

Define a map $\xi: T \to T$ by

$$\xi(i) = \begin{cases} 1 & if \ i = 1 \\ 2 & if \ i = 2 \\ 3 & if \ i = 3 \end{cases}$$

Then it is so clear that ξ is a derivation of T.

Definition 4.1.6

Let $(T, \circledast, 1)$ be a GK-algebra and $\xi: T \to T$ be a map of a GK-algebra, then ξ is called regular if $\xi(1)$ =1.

Note 4.1.7

In GK-algebra, we can observe that $i \wedge j = j \circledast (j \circledast i) = i \ \forall \ i, j \in T$.

Proposition 4.1.8

Let ξ be a self-map of GK algebra T, then

(i) If ξ is regular (GK-LR) derivation of T, then $\xi(i) = \xi(i) \wedge i \quad \forall \ i \in T$

(ii) If ξ is regular (GK-RL) derivation of T, then $\xi(i) = i \wedge \xi(i) \quad \forall \ i \in T$

Proof

(i) Let ξ be a regular (GK-LR) derivation of T. Then

$$\begin{aligned}
\xi(i) &= \xi(i \circledast 1) \\
&= (\xi(i) \circledast 1) \wedge \big(i \circledast \xi(1)\big) \\
&= \xi(i) \wedge (i \circledast \xi(1)) \\
&= \xi(i) \wedge (i \circledast 1) \\
&= \xi(i) \wedge i
\end{aligned}$$

(ii) Let ξ be a regular (GK-RL) derivation of T, then

$$\begin{aligned}
\xi(i) &= \xi(i \circledast 1) \\
&= (i \circledast \xi(1)) \wedge (\xi(i) \circledast 1) \\
&= (i \circledast 1) \wedge (\xi(i) \circledast 1) \\
&= i \wedge \xi(i)
\end{aligned}$$

Conversely,

Let ξ be a (GK-RL) derivation of T and $\xi(i) = i \wedge \xi(i) \ \forall \ i \in T$, then

we get, $\xi(1) = 1 \wedge \xi(1)$

$$\begin{aligned}
&= \xi(1) \circledast (\xi(1) \circledast 1) \qquad \because i \wedge j = j \circledast (j \circledast i) \\
&= \xi(1) \circledast \xi(1) \\
&= 1.
\end{aligned}$$

Hence ξ is regular.

Lemma 4.1.9

Let $(T, \circledast, 1)$ *be* a GK-algebra and ξ be a (GK-LR) derivation of T. Then the following hold $\forall\, i, j \in T$

(i) $\xi(i \circledast j) = \xi(i) \circledast j.$

(ii) If ξ is regular then $\xi(i) \leq i.$

Proof

(i) Let $(T, \circledast, 1)$ be a GK algebra and ξ be a (GK-LR) derivation of T.

Then, $\xi(i \circledast j) = (\xi(i) \circledast j) \wedge \big(i \circledast \xi(j)\big)$

$$= \big(i \circledast \xi(j)\big) \circledast \Big(\big(i \circledast \xi(j)\big) \circledast (\xi(i) \circledast j)\Big)$$

$$= \xi(i) \circledast j$$

$\therefore \xi(i \circledast j) = \xi(i) \circledast j.$

(ii) Let ξ be a regular derivation of T.

Then ξ (1) =1.

Now $\xi(i \circledast i) = \xi(1)$

$\xi(i) \circledast i = 1$

$\therefore \xi(i) \leq i.$

Lemma 4.1.10

Let $(T, \circledast, 1)$ be a GK algebra and ξ be a (GK-RL) derivation of T.

Then,

(i) $\xi(i \circledast j) = i \circledast \xi(j)$

(ii) If ξ is regular then $i \leq \xi(i)$

Proof

(i) Let $(T, \circledast, 1)$ be a GK algebra and ξ be a (GK-RL) derivation of T.

Then, $\xi(i \circledast j) = (i \circledast \xi(j)\,) \wedge (\xi(i) \circledast j)$

$$= (\xi(i) \circledast j) \circledast \left((\xi(i) \circledast j) \circledast (i \circledast \xi(j))\right)$$

$$= i \circledast \xi(j)$$

$$\therefore \xi(i \circledast j) = i \circledast \xi(j).$$

(ii) Let ξ be a regular derivation of T.

Then ξ (1) =1.

Now, $\xi(i \circledast i) = \xi(1)$

$$i \circledast \xi(i) = 1$$

Therefore, $i \leq \xi(i)$.

Note 4.1.11

(i) From the above lemma 4.1.10

$$\xi(i \circledast j) = \xi(i) \circledast j$$

and $\xi(i \circledast j) = i \circledast \xi(j)$

$\Rightarrow \xi(i \circledast j) = \xi(i) \circledast j = i \circledast \xi(j)$

(ii) Let ξ be the regular derivation then by lemma 4.1.10

$\xi(i) \leq i$ and $i \leq \xi(i)$

$\Rightarrow i = \xi(i)$.

Remark 4.1.12

A map $\xi: T \to T$ is regular derivation of T then $\xi(i) = i \ \forall \ i \in T$.

Lemma 4.1.13

Let $\xi: T \to T$ be a derivation of T. Then ξ is a regular derivation if ξ is either a (GK-LR) derivation or a (GK-RL) derivation.

Proof

Let ξ is (GK-LR) derivation, then for all $i \in T$, $\xi(i) \circledast i = 1$

Now $\xi(1) = \xi(i \circledast i)$

$$= \xi(i) \circledast i$$

$$\therefore \xi(1) = 1.$$

$\therefore \xi$ is regular.

Now if ξ is (GK-RL) derivation, then for all $i \in T$,

$i \circledast \xi(i) = 1$

Now, $\xi(1) = \xi(i \circledast i)$

$= i \circledast \xi(i)$

$\therefore \xi(1) = 1.$

$\therefore \xi$ is regular.

Theorem 4.1.14

Let $(T, \circledast, 1)$ be a GK algebra and ξ be a regular (GK-RL) derivation of T. Then the following hold, $\forall\, i, j \in T$.

(i) $\xi(i) = i$

(ii) $\xi(i) \circledast j = i \circledast \xi(j)$

(iii) $\xi(i \circledast j) = \xi(i) \circledast j = i \circledast \xi(j) = \xi(i) \circledast \xi(j)$

Proof

(i) Since ξ is regular (GK-RL) derivation of T, we have

$\xi(i) = \xi(i \circledast 1)$

$= i \circledast \xi(1)$

$= i \circledast 1$

$= i$

$\therefore \xi(i) = i.$

(ii) Since ξ is regular (GK-RL) derivation of T, then we have

$\xi(i \circledast j) = i \circledast \xi(j)$

$i \circledast j = i \circledast \xi(j) \qquad \rightarrow (1)$

and in (GK-LR) derivation

$$\xi(i \circledast j) = \xi(i) \circledast j$$

$$i \circledast j = \xi(i) \circledast j \qquad \rightarrow (2)$$

From (1) & (2)

$$\xi(i) \circledast j = i \circledast j = i \circledast \xi(j).$$

(iii) Since $\xi(i) = i \ \forall \ i \in T$

$$\xi(i \circledast j) = \xi(i) \circledast j = \xi(i) \circledast \xi(j)$$

$$\xi(i \circledast j) = i \circledast \xi(j) = \xi(i) \circledast \xi(j)$$

$$\Rightarrow \xi(i \circledast j) = \xi(i) \circledast j = i \circledast \xi(j) = \xi(i) \circledast \xi(j).$$

Lemma 4.1.15

Let $(T,\circledast,1)$ be a GK algebra and ξ be a derivation on T. If $i \leq j \ \forall \ i,j \ \in \ T$ then ξ (i) = ξ (j).

Proof

In GK algebra, $i \circledast j = j \circledast i = 1 \Leftrightarrow i \leq j$.

Then $\xi(j) = \xi(j \circledast 1)$

$$= \xi\big(j \circledast (j \circledast i)\big)$$

$$= \xi(i).$$

Proposition 4.1.16

Let ξ be a derivation on GK algebra and let $i \in T$, then

$$i \circledast \big(i \circledast \xi(i)\big) = \xi(i) \circledast (\xi(i) \circledast i).$$

Proof

We know that $\xi(i) = \xi(i) \wedge i$

$$i \circledast \xi(i) = i \circledast (\xi(i) \wedge i)$$

$$= i \circledast (i \circledast \xi(i))) \qquad \because i \wedge j = j \circledast (j \circledast i)$$

and $i \circledast \xi(i) = i \circledast (i \wedge \xi(i))$

$$= i \circledast (\xi(i) \circledast (\xi(i) \circledast i)) \qquad \because i \wedge j = j \circledast (j \circledast i)$$

$$\Rightarrow i \circledast \left(i \circledast \left(i \circledast \xi(i)\right)\right) = i \circledast \left(\xi(i) \circledast (\xi(i) \circledast i)\right)$$

By cancellation law,

$$i \circledast \left(i \circledast \xi(i)\right) = \xi(i) \circledast (\xi(i) \circledast i).$$

Lemma 4.1.17

If ξ is a regular (GK-RL) derivation on GK algebra, then $\xi\left(i \circledast \xi(i)\right) = 1$.

Proof

Since ξ is a regular (GK-RL) derivation on GK algebra, $i \circledast \xi(i)$=1.

$\therefore \xi\left(i \circledast \xi(i)\right) = \xi(1) = 1$

$\therefore \xi\left(i \circledast \xi(i)\right) = 1.$

Lemma 4.1.18

If ξ is a regular (GK-LR) derivation on GK algebra, then $(\xi(i) \circledast i) = 1$.

Proof

Since ξ is a regular (GK-LR) derivation on GK algebra, $\xi(i) \circledast i$=1.

$\therefore \xi(\xi(i) \circledast i) = \xi(1) = 1$

$\therefore \xi(\xi(i) \circledast i) = 1.$

Definition 4.1.19

Let ξ_1, ξ_2 be a self-map on GK algebra T. We define $\xi_1 \circ \xi_2$ as follows

$$(\xi_1 \circ \xi_2)(i) = \xi_2(\xi_1(i))$$

Lemma 4.1.20

Let ξ_1, ξ_2 be self-maps on a GK algebra. Let ξ_1, ξ_2 be two (GK-LR) derivations on T. Then $\xi_1 \circ \xi_2$ is also a (GK-LR) derivation on T.

Proof

Given ξ_1, ξ_2 is two (GK-LR) derivations on T.

By lemma 4.1.9, we know that

$\xi_1(i \circledast j) = \xi_1(i) \circledast j$ and $\xi_2(i \circledast j) = \xi_2(i) \circledast j$

Now, $(\xi_1 \circ \xi_2)(i \circledast j) = \xi_2(\xi_1(i \circledast j))$

$= \xi_2(\xi_1(i) \circledast j)$

$= \xi_2(\xi_1(i)) \circledast j$

$= (\xi_1 \circ \xi_2)(i) \circledast j$

Hence $\xi_1 \circ \xi_2$ is a (GK-LR) derivation on GK algebra.

Lemma 4.1.21

Let ξ_1, ξ_2 be self-maps on a GK algebra. Let ξ_1, ξ_2 be two (GK-RL) derivations on T. Then $\xi_1 \circ \xi_2$ is also a (GK-RL) derivation on T.

Proof

Given ξ_1, ξ_2 is two (GK-RL) derivations on T.

Now, $(\xi_1 \circ \xi_2)(i \circledast j) = \xi_2(\xi_1(i \circledast j))$

$= \xi_2(i \circledast \xi_1(j))$

$= i \circledast \xi_2(\xi_1(j))$

$= i \circledast (\xi_1 \circ \xi_2)(j)$.

Hence $\xi_1 \circ \xi_2$ is a (GK-RL) derivation on GK algebra.

By the above two lemmas 4.1.20 and 4.1.21, we get the following theorem.

Theorem 4.1.22

Let $(T, \circledast, 1)$ be a GK algebra and ξ_1, ξ_2 be two derivations on T, then

$$\xi_1 \circ \xi_2 = \xi_2 \circ \xi_1.$$

Proof

Since ξ_1, ξ_2 be two derivations on T, ξ_1, ξ_2 are both (GK-LR) and (GK-RL) derivations on T.

Now, $(\xi_1 \circ \xi_2)(i \circledast j) = \xi_2(\xi_1(i \circledast j))$

$= \xi_2(\xi_1(i) \circledast j)$

$= \xi_1(i) \circledast \xi_2(j).$ ___________(1)

Also, $(\xi_2 \circ \xi_1)(i \circledast j) = \xi_1(\xi_2(i \circledast j))$

$= \xi_1(i \circledast \xi_2(j))$

$=\xi_1(i) \circledast \xi_2(j).$ ___________(2)

From (1) & (2) $(\xi_1 \circ \xi_2)(i \circledast j)=(\xi_2 \circ \xi_1)(i \circledast j)$

This gives that $(\xi_1 \circ \xi_2)=(\xi_2 \circ \xi_1)$.

Definition 4.1.23

Let ξ_1, ξ_2 be a self-map on a GK algebra T. We define $\xi_1 \circledast \xi_2 : T \to T$ as follows $(\xi_1 \circledast \xi_2)(i) = \xi_2(i) \circledast \xi_1(i) \forall i \in T$.

Theorem 4.1.24

Let $(T, \circledast, 1)$ be a GK algebra and ξ_1, ξ_2 be two derivations of T, then

$\xi_1 \circledast \xi_2 = \xi_2 \circledast \xi_1$.

Proof

$(\xi_1 \circ \xi_2)(i \circledast j) = \xi_2(\xi_1(i \circledast j))$

$= \xi_2(\xi_1(i) \circledast j)$

$= \xi_1(i) \circledast \xi_2(j).$ ___________(1)

$(\xi_1 \circ \xi_2)(i \circledast j) = \xi_2(\xi_1(i \circledast j))$

$= \xi_2(i \circledast \xi_1(j))$

$= \xi_2(i) \circledast \xi_1(j).$ ___________(2)

From the above

$\xi_1(i) \circledast \xi_2(j)=\xi_2(i) \circledast \xi_1(j).$ ___________(3)

Substituting j = i in (3)

$\xi_1(i) \circledast \xi_2(i)=\xi_2(i) \circledast \xi_1(i)$

By definition 4.1.23,

$(\xi_2 \circledast \xi_1)(i) = (\xi_1 \circledast \xi_2)(i)$

This gives $(\xi_1 \circledast \xi_2) = (\xi_2 \circledast \xi_1)$.

Definition 4.1.25

Let LD (ξ) denotes the set of all (GK-LR) derivations on T. Define the operation ⋏ on LD (ξ) as follows. For $\xi_1, \xi_2 \in$ LD (ξ), define

$(\xi_1 \curlywedge \xi_2)(i) = \xi_1(i) \curlywedge \xi_2(i)$ for all $i \in T$.

Lemma 4.1.26

If ξ_1 and ξ_2 are (GK-LR) derivations on T, then $\xi_1 \curlywedge \xi_2$ is also a (GK-LR) derivation on T.

Proof

To Prove: $(\xi_1 \curlywedge \xi_2)(i \circledast j) = (\xi_1 \curlywedge \xi_2)(i) \circledast j$

$(\xi_1 \curlywedge \xi_2)(i \circledast j) = \xi_1(i \circledast j) \curlywedge \xi_2(i \circledast j)$

$= (\xi_1(i) \circledast j) \curlywedge (\xi_2(i) \circledast j)$

$= (\xi_2(i) \circledast j) \circledast ((\xi_2(i) \circledast j) \circledast (\xi_1(i) \circledast j))$

$= \xi_1(i) \circledast j$----------------(1)

$(\xi_1 \curlywedge \xi_2)(i) \circledast j = (\xi_1(i) \curlywedge \xi_2(i)) \circledast j$

$= (\xi_2(i) \circledast (\xi_2(i) \circledast \xi_1(i))) \circledast j$

$= \xi_1(i) \circledast j$ --------------(2)

From (1) and (2), $(\xi_1 \curlywedge \xi_2)(i \circledast j) = (\xi_1 \curlywedge \xi_2)(i) \circledast j$.

Hence, ξ_1 and ξ_2 are (GK-LR) derivations on T, then $\xi_1 \curlywedge \xi_2$ is also a (GK-LR) derivation on T.

Lemma 4.1.27

The operator ⋏ defined on LD (ξ) is associative.

Proof

Let T be a GK algebra.

Let ξ_1, ξ_2, ξ_3 are (GK-LR) derivations in GK algebra.

To Prove: $(\xi_1 \curlywedge \xi_2) \curlywedge \xi_3 = \xi_1 \curlywedge (\xi_2 \curlywedge \xi_3)$

Now, $((\xi_1 \curlywedge \xi_2) \curlywedge \xi_3)(i \circledast j) = (\xi_1 \curlywedge \xi_2)(i \circledast j) \curlywedge \xi_3 (i \circledast j)$

$= (\xi_1(i) \circledast j) \curlywedge (\xi_3(i) \circledast j)$

$= (\xi_3(i) \circledast j) \circledast ((\xi_3(i) \circledast j) \circledast (\xi_1(i) \circledast j))$

$= \xi_1(i) \circledast j$ -------------- (1)

Now, $(\xi_1 \curlywedge (\xi_2 \curlywedge \xi_3))(i \circledast j) = \xi_1 (i \circledast j) \curlywedge ((\xi_2 \curlywedge \xi_3)(i \circledast j))$

$= (\xi_1(i) \circledast j) \curlywedge (\xi_2(i) \circledast j)$

$= (\xi_2(i) \circledast j) \circledast ((\xi_2(i) \circledast j) \circledast (\xi_i(i) \circledast j))$

$= \xi_1(i) \circledast j$ -------------- (2)

From (1) and (2), $(\xi_1 \curlywedge \xi_2) \curlywedge \xi_3 = \xi_1 \curlywedge (\xi_2 \curlywedge \xi_3)$.

4.2 Summary

In the algebraic structure, derivation always takes a part of very enthralling and paramount topic of research in the field of Mathematics. In this Chapter, the notion of (GK-LR) (GK-RL) respectively) derivations of a GK algebra initiated and attained some remarkable results such as ξ_1, ξ_2 are self-maps on a GK algebra, ξ_1, ξ_2 two (GK-RL and GK- LR) derivations on T. Then $\xi_1 \circ \xi_2$ is also a (GK-RL and GK-LR) derivation on T.

CHAPTER 5

SYMMETRIC BI DERIVATION OF

GK ALGEBRA†

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CHAPTER 5

SYMMETRIC BI DERIVATION OF GK ALGEBRA

In this Chapter, the concept of symmetric bi derivation (GK-LR derivation and GK-RL derivation) of GK algebra is initiated and also the trace of GK algebra and regular, component wise regular in GK algebra are investigated and obtained some of its interesting properties which is related to them.

5.1 Symmetric bi derivation of GK algebra

This section explored about the concept of symmetric bi derivation of GK algebra. It is scrutinized about the left and right symmetric bi derivation of GK algebra and attained enthralling results of them.

Definition 5.1.1

Let $(T, \circledast, 1)$ be a GK algebra. A mapping Ω:T×T→T is said to be a left right symmetric bi derivation (simply GK- LR symmetric bi derivation) of T, if it is satisfying the following identity $\Omega(i \circledast j, k) = (\Omega(i, k) \circledast j) \wedge (i \circledast \Omega(j, k))$ for, i, j, k $\in T$.

Definition 5.1.2

Let $(T, \circledast, 1)$ be a GK algebra. A mapping Ω:TxT→T is said to be a right left symmetric bi derivation (simply GK- RL symmetric bi derivation) of T, if it is satisfying the following identity

$$\Omega(i, j \circledast k) = (\Omega(i, j) \circledast k) \wedge (j \circledast \Omega(i, k)) \text{ for i, j, k} \in T.$$

In general, if Ω is both GK-LR and GK-RL symmetric bi derivation then it is called as Ω is symmetric bi derivation.

Definition 5.1.3

Let T be a GK algebra. A map Ω:T×T→T is said to be symmetric if

$\Omega(i,j) = \Omega(j,i)$ ∀ pairs of i, j∈ T.

Definition 5.1.4

Let T be a GK algebra and the mapping Ω:T×T→T be a symmetric mapping.

A map $\delta: T \to T$ be defined as $\delta(i) = \Omega(i,i)$ is called trace of Ω.

Example 5.1.5

Consider the following Cayley's table for GK algebra

⊛	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Table 5.1

Define a mapping Ω:T×T→T by

$$\Omega(i,j) = \begin{cases} 1, (i,j) = (1,1), (2,2), (3,3), (4,4) \\ 2, (i,j) = (1,2), (2,1), (3,4), (4,3) \\ 3, (i,j) = (1,3), (2,4), (3,1), (4,2) \\ 4, (i,j) = (1,4), (2,3), (3,2), (4,1) \end{cases}$$

From this Ω is symmetric bi derivation of T.

Remark 5.1.6

In above example, $\Omega(i,i) = \{1\}$ $when$ $i = 1,2,3,4$ is called trace of Ω.

Definition 5.1.7

Let T be a GK algebra. The map Ω:T×T→T be a symmetric mapping. Ω is called component wise regular if $\Omega(i,1) = \Omega(1,i) = 1 \; for \; some \; i \in T$. In specific if $\Omega(1,1) = \delta(1) = 1$ then $\Omega \; is \; called \;\; \delta - regular$.

Proposition 5.1.8

Let $(T, \circledast, 1)$ be a GK algebra. Let Ω be an GK-LR symmetric bi derivation on T. Then the following holds

(i) $\Omega(i,j) = \Omega(i,j) \wedge \big(i \circledast \Omega(1,j)\big) \; for \; all \;\; i,j \in T$.

(ii) $\Omega(1,i) = \delta(i) \circledast i$ where δ is the trace of Ω.

(iii) $\Omega(1,j) = \Omega(i,j) \circledast i \;\; \forall \;\; i,j \;\; \in T$.

(iv) $\Omega(j,1) = \Omega(j,1) \wedge j \; \forall \; j \; in \; T \; if \;\; \Omega \;\; is \; \delta - regular$.

(v) $\Omega(j,1) = 1 \;\; \forall \; j \; in \; T \; if \;\; \Omega \;\; is \; component \; wise \; regular$

Proof

(i) Let us consider i,j in T.

By the definition of GK-LR symmetric bi derivation,

we have, $\Omega(i,\mathrm{j}) = \Omega(i \circledast 1, j)$

$= (\Omega(i,j) \circledast 1) \wedge (i \circledast \Omega(1,j))$

By axiom (ii) of GK algebra

$= (\Omega(i,j)) \wedge (i \circledast \Omega(1,j))$

(ii) Let i, j in T

Now, $\Omega(1,i) = \Omega(i \circledast i, i)$

$= (\Omega(i,i) \circledast i) \wedge (i \circledast \Omega(\mathrm{i},i))$

$= (\delta(i) \circledast i) \wedge \big(i \circledast \delta(i)\big)$

$= \big(i \circledast \delta(i)\big) \circledast (\big(i \circledast \delta(i)\big) \circledast (\delta(i) \circledast i))$

$= (\delta(i) \circledast i)$

(iii) Let i, j in T.

We have, $\Omega(1, j) = \Omega(i \circledast i, j)$

$= (\Omega(i, j) \circledast i) \wedge (i \circledast \Omega(\mathrm{i}, j))$

$= (i \circledast \Omega(\mathrm{i}, j)) \circledast ((i \circledast \Omega(\mathrm{i}, j) \circledast (\Omega(\mathrm{i}, j) \circledast i))$

$= \Omega(\mathrm{i}, j) \circledast i$

(iv) Let i, j in T.

Now, $\Omega(j, 1) = \Omega(j \circledast 1,1)$

$= (\Omega(j, 1) \circledast 1) \wedge (j \circledast \Omega(1,1))$

$= (\Omega(\mathrm{j}, 1)) \wedge (j \circledast \delta(1))$

$= \Omega(\mathrm{j}, 1) \wedge (j \circledast 1)$

$= \Omega(\mathrm{j}, 1) \wedge j$

(v) Let i, j in T

Now, $\Omega(j, 1) = \Omega(j \circledast 1,1)$

$= (\Omega(j, 1) \circledast 1) \wedge (j \circledast \Omega(1,1))$

$= (\Omega(\mathrm{j}, 1)) \wedge (j \circledast \delta(1))$

$= \Omega(\mathrm{j}, 1) \wedge (j \circledast 1)$

$= \Omega(\mathrm{j}, 1) \wedge j$

$= 1 \wedge j$

$= 1$ since $i \wedge j = i$.

Proposition 5.1.9

Let $(T, \circledast, 1)$ be a GK algebra. Let Ω be an GK-RL symmetric bi derivation on T.

Then the following holds

(i) $\Omega(i, j) = \Omega(i, j) \wedge \left(i \circledast \Omega(1, j)\right)$ *for all* $i, j \in T$.

(ii) $\Omega(i, 1) = \delta(i) \circledast i$ where δ is the trace of Ω.

(iii) $\Omega(1, j) = \Omega(i, j) \circledast i \ \forall \ i, j \in T$.

(iv) $\Omega\,(j,1) = \Omega(j,1) \wedge j\ \forall\, j\ in\ T\ if\ \ \Omega\ \ is\ \delta - regular$.

(v) $\Omega(j,1) = 1\ \ \forall\, j\ in\ T\ if\ \ \Omega\ \ is\ component\ wise\ regular$

Proof

(i) Let us consider i, j in T.

By the definition of GK-RL symmetric bi derivation,

We have, $\Omega(i,j) = \Omega(i, j \circledast 1)$

$= (\Omega(i,j) \circledast 1) \wedge (j \circledast \Omega(\mathrm{i}, 1))$

By axiom (ii) of GK algebra

$= (\Omega(i,j)) \wedge (j \circledast \Omega(\mathrm{i}, 1))$

$= \ \Omega(i,j) \wedge (j \circledast 1)$

$= \ \Omega(\mathrm{i},\mathrm{j}) \wedge \mathrm{j}$

(ii) Let i, j in T

Now, $\Omega(i,1) = \Omega(i, i \circledast i)$

$= (\Omega(i,i) \circledast i) \wedge (i \circledast \Omega(\mathrm{i}, i))$

$= (\delta(i) \circledast i) \wedge \big(i \circledast \delta(i)\big)$

$= \big(i \circledast \delta(i)\big) \circledast \big(\big(i \circledast \delta(i)\big) \circledast (\delta(i) \circledast i)\big)$

$= (\delta(i) \circledast i)$

(iii) Let i, j in T

We have, $\Omega(j,1) = \Omega(j, i \circledast i)$

$= (\Omega(j,i) \circledast i) \wedge (i \circledast \Omega(\mathrm{j},\mathrm{i}))$

$= (i \circledast \Omega(\mathrm{j},\mathrm{i})) \circledast ((i \circledast \Omega(\mathrm{j},\mathrm{i}) \circledast (\Omega(\mathrm{j},\mathrm{i}) \circledast i))$

$= \Omega(\mathrm{j},\mathrm{i}) \circledast i$

(iv) Let i, j in T.

$\Omega(1,j) = \Omega(1, j \circledast 1)$

$= (\Omega(1,j) \circledast 1) \wedge (j \circledast \Omega(1{,}1))$

$= (\Omega(1,\mathrm{j})) \curlywedge (j \circledast \delta(1))$

$= \Omega(1,\mathrm{j}) \curlywedge (j \circledast 1)$

$= \Omega(1,\mathrm{j}) \curlywedge j$

(v) Let i, j in T

$\Omega(j,1) = \Omega(j \circledast 1,1)$

$= (\Omega(j,1) \circledast 1) \curlywedge (j \circledast \Omega(1,1))$

$= (\Omega(\mathrm{j},1)) \curlywedge (j \circledast \delta(1))$

$= \Omega(\mathrm{j},1) \curlywedge (j \circledast 1)$

$= \Omega(\mathrm{j},1) \curlywedge j$

$= 1 \curlywedge j$ = j$\circledast$ $(j \circledast 1)$=1.

Proposition 5.1.10

Let T be the GK algebra and δ be the trace of the GK- LR symmetric bi derivation on T. Then

(i) $\delta(1) = \Omega(i,1) \circledast i$.

(ii) $If\ \Omega(i,1) = \Omega(j,1)\ \ \forall\ i,j\ \in T\ \ then\ \ \delta\ is\ 1-1.$

(iii) δ is regular if and only if $\Omega(i,1) = i$.

Proof

(i) Let $i \in T$. We know that $i \circledast i = 1$

We have, $\delta(1) = \Omega(1,1)$

$= \Omega(i \circledast i,1)$

$= (\Omega(i,1) \circledast i) \curlywedge (i \circledast \Omega(i,1))$

$= (\Omega(i,1) \circledast i)$

(ii) Let i, j ∈ T such that $\delta(i) = \delta(j)$.

We have, $\delta(1) = \Omega(i,1) \circledast i$

and $\delta(1) = \Omega(j,1) \circledast j$.

This implies that $\Omega(i,1) \circledast i = \Omega(j,1) \circledast j$.

Since $\Omega(i,1) = \Omega(j,1)$ and by using cancellation law, we get i=j.

Hence, we get δ is 1-1.

(iii) Let δ be regular.

We have $\delta(1) = \Omega(i,1) \circledast i$

Since δ is regular, $\delta(1) = 1$ *implies* $\Omega(i,1) \circledast i = 1$.

By axiom (iii) of GK algebra we have $\Omega(i,1) = i$

Conversely, Let $\Omega(i,1) = i$ for some i in T.

$\Rightarrow \Omega(i,1) \circledast i = i \circledast i$

$\Rightarrow \Omega(i,1) \circledast i = 1$

$\Rightarrow \delta(1) = 1$

Hence δ is regular.

Proposition 5.1.11

Let T be the GK algebra and δ be the trace of the GK-RL symmetric bi derivation on T. Then

(i) $\delta(1) = \Omega(1,i) \circledast i$.

(ii) $\delta(i) = \delta(i) \wedge (i \circledast \Omega(i,1))$

(iii) $If\ \Omega(1,i) = \Omega(1,j)\ \ \forall\ i,j \in T\ then\ \delta\ is\ 1-1$.

(iv) δ is regular if and only if $\Omega(1,i) = i$.

Proof

(i) Let $i \in T$. We know that $i \circledast i = 1$

we have, $\delta(1) = \Omega(1,1)$

$= \Omega(1, i \circledast i)$

$= (\Omega(1,i) \circledast i) \wedge (i \circledast \Omega(1,i))$

$= (\Omega(1,i) \circledast i)$

(ii) Let i in T.

$$\begin{aligned}\delta(i) &= \Omega(i,i)\\ &= \Omega(i, i \circledast 1)\\ &= (\Omega(i,i) \circledast 1) \wedge (i \circledast \Omega(i,1))\\ &= (\delta(i) \circledast 1) \wedge (i \circledast \Omega(i,1))\\ &= \delta(i) \wedge (i \circledast \Omega(i,1))\end{aligned}$$

If it is component wise regular, we get $\delta(i) \wedge i$.

(iii) Let i, j ∈ T such that $\delta(i) = \delta(j)$.

We have,$\delta(1) = \Omega(1,i) \circledast i$

and $\delta(1) = \Omega(1,j) \circledast j$.

This implies that $\Omega(1,i) \circledast i = \Omega(1,j) \circledast j$.

Since $\Omega(1,i) = \Omega(1,j)$ and by using cancellation law, we get i=j.

Hence, we get δ is 1-1.

(iv) Let δ be regular.

We have $\delta(1) = \Omega(1,i) \circledast i$

Since δ is regular, $\delta(1) = 1 \ implies \ \Omega(1,i) \circledast i = 1$.

By axiom (iii) of GK algebra we have $\Omega(1,i) = i$

Conversely,

Let $\Omega(1,i) = i$ for some i in T.

$\Rightarrow \ \Omega(1,i) \circledast i = i \circledast i$

$\Rightarrow \ \Omega(1,i) \circledast i = 1$

$\Rightarrow \ \delta(1) = 1$

$Hence \ \delta \ is \ regular.$

5.2 Summary

In this Chapter, the concept of symmetric bi derivation (GK-LR derivation and GK-RL derivation) of GK algebra is established and also, brought some important and at the same time interesting results about the trace of GK algebra and regular, component wise regular in GK algebra. Some theorems which is to be needed for further studies are derived.

CHAPTER 6

FUZZY SUB ALGEBRA AND ANTI-FUZZY SUB ALGEBRA OF GK ALGEBRA†

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CHAPTER 6

FUZZY SUB ALGEBRA AND ANTI-FUZZY SUB ALGEBRA OF GK ALGEBRA

This Chapter is separated into three sections.

In the first section, the newly defined algebraic structure GK-algebra is fuzzified and fuzzy GK subalgebra of GK algebra is defined, and by using this, some of its features are investigated and attained captivating results.

In the second section, fuzzy GK ideal is established and discussed roughly about its features and also, explored the Cartesian product of fuzzy GK algebra.

In the third section, initiated the concept of Anti-fuzzy GK sub algebra, Anti-fuzzy GK ideals. The properties of anti-fuzzy GK algebra is discussed and obtained some enthralling results.

6.1 Fuzzy sub algebra of GK algebra

This section explored the concept of fuzzified newly initiated algebraic structure of GK algebra.

Definition 6.1.1

A fuzzy subset ρ_{gk} of a GK algebra $(T, \circledast, 1)$ is called a fuzzy GK sub algebra of T, if the following conditions are satisfied

$$\rho_{gk}(i \circledast j) \geq min\{\rho_{gk}(i), \rho_{gk}(j)\} \text{ for all i,j in T.}$$

Example 6.1.2

Consider T= {1, 2, 3, 4} is a GK algebra

$\circledast$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Table 6.1

Define a mapping $\rho_{gk}: T \to [0,1]$ by

$$\rho_{gk}(i) = \begin{cases} 0.9 & if \ \ i = 1,2 \\ 0.5 \ \ if & otherwise \end{cases}$$

Then ρ_{gk} is a fuzzy GK sub algebra of T.

Theorem 6.1.3

Intersection of any two fuzzy GK sub algebras of T is again a fuzzy GK algebra.

Proof

Let ρ_{gk} and σ_{gk} be any two fuzzy GK sub algebras of T. Then,

$$(\rho_{gk} \sqcap \sigma_{gk})(i \circledast j) = \min\{\rho_{gk}(i \circledast j), \sigma_{gk}(i \circledast j)\}$$

$$\geq \min\{min\{\rho_{gk}(i), \rho_{gk}(j)\}, \min\{\sigma_{gk}(\mathrm{i}), \sigma_{gk}(\mathrm{j})\}\}$$

$$= \min\{\min\{\rho_{gk}(i), \sigma_{gk}(i)\}, \min\{\rho_{gk}(\mathrm{j}), \sigma_{gk}(\mathrm{j})\}\}$$

$$= \min\{(\rho_{gk} \sqcap \sigma_{gk})(\mathrm{i}), (\rho_{gk} \sqcap \sigma_{gk})(j)\}$$

$$(\rho_{gk} \sqcap \sigma_{gk})(i \circledast j) \geq \min\{(\rho_{gk} \sqcap \sigma_{gk})(\mathrm{i}), (\rho_{gk} \sqcap \sigma_{gk})(j)\} \ \forall \ \mathrm{i, j} \in \mathrm{T}.$$

Hence $\rho_{gk} \sqcap \sigma_{gk}$ is fuzzy sub algebra of T.

Definition 6.1.4

Let ρ_{gk} be any fuzzy subset of a GK algebra and let $y \in [0,1]$. The set $\Gamma(\rho_{gk}, y) = \{i \in T: \rho_{gk}(i) \geq y\}$ is called a level subset of ρ_{gk} in T.

Lemma 6.1.5

Let $(T, \circledast, 1)$ be a GK algebra. Let ρ_{gk} be a fuzzy GK sub algebra of T. Let $\tau \in [0,1]$. Then,

(i) if and only if $\Gamma(\rho_{gk}, \tau)$ is either $\emptyset$ or a GK sub algebra of T.

(ii) $\rho_{gk}(1) \geq \rho_{gk}(i) \quad for\ all \quad i \in T.$

Proof

(i) For any $\tau \in [0,1]$, assume that $\Gamma(\rho_{gk}, \tau)$ is non-empty.

Let $i, j \in \Gamma(\rho_{gk}, \tau)$. Then $\rho_{gk}(i) \geq \tau$ and $\rho_{gk}(j) \geq \tau$.

We need to prove $\Gamma(\rho_{gk}, \tau)$ is a GK sub algebra, for that we have to prove $i \circledast j \in \Gamma(\rho_{gk}, \tau)$.

i.e., we need to prove $\rho_{gk}(i \circledast j) \geq \tau$.

Now , $\rho_{gk}(i \circledast j) \geq min\{\rho_{gk}(i), \rho_{gk}(j)\}$

$\geq \min\{\tau, \tau\} = \tau$

$\therefore \rho_{gk}(i \circledast j) \geq \tau$

Hence $\Gamma(\rho_{gk}, \tau)$ is a GK sub algebra.

Conversely, assume that $\Gamma(\rho_{gk}, \tau)$ is a GK sub algebra of T.

Let $i, j \in T$. Take τ = min $\{\rho_{gk}(i),\ \rho_{gk}(j)\}$

Then by assumption $\Gamma(\rho_{gk}, \tau)$ is a GK sub algebra of T, $(i \circledast j) \in \Gamma(\rho_{gk}, \tau)$

$\rho_{gk}(i \circledast j) \geq \tau$ = min $\{\rho_{gk}(i), \rho_{gk}(j)\}$

Hence $\Gamma(\rho_{gk}, \tau)$ is a fuzzy GK Sub algebra of T.

(ii) To prove $\rho_{gk}(1) \geq \rho_{gk}(i)$

$\rho_{gk}(1) = \rho_{gk}(i \circledast i)$

$\geq \min\{\rho_{gk}(i), \rho_{gk}(i)\} = \rho_{gk}(i)$

$Hence\ \rho_{gk}(1) \geq \rho_{gk}(i) for\ all\ i \in T.$

$Hence\ the\ proof.$

Theorem 6.1.6

If ρ_{gk_1} and ρ_{gk_2} are fuzzy GK sub algebras of T, then $\rho_{gk_1} \times \rho_{gk_2}$ is a fuzzy GK algebra of $T \times T$.

Proof

For any (i_1, i_2) and $(j_i, j_2) \in T \times T$.

Now, $\rho_{gk}\big((i_1, i_2) \circledast (j_1, j_2)\big) = \rho_{gk}(i_1 \circledast j_1, i_2 \circledast j_2)$

$= (\rho_{gk_1} \times \rho_{gk_2})\ (i_1 \circledast j_1, i_2 \circledast j_2)$

$= \min\ \{\rho_{gk_1}(i_1 \circledast j_1), \rho_{gk_2}\ (i_2 \circledast j_2)\}$

$\geq min\ \{min(\rho_{gk_1}(i_1), \rho_{gk_1}(j_1)), min(\rho_{gk_2}(i_2), \rho_{gk_2}(j_2)\}$

$= min\ \{min(\rho_{gk_1}(i_1), \rho_{gk_2}(i_2)), min(\rho_{gk_1}(j_1), \rho_{gk_2}(j_2)\}$

$= \min\{\ ((\rho_{gk_1} \times \rho_{gk_2})\quad (i_1 \circledast i_2), (\rho_{gk_1} \times \rho_{gk_2})(j_1 \circledast j_2)\}$

$= \min\{\rho_{gk}\ (i_1 \circledast i_2), \rho_{gk}(j_1 \circledast j_2)\}$

Hence ρ_{gk} is a fuzzy GK sub algebra of $T \times T$.

Theorem 6.1.7

Let $\Gamma(\rho_{gk}, p) and\ \Gamma(\rho_{gk}, q)$be level sub algebras in fuzzy GK algebras are equal if and only if there is no $i \in T$ such that $p \leq \rho_{gk}(i) < q$.

Proof

Let $\Gamma(\rho_{gk}, p) = \Gamma(\rho_{gk}, q)$for $p < q$.

If there exist $i \in T$ such that $p \leq \rho_{gk}(i) < q$, then $\Gamma(\rho_{gk}, p) \subset \Gamma(\rho_{gk}, q)$, which tends to contradiction.

Conversely,

Let us assume, there is no $i \in T$ such that $p \leq \rho_{gk}(i) < q$, since

$p < q, \Gamma(\rho_{gk}, p) \subset \Gamma(\rho_{gk}, q)$

If i $\in \Gamma(\rho_{gk}, q)$ then $\rho_{gk}(i) \geq q$ and so $\rho_{gk}(i) \geq p$, because $\rho_{gk}(i) \notin (p, q)$

Hence i $\in \Gamma(\rho_{gk}, p)$, this implies $\Gamma(\rho_{gk}, q) \subseteq \Gamma(\rho_{gk}, p)$.

Hence the proof.

6.2 Fuzzy ideals of GK algebra

This section explained about the Fuzzy GK ideal with necessary illustration and talked over about its characteristics and attained some fascinating results of the same.

Definition 6.2.1

Let T be a GK algebra. A fuzzy set $\rho_{gk}\ in\ T$ is called fuzzy GK ideal of T if it satisfies the following conditions.

(i) $\rho_{gk}(1) \geq \rho_{gk}(i)$

(ii) $\rho_{gk}(i \circledast k) \geq \min\{\rho_{gk}(j \circledast k), \rho_{gk}(j \circledast i)\}\ \forall\, i, j, k \in T.$

Example 6.2.2

Consider the above Example (6.1.2). This is an example of fuzzy GK ideal.

Theorem 6.2.3

In GK-algebra, the intersection of family of sets on fuzzy GK-ideals is also a fuzzy GK-ideal.

Proof

Let $\{\rho_{gk_i}\}$ be a set of all fuzzy GK ideals of GK algebras T.

Then for any $, j, k \in T$,

$(\ \sqcap\rho_{gk_i})$ (1) = Inf $(\rho_{gk_i}$ (1))

$\geq$ Inf $(\rho_{gk_i}$ (i))

= $(\sqcap\rho_{gk_i})$ (i)

$(\sqcap\rho_{gk_i})\ (i \circledast k)$ = Inf $((\sqcap\rho_{gk_i}\ (i \circledast k))$

$\geq$ Inf {min $\{(\sqcap\rho_{gk_i}\ (j \circledast k), (\sqcap\rho_{gk_i}\ (j \circledast i)\}$

$= \min\ \{\text{Inf}\ (\sqcap\rho_{gk_i}\ (j \circledast k)),\ \text{Inf}\ (\sqcap\rho_{gk_i}\ (j \circledast i))\}$

$= \min\ \{(\sqcap\rho_{gk_i}\ (j \circledast k\ )\ ),\ (\sqcap\rho_{gk_i}\ (j \circledast i))\}$

Hence the proof.

Theorem 6.2.4

Every fuzzy GK ideal of a GK-algebra T is order overturn.

Proof

Let ρ_{gk} be a fuzzy GK ideal of a GK algebra T.

Let i, j $\in T$ be such that $i \leq j$ then $i \circledast j = j \circledast i = 1$.

Now, we know that $i \circledast 1 = i$.

$\rho_{gk}(i) = \rho_{gk}(i \circledast 1) \geq \min\{\rho_{gk}(j \circledast 1), \rho_{gk}(j \circledast i)\}$

$\geq \min\{\rho_{gk}(j), \rho_{gk}(1)\}$

$\geq \rho_{gk}(j)$

Therefore ρ_{gk} is order overturn.

Theorem 6.2.5

If ρ_{gk} is a fuzzy ideal of GK algebra $(T, \circledast 1) and \rho_{gk_\tau}(i) = \min\{\tau, \rho_{gk}(i)\} \forall\ i \in T$ and $\tau \in [0,1]$ then $\rho_{gk_\tau}(i)$ is fuzzy GK ideal of T.

Proof

Let ρ_{gk} be a fuzzy ideal of GK algebra and $\tau \in [0,1]$.

Therefore $\rho_{gk}(1) \geq \rho_{gk}(i)\ \forall\ i \in T$.

Now, $\rho_{gk_\tau}(1) = \min\{\tau, \rho_{gk}(1)\} \geq \min\{\tau, \rho_{gk}(i)\} = \rho_{gk_\tau}(i)\ \ \forall\, i \in T$.

and we know that

$\rho_{gk}(i \circledast k) \geq \min\{\rho_{gk}(j \circledast k), \rho_{gk}(j \circledast i)\}$

Now ,

$$\begin{aligned}\rho_{gk_\tau}(i \circledast k) &= \min\{\tau, \rho_{gk}(i \circledast k)\}\\ &\geq \min\{\tau, \min(\rho_{gk}(j \circledast k), \rho_{gk}(j \circledast i))\}\\ &= \min\{\min\left(\tau, \rho_{gk}(j \circledast k)\right), \min(\tau, \rho_{gk}(j \circledast i))\}\\ &= \min\{\rho_{gk_\tau}(j \circledast k), \rho_{gk_\tau}(j \circledast i)\}\end{aligned}$$

Hence $\rho_{gk_\tau}(i)$ is fuzzy GK ideal of T.

Proposition 6.2.6

Let ρ_{gk} be fuzzy GK ideal of GK algebra. If the inequality $j \circledast i \leq k$ holds in T, then $\rho_{gk}(i) \geq \min\{\rho_{gk}(j), \rho_{gk}(k)\} \forall\, i, j, k \in T$.

Proof

Assume that the inequality $j \circledast i \leq k$ holds in T,

Then by theorem ,$\rho_{gk}(j \circledast i) \geq \rho_{gk}(k)$------------- (1)

By the definition fuzzy GK ideal

$\rho_{gk}(i \circledast k) \geq \min\{\rho_{gk}(j \circledast k), \rho_{gk}(j \circledast i)\}$

Put k=1

Then $\rho_{gk}(i \circledast 1) \geq \min\{\rho_{gk}(j \circledast 1), \rho_{gk}(j \circledast i)\}$

$\rho_{gk}(i) \geq \min\{\rho_{gk}(j), \rho_{gk}(j \circledast i)\}$ ------------ (2)

From (1) and (2),

$\rho_{gk}(i) \geq \min\{\rho_{gk}(j), \rho_{gk}(k)\}$.

Definition 6.2.7

Let ρ_{gk} *and* σ_{gk} be fuzzy subsets of a set T. The Cartesian product of ρ_{gk} *and* σ_{gk} is defined by

$$(\rho_{gk} \times \sigma_{gk})(i,j) = \min\{\rho_{gk}(i), \sigma_{gk}(j)\} \, \forall \, i,j \in T$$

Theorem 6.2.8

Let ρ_{gk} *and* σ_{gk} be fuzzy GK ideals of GK algebra X. Then $\rho_{gk} \times \sigma_{gk}$ is a fuzzy GK ideal of $T \times T$.

Proof

Let us consider$(i,j) \in T \times T$

$$(\rho_{gk} \times \sigma_{gk})(1,1) = \min\{\rho_{gk}(1), \sigma_{gk}(1)\}$$

$$\geq \min\{\rho_{gk}(i), \sigma_{gk}(j)\} = (\rho_{gk} \times \sigma_{gk})(i,j)$$

Now let $(i_1, i_2), (j_1, j_2), (k_1, k_2) \in T \times T$

$$(\rho_{gk} \times \sigma_{gk})(i_1 \circledast k_1, i_2 \circledast k_2) = \min\{\rho_{gk}(i_1 \circledast k_1), \sigma_{gk}(i_2 \circledast k_2)\}$$

$$\geq \min\{\min\{\rho_{gk}(j_1 \circledast k_1), \rho_{gk}(j_1 \circledast i_1)\}, \min\{\sigma_{gk}(j_2 \circledast k_2), \sigma_{gk}(j_2 \circledast i_2)\}\}$$

$$= \min\{\min\{\rho_{gk}(j_1 \circledast k_1), \sigma_{gk}(j_2 \circledast k_2)\}, \min\{\rho_{gk}(j_1 \circledast i_1), \sigma_{gk}(j_2 \circledast i_2)\}\}$$

$$= \min\{(\rho_{gk} \times \sigma_{gk})(j_1 \circledast k_1, j_2 \circledast k_2), (\rho_{gk} \times \sigma_{gk})(j_1 \circledast i_1, j_2 \circledast i_2)\}$$

Therefore $\rho_{gk} \times \sigma_{gk}$ is a fuzzy GK ideal of $T \times T$.

Theorem 6.2.9

Let ρ_{gk} *and* σ_{gk} be fuzzy subsets of GK algebra T such that $\rho_{gk} \times \sigma_{gk}$ is a fuzzy GK ideal of $T \times T$. Then for all $i \in T$,

(i) either $\rho_{gk}(1) \geq \rho_{gk}(i)$ *or* $\sigma_{gk}(1) \geq \sigma_{gk}(i)$

(ii) $\rho_{gk}(1) \geq \rho_{gk}(i) \, \forall \, i \in T$ *then either* $\sigma_{gk}(1) \geq \rho_{gk}(i)$ *or* $\sigma_{gk}(1) \geq \sigma_{gk}(i)$.

(iii) $If\ \sigma_{gk}(1) \geq \sigma_{gk}(i)\ \forall\ i \in T, then\ either\ \ \rho_{gk}(1) \geq \rho_{gk}(i)\ or\ \rho_{gk}(1) \geq \sigma_{gk}(i)$.

(iv) $either\ \rho_{gk}\ or\ \sigma_{gk}$ is a fuzzy GK ideal of T.

Proof

(i) Suppose that $\rho_{gk}(i) > \rho_{gk}(1) and \sigma_{gk}(j) > \sigma_{gk}(1)$ for some $j \in T$.

Then $\left(\rho_{gk} \times \sigma_{gk}\right)(i,j) = \min\left\{\rho_{gk}(i), \sigma_{gk}(j)\right\}$

$$> \min\{\ \rho_{gk}(1), \sigma_{gk}(1)\} = (\rho_{gk} \times \sigma_{gk})(1,1)$$

This is a contradiction, since $\rho_{gk} \times \sigma_{gk}$ is a fuzzy GK ideal of $T \times T$.

Hence, we obtain (i).

(ii) Assume that $i, j \in T$

$\rho_{gk}(i) > \sigma_{gk}(1)\ \ and\ \sigma_{gk}(j) > \sigma_{gk}(1)$

Then we have, $\left(\rho_{gk} \times \sigma_{gk}\right)(1,1) = \min\{\rho_{gk}(1), \sigma_{gk}(1)\}$

$$> \min\left\{\sigma_{gk}(1), \sigma_{gk}(1)\right\} = \sigma_{gk}(1)$$

This implies that $\left(\rho_{gk} \times \sigma_{gk}\right)(i,j) = \min\{\rho_{gk}(i), \sigma_{gk}(j)\}$

$$> \min\ \left\{\sigma_{gk}(1), \sigma_{gk}(1)\right\} = \sigma_{gk}(1)$$

$$> \left(\rho_{gk} \times \sigma_{gk}\right)(1,1)$$

This is a contradiction.

Hence, we obtain (ii)

(iii) By the similar way to part (ii)

(iv) In (i) we have

Either $\rho_{gk}(1) \geq \rho_{gk}(i)\ \ or\ \sigma_{gk}(1) \geq \sigma_{gk}(i)\ \forall\ i \in T$.

We assume that $\sigma_{gk}(1) \geq \sigma_{gk}(i)$, without loss of generality,

It is from (iii) such that

Either $\rho_{gk}(1) \geq \rho_{gk}(i)\ or\ \ \rho_{gk}(1) \geq \sigma_{gk}(i)$

If $\rho_{gk}(1) \geq \sigma_{gk}(i)$ for any $i \in T$, then

$$\left(\rho_{gk} \times \sigma_{gk}\right)(1, i) = \min\left\{\rho_{gk}(1), \sigma_{gk}(i)\right\} = \sigma_{gk}(i) \text{------------ (1)}$$

Now we have to prove σ_{gk} is a fuzzy GK ideal.

For that, let us consider $(i_1, i_2), (j_1, j_2), (k_1, k_2) \in T \times T, we\ have$

Since $\rho_{gk} \times \sigma_{gk}$ is a fuzzy GK ideal of $T \times T$, we have

$$\left(\rho_{gk} \times \sigma_{gk}\right)(i_1 \circledast k_1, i_2 \circledast k_2)$$

$$\geq \min\{(\rho_{gk} \times \sigma_{gk})(j_1 \circledast k_1, j_2 \circledast k_2), (\rho_{gk} \times \sigma_{gk})(j_1 \circledast i_1, j_2 \circledast i_2)\}$$

Now, if we take $i_1 = j_1 = k_1 = 1$, then

$$\left(\rho_{gk} \times \sigma_{gk}\right)(1, i_2 \circledast k_2)$$

$$\geq \min\{(\rho_{gk} \times \sigma_{gk})(1, j_2 \circledast k_2), (\rho_{gk} \times \sigma_{gk})(1, j_2 \circledast i_2)\}$$

$Since$ by (1), LHS becomes,

$$\sigma_{gk}(i_2 \circledast k_2)$$

$$\geq \min\{(\rho_{gk} \times \sigma_{gk})(1, j_2 \circledast k_2), (\rho_{gk} \times \sigma_{gk})(1, j_2 \circledast i_2)\}$$

$$\geq \min\{\min\{\rho_{gk}(1), \sigma_{gk}(j_2 \circledast k_2)\}, \min\{\rho_{gk}(1), \sigma_{gk}(j_2 \circledast i_2)\}$$

$$\geq \min\left\{\sigma_{gk}(j_2 \circledast k_2), \sigma_{gk}(j_2 \circledast i_2)\right\}$$

$$\sigma_{gk}(i_2 \circledast k_2) \geq \min\{\sigma_{gk}(j_2 \circledast k_2), \sigma_{gk}(j_2 \circledast i_2)\}$$

This proves that σ_{gk} is a fuzzy GK ideal of T.

Now we consider $\rho_{gk}(1) \geq \rho_{gk}(i)$.

Suppose let us consider

$\rho_{gk}(1) < \rho_{gk}(j)$for some $j \in T$

Then $\sigma_{gk}(1) \geq \sigma_{gk}(j) > \rho_{gk}(1)$

Since$\rho_{gk}(1) \geq \rho_{gk}(i)\ \forall\ i \in T, then \sigma_{gk}(1) \geq \rho_{gk}(i)$

Hence $\left(\rho_{gk} \times \beta\right)(i, 1) = \min\left\{\rho_{gk}(i), \beta(1)\right\} = \rho_{gk}(i)$--------- (2)

Taking $i_2 = j_2 = k_2 = 1$ in (1)

$(\rho_{gk} \times \sigma_{gk})(i_1 \circledast k_1, 1)$

$\geq \min\{(\rho_{gk} \times \sigma_{gk})\,(j_1 \circledast k_1, 1), (\rho_{gk} \times \sigma_{gk})(j_1 \circledast i_1, 1)\}$

By (2)

$\rho_{gk}(i_1 \circledast k_1) \geq \min\{(\rho_{gk} \times \sigma_{gk})\,(j_1 \circledast k_1, 1), (\rho_{gk} \times \sigma_{gk})(j_1 \circledast i_1, 1)\}$

$\geq \min\{\min\{\rho_{gk}((j_1 \circledast k_1), \sigma_{gk}(\,1)\}, \min\{\rho_{gk}(j_1 \circledast i_1), \sigma_{gk}(1)\}$

$\geq \min\{\rho_{gk}((j_1 \circledast k_1), \rho_{gk}(j_1 \circledast i_1)\}$

$\rho_{gk}(i_1 \circledast k_1) \geq \min\{\rho_{gk}((j_1 \circledast k_1), \rho_{gk}(j_1 \circledast i_1)\}$

This proves that ρ_{gk} is a fuzzy GK ideal of GK algebra.

Therefore $either \rho_{gk} or \sigma_{gk}$ is a fuzzy GK ideal of GK algebra T.

6.3 Anti-fuzzy GK sub algebra and anti-fuzzy GK ideal

In this section, the theory of Anti-fuzzy GK sub algebra and anti-fuzzy GK ideal are established and analyzed its properties. The lower-level set of GK algebra is initiated and discussed some of its aspects in this section.

Definition 6.3.1

A fuzzy set ρ_{gk}in GK algebra T is said to be an anti-fuzzy sub algebra of T if $\rho_{gk}(i \circledast j) \leq$ max { ρ_{gk} (i), ρ_{gk} (j)}, for all i, j∈T.

Theorem 6.3.2

Let $\rho_{gk}$is an anti-fuzzy sub algebra of GK algebra. Prove that

$$\rho_{gk}(1) \leq \rho_{gk}(i)\ for\ any\ i\ in\ T.$$

Proof

We know that ,$i \circledast i = 1\ from$ the definition of GK algebra

Now, $\rho_{gk}(1) = \rho_{gk}(i \circledast i)$

$\leq \max\{\rho_{gk}\,(i), \rho_{gk}(i)\,\}$

$\leq \rho_{gk}(i).$

Therefore $\rho_{gk}(1) \leq \rho_{gk}(i)$.

Definition 6.3.3

Let ρ_{gk} be any fuzzy subset of a GK algebra and let $q \in [0,1]$. The set $\Gamma(\rho_{gk}, q) = \{i \in T: \rho_{gk}(i) \leq q\}$ is called a lower-level subset of ρ_{gk} in T.

Theorem 6.3.4

A fuzzy set ρ_{gk} in GK algebra is an anti-fuzzy sub algebra if and only if for every q in [0,1], $\Gamma(\rho_{gk}, q)$ is either $\emptyset$ or a sub algebra of T.

Proof

Let us assume ρ_{gk} is an anti -fuzzy sub algebra of T and also lower level subset is non-empty. Then for any $i, j \in \Gamma(\rho_{gk}, q)$

we have, $\rho_{gk}(i \circledast j) \leq$ max { ρ_{gk} (i), ρ_{gk} (j)} $\leq q$

Therefore $i \circledast j \in \Gamma(\rho_{gk}, q)$.

Hence $\Gamma(\rho_{gk}, q)$ is a sub algebra.

Conversely, Now consider $i, j \in T$

Take q = max { ρ_{gk} (i), ρ_{gk} (j)}.

Since $\Gamma(\rho_{gk}, q)$ is a sub algebra of T,

$\Rightarrow i \circledast j \in \Gamma(\rho_{gk}, q)$.

Therefore $\rho_{gk}(i \circledast j) \leq q$ =max { ρ_{gk} (i), ρ_{gk} (j)}

Hence ρ_{gk} is an anti-fuzzy sub algebra.

Definition 6.3.5

Let T be a GK algebra. A fuzzy set ρ_{gk} $in\ T$ is called anti-fuzzy GK ideal of T if it satisfies the following conditions.

(i) $\rho_{gk}(1) \leq \rho_{gk}(i)$

(ii) $\rho_{gk}(i \circledast k) \leq \max\{\rho_{gk}(j \circledast k), \rho_{gk}(j \circledast i)\}$ $\forall\, i, j, k \in T$.

Definition 6.3.6

Let (T,$\circledast_T, 1$) and (P,$\circledast_P, 1'$) be a GK algebra. Then the mapping $\sigma: T \to P$ of GK algebra is called anti- homomorphism if $\sigma(i \circledast_T j) = \sigma(j) \circledast_P \sigma(i)\ \forall\, i, j \in T$.

Definition 6.3.7

Let $\sigma: T \longrightarrow T$ be an endomorphism and ρ_{gk} be a fuzzy set in T. We define fuzzy set in T by $(\rho_{gk})_\sigma$ in T as $\left(\rho_{gk}\right)_\sigma(i) = \left(\rho_{gk}\right)\left(\sigma(i)\right)$ $for\ every\ i \in T$.

Theorem 6.3.8

Let ρ_{gk} be an anti-fuzzy GK ideal of GK algebra of T and if $\leq j$, then $\rho_{gk}(i) \leq \rho_{gk}(j)$, for all $i, j \in T$.

Proof

Let us consider $i \leq j$, then $i \circledast j = 1 = j \circledast i$,

and $\rho_{gk}(i \circledast 1) = \rho_{gk}(1) \leq max\,\{\rho_{gk}(j \circledast 1), \rho_{gk}(j \circledast i)\}$

$$= max\,\{\rho_{gk}(j), \rho_{gk}(1)\} = \rho_{gk}(j).$$

Hence $\rho_{gk}(x) \leq \rho_{gk}(y)$.

Theorem 6.3.9

Let ρ_{gk} be an anti-fuzzy GK-ideal of GK algebra T. If the inequality $j \circledast i \leq k$ carry in T, then $\rho_{gk}\,(i) \leq max\,\{\rho_{gk}\,(j), \rho_{gk}\,(k)\}$.

Proof

Let us consider the inequality $j \circledast i \leq k$ carry in T.

By theorem 6.3.8 $\rho_{gk}(j \circledast i) \leq \rho_{gk}(k)$ ------------- (1)

By definition of anti-fuzzy ideal of GK algebra

$\rho_{gk}(i \circledast k) \leq max\,\{\rho_{gk}(j \circledast k), \rho_{gk}(j \circledast i)\}$

Put $k = 1$,

then $\rho_{gk}(i \circledast 1) = \rho_{gk}(i) \leq \max\{\rho_{gk}(j \circledast 1), \rho_{gk}(j \circledast i)\}$

$$= max\{\rho_{gk}(j), \rho_{gk}(j \circledast i)\} \text{ ------------ (2)}$$

From (1) and (2), we get

$\rho_{gk}(i) \leq max\{\rho_{gk}(j), \rho_{gk}(k)\}$,for all $i, j, k \in T$

6.4 Summary

The fuzzification of GK algebra was introduced in this chapter. The properties of fuzzy GK algebra are analyzed and attained some results which are very interesting. The content of fuzzy GK ideal was explored and derived the results which are related to their aspects. Finally, anti-fuzzy GK algebra, anti-fuzzy GK ideal are introduced and attained the paramount results about its aspects.

CONCLUSION

CONCLUSION

In this research work, the content of newly constructed algebraic structure namely GK algebra has been introduced. It has been explored that this algebraic structure, GK algebra is different from all other algebraic structures that are already defined. Especially it is shown that GK algebra is totally different from BE algebra and CI algebra, with sufficient illustrations. It is proved that the GK algebra is satisfied the associative law, self-distributive law, Commutativity law and also investigated its properties. The GK ideal, kernel of GK algebra, anti-homomorphism are defined and attained remarkable results. The study of multipliers (left and right) on GK algebra have been explored. The direct product of GK algebras is initiated and investigated its nature. In this the notion of (GK-LR) (GK-RL) respectively) derivations of a GK algebra initiated and attained some remarkable results such as Let ξ_1, ξ_2 be self-maps on a GK algebra. Let ξ_1, ξ_2 be two (GK-RL) derivations on T. Then $\xi_1 \circ \xi_2$ is also a (GK-RL) derivation on T. The concept of symmetric bi derivation (GK-LR derivation and GK-RL derivation) of GK algebra has been developed. The fuzzification of GK algebra has been introduced. The properties of fuzzy GK algebra are analyzed and attained some results which are very interesting. Finally, anti-fuzzy GK algebra, anti-fuzzy GK ideal are introduced and attained the paramount results about its aspects.

FUTURE SCOPE

FUTURE SCOPE

The new algebraic structure GK algebra and its characteristics have been explored in this thesis and also fuzzy structure of GK algebra and its aspects are exhibited. Hope that this work would be a point of departure for further study of the theory of GK algebra.

This work can be ensued in the following way:

- In this study, the derivation of GK algebra and its properties have been discussed. In connection with, concentrate on the concept of derivations such as Jordan derivation in GK algebra, T- derivation in GK algebra, (α, β) derivations in GK algebra and anti-symmetric bi derivation in GK algebra etc., to get more results.
- The fuzzy structure of GK algebra can be developed in the concept of the intuitionistic fuzzy GK algebra, multi fuzzy sub algebra and multi fuzzy ideals of GK algebra and analyzed its related aspects.
- The theory of Pseudo GK algebra, Soft GK algebra and Neutrosophic structure in GK algebra can be studied.
- This work can be carried on the topological spaces in GK algebra and their properties can be expounded.

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The Structure of GK Algebras

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Abstract: In this paper, the new notion which is called GK algebra from a non-empty set is introduced. The basic properties of GK algebra are analyzed.
Keywords: GK algebra, commutative, associative, self-distributive, subalgebra.

I. INTRODUCTION

In 1966, the concept of BCK and BCI algebras are introduced by Iseki [3]. Since Kim and Yon [8] studied on dual BCK algebras and MV algebra, it is known that BCK algebras is a proper subclass of BCI algebras. The concept of BE algebra which is a generalization dual BCK was introduced by Kim and Y.H. Kim [7]. Meng [9] introduced the concept of CI algebra as a generalization of BE algebra and also discussed about some of its properties and relations with BE algebras.

II. PRELIMINARIES

A. *Definition:2.1 [7]* An algebra $(X,*,1)$ of type (2,0) is said to be a BE-algebra if it satisfies the following
1) $x*x=1$
2) $x*1=1$
3) $1*x=x$
4) $x*(y*z)=y*(x*z)$ for all $x,y,z \in X$

B. *Definition:2.2 [9]* A CI-algebra is an algebra $(X,*,1)$ of type (2,0) satisfying the following axioms
1) $x*x=1$
2) $1*x=x$
3) $x*(y*z) = y*(x*z)$ for all $x,y,z \in X$

C. *Proposition:2.3 [7]* If $(X,*,1)$ is a BE-algebra, then $x*(y*x) = 1$
D. *Definition:2.4 [7]* A BE-algebra $(X,*,1)$ is said to be self distributive if $x*(y*z)=(x*y)*(x*z)$ for all $x, y, z \in X$.
E. *Proposition:2.5 [9]* Any CI algebra X satisfies the condition $y*((y*x)*x)=1$ for any $x, y \in X$,

III. THE NOTION AND ELEMENTARY PROPERTIES OF GK ALGEBRA.

A. *Definition:3.1*

A non-empty set X with fixed constant 1 and a binary operation $*$ is called GK-algebra if it satisfying the following axioms
$x * x =1$
(ii) $x * 1 =x$
(iii) $x * y =1$ and $y * x =1$ implies $x = y$
(iv) $(y * z) *(x * z) = y* x$
(v) $(x * y) * (1 * y) = x$ for all $x, y, z \in X$

B. *Example:3.2*

Consider the set $X=\{1,2,3\}$. The binary operation $*$ is defined as follows

Table:1

*	1	2	3
1	1	3	2
2	2	1	3
3	3	2	1

$\therefore (X,*,1)$ is a GK-algebra.

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C. Remark:3.3

1) A GK algebra need not be a BE algebra, for $2*1=2\neq1, 3*1=3\neq1$.
2) A GK algebra need not be a CI algebra, for $1*3=2\neq3, 1*2=3\neq2$.
3) A GK algebra is said to be a CI algebra if it satisfies the additional relations, $1*x = x$ and $x*(y*z)=y*(x*z)$
4) A GK algebra is said to be a BE algebra if it satisfies the additional relations, $x*1 = 1$, $1*x = x$ and $x*(y*z) = y*(x*z)$.

D. Theorem:3.4 Let $(X,*,1)$ be a GK-algebra. Then

1) $1*(1*x) = x$
2) $(x*y)*1 = (x*1)*(y*1)$
3) $y*(1*(1*y)) = 1$
4) If $1*x = 1*y$ then $x = y$ for any $x,y\in X$
5) $x*(1*x))*x = x$ for any $x\in X$
6) $x*(x*y) = x = y = y*(x*x)$ for any $x,y\in X$
7) $x * (y*x) = x = y = y*(x*x)$ for any $x,y\in X$
8) $1*(x*y)=y*x$

Proof:

a) In axiom (v) $(x*y)*(1*y)=x$ of GK-algebra,
replacing y by x ,
we have $(x*x)*(1*x) = x$
$\Rightarrow 1*(1*x) = x$ by axiom (i) of definition:3.1

b) By axiom (ii) $x*1=1$ of GK algebra
we have $(x*y)*1 = x*y$
$=(x*1)*(y*1)$ by axiom(ii) of definition:3.1

c) In theorem 3.4 (i), we have $1*(1*x) = x$
Now $y*(1*(1*y)) = y*y = 1$ by axiom (i) of definition:3.1

d) Let $1*x = 1*y$
Now $x = 1*(1*x)$ by theorem 3.4 (i)
$=1*(1*y)$ since $1*x = 1*y$
$=y$ by theorem 3.4 (i)

e) $(x*(1*x))*x = (x*(1*x))*(1*(1*x))$ by theorem 3.4 (i)
$=x$ by axiom (v) of definition 3.1

f) $x*(x*y) = x*1$ by axiom (iii) of definition 3.1
$= x$ by axiom (ii) of definition 3.1
$= y$ by axiom (iii) of definition 3.1
$= y*1$ by axiom (i) of definition 3.1
$= y*(x*x)$ by axiom (i) of definition 3.1

g) The proof is similar to previous proof of (vi).

h) In axiom (iv) $(y*z)*(x*z)=y*x$ of GK algebra,
replacing z by y , we have $(y*y)*(x*y)=y*x$
$\Rightarrow 1*(x*y)=y*x$ by axiom (i) of definition 3.1 .

E. Theorem:3.5

Left and Right cancellation law holds in GK-algebra

1) Right cancellation law : if $x*y = z*y$ then $x = z$
2) Left cancellation law : if $z*x = z*y$ then $x = y$

Proof:

a) Let us assume that $x*y = z*y$

Then, $x = (x*y)*(1*y)$ by axiom (v) of definition 3.1

$= (z*y)*(1*y)$

$= z$ by axiom (v) of definit ion 3.1

b) Assume that $z*x = z*y$

Now, $z*(z*x) = x*(z*z)$ by theorem 3.4 (vi)

$= x*1$ by axiom (i) of definition 3.1

$= x$ by axiom (ii) of definition 3.1

and, $z*(z*y) = y*(z*z)$ by theorem 3.4 (vi)

$= y*1$ by axiom (i) of definition 3.1

$= y$ by axiom (ii) of definition 3.1

Since $z*x = z*y$ implies $x = y$.

F. Theorem:3.6

Let $(X,*,1)$ be a group with respect to $x*y = xy^{-1}$, then $(X,*,1)$ is a GK algebra.

Proof:

We see that $x*x = xx^{-1}=1$

and $x*1 = x1^{-1} = x$

For any $x,y \in X$, we have $x*y = xy^{-1}$

when $x = y$, then $x*y= xy^{-1} =xx^{-1} =1 =yy^{-1} =yx^{-1} = y*x$.

For any $x,y,z \in X$,we have $(y*z)*(x*z) = (yz^{-1}) (xz^{-1})^{-1}$

$=(yz^{-1}) (zx^{-1})$

$=y(zz^{-1})x^{-1}$

$=yx^{-1}$

$=y*x$

For any $x,y \in X$, $(x*y)*(1*y) = (xy^{-1}) (1y^{-1})^{-1}$

$=(xy^{-1}) (y)$

$=x (y^{-1}y)$

$=x$

Hence $(X,*,1)$ is a GK-algebra.

G. Definition:3.7

A GK algebra X is said to be associative if it satisfies $(x*y)*z = x*(y*z)$ for all $x,y,z \in X$. **Theorem:3.8** Every Gk algebra $(X,*,1)$ satisfying the associative law is group under the operation "$*$".

Proof:

Putting $x = y = z$ in the associative law $(x*y)*z = x*(y*z)$

we have $(x*x)*x = x*(x*x)$

$\Rightarrow 1*x = x*1$ by axiom (i) of definition 3.1

$=x$ by axiom (ii) of definition 3.1

$\Rightarrow 1*x=x*1=x$

This means that 1 acts as the identity element in X. By axiom (i) of definition 3.1, every element x of X has its own inverse.

Now, $(y*z)*(x*z) = y*(z*(x*z))$

$= y*(x*(z*z))$

$= y*x$

and $(x*y)*(1*y) = x *(y*(1*y))$

$= x*((y*1)*y)$

$= x*(y*y)$

$=x*1$

$= x.$

Therefore $(X,*,1)$ is a group.

H. Definition:3.9

A GK-algebra $(X,*,1)$ is a self-distributive if the operation $*$ is

1) Right distributive law $(x*y)*z = (x*z)*(y*z)$ for all $x,y,z \in X$.
2) Left distributive law $x*(y*z) = (x*y)*(x*z)$ for all $x,y,z \in X$.

I. Definition:3.10

A GK algebra X is said to be commutative if it satisfies for all $x,y \in X$, $(x*y)*y = (y*x)*x$.

J. Proposition:3.11

Let X be a GK algebra. If $x \neq y$ and $x * y = 1$ then $y*x \neq 1$.

K. Proposition:3.12

Let $(X,*,1)$ be a GK algebra. Then for any $x,y,z \in X$,

1) $x*(x*(y*x)) = 1$
2) $y*(y*(x*y)) = 1$
3) $(x*y)*x = (y*x)*y$
4) $(x*y)*y = (y*x)*x$
5) $(x*y)*x = (x*x)*y$
6) $(x*y)*y = (y*y)*x$

*Proof***:**

a) Let us consider $x*(x*(y*x))$
$= x*(x*1)$ by axiom (iii) of definition 3.1
$= x*x$ by axiom (ii) of definition 3.1
$= 1$ by axiom (i) of definition 3.1.

b) The proof is similar to proof (i) in proposition 3.12 .

c) Consider $(x*y)*x$
$=1*x$
$=1*y$ by axiom (iii) of definition 3.1
$=(x*y)*y$ by axiom (iii) of definition 3.1
$=(y*x)*y$ by axiom (iii) of definition 3.1

d) Consider $(x*y)*y$
$=1*y$
$=1*x$ by axiom (iii) of definition 3.1
$=(x*y)*x$ by axiom (iii) of definition 3.1
$=(y*x)*x$ by axiom (iii) of definition 3.1

This proof shows that the commutativity of GK algebra.

(v) Consider $(x*y)*x = 1*x = 1*y = (x*x)*y$ by axiom(i) & (iii) of definition 3.1

(vi) Proof is similar to (v) in proposition 3.12.

L. Theorem:3.13

In GK algebra X, for any $x,y,z \in X$ if associativity holds then the following are equivalent

1) $x*(y*z) = (x*z)*y$
2) $(y*z)*(x*z) = y*x$

Proof

(i)⇒(ii) Assume $x*(y*z) = (x*z)*y$

Then $(y*z)*(x*z) = ((y*z)*z)*x$
$=(y*(z*z))*x$
$=(y*1)*x$ by axiom (i) of definition 3.1

$=y*x$ by axiom (ii) of definition 3.1

(ii)⇒(i) Assume $(y*z)*(x*z) = y*x$

Then $x*(y*z) = (x*z)*((y*z)*z)$

$= (x*z)*(y*(z*z))$

$= (x*z)*(y*1)$ by axiom (i) of definition 3.1

$= (x*z)*y.$ by axiom (ii) of definition 3.1

M. Definition:3.14

Let $(X,*,1)$ be a GK-algebra. A non-empty subset A of X is called a subalgebra of X if

$x*y \in A$ for any $x,y \in A$.

N. Theorem:3.15

Let $(X,*,1)$ be a GK algebra and $A \neq \varphi$, $A \subseteq X$ then the following are equivalent

1) A is a subalgebra of X

2) $x*(1*y)$, $1*y \in A$ for any $x,y \in A$

Proof:

(i)⇒(ii) Let A be a subalgebra of X. Since A is a subset of X which is non-empty there exists an element $x \in A$ such that $x*x = 1 \in A$.

Since X is closed under '$*$', $y \in A$, $1*y \in A \Rightarrow x*(1*y) \in A$.

(ii)⇒(i) Since $x*y = x*(1*(1*y))$ by theorem 3.4 (i)

which implies $x*y \in A$ for any $x,y \in A$.

∴ A is a subalgebra of X.

IV. CONCLUSION

In this paper the notion of GK algebra is introduced and studied about some of their properties. It may lead our future study of GK algebra such as homomorphism of GK algebra, filter of GK algebra and Ideal theory on GK algebra.

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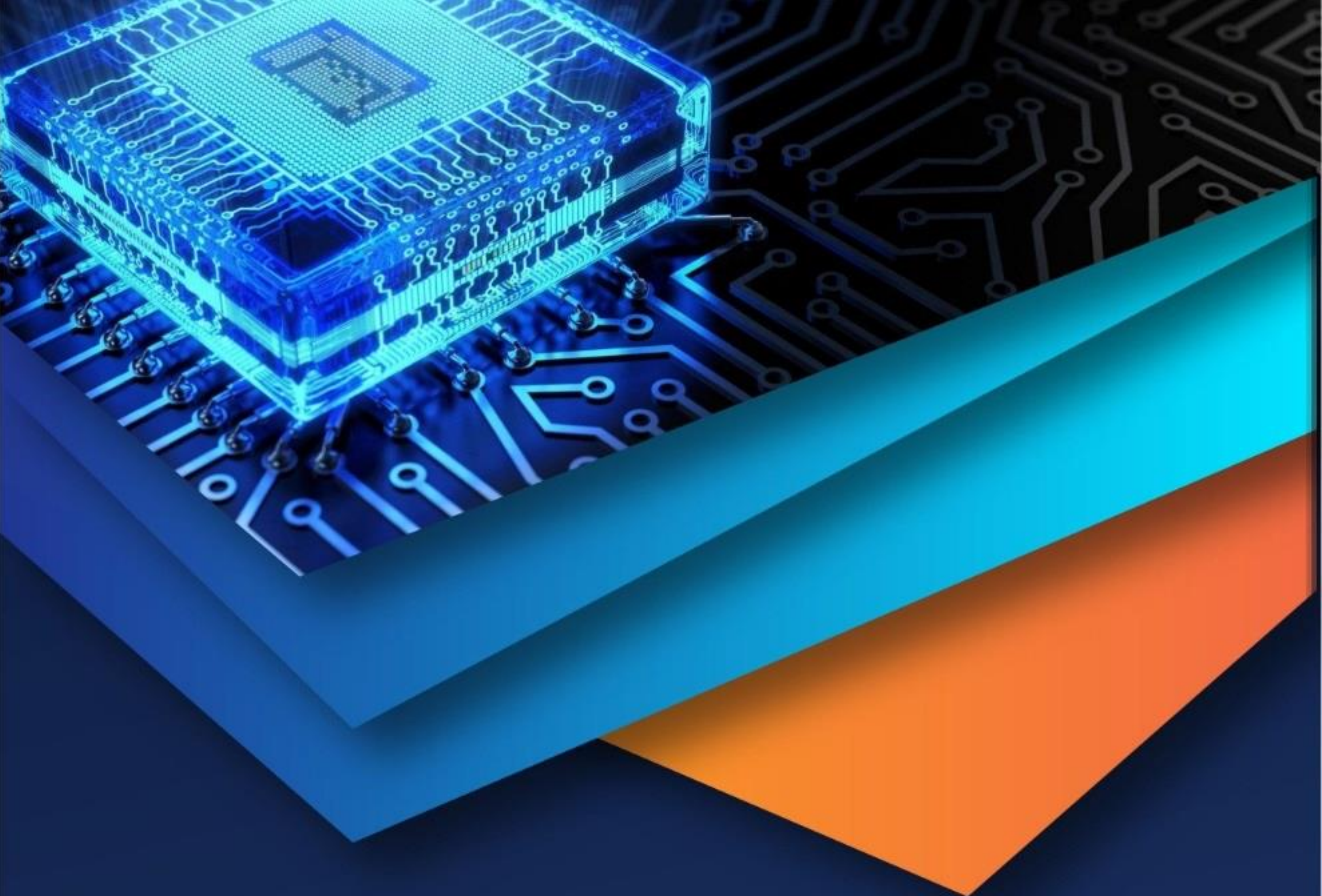

Derivations on GK algebra

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Abstract: **In this paper, we introduce the concept of derivation of GK algebra and also obtain some properties about this concept.**

I. INTRODUCTION

The structure of GK algebra was introduced by us in 2018[4].A GK algebra is an algebra of non-empty set X together with a binary operation * and a constant 1,satisfying the following axioms

(i) $x*x=1$
(ii) $x*1=x$
(iii) $x*y=1$ and $y*x=1$ implies $x=y$
(iv) $(y*z)*(x*z)=y*x$
(v) $(x*y)*(1*y)=x$ for all x,y,z in X.

Let the commutative ring R with identity.Let X be an algebra over R. An R linear mapping $f:X\to X$ is called a derivation if $f(xy)=f(x)y+xf(y)$ for all x,y in X.The derivation of ring is wide area, many researchers started their work in this concept[1],[2],[3]. Inspired by these works, in a same way, we introduce the concept of derivation on GK algebra and discuss some properties in this paper.

II.PRELIMINARIES

A. *Definition:*2.1[4]

A non-empty set X with fixed constant 1and a binary operation *is called GK algebra if it satisfying the following axioms

(i) $x*x=1$
(ii) $x*1=x$
(iii) $x*y=1$ and $y*x=1$ implies $x=y$
(iv) $(y*z)*(x*z)=y*x$
(v) $(x*y)*(1*y)=x$ for all x,y,z in X.

B. *Definition:*2.2[4]

GK algebra X is said to be commutative if it satisfies for all x,y in X, $(x*y)*y=(y*x)*x$.

C. *Definition*:2.3[3]

Let X be a d-algebra. A map $\theta:X\to X$ is a left-right derivation (l,r)- derivation of X it satisfies the identity $\theta(x*y)=(\theta(x)*y)\wedge(x*\theta(y))$ for all x, y in X. If θ satisfies the identity $\theta(x*y)=(x*\theta(y))\wedge(\theta(x)*y)$ for all x,y in X then θ is a right-left derivation (r,l) derivation of X.

D. *Definition:*2.4[2]

Let $(X,*,0)$ be a TM algebra.A self map $d:X\to X$ is said to be a (l,r) derivation of X if $d(x*y)=(d(x)*y)\wedge(x*d(y))$.A self map $d:X\to X$ is said to be (r,l) derivation on X if $d(x*y)=(x*d(y))\wedge(d(x)*y)$.

III.DERIVATIONS OF GK ALGEBRA.

A. *Definition:*3.1

Let (X,∗,1) be a GK algebra. A map $\eta: X \to X$ is called a left-right derivation (briefly (LR) derivation) of X if $\eta(x * y) = (\eta(x) * y) \wedge (x * \eta(y)) \forall x, y \in X$.

B. *Definition:*3.2

Let $(X,*,1)$ be a GK-algebra. A map $\eta: X \to X$ is called a right-left derivation (briefly (RL) derivation) of X if $\eta(x * y) = (x * \eta(y)) \wedge (\eta(x) * y) \forall x, y \in X$.

C. *Remark:*3.3

A map $\eta: X \to X$ is called a derivation of X if η is both a (LR) derivation and a (RL) derivation of X.

D. *Note*:3.4

Let $(X,*,1)$ be a GK-algebra ,x,y∈X. We denote $x \wedge y = y * (y * x)$.

E. *Example:*3.5

Let $X = \{1,2,3\}$ be a GK-algebra. The operation ∗ is defined as follows

∗	1	2	3
1	1	3	2
2	2	1	3
3	3	2	1

Define a map $\eta: X \to X$ by

$$\eta(x) = \begin{cases} 1 & if \ \ x = 1 \\ 2 & if \ x = 2 \\ 3 & if \ x = 3 \end{cases}$$

Then it is clear that η is a derivation of x.

F. *Definition:*3.6

Let $(X,*,1)$ be a GK-algebra and $\eta: X \to X$ be a map of a GK-algebra, then η is called regular if η (1)=1.

G. ***Note*:3.7**

In GK-algebra[4], we can observe that $x \wedge y = y * (y * x) = x \ \forall x, y \in X$.

H. *Proposition:*3.8

Let η be a self-map of GK algebra X,then

(a) If η is regular (LR) derivation of X, then $\eta(x) = \eta(x) \wedge x \ \ \forall \ x \in X$

(b) If η is regular (RL) derivation of X, then $\eta(x) = x \wedge \eta(x) \ \ \forall \ x \in X$

***Proof*:**

(a) Let η be a regular (LR) derivation of X. Then $\eta(x) = \eta(x * 1)$

$$= (\eta(x) * 1) \wedge \big(x * \eta(1)\big)$$

$$= \eta(x) \wedge (x * \eta(1))$$

$$= \eta(x) \wedge (x * 1)$$

$$= \eta(x) \wedge x$$

(b) Let η be a regular (RL) derivation of X, then

$$\eta(x) = \eta(x * 1)$$

$$= (x * \eta(1)) \wedge (\eta(x) * 1)$$

$$= (x * 1) \wedge (\eta(x) * 1)$$

$= x \wedge \eta(x)$

Conversely ,

Let η be a (RL) derivation of X and $\eta(x) = x \wedge \eta(x) \forall x \in X$,then we get

$\eta(1) = 1 \wedge \eta(1)$

$= \eta(1) * (\eta(1) * 1)$ $\qquad \because x \wedge y = y * (y * x)$

$= \eta(1) * \eta(1)$

$= 1.$

Hence η is regular.

I. *Lemma*:3.9

Let $(X,*,1)$ *be* a GK-algebra and η be a (LR) derivation of X. Then the following hold $\forall\, x, y \in X$

(a) $\eta(x * y) = \eta(x) * y$

(b) If η is regular then $\eta(x) \leq x$

***Proof*:**

(a) Let $(X,*,1)$ be a GK algebra and η be a (LR) derivation of X.

Then

$$\eta(x * y) = (\eta(x) * y) \wedge \big(x * \eta(y)\big)$$
$$= \big(x * \eta(y)\big) * \Big(\big(x * \eta(y)\big) * (\eta(x) * y)\Big)$$
$$= \eta(x) * y$$
$$\therefore \eta(x * y) = \eta(x) * y.$$

(b) Let η be a regular derivation of X.

Then η (1)=1.

Now

$\eta(x * x) = \eta(1)$

$\eta(x) * x = 1$

$\eta(x) \leq x.$

J. *Lemma:*3.10

Let $(X,*,1)$ be a GK algebra and η be a (RL) derivation of X. Then

(a) $\eta(x * y) = x * \eta(y)$

(b) If η is regular then $x \leq \eta(x)$

***Proof*:**

(a) Let $(X,*,1)$ be a GK algebra and η be a (RL) derivation of X.

Then

$$\eta(x * y) = (x * \eta(y)\,) \wedge (\eta(x) * y)$$
$$= (\eta(x) * y) * \big((\eta(x) * y) * (x * \eta(y))\big)$$
$$= x * \eta(y)$$
$$\therefore \eta(x * y) = x * \eta(y).$$

(b) Let η be a regular derivation of X.

Then η (1)=1.

Now

$\eta(x * x) = \eta(1)$

$x * \eta(x) = 1$

$x \leq \eta(x).$

K. *Note*:3.11

(a) From the above lemma:3.9

$\eta(x * y) = \eta(x) * y$

And $\eta(x * y) = x * \eta(y)$

$\Rightarrow \eta(x * y) = \eta(x) * y = x * \eta(y)$

(b) Let η be the regular derivation then by lemma

$\eta(x) \leq x$ and $x \leq \eta(x)$
$\Rightarrow \; x = \eta(x)$.

L. ***Remark*:3.12** A map $\eta: X \rightarrow X$ is regular derivation of X then $\eta(x) = x \;\forall\; x \in X$

M. *Lemma*:3.13

Let $\eta: X \rightarrow X$ be a derivation of X. Then η is a regular derivation if η is either a (LR) derivation or a (RL) derivation.

***Proof*:**

Let η is (LR) derivation, then for all $x \in X$, $\eta(x) * x = 1$

Now $\eta(1) = \eta(x * x)$
$= \eta(x) * x$
$\therefore \eta(1) = 1.$
$\therefore$ η is regular.

Now if η is (RL) derivation, then for all $x \in X$, $x * \eta(x) = 1$

Now $\eta(1) = \eta(x * x)$
$= x * \eta(x)$
$\therefore \eta(1) = 1.$
$\therefore$ η is regular.

N. *Theorem:*3.14

Let $(X, *, 1)$ be a GK algebra and η be a regular (RL) derivation of X. Then the following hold, $\forall\, x, y \in X$.

(a) $\eta(x) = x$
(b) $\eta(x) * y = x * \eta(y)$
(c) $\eta(x * y) = \eta(x) * y = x * \eta(y) = \eta(x) * \eta(y)$

Proof:-

(a) Since η is regular (RL) derivation of X, we have

$\eta(x) = \eta(x * 1)$
$= x * \eta(1)$
$= x * 1$
$= x$
$\therefore \eta(x) = x.$

(b) Since η is regular (RL) derivation of X, then we have

$\eta(x * y) = x * \eta(y)$

$x * y = x * \eta(y) \quad \rightarrow (1)$

And in (LR) derivation

$\eta(x * y) = \eta(x) * y$

$x * y = \eta(x) * y \quad \rightarrow (2)$

From (1) & (2)

$\eta(x) * y = x * y = x * \eta(y).$

(c) Since $\eta(x) = x \;\forall\; x \in X$

$$\eta(x*y)=\eta(x)*y=\eta(x)*\eta(y)$$

$$\eta(x*y)=x*\eta(y)=\eta(x)*\eta(y)$$

$$\Rightarrow \eta(x*y)=\eta(x)*y=x*\eta(y)=\eta(x)*\eta(y).$$

O. *Lemma*:3.15

Let $(X,*,1)$ be a GK algebra and η be a derivation on X. If $x \le y$ $\forall$ $x,y \in X$ then η (x)= η (y).

***Proof*:**

In GK algebra , $x*y=y*x=1 \Leftrightarrow x \le y$.

Then $\eta\,(y)=\eta\,(y*1)$

$=\eta\,\big(y*(y*x)\big)$

$=\eta\,(x)$.

P. *Proposition:*3.16

Let η be a derivation on GK algebra and let $x \in X$, then $x*\big(x*\eta\ (x)\big)=\eta\ (x)*(\eta\ (x)*x)$.

***Proof*:**

We know that $\eta\,(x)=\eta\,(x)\wedge x$

$x*\eta\,(x)$ $=x*(\eta\,(x)\wedge x)$

$=x*(x*\eta\ (x)))$ $\qquad \because x\wedge y=y*(y*x)$

and

$x*\eta\ (x)$ $=x*(x\wedge\eta\ (x))$

$=x*(\eta\ (x)*(\eta\ (x)*x))$ $\qquad \because x\wedge y=y*(y*x)$

$\Rightarrow x*\Big(x*\big(x*\eta\ (x)\big)\Big)=x*\big(\eta\ (x)*(\eta\ (x)*x)\big)$

By cancellation law,

$x*\big(x*\eta\ (x)\big)=\eta\ (x)*(\eta\ (x)*x)$.

Q. *Lemma:*3.16

If η is a regular (RL) derivation on GK algebra, then $\eta\ \big(x*\eta\ (x)\big)=1$.

***Proof*:**

Since η is a regular (RL) derivation on GK algebra, $x*\eta\ (x)$=1.

$\therefore \eta\ \big(x*d(x)\big)=\eta\ (1)=1$

$\therefore \eta\ \big(x*\eta\ (x)\big)=1$.

R. Lemma:3.17

If η is a regular (LR) derivation on GK algebra, then $\eta\,(\eta\,(x)*x)=1$.

Proof:

Since η is a regular (LR) derivation on GK algebra, $\eta\,(x)*x$=1.

$\therefore \eta\,(\eta\,(x)*x)=\eta\,(1)=1$

$\therefore \eta\,(\eta\,(x)*x)=1$.

S. *Definition:*3.18

Let η_1,η_2 be a self maps on a GK algebra X . We define $\eta_1{}^\circ\eta_2$ as follows

$(\eta_1{}^\circ\eta_2)(x)=\eta_2(\eta_1(x))$

T. *Lemma*:3.19

Let η_1,η_2 be self maps on a GK algebra. Let η_1,η_2 be two (LR) derivations on X. Then $\eta_1{}^\circ\eta_2$ is also a (LR) derivation on X.

Proof:

Given η_1, η_2 is two (LR) derivations on X.

By lemma (3.1),

We know that

$\eta_1(x * y) = \eta_1(x) * y$

And

$\eta_2(x * y) = \eta_2(x) * y$

Now

$(\eta_1 \circ \eta_2)(x * y) = \eta_2(\eta_1(x * y))$

$= \eta_2(\eta_1(x) * y)$

$= \eta_2(\eta_1(x)) * y$

$= (\eta_1 \circ \eta_2)(x) * y$

Hence $\eta_1 \circ \eta_2$ is a (LR) derivation on GK algebra.

U. *Lemma*:3.20

Let η_1, η_2 be self maps on a GK algebra. Let η_1, η_2 be two (RL) derivations on X. Then $\eta_1 \circ \eta_2$ is also a (RL) derivation on X.

Proof:

Given η_1, η_2 is two (RL) derivations on X.

Now

$(\eta_1 \circ \eta_2)(x * y) = \eta_2(\eta_1(x * y))$

$= \eta_2(x * \eta_1(y))$

$= x * \eta_2(\eta_1(y))$

$= x * (\eta_1 \circ \eta_2)(y).$

Hence $\eta_1 \circ \eta_2$ is a (RL) derivation on GK algebra.

By the above two lemmas 3.7 and 3.8 , we get the following theorem.

V. *Theorem*:3.21

Let $(X,*,1)$ be a GK algebra and η_1, η_2 be two derivations on X, then $\eta_1 \circ \eta_2 = \eta_2 \circ \eta_1$.

Proof:

Since η_1, η_2 be two derivations on X, η_1, η_2 are both (LR) and (RL) derivations on X.

Now

$(\eta_1 \circ \eta_2)(x * y) = \eta_2(\eta_1(x * y))$

$= \eta_2(\eta_1(x) * y)$

$= \eta_1(x) * \eta_2(y).$ ____________(1)

Also

$(\eta_2 \circ \eta_1)(x * y) = \eta_1(\eta_2(x * y))$

$= \eta_1(x * \eta_2(y))$

$= \eta_1(x) * \eta_2(y).$____________(2)

From (1) & (2)

$(\eta_1 \circ \eta_2)(x * y) = (\eta_2 \circ \eta_1)(x * y)$

This gives that $(\eta_1 \circ \eta_2) = (\eta_2 \circ \eta_1)$.

W. *Definition*:3.22

Let η_1, η_2 be a self maps on a GK algebra X . We define $\eta_1 * \eta_2 : X \to X$ as follows

$(\eta_1 * \eta_2)(x) = \eta_2(x) * \eta_1(x) \forall x \in X.$

X. *Theorem*:3.23

Let $(X,*,1)$ be a GK algebra and η_1, η_2 be two derivations of X, then $\eta_1 * \eta_2 = \eta_2 * \eta_1$.

Proof:

$(\eta_1 \circ \eta_2)(x * y) = \eta_2(\eta_1(x * y))$

$= \eta_2(\eta_1(x) * y)$

$= \eta_1(x) * \eta_2(y).$ ____________(1)

$(\eta_1{}^\circ\eta_2)(x * y) = \eta_2(\eta_1(x * y))$

$= \eta_2(x * \eta_1(y))$

$= \eta_2(x) * \eta_1(y).$ ____________(2)

From the above

$\eta_1(x) * \eta_2(y) = \eta_2(x) * \eta_1(y).$______________(3)

Substituting y = x in (3)

$\eta_1(x) * \eta_2(x) = \eta_2(x) * \eta_1(x)$

By definition

$(\eta_2 * \eta_1)(x) = (\eta_1 * \eta_2)(x)$

This gives $(\eta_1 * \eta_2) = (\eta_2 * \eta_1)$.

IV. CONCLUSION

In this paper, the concept of derivation on GK algebra discussed and also studied about some related interesting properties of derivation on GK algebra. In future we plan to study about fuzziness in the GK algebra.

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A Note on Multipliers in GK Algebra

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***Abstract*: In this paper, we introduce the concept of multipliers in GK algebra and also we discuss about the properties of the regular multiplier of GK algebra. We also introduce the kernel of multipliers of GK algebra.**

I.INTRODUCTION

In 1971, R.Larsen [3] introduced the theory of multipliers. In continuation of this, in 1980 W.H.Cornish[5] introduced the concept of multipliers in implicative BCK algebras. After that many researchers have applied this concept in their algebraic structure and brought some interesting properties of multipliers. Motivated by their work ,in this paper we discuss about multiplier in GK algebra[2] and Kernel of multiplier in GK algebra and also discuss some properties of regular multiplier of GK algebra.

II.MULTIPLIERS IN GK ALGEBRA

2.1 Definition

Let $(\mathcal{X},*,1)$ *be an GK algebra. A self map* Δ *is called a right multipliers of* $\mathcal{X}$ *if* $\Delta(m*n) = m*\Delta(n)$ *for all* $m,n \in \mathcal{X}$.

2.2 Example

Consider $\mathcal{X} = \{1,2,3\}$ in which '$*$' is defined by

$*$	1	2	3
1	1	3	2
2	2	1	3
3	3	2	1

Then $\mathcal{X}$ is an GK algebra.

Define a mapping $\Delta: \mathcal{X} \to \mathcal{X}$ by

$$\Delta(\mathrm{m}) = \begin{cases} 1 & if \quad x = 1 \\ 2 & if \; x = 2 \\ 3 & if \; x = 3 \end{cases}$$

It is clearly known that Δ is a right multiplier of GK algebra.

2.3 Definition

Let $(\mathcal{X},*,1)$ be an GK algebra. A self map Δ is called a left multipliers of $\mathcal{X}$ if $\Delta(m*n) = \Delta(m)*n \;\; for\, all \;\; m,n \in \; \mathcal{X}$.

Note: The above said example is also an example of the left multiplier of GK algebra.

2.4 Definition:

A map Δ of an GK algebra $\mathcal{X}$ is said to be regular if $\Delta(1) = 1$.

2.5 Proposition

Let Δ be a left multiplier of $\mathcal{X}$, then

(i) For every m in $\mathcal{X}$, $\Delta(1) = \Delta(m)*m$.
(ii) Δ is 1-1.

Proof:

(i) Let $\mathrm{m} \in \mathcal{X}$. Then $m*m = 1$.
We have $\Delta(1) = \Delta(\mathrm{m}*\mathrm{m}) = \Delta(m)*m \; for\, all \; m \in \mathcal{X}$.
(ii) Let $m,n \in \mathcal{X}$ such that $\Delta(m) = \Delta(n)$.
Then by (i), we have $\Delta(1) = \Delta(\mathrm{m}*\mathrm{m}) = \Delta(m)*m$ and $\Delta(1) = \Delta(\mathrm{n}*\mathrm{n}) = \Delta(n)*n$.
Then $\Delta(m)*m = \Delta(n)*n$.
By cancellation law, $m = n$.
$\therefore \; \Delta \; is\; 1-1$

2.6 Proposition

Let Δ be a right multiplier of $\mathcal{X}$, then

(i) For every m in $\mathcal{X}$, $\Delta(1) = m*\Delta(m)$.
(ii) Δ is 1-1.

Proof:

(i) Let $\mathrm{m} \in \mathcal{X}$. Then $m*m = 1$.

We have $\Delta(1) = \Delta(\mathrm{m} * \mathrm{m}) = m * \Delta(m)$ $for\ all\ m \in \mathcal{X}$.

(ii) Let $m, n \in \mathcal{X}$ such that $\Delta(m) = \Delta(n)$.

Then by (i), we have $\Delta(1) = \Delta(\mathrm{m} * \mathrm{m}) = m * \Delta(m)$ and

$\Delta(1) = \Delta(\mathrm{n} * \mathrm{n}) = n * \Delta(n)$. Then $m * \Delta(m) = n * \Delta(n)$.

By cancellation law, $m = n$.

$\therefore\ \Delta\ is\ 1 - 1$.

2.7 Theorem:

Let Δ be a left multiplier of $\mathcal{X}$. Then $\Delta(m) = m$ iff Δ is regular.

Proof:

Let Δ is regular. Since $\Delta(1) = 1$.

Then we have $\Delta(1) = \Delta(\mathrm{m} * \mathrm{m}) = \Delta(m) * m = 1$.

By definition of GK algebra, $\Delta(m) = m$.

Conversely, let $\Delta(m) = m\ for\ m\ in\ \mathcal{X}$.

It is clear that $\Delta(1) = 1$.

Hence Δ is regular.

2.8 Proposition

Let $\mathcal{X}$ be GK algebra and let Δ be a left multiplier of $\mathcal{X}$.

If $\Delta(m) * m = 1\ for\ every\ \mathcal{X}$, then Δ is regular.

Proof:

Let $\Delta(m) * m = 1$ and let Δ be a left multiplier of $\mathcal{X}$.

By definition of GK algebra,

We have $\Delta(1) = \Delta(\mathrm{m} * \mathrm{m}) = \Delta(m) * m = 1$.

Hence Δ is regular.

2.9 Proposition

Let Δ be a left multiplier of $\mathcal{X}$. Then the following holds

(i) If $\exists$ an element $m \in \mathcal{X}$ $\ni$: $\Delta(m) = m$, Δ is the identity.

(ii) If $\exists$ an element $m \in \mathcal{X}$ $\ni$: $\Delta(n) * m = 1\ for\ every\ n \in \mathcal{X}$ then $\Delta(n) = m$.

Proof:

(i) Let $\Delta(m) = m$ for some $m \in \mathcal{X}$.
Then $\Delta(m) * m = m * m$
$\Rightarrow \Delta(m) * m = 1$.
Hence $\Delta(1) = 1$ by the proposition(2.7)
Which implies that Δ is regular.

(ii) By the definition of GK algebra,
$\Delta(\mathrm{m} * \mathrm{n}) = \Delta(\mathrm{n} * \mathrm{m}) = \Delta(1)$
$\Rightarrow \Delta(m) * n = \Delta(n) * m = \Delta(1)$
$\Rightarrow \Delta(m) * n = 1$
$\Rightarrow \Delta(m) = n$.

2.10 Proposition

Let $\mathcal{X}$ be a GK algebra and Δ be a left multiplier of $\mathcal{X}$. Then

$\Delta(\Delta(m) * m) = 1 \; \forall \, m \in \mathcal{X}$

Proof:

Let $m \in \mathcal{X}$. Then we have $\Delta(\Delta(m) * m) = \Delta(m) * \Delta(m) = 1$.

2.11 Proposition

Let $\mathcal{X}$ be a GK algebra and let Δ be a regular multiplier. Then the self map Δ is an identity map if it satisfies left multiplier is equal to right multiplier that is $\Delta(m) * n = m * \Delta(n) \; \forall \, m, n \in \mathcal{X}$.

Proof:

Since Δ is regular, we have $\Delta(1) = 1$.

Let $\Delta(m) * n = m * \Delta(n) \; \forall \, m, n \in \mathcal{X}$

Then $\Delta(m) = \Delta(m * 1) = \Delta(m) * 1 = m * \Delta(1) = m * 1 = m$.

Hence Δ is an identity map.

2.12 Definition

Let Δ be a multiplier of GK algebra. A set defined by $\mathcal{H}_\Delta(\mathcal{X})$ by

$\mathcal{H}_\Delta(\mathcal{X}) = \{m \in \mathcal{X} \,/\, \Delta(m) = m\} \; \forall \, m \in \mathcal{X}$.

2.13 Proposition

Let $\mathcal{X}$ be a GK algebra and let Δ be a left multiplier on $\mathcal{X}$. If $n \in \mathcal{H}_\Delta(\mathcal{X})$, we have $m \wedge n \in \mathcal{H}_\Delta(\mathcal{X}) \; \forall \, m, n \in \mathcal{X}$.

Proof:

Let Δ be a left multiplier on $\mathcal{X}$ and let $n \in \mathcal{H}_\Delta(\mathcal{X})$.

Now $\Delta(m \wedge n) = \Delta\big(n * (n * m)\big)$

$= \Delta(n) * (n * m)$

$= n * (n * m)$

$= m \wedge n.$

Hence $m \wedge n \in \mathcal{H}_\Delta(\mathcal{X})$.

2.14 Proposition

Let $\mathcal{X}$ be a GK algebra and let Δ be a right multiplier on $\mathcal{X}$. If $n \in \mathcal{H}_\Delta(\mathcal{X})$, we have $m \wedge n \in \mathcal{H}_\Delta(\mathcal{X})\ \forall\, m, n \in \mathcal{X}$.

Proof:

Let Δ be a right multiplier on $\mathcal{X}$ and let $n \in \mathcal{H}_\Delta(\mathcal{X})$.

Now $\Delta(m \wedge n) = \Delta\big(n * (n * m)\big)$

$= n * \Delta(n * m)$

$= n * (n * \Delta(m))$

$= n * (n * m)$

$= m \wedge n.$

Hence $m \wedge n \in \mathcal{H}_\Delta(\mathcal{X})$.

2.15 Definition

Let $\mathcal{X}$ be an GK algebra and Δ_1, Δ_2 two self maps. We define a mapping

$\Delta_1 \circ \Delta_2 : \mathcal{X} \to \mathcal{X}$ *by* $(\Delta_1 \circ \Delta_2)(m) = \Delta_1(\Delta_2(m))\ \forall\, m \in \mathcal{X}$.

2.16 Proposition

Let $\mathcal{X}$ be an GK algebra and Δ_1, Δ_2 two right (left) multipliers of $\mathcal{X}$. The $\Delta_1 \circ \Delta_2$ is also right (left) multiplier of $\mathcal{X}$.

Proof:

Let $\mathcal{X}$ be an GK algebra and Δ_1, Δ_2 two right multipliers of $\mathcal{X}$. Then we have

$(\Delta_1 \circ \Delta_2)(m * n) = \Delta_1(\Delta_2(m * n))$

$= \Delta_1(m * \Delta_2(n))$

$= m * \Delta_1(\Delta_2(n))$

$= m * (\Delta_1 \circ \Delta_2)(n)$

Let $\mathcal{X}$ be an GK algebra and Δ_1, Δ_2 two left multipliers of $\mathcal{X}$. Then we have

$(\Delta_1 \circ \Delta_2)(m * n) = \Delta_1(\Delta_2(m * n))$

$= \Delta_1(\Delta_2(m)) * n$

$= (\Delta_1 \circ \Delta_2)(m) * n.$

2.17 Definition

Let $\mathcal{X}$ be an GK algebra and Δ_1, Δ_2 two self maps. We define $(\Delta_1 \wedge \Delta_2): \mathcal{X} \to \mathcal{X}$ by $(\Delta_1 \wedge \Delta_2)(m) = \Delta_1(m) \wedge \Delta_2(m).$

2.18 Proposition

Let $\mathcal{X}$ be an GK algebra and Δ_1, Δ_2 two left multiplier of $\mathcal{X}$. Then $\Delta_1 \wedge \Delta_2$ is also left multiplier of $\mathcal{X}$.

Proof:

Let $\mathcal{X}$ be an GK algebra and Δ_1, Δ_2 two multiplier of $\mathcal{X}$.

$(\Delta_1 \wedge \Delta_2)(m * n) = \Delta_1(m * n) \wedge \Delta_2(m * n)$

$= (\Delta_1(m) * n) \wedge (\Delta_2(m) * n)$

$= (\Delta_2(m) * n) * ((\Delta_2(m) * n) * (\Delta_1(m) * n))$

$= \Delta_1(m) * n$ ………………….(1)

$(\Delta_1 \wedge \Delta_2)(m) * n = (\Delta_1(m) \wedge \Delta_2(m)) * n$

$= \Delta_1(m) * n$ …………………..(2)

From (1)and (2)

$(\Delta_1 \wedge \Delta_2)(m * n) = (\Delta_1 \wedge \Delta_2)(m) * n.$

Hence $\Delta_1 \wedge \Delta_2$ is left multiplier.

2.19 Definition

For any $\omega \in \mathcal{Q}(\mathcal{X})$, the set of all multipliers ,we define the Kernel of ω as follows

$\mathcal{K}_\omega = \{m \in \mathcal{X} / \omega(m) = 1\}.$

2.20 Proposition

Let ω be a multiplier of $\mathcal{X}$ *and* $1-1$.Then $\mathcal{K}_\omega$ is {1}

Proof:

Let ω be one-to-one.

Let $m \in \mathcal{K}_{\omega}$. So $\omega(m) = 1 = \omega(1)$. Thus $m = 1$.

So $\mathrm{Ker}(\omega) = \{1\}$.

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Symmetric bi derivations in GK algebra

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Abstract: In this paper, we introduce the concept of symmetric bi derivation of GK algebra and also obtain some properties about this concept.

I.INTRODUCTION

Gy.Maska[2] was introduced the concept of symmetric biderivation,and later J.Vukman[4] was proved few results of symmetric bi-derivation on prime and semi prime rings. In 2011[3], sabahatt in Ilbira.Alev Firat and young Bae Jun was introduced the notion of symmetric bi derivation of BCI algebra. The authors[1] T.Ganeshkumar and M.chandramouleeswaran have introduced the concept of symmetric bi derivation of TM algebra. Afterwards few authors have applied the concept of symmetric bi derivation in their papers.Induced by these works, in our paper, we introduce the concept of symmetric bi derivation on GK algebra and also discuss about some interesting properties.

II.SYMMETRIC BI DERIVATION OF GK ALGEBRA

Definition:2.1 Let $(X,*,1)$ be a GK algebra. A mapping $\mathbb{N}$:XxX→X is said to be a left right symmetric bi derivation(simply LR symmetric bi derivation) of X, if it is satisfying the following identity

$\mathbb{N}(x * y, z) = (\mathbb{N}(x, z) * y) \wedge (x * \mathbb{N}(y, z))$ for x,y,z∈ X.

Definition:2.2 Let $(X,*,1)$ be a GK algebra. A mapping $\mathbb{N}$:XxX→X is said to be a right left symmetric bi derivation(simply RL symmetric bi derivation) of X, if it is satisfying the following identity

$\mathbb{N}(x, y * z) = (\mathbb{N}(x, y) * z) \wedge (y * \mathbb{N}(x, z))$ for x,y,z∈ X.

In general , if $\mathbb{N}$ is both LR and RL symmetric bi derivation then it is called as $\mathbb{N}$ is symmetric bi derivation.

Definition:2.3 Let X be a GK algebra. A map $\mathbb{N}$:XxX→X is said to be symmetric if $\mathbb{N}(x, y) = \mathbb{N}(y, x)$ ∀ pairs of x,y∈ X.

Definition:2.4 Let X be a GK algebra and the mapping $\mathbb{N}$:XxX→X be a symmetric mapping. A map $\eta: X \to X$ be defined as $\eta(x) = \mathbb{N}(x, x)$ is called trace of $\mathbb{N}$.

Example:2.5 Consider the following cayley's table for GK algebra

$*$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Define a mapping $\mathbb{N}: X \times X \to X$ by

$$\mathbb{N}(x,y) = \begin{cases} 1, & (x,y) = (1,1),(2,2),(3,3),(4,4) \\ 2, & (x,y) = (1,2),(2,1),(3,4),(4,3) \\ 3, & (x,y) = (1,3),(2,4),(3,1),(4,2) \\ 4, & (x,y) = (1,4),(2,3),(3,2),(4,1) \end{cases}$$

From this $\mathbb{N}$ is symmetric bi derivation of X.

Remark:2.6 In above example, $\mathbb{N}(x,x) = \{1 \ \ when \ \ x = 1,2,3,4$ is called trace of $\mathbb{N}$.

Definition:2.7 Let X be a GK algebra. The map $\mathbb{N}: X \times X \to X$ be a symmetric mapping. $\mathbb{N}$ is called component wise regular if $\mathbb{N}(x,1) = \mathbb{N}(1,x) = 1 \, for \, some \, x \in X$. In specific if $\mathbb{N}(1,1) = \eta(1) = 1$ then $\mathbb{N} \, is \, called \ \eta - regular.$

Proposition:2.8

Let $(X,*,1)$ be a GK algebra. Let $\mathbb{N}$ be an LR symmetric bi derivation on X. Then the following holds

(i) $\mathbb{N}(x,y) = \mathbb{N}(x,y) \wedge \big(x * \mathbb{N}(1,y)\big) \, for \, all \ x,y \in X.$
(ii) $\mathbb{N}(1,x) = \eta(x) * x$ where η is the trace of $\mathbb{N}$.
(iii) $\mathbb{N}(1,y) = \mathbb{N}(x,y) * x \ \forall \ x,y \in X.$
(iv) $\mathbb{N}(y,1) = \mathbb{N}(y,1) \wedge y \, \forall \, y \, in \, X \, if \ \mathbb{N} \ is \, \eta - regular$.
(v) $\mathbb{N}(y,1) = 1 \ \ \forall \, y \, in \, X \, if \ \mathbb{N} \ is \, component \, wise \, regular$

Proof:

(i) Let us consider x,y in X
By the definition of LR bi symmetric bi derivation,
We have,
$\mathbb{N}(\boldsymbol{x},\boldsymbol{y}) = \mathbb{N}(x*1,y)$
$\quad = (\mathbb{N}(x,y) * 1) \wedge (x * \mathbb{N}(1,y))$

By axiom (ii) of GK algebra
$= (\mathbb{N}(x,y)) \wedge (x * \mathbb{N}(1,y))$

(ii) Let x, y in X
Now,

$\mathbb{N}(\mathbf{1},\boldsymbol{x}) = \mathbb{N}(x * x, x)$
$= (\mathbb{N}(x,x) * x) \wedge (x * \mathbb{N}(\mathrm{x},x))$
$= (\eta(x) * x) \wedge \left(x * \eta(x)\right)$
$= \left(x * \eta(x)\right) * (\left(x * \eta(x)\right) * (\eta(x) * x))$
$= (\eta(x) * x)$

(iii) Let x, y in X
We have,
$\mathbb{N}(\mathbf{1},\boldsymbol{y}) = \mathbb{N}(x * x, y)$
$= (\mathbb{N}(x,y) * x) \wedge (x * \mathbb{N}(\mathrm{x},y))$
$= (x * \mathbb{N}(\mathrm{x},y)) * ((x * \mathbb{N}(\mathrm{x},y) * (\mathbb{N}(\mathrm{x},y) * x))$
$= \mathbb{N}(\mathrm{x},y) * x$

(iv) Let x, y in X

$\mathbb{N}(\boldsymbol{y},\mathbf{1}) = \mathbb{N}(y * 1,1)$

$= (\mathbb{N}(y,1) * 1) \wedge (y * \mathbb{N}(1,1))$

$= (\mathbb{N}(\mathrm{y},1)) \wedge (y * \eta(1))$
$= \mathbb{N}(\mathrm{y},1) \wedge (y * 1)$
$= \mathbb{N}(\mathrm{y},1) \wedge y$

(v) Let x, y in X

$\mathbb{N}(\boldsymbol{y},\mathbf{1}) = \mathbb{N}(y * 1,1)$

$= (\mathbb{N}(y,1) * 1) \wedge (y * \mathbb{N}(1,1))$

$= (\mathbb{N}(\mathrm{y},1)) \wedge (y * \eta(1))$
$= \mathbb{N}(\mathrm{y},1) \wedge (y * 1)$
$= \mathbb{N}(\mathrm{y},1) \wedge y$
$= 1 \wedge y$
$= 1$ since $x \wedge y = x$.

Proposition:2.9 Let $(X,*,1)$ be a GK algebra. Let $\mathbb{N}$ be an RL symmetric bi derivation on X. Then the following holds

(i) $\mathbb{N}(x,y) = \mathbb{N}(x,y) \wedge \left(x * \mathbb{N}(1,y)\right)$ *for all* $x,y \in X$.
(ii) $\mathbb{N}(1,x) = \eta(x) * x$ where η is the trace of $\mathbb{N}$.
(iii) $\mathbb{N}(1,y) = \mathbb{N}(x,y) * x \;\; \forall \; x,y \in X$.
(iv) $\mathbb{N}(y,1) = \mathbb{N}(y,1) \wedge y \; \forall \, y \; in \; X \; if \;\; \mathbb{N} \;\; is \, \eta - regular$.
(v) $\mathbb{N}(y,1) = 1 \;\; \forall \, y \; in \; X \; if \;\; \mathbb{N} \;\; is \; component \; wise \; regular$

Proof:

(i) Let us consider x,y in X
By the definition of RL bi symmetric bi derivation,
We have,
$\mathbb{N}(\boldsymbol{x},\boldsymbol{y}) = \mathbb{N}(x, y * 1)$

$=(\mathbb{N}(x,y)*1)\wedge(y*\mathbb{N}(x,1))$

By axiom (ii) of GK algebra

$=(\mathbb{N}(x,y))\wedge(y*\mathbb{N}(x,1))$

$=\mathbb{N}(x,y)\wedge(y*1)$

$=\mathbb{N}(x,y)\wedge y$

(ii) Let x, y in X

Now,

$\mathbb{N}(\mathbf{1},\boldsymbol{x})=\mathbb{N}(x,x*x)$

$=(\mathbb{N}(x,x)*x)\wedge(x*\mathbb{N}(x,x))$

$=(\eta(x)*x)\wedge\left(x*\eta(x)\right)$

$=\left(x*\eta(x)\right)*(\left(x*\eta(x)\right)*(\eta(x)*x))$

$=(\eta(x)*x)$

(iii) Let x, y in X

We have,

$\mathbb{N}(\boldsymbol{y},\mathbf{1})=\mathbb{N}(y,x*x)$

$=(\mathbb{N}(y,x)*x)\wedge(x*\mathbb{N}(y,x))$

$=(x*\mathbb{N}(y,x))*((x*\mathbb{N}(y,x)*(\mathbb{N}(y,x)*x))$

$=\mathbb{N}(y,x)*x$

(iv) Let x, y in X

$\mathbb{N}(\mathbf{1},\boldsymbol{y})=\mathbb{N}(1,y*1)$

$=(\mathbb{N}(1,y)*1)\wedge(y*\mathbb{N}(1,1))$

$=(\mathbb{N}(1,y))\wedge(y*\eta(1))$

$=\mathbb{N}(1,y)\wedge(y*1)$

$=\mathbb{N}(1,y)\wedge y$

(v) Let x, y in X

$\mathbb{N}(\boldsymbol{y},\mathbf{1})=\mathbb{N}(y*1,1)$

$=(\mathbb{N}(y,1)*1)\wedge(y*\mathbb{N}(1,1))$

$=(\mathbb{N}(y,1))\wedge(y*\eta(1))$

$=\mathbb{N}(y,1)\wedge(y*1)$

$=\mathbb{N}(y,1)\wedge y$

$=1\wedge y$

$=y*(y*1)$

$=1.$

Proposition:2.10 Let X be the GK algebra and η be the trace of the LR symmetric bi derivation on X. Then

(i) $\eta(1)=\mathbb{N}(x,1)*x.$

(ii) $If\ \mathbb{N}(x,1)=\mathbb{N}(y,1)\ \ \forall\ x,y\ \in X\ then\ \eta\ is\ 1-1.$

(iii) η is regular iff $\mathbb{N}(x,1)=x.$

Proof:

(i) Let $x\in X$. We know that $x*x=1$

We have,

$\boldsymbol{\eta}(\mathbf{1}) = \mathbb{N}(1,1)$
$= \mathbb{N}(\boldsymbol{x} * \boldsymbol{x}, \mathbf{1})$
$= (\mathbb{N}(\boldsymbol{x}, \mathbf{1}) * \boldsymbol{x}) \wedge (\boldsymbol{x} * \mathbb{N}(\boldsymbol{x}, \mathbf{1}))$
$= (\mathbb{N}(\boldsymbol{x}, \mathbf{1}) * \boldsymbol{x})$

(ii) **Let** x, y ∈ X such that $\eta(x) = \eta(y)$.
We have,
$\eta(1) = \mathbb{N}(x, 1) * x$
and
$\eta(1) = \mathbb{N}(y, 1) * y$.
This implies that
$\mathbb{N}(x, 1) * x = \mathbb{N}(y, 1) * y$.
Since $\mathbb{N}(x, 1) = \mathbb{N}(y, 1)$ and by using cancellation law, we get
x=y.
Hence we get η is 1-1.

(iii) Let η be regular.
We have
$\eta(1) = \mathbb{N}(x, 1) * x$
Since η is regular
$\eta(1) = 1$ *implies* $\mathbb{N}(x, 1) * x = 1$.
By axiom (iii) of GK algebra we have $\mathbb{N}(x, 1) = x$
Conversely ,

Let $\mathbb{N}(x, 1) = x$ for some x in X.
$\Rightarrow \mathbb{N}(x, 1) * x = x * x$
$\Rightarrow \mathbb{N}(x, 1) * x = 1$
$\Rightarrow \eta(1) = 1$
Hence η *is regular.*

Proposition:2.11 Let X be the GK algebra and η be the trace of the RL symmetric bi derivation on X. Then

(i) $\eta(1) = \mathbb{N}(1, x) * x$.
(ii) $\eta(x) = \eta(x) \wedge (x * \mathbb{N}(x, 1))$
(iii) *If* $\mathbb{N}(1, x) = \mathbb{N}(1, y)\ \ \forall\ x, y \in X$ *then* η *is* $1 - 1$.
(iv) η is regular iff $\mathbb{N}(1, x) = x$.

Proof:

(i) Let $x \in X$. We know that $x * x = 1$
We have,
$\boldsymbol{\eta}(\mathbf{1}) = \mathbb{N}(1,1)$
$= \mathbb{N}(\mathbf{1}, \boldsymbol{x} * \boldsymbol{x})$
$= (\mathbb{N}(\mathbf{1}, \boldsymbol{x}) * \boldsymbol{x}) \wedge (\boldsymbol{x} * \mathbb{N}(\mathbf{1}, \boldsymbol{x}))$
$= (\mathbb{N}(\mathbf{1}, \boldsymbol{x}) * \boldsymbol{x})$

(ii) Let x in X

$\boldsymbol{\eta}(\boldsymbol{x}) = \mathbb{N}(x, x)$

$= \mathbb{N}(\boldsymbol{x}, \boldsymbol{x} * \mathbf{1})$

$= (\mathbb{N}(\boldsymbol{x}, \boldsymbol{x}) * \mathbf{1}) \wedge (\boldsymbol{x} * \mathbb{N}(\boldsymbol{x}, \mathbf{1}))$

$= (\eta(x) * \mathbf{1}) \wedge (\boldsymbol{x} * \mathbb{N}(\boldsymbol{x}, \mathbf{1}))$

$= \eta(x) \wedge (\boldsymbol{x} * \mathbb{N}(\boldsymbol{x}, \mathbf{1}))$

If it is component wise regular , we get $\eta(x) \wedge x$.

(iii) **Let** x, y ∈ X such that $\eta(x) = \eta(y)$.
We have,
$\eta(1) = \mathbb{N}(1, x) * x$
and
$\eta(1) = \mathbb{N}(1, y) * y$.
This implies that
$\mathbb{N}(1, x) * x = \mathbb{N}(1, y) * y$.
Since $\mathbb{N}(1, x) = \mathbb{N}(1, y)$ and by using cancellation law, we get x=y.
Hence we get η is 1-1.

(iv) Let η be regular.
We have
$\eta(1) = \mathbb{N}(1, x) * x$
Since η is regular
$\eta(1) = 1$ *implies* $\mathbb{N}(1, x) * x = 1$.
By axiom (iii) of GK algebra we have $\mathbb{N}(1, x) = x$

Conversely ,
Let $\mathbb{N}(1, x) = x$ for some x in X.
$\Rightarrow \mathbb{N}(1, x) * x = x * x$
$\Rightarrow \mathbb{N}(1, x) * x = 1$
$\Rightarrow \eta(1) = 1$
Hence η is regular.

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FUZZY SUB ALGEBRA AND FUZZY IDEALS OF GK ALGEBRA

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Abstract: **In this paper we introduce the fuzzification of GK algebra. We discuss about the fuzzy sub algebra of GK algebra and also fuzzy GK ideals of GK algebra and then we discuss about fuzzy Cartesian product of Fuzzy GK algebra and some interesting theorems.**

I.INTRODUCTION

The notion of fuzzy sets was introduced by L.A.Zadeh [5] and the notion of fuzzy group was introduced by Rosenfeld[3]. Later inspired by their results, O.G.Xi [4] introduced the notion of fuzzy BCK algebras. Afterwards Y.B.Jun and J.Meng [2] was studied fuzzy BCK algebra. Nowadays many authors have introduced the fuzzification of their work. In this paper we introduce the concept of fuzzy GK algebra.

II.FUZZY SUBALGERA OF GK ALGEBRA

***Definition:*2.1** A fu**zzy subset $\boldsymbol{\mu}$** of a GK algebra $(X,*,1)$ is called a fuzzy GK subalgebra of X, if the following conditions are satisfied

$\mu(x * y) \geq min\{\mu(x), \mu(y)\}$ for all x,y in X.

***Example*:2.2**

Consider X={1,2,3,4} is a GK algebra

*	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Define a mapping $\mu: X \to [0,1]$ by

$$\mu(x) = \begin{cases} 0.9 & if \ \ x = 1,2 \\ 0.5 \ \ if & otherwise \end{cases}$$

Then μ is a fuzzy GK subalgebra of X.

***Theorem*:2.3** Intersection of any two fuzzy GK subalgebras of X is again a fuzzy GK algebra.

Proof:

Let μ and δ be any two fuzzy GK subalgebras of X.Then,

$(\mu \cap \delta)(x * y) = \min\{\mu(x * y), \delta(x * y)\}$

$\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\delta(x), \delta(y)\}\}$

$= \min\{\min\{\mu(x), \delta(x)\}, \min\{\mu(y), \delta(y)\}\}$

$= \min\{(\mu \cap \delta)(x), (\mu \cap \delta)(y)\}$

$(\mu \cap \delta)(x * y) \geq \min\{(\mu \cap \delta)(x), (\mu \cap \delta)(y)\} \ \forall x, y \in X.$

Hence $\mu \cap \delta$ is fuzzy subalgebra of X.

***Definition:*2.4**

Let μ be any fuzzy subset of a GK algebra and let $s \in [0,1]$. The set $U(\mu, s) = \{x \in X: \mu(x) > s\}$ is called a level subset of μ in X.

Lemma:2.5

Let $(x, *, 1)$ be a GK algebra.Let μ be a fuzzy GK subalgebra of X. Let $\gamma \in [0,1]$.Then,

(i) $U(\mu, \gamma)$ is either $\emptyset$ or a GK subalgebra of X

(ii) $\mu(1) \geq \mu(x) \quad for\ all \quad x \in X.$

Proof:

(i) For any $\gamma \in [0,1]$, assume that $U(\mu,\gamma)$ is non-empty.

Let $x, y \in U(\mu,\gamma)$. Then $\mu(x) \geq \gamma$ and $\mu(y) \geq \gamma$.

We need to prove $U(\mu,\gamma)$ is a GK subalgebra , for that we have to prove $x * y \in U(\mu,\gamma)$.

i.e., we need to prove $\mu(x * y) \geq \gamma$.

Now

$\mu(x * y) \geq min\{\mu(x), \mu(y)\}$

$\geq \min\{\gamma, \gamma\} = \gamma$

$\therefore \mu(x * y) \geq \gamma$

(ii) To prove $\mu(1) \geq \mu(x)$

$\mu(1) = \mu(x * x)$

$\geq \min\{\mu(x), \mu(x)\} = \mu(x)$

$Hence \quad \mu(1) \geq \mu(x) \quad for\ all \quad x \in X.$

Theorem:2.6 If χ_1 and χ_2 are fuzzy GK subalgebras of X, then $\chi_1 \times \chi_2$ is a fuzzy GK algebra of $X \times X$.

Proof:

For any (x_1,x_2) and (y_1,y_2) $\in X \times X$.

Now,

$\chi((x_1, x_2) * (y_1, y_2)) = \chi(x_1 * y_1, x_2 * y_2)$

$= (\chi_1 \times \chi_2)\ (x_1 * y_1, x_2 * y_2)$

$= \min\ \{(\chi_1(x_1 * y_1), \chi_2\ (x_2 * y_2)\}$

$\geq min\ \{min(\chi_1(x_1), \chi_1(y_1)),\ min(\chi_2(x_2), \chi_2(y_2))\}$

$= min\ \{min(\chi_1(x_1), \chi_2(x_2)),\ min(\chi_1(y_1), \chi_2(y_2))\}$

$$= \min\{ (\chi_1 \times \chi_2)\,(x_1, x_2),\ (\chi_1 \times \chi_2)(y_1 * y_2)\}$$

$$= \min\{\chi\,(x_1, x_2), \chi(y_1 * y_2)\}$$

Hence χ is a fuzzy GK subalgebras of $X \times X$.

III.FUZZY GK IDEALS

***Definition*:3.1** Let X be a GK algebra.A fuzzy set $\mu\ in\ X$ is called fuzzy GK ideal of X if it satisfies the following conditions.

(i) $\mu(1) \geq \mu(x)$

(ii) $\mu(x * z) \geq \min\{\mu(y * z), \mu(y * x)\}\ \ \forall\, x, y, z \in X.$

***Example:*3.2** Consider the above example (2.2). This is an example of fuzzy GK ideal.

Theorem:3.3

Every fuzzy GK ideal of a GK-algebra X is order reversing.

Proof:

Let μ be a fuzzy GK ideal of a GK algebra X.

Let x, y$\in X$ be such that $x \leq y$ then $x * y = y * x = 1$.

Now, we know that $x * 1 = x$.

$$\mu(x) = \mu(x * 1) \geq \min\{\mu(y * 1), \mu(y * x)\}$$

$$\geq \min\{\mu(y), \mu(1)\}$$

$$\geq \mu(y)$$

Therefore μ is order reversing.

***Theorem*:3.4** If μ is a fuzzy ideal of GK algebra $(X, * 1)$ and $\mu_\gamma(x) = \min\{\gamma, \mu(x)\}\, \forall x \in X$ and $\gamma \in [0,1]$ then $\mu_\gamma(x)$ is fuzzy GK ideal of X.

Proof:

Let μ be a fuzzy ideal of GK algebra and $\gamma \in [0,1]$.

Therefore $\mu(1) \geq \mu(x) \; \forall \; x \in X$.

Now, $\mu_\gamma(1) = \min\{\gamma, \mu(1)\} \geq \min\{\gamma, \mu(x)\} = \mu_\gamma(x) \quad \forall x \in X$.

And we know that

$\mu(x * z) \geq \min\{\mu(y * z), \mu(y * x)\}$

Now

$\mu_\alpha(x * z) = \min\{\gamma, \mu(x * z)\}$

$\geq \min\{\, \gamma, \min(\, \mu(y * z), \mu(y * x))\}$

$= \min \{\, \min\big(\gamma, \mu(y * z)\big), \min(\gamma, \mu(y * x))\}$

$= \min\{\mu_\gamma(y * z), \mu_\gamma(y * x)\}$

Hence $\mu_\gamma(x)$ is fuzzy GK ideal of X.

***Proposition*:3.5**

Let μ be fuzzy GK ideal of GK algebra. If the inequality $y * x \leq z$ holds in X, then

$\mu(x) \geq \min\{\mu(y), \mu(z)\} \; \forall \; x, y, z \in X$.

Proof:

Assume that the inequality $y * x \leq z$ holds in X,

Then by theorem 3.3

$\mu(y * x) \geq \mu(z)$-------------(1)

By the definition fuzzy GK ideal

$\mu(x * z) \geq \min\{\mu(y * z), \mu(y * x)\}$

Put z=1

Then $\mu(x * 1) \geq \min\{\mu(y * 1), \mu(y * x)\}$

$\mu(x) \geq \min\{\mu(y), \mu(y * x)\}$ ------------(2)

From (1) and (2),

$\mu(x) \geq \min\{\mu(y), \mu(z)\}$.

Definition:3.6

Let $\mu \; and \; \beta$ be fuzzy subsets of a set S. The Cartesian product of $\mu \; and \; \beta$ is defined by

$(\mu \times \beta)(x, y) = \min\{\mu(x), \beta(y)\} \, \forall x, y \in S$

***Theorem*:3.7**

Let $\mu \; and \; \beta$ be fuzzy GK ideals of GK algebra X. Then $\mu \times \beta$ is a fuzzy GK ideal of $X \times X$.

Proof:

Let us consider

$(x, y) \in X \times X$

$(\mu \times \beta)(1,1) = \min\{\mu(1), \beta(1)\}$

$\geq \min\{\mu(x), \beta(y)\} = (\mu \times \beta)(x, y)$

Now let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$

$(\mu \times \beta)(x_1 * z_1, x_2 * z_2) = \min\{\mu(x_1 * z_1), \beta(x_2 * z_2)\}$

$\geq \min\{\min\{\mu\,(y_1 * z_1), \mu(y_1 * x_1)\}, \min\{\beta(y_2 * z_2), \beta(y_2 * x_2)\}\}$

$= \min\{\min\{\mu\,(y_1 * z_1), \beta(y_2 * z_2)\}, \min\{\mu(y_1 * x_1), \beta(y_2 * x_2)\}\}$

$= \min\{(\mu \times \beta)\,(y_1 * z_1, y_2 * z_2), (\mu \times \beta)(y_1 * x_1, y_2 * x_2)\}$

Therefore $\mu \times \beta$ is a fuzzy GK ideal of $X \times X$.

Theorem:3.8

Let $\mu\ and\ \beta$ be fuzzy subsets of GK algebra X such that $\mu \times \beta$ is a fuzzy GK ideal of $X \times X$.Then for all $x \in X$,

(i) Either $\mu(1) \geq \mu(x)\ or\ \beta(1) \geq \beta(x)$

(ii) $\mu(1) \geq \mu(x) \forall x \in X\ \ then\ either\ \beta(1) \geq \mu(x)\ or\ \beta(1) \geq \beta(x).$

(iii) $If\ \beta(1) \geq \beta(x) \forall x \in X, then\ either\ \mu(1) \geq \mu(x)\ or\ \mu(1) \geq \beta(x).$

(iv) $either\ \mu\ or\ \beta$ is a fuzzy GK ideal of X.

Proof:

(i) Suppose that $\mu(x) > \mu(1)\ and\ \beta(y) > \beta(1)$ for some $y \in X$.

Then

$(\mu \times \beta)(x, y) = \min\{\mu(x), \beta(y)\}$

$> \min\{\ \mu(1), \beta(1)\} = (\mu \times \beta\ )(1,1)$

This is a contradiction ,since $\mu \times \beta$ is a fuzzy GK ideal of $X \times X$.

Hence we obtain (i).

(ii) Assume that $x, y \in X$

$\mu(x) > \beta(1)\ and\ \beta(y) > \beta(1)$

Then we have

$(\mu \times \beta)(1,1) = \min\{\mu(1), \beta(1)\}$

$> \min\{\beta(1), \beta(1)\} = \beta(1)$

This implies that

$(\mu \times \beta)(x, y) = \min\{\mu(x), \beta(y)\}$

$> \min\{\beta(1), \beta(1)\} = \beta(1)$

$> (\mu \times \beta)(1,1)$

This is a contradiction.

Hence we obtain (ii)

(iii) By the similar way to part (ii)

(iv) In (i) we have

Either $(1) \geq \mu(x)$ or $\beta(1) \geq \beta(x) \forall x \in X$.

We assume that $\beta(1) \geq \beta(x)$, without loss of generality,

It is from (iii) such that

Either $\mu(1) \geq \mu(x)$ or $\mu(1) \geq \beta(x)$

If $\mu(1) \geq \beta(x)$ for any $x \in X$, then

$(\mu \times \beta)(1, x) = \min\{\mu(1), \beta(x)\} = \beta(x) ------(1)$

Now we have to prove β is a fuzzy GK ideal .

For that, let us consider $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X, we\ have$

Since $\mu \times \beta$ is a fuzzy GK ideal of $X \times X$, we have

$(\mu \times \beta)(x_1 * z_1, x_2 * z_2) \geq \min\{(\mu \times \beta)\,(y_1 * z_1, y_2 * z_2), (\mu \times \beta)(y_1 * x_1, y_2 * x_2)\}$

Now,if we take $x_1 = y_1 = z_1 = 1$, then

$(\mu \times \beta)(1, x_2 * z_2) \geq \min\{(\mu \times \beta)\,(1, y_2 * z_2), (\mu \times \beta)(1, y_2 * x_2)\}$

$Since$ by (1), LHS becomes

$\beta(x_2 * z_2) \geq \min\{(\mu \times \beta)\,(1, y_2 * z_2), (\mu \times \beta)(1, y_2 * x_2)\}$

$\geq \min\{\min\{\mu(1), \beta(y_2 * z_2)\}, \min\{\mu(1), \beta(y_2 * x_2)\}$

$\geq \min\{\beta(y_2 * z_2), \beta(y_2 * x_2)\}$

$\beta(x_2 * z_2) \geq \min\{\beta(y_2 * z_2), \beta(y_2 * x_2)\}$

This proves that β is a fuzzy GK ideal of X.

Now we consider $\mu(1) \geq \mu(x)$.

Suppose let us consider

$\mu(1) < \mu(y)$ for some y$\in X$

Then $\beta(1) \geq \beta(y) > \mu(1)$

Since $\mu(1) \geq \mu(x) \forall x \in X, then\ \beta(1) \geq \mu(x)$

Hence $(\mu \times \beta)(x, 1) = \min\{\mu(x), \beta(1)\} = \mu(x)$-------------(3)

Taking $x_2 = y_2 = z_2 = 1$ in (2)

$(\mu \times \beta)(x_1 * z_1, 1) \geq \min\{(\mu \times \beta)\,(y_1 * z_1, 1), (\mu \times \beta)(y_1 * x_1, 1)\}$

By (3)

$\mu(x_1 * z_1) \geq \min\{(\mu \times \beta)\,(y_1 * z_1, 1), (\mu \times \beta)(y_1 * x_1, 1)\}$

$\geq \min\{\,\min\{\mu((y_1 * z_1), \beta(\,1)\}, \min\{\mu(y_1 * x_1), \beta(1)\}$

$\geq \min\{\mu((y_1 * z_1), \mu(y_1 * x_1)\}$

$\mu(x_1 * z_1) \geq \min\{\mu((y_1 * z_1), \mu(y_1 * x_1)\}$

This proves that μ is a fuzzy GK ideal of GK algebra.

Therefore $either\ \mu\ or\ \beta$ is a fuzzy GK ideal of GK algebra X.

IV.CONCLUSION

In this paper we introduced the concept of fuzzy GK sub algebras of GK algebra. We discussed about fuzzy GK ideal and concept of Cartesian product of fuzzy GK algebra and some of the interesting results were also discussed.

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STUDY OF ANTI-FUZZY GK SUB ALGEBRA AND ANTI-FUZZY GK IDEAL

J.KAVITHA AND R.GOWRI

Abstract. In this paper, we establish the theory of Anti-fuzzy GK sub algebra and Anti-fuzzy GK ideals. We defined lower-level set of GK algebra and discussed some of its aspects in this paper.

1. INTRODUCTION

In 1991, the fuzzification of BCK algebras was introduced by O.G. Xi [10] discussed its characteristics and its properties. In 1993, the concept of Fuzzy BCI algebra was introduced by B. Ahamed [1], in this study he explored the properties of Fuzzy BCI algebras. In 2003, Ahn and Bang [2] introduced fuzzified B algebra and in this article, they classified the sub algebras by their family of level sets. Many authors [3-7] have introduced new algebraic structures and fuzzified the same and obtained many interesting results and also derived new concepts of that new algebraic structure. Inspiring by these kinds of articles, we introduced new algebraic structure namely GK algebra [8]and fuzzified [9] it. In this paper we discuss about Anti-fuzzy GK sub algebra and Anti-fuzzy GK ideal and brought very interesting results.

2. ANTI-FUZZY GK SUB ALGEBRA AND ANTI-FUZZY GK IDEAL

Definition 2.1. A fuzzy set ρ_{gk} in GK algebra T is said to be an anti-fuzzy sub algebra of T if

$$\rho_{gk}(i \circledast j) \leq max\{\rho_{gk}(i), \rho_{gk}(j)\}, \text{ for all } i, j \in T.$$

Theorem 2.2. *Let ρ_{gk} is an anti-fuzzy sub algebra of GK algebra. Prove that $\rho_{gk}(1) \leq \rho_{gk}(i)$, for any i in T.*

Proof. We know that $i \circledast j = 1$ from the definition of GK algebra
Now, $\rho_{gk}(1) = \rho_{gk}(i \circledast j)$
$\leq max\{\rho_{gk}(i), \rho_{gk}(j)\} \leq \rho_{gk}(i)$
Therefore $\rho_{gk}(1) \leq \rho_{gk}(i)$. □

Definition 2.3. Let ρ_{gk} be any fuzzy subset of a GK algebra and let $q \in [0, 1]$. The set $\Gamma(\rho_{gk}, q) = \{i \in T : \rho_{gk} \leq q\}$ is called a lower-level subset of ρ_{gk} in T.

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Theorem 2.4. *A fuzzy set ρ_{gk} in GK algebra is an anti-fuzzy sub algebra if and only if for every q in $[0,1]$, $\Gamma(\rho_{gk}, q)$ is either ϕ or a sub algebra of T.*

Proof. Let us assume ρ_{gk} is an anti-fuzzy sub algebra of T and also lower-level subset is non-empty. Then for any $i, j \in \Gamma(\rho_{gk}, q)$
we have, $\rho_{gk}(i \circledast j) \leq max\{\rho_{gk}(i), \rho_{gk}(j)\} \leq q$.
Therefore, $i \circledast j \in \Gamma(\rho_{gk}, q)$.
Hence $\Gamma(\rho_{gk}$ is a sub algebra.
Conversely,
Now,Consider $i, j \in T$.
Take $q = max\{\rho_{gk}(i), \rho_{gk}(j)\}$.
Since $\Gamma(\rho_{gk}, q)$ is a sub algebra of T,
$\Rightarrow i \circledast j \in \Gamma(\rho_{gk}, q)$
Therefore $\rho_{gk}(i \circledast j) \leq q = max\{\rho_{gk}(i), \rho_{gk}(j)\}$
Hence ρ_{gk} is an anti-fuzzy sub algebra. □

Definition 2.5. Let T be a GK algebra. A fuzzy set ρ_{gk} in T is called anti-fuzzy GK ideal of T if it satisfies the following conditions.

(i) $\rho_{gk}(1) \leq \rho_{gk}(i)$

(ii) $\rho_{gk}(i \circledast k) \leq max\{\rho_{gk}(j \circledast k), \rho_{gk}(j \circledast i)\}$

for all $i, j, k \in T$.

Definition 2.6. Let $(T, \circledast_T, 1)$ and $(P, \circledast_P, 1')$ be a GK algebra. Then the mapping $\sigma : T \rightarrow P$ of GK algebra is called anti-homomorphism if $\sigma(i \circledast_T j) = \sigma(j) \circledast_P \sigma(i)$ for all $i, j \in T$.

Definition 2.7. Let $\sigma : T \rightarrow T$ be an endomorphism and ρ_{gk} be a fuzzy set in T. We define fuzzy set in T by $(\rho_{gk})_\sigma$ in T as $(\rho_{gk})_\sigma(i) = (\rho_{gk})(\sigma(i))$ for every $i \in T$.

Theorem 2.8. *Let ρ_{gk} be an anti-fuzzy GK ideal of GK algebra of T and if $i \leq j$, then $\rho_{gk}(i) \leq \rho_{gk}(j)$, for all $i, j \in T$.*

Proof. Let us consider $i \leq j$, then $i \circledast j = 1 = j \circledast i$,
and $\rho_{gk}(i \circledast 1) = \rho_{gk}(1) \leq max\{\rho_{gk}(j \circledast 1), \rho_{gk}(j \circledast i)\}$
$= max\{\rho_{gk}(j), \rho_{gk}(1)\} = \rho_{gk}(j)$.
Hence $\rho_{gk}(x) \leq \rho_{gk}(y)$.

Theorem 2.9. *Let ρ_{gk} be an anti-fuzzy GK-ideal of GK algebra T. If the inequality $j \circledast i \leq k$ carry in T, then $\rho_{gk}(i) \leq \max\{\rho_{gk}(j), \rho_{gk}(k)\}$.*

Proof. Let us consider the inequality $j \circledast i \leq k$ carry in T.
By theorem 2.8, $\rho_{gk}(j \circledast i) \leq \rho_{gk}(k)$ ....(1)
By definition of anti-fuzzy ideal of GK algebra
$\rho_{gk}(i \circledast k) \leq max\{\rho_{gk}(j \circledast k), \rho_{gk}(j \circledast i)\}$
Put $k = 1$, then $\rho_{gk}(i \circledast 1) = \rho_{gk}(i) \leq max\{\rho_{gk}(j \circledast 1), \rho_{gk}(j \circledast i)\}$
$= max\{\rho_{gk}(j), \rho_{gk}(j \circledast i)\}$.............(2)
From (1) and (2), we get
$\rho_{gk}(i) \leq \max\{\rho_{gk}(j), \rho_{gk}(k)\}$, for all $i, j, k \in T$.

Conclusion. In this article, we defined and discussed about Anti-fuzzy GK sub algebra and Anti-fuzzy GK ideal and also derived some important results. In future we planned to work the concept of algebraic structure of GK algebra with soft set, Neutrosophic set for obtaining new kind of results.

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RESEARCH ARTICLE

Direct Product of GK Algebra

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Abstract

Objectives: To find the direct product of an algebraic structure namely as GK algebra. **Methods/Findings**: We derive some important results in which direct product of two GK algebra is again GK algebra as a particular case and also, derive the general case of the same then after investigate the direct product of kernel of GK algebra.

Keywords: Direct Product; Kernel; isomorphism; Homomorphism; GK algebra

1 Introduction

BCK-algebras and BCI-algebras are abridged to two B-algebras. The BCK algebra was coined in 1966 by the Japanese mathematicians, Y. Imai and K. Iseki[1]. Two B-algebras are created from two different provenances. In 2007, the new algebraic structure which is said to be BF algebra, was explored by Andrze J Walendziak[2] which is a generalization of BCI/BCK/B-algebras. In 2008, the generalization of B algebra called as BG algebra initiated by Kim & Kim[3]. In 2009, another algebra which is generalization of BE algebra and dual BCK/BCI/BCH algebras, namely CI algebra was initiated by Biao long Meng[4].

Direct product plays an important role in algebraic structures. In 2019, Slamet Widianto, Sri Gemawati, Kartini[5–7] were discussed about the Direct product of BG algebra. Likewise, many authors have discussed this topic in their work. Motivated by these, in this paper we discuss about direct product of GK algebra and obtain its some interesting results. In 2018, we introduced the new algebraic structure namely GK algebra[8] and discussed about its characteristics and investigated some results. In this paper we discuss about the direct product of GK algebra and investigate its properties.

2 Direct product of GK algebra

2.1 Definition

Let (M,$\circledast$,1_M) and (N,$\circledast$,1_N) be GK algebras. Direct product $M \times N$ is defined as a structure $M \times N = (M \times N; \otimes; (1_M; 1_N))$, where $M \times N$ is the set $\{(m,n)/m \in M, n \in N\}$

and $\otimes$ is given by

$$(m_1,n_1)\otimes(m_2,n_2)=(m_1\circledast m_2,n_1\circledast n_2)$$

This shows that the direct product of two sets of GK algebra M and N is denoted by $M\times N$, which each (m,n) is an ordered pair.

2.2 Theorem

Direct product of any two GK algebras is again a GK algebra.

Proof:

Let M and N be GK algebras, let $m_1,m_2\in M\ and\ n_1,n_2\in N$

We know that $M\times N=(M\times N;\otimes;(1_M;1_N))$

Since $1_M\in M, 1_N\in N$

This implies that $(1_M,1_N)\in M\times N$

$\therefore\ M\times N\ \ is\ non-empty.$

Now let us prove it is GK algebra.

Let $m_1,m_2\in M\ and\ n_1,n_2\in N$

1. $(m_1,n_1)\otimes(m_1,n_1)=(m_1\circledast m_1,n_1\circledast n_1)$
   = $(1_M,1_N)$ by definition of GK algebra
2. $(m_1,n_1)\otimes(1_M,1_N)=(m_1\circledast 1_M,n_1\circledast 1_N)$
   = (m_1,n_1) by definition GK algebra
3. If $(m_1,n_1)\otimes(m_2,n_2)=(1_M,1_N)$ and $(m_2,n_2)\otimes(m_1,n_1)=(1_M,1_N)$
   then $(m_1\circledast m_2,n_1\circledast n_2)=(1_M,1_N)$
   $\Longrightarrow\ m_1\circledast m_2=1_M\ \ and\ \ n_1\circledast n_2=1_N$
   $\Longrightarrow\ m_1=m_2\ \ \ and\ \ n_1=n_2$ by definition GK algebra.
4. $[(m_2,n_2)\otimes(m_3,n_3)]\otimes[(m_1,n_1)\otimes(m_3,n_3)]$
   $\Longrightarrow(m_2\circledast m_3,n_2\circledast n_3)\otimes(m_1\circledast m_3,n_1\circledast n_3)$
   $\Longrightarrow\{[(m_2\circledast m_3)\circledast(m_1\circledast m_3)]\circledast[(n_2\circledast n_3)\circledast(n_1\circledast n_3)]\}$
   $\Longrightarrow(m_2\circledast m_1,n_2\circledast n_1)$
   $\Longrightarrow(m_2,n_2)\otimes(m_1,n_1).$
5. $[(m_1,n_1)\otimes(m_2,n_2)]\otimes[(1_M,1_N)\otimes(m_2,n_2)]$
   $\Longrightarrow[(m_1\circledast m_2),(n_1\circledast n_2)]\otimes[(1_M\circledast m_2),(1_N\circledast n_2)]$
   $\Longrightarrow((m_1\circledast m_2)\circledast(1_M\circledast m_2)],[(n_1\circledast n_2)\circledast(1_N\circledast n_2)]$
   $\Longrightarrow\ \ (m_1\circledast 1_M,n_1\circledast 1_N)$
   $\Longrightarrow\ \ (m_1,n_1)$

Hence $M\times N$ is a GK algebra.

2.3 Theorem

Let $\{M_i\ /(M_i;\circledast;1)\ :i=1,2,3\ldots.n\}\ \ and\ \ \{N_i\ /(N_i;\circledast;1)\ :i=1,2,3\ldots.n\}$ be the family of GK algebras and let $\zeta_i:\ M_i\longrightarrow N_i,\ i=1,2,3\ldots..n$ be the set of isomorphism.

If $\zeta\ from\ \prod_1^n M_i\longrightarrow\prod_1^n N_i$ given by $\zeta\,(m_i),(i=1,2,3\ldots n)=\zeta_i\,(m_i)\,,\ i=1,2,3\ldots n$, then ζ is also an isomorphism.

Proof :

Let $\{M_i\ /(M_i;\circledast;1)\ :i=1,2,3\ldots.n\}\ \ and\ \ \{N_i\ /(N_i;\circledast;1)\ :i=1,2,3\ldots.n\}$ be the family of GK algebras and let $\zeta_i:\ M_i\longrightarrow N_i,\ i=1,2,3\ldots..n$ be the set of isomorphism.

Let $\zeta\ from\ \prod_1^n M_i\longrightarrow\prod_1^n N_i$ given by $\zeta\,(m_i),(i=1,2,3\ldots n)=\zeta_i\,(m_i)\,,\ i=1,2,3\ldots n.$

We have to prove ζ is an isomorphism.

If $(m_i,n_i)\in\prod_1^n M_i$ then $\zeta\,[(m_1,m_2,\ldots..m_n)\otimes(n_1,n_2,\ldots\ldots n_n)]$

$=\zeta\,[m_1\circledast n_1,m_2\circledast n_2\ldots\ldots m_n\circledast n_n]$

$=(\zeta_1(m_1\circledast n_1),\zeta_2(m_2\circledast n_2)\ldots\ldots\zeta_n(m_n\circledast n_n))$

$=((\zeta_1(m_1)\circledast\zeta_1(n_1)),\ (\zeta_2(m_2)\circledast\zeta_2(n_2))\ \ldots\ldots(\zeta_n(m_n)\circledast\zeta_n(n_n))$

$=(\zeta_1(m_1)\,,\ \zeta_2(m_2),\ldots\ldots.\ \zeta_n(m_n)]\ \ \otimes\ \ (\zeta_1(n_1)\,,\ \zeta_2(n_2),\ldots\ldots.\ \zeta_n(n_n)]$

$=\zeta\,(m_1,m_2,\ldots..m_n)\ \ \ \otimes\ \ \ \zeta\,(n_1,n_2,\ldots\ldots n_n)$

This implies that ζ is a homomorphism.
We have to prove ζ is onto, we have ζ_i is onto, where i=1,2,3….n.
Let $(n_1,n_2,\ldots\ldots n_n) \in N_1 \times N_2 \times \ldots. \times N_n$
$\Longrightarrow$ *Since* ζ *is onto*, $n_i \in N_i$, *there exists* $m_i \in M_i$ *such that* $\zeta_i(m_i) = n_i$ *for* $i = 1,2,3\ldots n$
$\Longrightarrow (n_1,n_2,\ldots\ldots n_n) = [(\zeta_1(m_1),\ \zeta_2(m_2),\ldots\ldots.\ \zeta_n(m_n)] =\ \zeta(m_1,m_2,\ldots..m_n)$
$\Longrightarrow\ \zeta$ *is onto*.
Now, to prove ζ *is* $1-1$.
$\zeta(m_1,m_2,\ldots..m_n) =\ \zeta(n_1,n_2,\ldots\ldots n_n)$
$[(\zeta_1(m_1),\ \zeta_2(m_2),\ldots\ldots.\ \zeta_n(m_n)]\ =\ [(\zeta_1(n_1),\ \zeta_2(n_2),\ldots\ldots.\ \zeta_n(n_n)]$
$\Longrightarrow\ \zeta_i(m_i) = \zeta_i(n_i)$
$\Longrightarrow\ m_i = n_i$, where i=1,2,3….n , since ζ_i is 1-1.
$\Longrightarrow (m_1,m_2,\ldots..m_n) = (\ n_1,n_2,\ldots\ldots n_n)$
$\Longrightarrow\ \zeta$ *is* $1-1$.
Hence ζ is an isomorphism.

2.4 Theorem

Let $M_i,N_i,\ i=1,2$ be GK algebras. consider the mapping $\zeta_1 : M_1 \longrightarrow N_1$ *and* $\zeta_2 : M_2 \longrightarrow N_2$ where ζ_1,ζ_2 are homomorphisms. If the map $\zeta : M_1 \times M_2 \longrightarrow N_1 \times N_2$ given by $\zeta(m_1,m_2) = (\zeta_1(m_1),\zeta_2(m_2))$, then

1. ζ is a homomorphism.
2. Ker $\zeta = ker\zeta_1 \times ker\zeta_2$.

Proof:
Let us consider the mapping $\zeta_1 : M_1 \longrightarrow N_1$ *and* $\zeta_2 : M_2 \longrightarrow N_2$ where ζ_1,ζ_2 are homomorphisms.
If the map $\zeta : M_1 \times M_2 \longrightarrow N_1 \times N_2$ given by $\zeta(m_1,n_1) = (\zeta_1(m_1),\zeta_2(n_1))$,
for $m_1,m_2 \in M_1$ *and* $n_1,n_2 \in M_2$ then

- $\zeta[(m_1,n_1)\otimes(m_2,n_2)] = \zeta(m_1 \circledast m_2, n_1 \circledast n_2)$
  = $(\zeta_1(m_1 \circledast m_2),\zeta_2(n_1 \circledast n_2))$
  = $(\zeta_1(m_1)\circledast\zeta_1(m_2),\zeta_2(n_1)\circledast\zeta_2(n_2))$
  = $(\zeta_1(m_1),\zeta_2(n_1))\ \otimes (\zeta_1(m_2),\zeta_2(n_2))$
  = $\zeta_1(m_1,n_1)\otimes\zeta_2(m_2,n_2)$

Therefore ζ is a homomorphism.

- Let $(m,n) \in \ker\zeta \Leftrightarrow \zeta(m,n) = (1_{M_1},1_{M_2})$
  $\Longleftrightarrow (\ \zeta_1(m),\zeta_2(n)) = (1_{M_1},1_{M_2})$
  $\Longleftrightarrow\ \zeta_1(m) = 1_{M_1},\zeta_2(n) = 1_{M_2}$
  $\Longleftrightarrow m \in ker\zeta_1\ , n \in ker\zeta_2$
  $\Longleftrightarrow\ \ (m,n)\ \in\ \ ker\zeta_1 \times ker\zeta_2.$

Hence Ker $\zeta = ker\zeta_1 \times ker\zeta_2$.

3 Conclusion

In this article we discussed about the concept of the direct product of GK algebra. We derived the finite form of direct product of GK algebra is isomorphism and also, we investigated and applied the concept of direct product of GK algebra in GK homomorphism and GK kernel, then obtained interesting results.

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