A STUDY ON FUZZY OPTIMIZATION PROBLEMS

Thesis submitted to the Bharathidasan University, Tiruchirappalli, in partial fulfilment of the requirements for the Degree of

DOCTOR OF PHILOSOPHY IN MATHEMATICS

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(Ref. No. 25312/Ph.D.K1/Mathematics/P-T(3-5) Jan - 2018)

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is a research work done by **Mr. B. Christopar Raj**, a part-time scholar of Ph.D., Degree in the Department of Mathematics, T.B.M.L. College, Porayar, South India.

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CERTIFICATE ON PLAGIARISM CHECK

It is certified that the thesis in respect of **B. Christopar Raj, (Reg. No. 25312/Ph.D.K1/Mathematics/P-T(3-5) Jan - 2018)** entitled **"A STUDY ON FUZZY OPTIMIZATION PROBLEMS"** has been subjected to Plagiarism Check in the Ouriginal Software and the percentage of significance is 2%. The report is enclosed in the thesis.

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Acknowledgments

First and Foremost, praises and thanks to the **God Almighty**, for His showers of blessings throughout my research work to complete the research successfully.

I would like to express my deep and sincere gratitude to my research supervisor **Dr. D. Stephen Dinagar**, M.Sc., M.Phil.,Ph.D., Associate Professor of Mathematics, T.B.M.L. College, Porayar, South India, for giving me the opportunity to begin my research and proving invaluable guidance throughout this research, his dynamism, vision, sincerity and motivation have deeply inspired me. He has taught me the methodology to carry out the research work and present the research report as clearly as possible. It was a great privilege and honour to work under his guidance. He has constantly advised me to focus on my goal, his observations and comments extremely helped me to establish the overall direction of the research and to move forward with investigation in depth. I am extremely grateful and thank him for his noble-touch and understanding. I am extending my heartfelt thanks to his beloved wife **Mrs. A. Reeta Gunavathy** for her patience during our research discussion.

I am grateful to express my thanks to my Doctoral Committee members **Dr. D. Rajan** and **Dr. D. Lilly Esthar Rani** for sharing their time and valuable comments on my work generously.

I would like to express my sincere thanks to **The Management**, the Secretary and the Correspondent of T.B.M.L. College, for providing all the necessary facilities to pursue the research programme.

I would like to record my heartfelt thanks to **Dr. Gene George**, M.Sc., M.Phil., Ph.D., Principal, T.B.M.L. College, Porayar, who permitted me to undergo my research in the Department of Mathematics as a part-time research scholar.

I wish to acknowledge my heartfelt thanks to **Dr. Thangaraj Beaula**, M.Sc., M.Phil., Ph.D., Head of the Department of Mathematics, T.B.M.L. College, Porayar, for her kind and constant encouragement.

I take up this opportunity to thank all the staff members of the **Department of Mathematics**, T.B.M.L. College, Porayar, who have supported me in various aspects towards the completion of this thesis.

I wish to express my sincere thanks to **Dr. J. Joseline Manora**, M.Sc., M.Phil., Ph.D, who encourged me and introduced my guide.

I would like to record my heartfelt thanks to **Miss. E. Fany Helena** and **Miss. A. Hetchiyal Sangeetha Priyam** Research Scholars, Department of Mathematics, for their help to complete my work.

It is my bounded duty to record my gratitude to my brother **Mr. T. Devaraj**, M.A., M.Phil., B.Ed., Assistant Professor of English, Raja Serfoji Government College, Thanjavur for his immense support to enhance the quality of the thesis and also I thank to friend **Dr. N. Ilayaraja**, Assistant Professor of Engligh, Arignar Anna Government Arts and Science College, Karaikal for his valuable suggestions.

I would like to express special thanks to **Mr. D. Sriram Dhasarathan** B.Tech., for aligning my thesis in LaTeX format.

I deeply thank my family members for their prayers, support and encouragement.

I thank the various international journals who have published my research articles.

Finally, I would like to thank all the good and noble-minded persons who have supported me to complete my research work successfully.

(B. Christopar Raj)

List of Publications

1. Stephen Dinagar. D., and Christopar Raj. B., "A Method for Solving Fully Fuzzy Transportation Problem with Generalized Quadrilateral Fuzzy Numbers", **Malaya Journal of Matematik**, **S(1)**: 24 – 27, (2019).

2. Stephen Dinagar. D., and Christopar Raj. B., "A Novel approach for solving Fully Fuzzy Transportation Problems using Generalized Quadrilateral Fuzzy Numbers", **American International Journal of Research in Science, Technology, Engineering and Mathematics**, 199–204, (2019).

3. Stephen Dinagar. D., and Christopar Raj. B., "A Distinct method for Solving Fuzzy Assignment Problems using Generalized Quadrilateral Fuzzy Numbers", **Mathematical Analysis and Computing, Springer Proceedings in Mathematics and Statistics**, **344**: 221–229, (2021).

4. Stephen Dinagar. D., and Christopar Raj. B., "Stephen's Algorithm for Solving Assignment Problems", **Advances and Applications in Mathematical Sciences**, **20(5)**: 887–894, (2021).

5. Stephen Dinagar. D., and Christopar Raj. B., “A Dynamic Programming approach for solving travelling salesman problem with quadrilateral fuzzy number”, **Malaya Journal of Matematik**, **S(1)**: 448–450, (2021).

6. Stephen Dinagar. D., and Christopar Raj. B., “DSD Approach for Solving Travelling Salesman Problem in Fuzzy Environment”– Communicated.

7. Stephen Dinagar. D., and Christopar Raj. B., “Solving Travelling Salesman Problem by BCR Approach in Fuzzy Environment” – Communicated.

List of Notations

AP	Assignment Problem
CPM	Critical Path Method
DPA	Dynamic Programming Algorithm
FAP	Fuzzy Assignment problem
FTP	Fuzzy Transportation Problem
FFTP	Fully Fuzzy Transportation Problem
FTSP	Fuzzy Travelling Salesman Problem
FLPP	Fuzzy Linear Programming Problem
FFLP	Fully Fuzzy Linear Programming Problem
FFLFPP	Fully Fuzzy Linear Fractional Programming Problem
GQFNs	Generalized Quadrilateral Fuzzy Numbers
GIVFNS	Generalized Interval Valued Fuzzy Numbers
HA	Hungarian Algorithm
HHO	Harris Hawk Optimization
IZPM	Improved Zero Point Method

MIVFNS	More Generalized Interval Valued Fuzzy Numbers
MOLP	Multi-Objective Linear Programming
OR	Operations Research
PERT	Project Evaluation Review Techniques
TP	Transportation Problem
TSP	Travelling Salesman Problem
VAM	Vogel's Approximation Method

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Abstract

A Study on Fuzzy Optimization Problems

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In recent years, fuzzy optimization has grown in various developing fields and has brought remarkable changes in business, research, science and technology. Fuzzy optimization problem is the problem of finding the best solution among all feasible solutions. Theoretically, the parameters of optimization problems are represented as precise value. But in real life situation, it is impossible to express the parameters as précised due to several factors. To overcome the vagueness, fuzzy concepts have been introduced. In this thesis, sub-classes and special classes of Linear Programming Problem such as Transportation Problem, Assignment Problem and Travelling Salesman Problem are discussed in fuzzy environment and parameters of the problem are represented as Generalized Quadrilateral Fuzzy Numbers. The working procedure and results are dwelt upon the concept of Generalized Quadrilateral Fuzzy Numbers. These concepts bring some new algorithms and approaches to find the optimal solution for fuzzy optimization problems. Fuzzy optimization provides a useful solution to all real life problem and all real life situations.

This thesis entitled **"A Study on Fuzzy Optimization Problems"** consists of seven chapters which enfold the study of subclasses of Linear Programming models in fuzzy environment such as fuzzy Transportation Problem, Fuzzy Assignment Problem and Fuzzy Travelling Salesman Problem with a new fuzzy number Generalized Quadrilateral Fuzzy Numbers and some arithmetic operations.

Chapter 1 is the introductory in nature which explicates the fundamental theoretical background of this thesis. Moreover, it also records the chronological details of literature survey on the topic of the thesis. This part also explains briefly about the motivation and scope of the research.

Chapter 2 deals with some relevant definitions and fundamentals of fuzzy numbers such as definition of fuzzy sets, fuzzy numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, generalized trapezoidal fuzzy numbers and etc., and their graphical representations. Furthermore, it also provides some information about Fuzzy Linear Programming, sub-classes and special classes of Fuzzy Linear Programming Problem.

Chapter 3 presents, a new fuzzy number named as Generalized Quadrilateral Fuzzy Numbers, membership function, arithmetic operations and its graphical representation. Moreover, it also presents the working procedure for Fully Fuzzy Transportation Problem and working procedure is applied on some numerical examples.

Chapter 4 deals with a new algorithm named as Stephen's Algorithm is proposed to solve Assignment Problem and a distinct method is presented to solve Fuzzy Assignment Problem. The efficiency of these methods is examined and the results are compared with existing algorithm.

Chapter 5 gives working procedure of Dynamic Programming Algorithm to solve both Travelling Salesman Problem and Fuzzy Travelling Salesman Problem with some additional steps.

Chapter 6 provides a new proposed approach named as DSD approach to solve both Travelling Salesman Problem and Fuzzy Travelling Salesman Problem with four cities. The results of the DSD Approach are compared with Dynamic Programming Algorithm.

Chapter 7 deals with a new proposed approach named as BCR approach to solve Travelling Salesman Problem and Fuzzy Travelling Salesman Problem with some illustrations for five cities. At the end of this chapter comprise the overall summary and its result of thesis.

Chapter 1

Introduction

Abstract

In this chapter, the theoretical background of the thesis is integrated. It deals with the details of chronological survey of the state of art. The development of optimization theory, linear programming problem, sub-classes and special classes of linear programming problem and fuzzy theory are discussed. The part bears a special attention to a section on motivation and to the scope of the thesis.

1.1 Introduction to Operations Research (OR)

Operations Research (OR) is one of the fastest developing fields in Science and Technology and also a discipline that deals with the development and applications of advanced analytical methods to improve decision-making. It is often considered to be a sub-field of mathematical sciences. Operation Research is the study of optimization techniques. No science has ever been born on a specific day. OR has no exception.

The development and invention of OR begun during the World War II. During the World War II, the military organization in England called on teams of scientist to study the strategic and tactical problems of air and land defence and gave the objective to find out the most effective allocation of limited military resources to various military operations. The group of scientists formed the first OR team. The name Operations Research was apparently coined in 1940, because the team was carrying out research on military operations. The impact of the success of the military teams attracted the attention of industrial people, who were seeking solution to their problems in other countries. In 1950, the concept was extended to academic study in American Universities. OR societies and activities were established in many countries.

In India the OR unit was arrived in 1949 at the Hyderabad Regional Research Laboratory. After that, some Indian organizations such as Airlines, Railways, Defense Organizations, Fertilizer Corporation of India, Hindustan Steel Ltd., Tata Iron and Steel Co., CSIR, BHEL, ONGC, etc. started to use OR techniques.

The word 'optimum' comes from a Latin word 'optimus' which means finding best one. Optimization is the process of finding best solution among all possible solution to the problem. In mathematics, optimization is meant for maximization or minimization. In real life situation, people are interested to maximize the profit and minimize the loss in all aspects. Especially, People in the field of business think about optimization in order to gain more profit. This is the basic principle of optimization.

Optimization problem is the problem of finding the best solution among all feasible solutions. Linear Programming is a technique for finding an optimum solution of linear functions subject to linear constraints. The term linear means that all variables which are occurring in the objective function and constraints are in the first degree. Programming is another word for ‘planning’ and it refers to the process of determination of a particular plan of action among several alternatives. Various techniques are available to solve linear programming problem such as graphical method, simplex method, Big-M method, revised simplex methods, etc.

Linear Programming Problem (LPP) has some subclasses. Transportation problem (TP) is one of the subclasses of LPPs. Assignment Problem (AP) is a special case of transportation problem. Travelling Salesman Problem (TSP) is similar to Assignment problem with some additional restrictions. Dynamic programming is the advanced model of OR and these models are used to make a series of interrelated decisions for multistage problems. Dynamic Programming Algorithm (DPA) is one of the best algorithms to solve travelling salesman problem.

1.2 Introduction to Fuzzy Concepts

Recent developments of Fuzzy set theory and its application have brought an exponential growth in research fields. Fuzzy set theory has numerous applications in various fields such as decision making, automata theory, artificial intelligence, neural networks, robotics, medical diagnosis and etc. Within a short span of time it has pervaded almost all fields and its applications are percolated down to customer goods level. Apart from that, it is being applied on a major scale in industries through intelligent robots for machine building (cars, engines, turbines, ships, etc.) and also used for military purpose.

The main contribution of fuzzy concepts in mathematics is its ability to represent vague information. It is used to model the systems that are difficult to exactly define. Fuzzy set theory incorporates vagueness and subjectivity. Many actions and strategies are designed to handle the uncertainties of the human behavior in decision-making.

Fuzzy set theory was introduced by Prof. Lotfi. A. Zadeh in 1965. Thousands of research papers were published in various journals which are devoted to theoretical and applications of fuzzy sets. It is a powerful way to quantitatively represent and manipulate imprecise decision-making problems. Since the vague parameters are treated as imprecise rather than precise values, the process is more powerful and results are more creditable. Fuzzy mathematics emerges as a tool to model processes the complex problems which are too complex for traditional techniques and information are inaccurate and unclear.

Fuzzy logic captures an inherent property of human communications which are not accurate, concise, perfectly clear, and crisp. Developments in fuzzy set theory have the potential to make it possible to offer a formal treatment of vagueness of natural language concepts. Fuzzy logic resembles the humans to make decisions and inferences. In fuzzy processing, there are three major components. They are fuzzification, fuzzy inference, and defuzzification. Fuzzification is the process of the input variables transformed into fuzzy numbers sets. Fuzzy inference is a set of fuzzy if-then-else rules used to process diffuse inputs and generate fuzzy outputs; that is, fuzzy inference interprets input vector values and based on set of rules, generates an output vector.

Defuzzification is the process of weighing and averaging out all fuzzy values into a single output or single decision. It is very useful to see the applicability of this tool for quantifying human reliability. Many authors have made contributions for developing new models with fuzzy quantification methodologies or using fuzzy techniques or methodologies for quantifying existing models.

1.3 Uncertainty

The management of uncertainties is one of the main issues for decision makers. Most of the decisions that are to be made by engineers and governors are subject to lack of information because of uncertainty. Some information is not always accessible and inadequate at the time of decision making. Expert's opinion is essential in many decisions-making models, where the parameters and their relationships are complex. There are numerous decision-making tools which are developed for dealing with the uncertain parameters.

Decision-making methods and approaches are created using the concept of a human being's expertise. So, it is necessary for developing models to be estimated and imitated to the human decision-making systems. The result of the study helps us to make decision models which can be used as human-thinking. The major issues of decision-making models are uncertainties of human behaviours and insufficient information. They are demonstrated in many forms:

- Fuzzy (not crisp, unclear, imprecise, estimated),
- Vague (not specific, unstructured),
- Ambiguous (too many choices),
- Natural variability (contradictory, random, chaotic, unpredictable).

Decision-making models are using the fuzzy logic to handle uncertainties of human behaviour in decision-making. In twentieth century, finding the alternatives for probability theory and traditional logic lead the way to handle uncertainties. Since fuzzy logic is the derivative of the probability theory, it is more suitable to apply in the fields of engineering, management, social and governance. In practice, most of the decision-making models not required exact outputs. In this case, we can make use of fuzzy theory, which can handle uncertainties and non-linearities. Fuzzy reliability theory carries a new perceptive of decision-making models, which are implemented with higher efficiency and reliability.

1.4 Impreciseness and Vagueness

Impreciseness and vagueness are facets of uncertainty and mean that an object or some of its features cannot be recorded or described precisely. Hence both of them are considered in contrast with randomness, which describes variability, being another kind of uncertainty, whereas probability theory and mathematical statistics deal with the behaviour of (perhaps hypothetical) populations, impreciseness and vagueness concerned with each single piece of information, called a datum. For handling such type of data, they must be described as mathematical items. The "classical" model for impreciseness is given by set - value - description. However, as it is known to us, the main problem in application of interval mathematics is fixing precise ends of the intervals. Moreover, data can be represented by verbal descriptions. They are called vague data.

1.5 Fuzzy Numbers

Fuzzy numbers have been introduced by Zadeh in order to deal with imprecise numerical quantities in a practical way. Since then, several authors have investigated properties and proposed applications of fuzzy numbers. The solution of fuzzy equations presents difficulties which are due to the fact that addition and multiplication of fuzzy numbers are non invertible operations. Consequently, the class of all fuzzy numbers is not a group under addition or under multiplication. Fuzzy numbers and their fuzzy operations are fundamentals of fuzzy number theory and fuzzy sets theory to model the fuzzy intelligent systems and computational models.

The domains of number theories in mathematics have been continuously expanding from binary numbers ($\mathbb{B}$), natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$), and real numbers ($\mathbb{R}$) to fuzzy numbers ($\mathbb{F}$) and hyperstructures ($\mathbb{H}$). It demonstrates an interesting course of advances in the human ability of abstraction and quantification in order to deal with the real-world entities and their perceptive representations in the brain. The difference between $\mathbb{N}$ and $\mathbb{Z}$ is that the members of the former are divisible by one, while those of the latter may not. The characteristic of the domain of fuzzy numbers $\mathbb{F}$ is a 2-dementia hyperstructure,$\mathbb{F} = \mathbb{R} \times \mathbb{R} = \{(\mathbb{R}, \mathbb{R})\}$, with a crisp set of member elements in $(-\infty, +\infty)$ and an associate crisp set of degrees of membership in [0, 1] for each of the members.

Fuzzy arithmetic is a system of fuzzy operations on fuzzy numbers. A theory of fuzzy arithmetic is presented towards a fuzzy mathematical structure for fuzzy inference and cognitive computation. The mathematical models of fuzzy numbers and their algebraic properties enable rigorous modeling of fuzzy entities in fuzzy systems and efficient manipulation of fuzzy variables in the fuzzy analysis, fuzzy inference, and fuzzy computing.

1.6 Fuzzy Optimization

Fuzzy optimization is a rapidly growing field and has increasing acceptance on scientific communities. Fuzzy optimization means the optimization in fuzzy environment. Classical optimization problems require well-defined parameters. But some real life problems, some parameters are roughly estimated. Fuzzy optimization offers the opportunity to handle such type of parameters. In this thesis, the transportation problem, assignment problem and travelling salesman problem are solved in fuzzy environment and some new solving techniques are discussed.

1.7 Review of Literature

This section reviews the background and development of Fuzzy Numbers, Linear Programming, Optimization, Transportation Problem, Assignment Problem and Travelling Salesman Problem in both crisp and fuzzy environment.

1.7.1 Fuzzy Numbers

Developments in fuzzy set theory have made a tremendous change in treatment of vagueness of natural language concepts. Zadeh [98] introduced the concept of Fuzzy sets and Fuzzy logic, which played an important role in the growth of fuzzy set theory. He characterized the set by a membership function which assigns to each object a grade of membership ranging between zero and one and also established the notions of inclusion, union, intersection, complement, relation, convexity etc. in the context of fuzzy sets. Zimmerman [100] studied the fuzzy concepts and its applications. Klir and Yuan [45] facilitated education in the areas of fuzzy set theory and fuzzy logic. This book reflects a broad variety of applications of fuzzy sets and fuzzy logic. To overcome the shortcomings of ranking the fuzzy numbers in earlier techniques, Abbasbandy and Babak Asady [1] modified the distance bases approach called sign distance. Nagoor Gani and Mohamed Assarudeen [67] modified some arithmetic operations on triangular fuzzy numbers. The modified operations yield an exact inverse of addition and multiplication operators. Stephen Dinagar and Abirami [86, 87] investigated some new algebraic arithmetic operations of the Generalized Interval Valued Fuzzy Numbers [GIVFNS] and studied the properties of single weighted and double weighted fuzzy numbers.

1.7.2 Linear programming and Optimization

Linear programming techniques play a major role to solve optimization problems. Charnes and Cooper [13] made an effort to show how linear programming models and methods of optimization are applied on manufacturing operation in 1953. Various techniques are available to solve linear programming problem such as graphical method, simplex method, Big-M method, revised simplex methods, etc. In 1947, Dantzig [17] devised the Simplex method to solve linear programming problems and introduced the basic theory of linear inequalities.

Zimmermann [99] employed a linear programming approach to the linear vector maximum problem. Tanaka [90] formulated a fuzzy linear programming problem based on comparing the fuzzy numbers. To obtain a reasonable solution for linear programming problem, Tanaka and Asai [91] considered the ambiguity of parameters. Buckley [9] introduced possibilistic programming which is an alternative to the stochastic programming. Mario Fedrizzi et al., [30] emphasis various concepts, problem classes, issues, etc. related to fuzzy mathematical programming and fuzzy optimization. Tong Shaocheng [79] focused two kinds of linear programming called as interval number and fuzzy linear programming in which the interval number coefficients are approached by taking maximum value range and minimum value range inequalities as constraint conditions and it is reduced into two classical linear programming problems.

Kuchta [46] recommended a definition of optimal solution for transportation problem with fuzzy cost coefficients using the general linear programming problem approach. Lodwick and David Jamison [54] described interval-based methods for solving constrained fuzzy optimization problems and focused to explore some relationships between fuzzy set theory and interval analysis.

Chanas and Kuchta [10] made a survey on selected discrete fuzzy optimization problems. The selected problems are transportation and assignment problems, maximal and minimal flow and shortest route in network, CPM and PERT method, selection of scheduling problems and knapsack and 0–1 problems. Fang et al., [29] introduced a cutting plane method for solving linear programming problems with fuzzy coefficients in constraint. John Chinneck and Ramadan [16] approached the linear programming in specified intervals in which the interval range of the optimized objective function and coefficient settings give some possibility extremes.

In many real life problems the decision parameters are fuzzy numbers. Such problems are solved by either possiblistic programming or programming methods. But all these methods have short comings. To overcome the short comings, Maleki et al., [55] introduced an effective method to solve fuzzy linear programming problem.

Ganesan and Veeramani [31] presented a new kind of arithmetic operation for symmetric trapezoidal fuzzy numbers to solve FLPP without converting them to crisp linear programming problems. Tang et al., [92] devoted a comprehensive understanding of fuzzy optimization theories and methods. Allahviranloo et al., [4] proposed a ranking function for defuzzifying the FLPP. In this approach symmetric triangular fuzzy numbers are applied on objective function and coefficient matrix.

Veeramani and Duraisamy [95] used the concept of nearest symmetric triangular fuzzy number approximation with expected interval. In this technique, fully fuzzy linear programming is transformed into two crisp linear problems. First part is used to calculate the centre objective value and second part is used to obtain the margin from the objective of the principal problem. Finally approximate fuzzy solution of FFLP is calculated. Mohanaselvi and Ganesan [58] investigated a new algorithm, based on the fuzzy ideal and fuzzy negative ideal solution of each single fuzzy objective function for fully fuzzy multi objective linear programming problems. Cheng et al., [15] introduced an approach which compromises programming to solve fully fuzzy linear programming by transforming the equality constraints into crisp inequality using the similarity measure, which is interpreted by the feasibility degree of constraints then fuzzy objective function is transformed into two crisp objectives functions.

The modified operations yield an exact inverse of addition and multiplication operators. Nasseri and Behmanesh [70] achieved some interesting fundamental results which lead the solution of fuzzy linear programming model without converting the problems to the crisp linear programming models.

Shapla Shirin and Kamrunnahar [81] reviewed three methods such as Bellman-Zadeh's method, Zimmerman's Method, and Fuzzy Version of Simplex method, are compared with each other to solve optimization problem. Moumita Deb and Pijus Kanti De [18] discussed graded mean integration representation method to solve Fully Fuzzy Linear Fractional Programming Problem (FFLFPP) with trapezoidal fuzzy numbers. To find the critical path for the project scheduling problem, TOPSIS Ranking method is utilized to rank More Generalized Interval Valued Fuzzy Numbers (MIVFNS). Qiumei Liu and Xin Gao [53] proposed two methods, Multi-Objective Linear Programming (MOLP) method and two-phase method, to solve FFLP problem.

Xiaofeng Xu et al., [96] presented an optimal allocation model of fuzzy resources for multistage random logistics with six-point trapezoidal fuzzy numbers and membership function. Nagoor Gani and Yogarani [66] developed an algorithm based on a polynomial order even penalty function, which gives a better solution to the fuzzy linear programming problem.

1.7.3 Transportation problem

Transportation problem deals with shipping a single commodity from various sources of supply to various destinations of demand with minimum total transportation cost. Hitchcock [38] developed some techniques to solve transportation problem in 1941. Charnes and Cooper [14] employed to explain the "stepping stone" method for transportation problems in 1954. Chanas and Kuchta [11] proposed an algorithm which solves the transportation problem with fuzzy supply and demand values with integrality condition. Nagoor Gani and Abdul Razak [65] used parametric approach to obtain the fuzzy solution for transportation problem.

Stephen Dinagar and Palanivel [85] proposed a modified distribution method to obtain optimal solution with the help of trapezoidal fuzzy numbers for fuzzy transportation problem. Amarpreet Kaur and Amit Kumar [44] introduced a new method to solve transportation problems with generalized trapezoidal fuzzy numbers. Edward Samuel and Venkatachalapathy [23] modified the Vogel's approximation method for fuzzy transportation problems. This method provides the nearest optimal solution to the transportation problems.

Nagoor Gani [64] et al., adopted Arsham - Khan's simplex algorithm to solve fuzzy transportation problem with fuzzy parameters without using any artificial variables. Abdul Quddoos et al., [75] constructed a new method to solve TP. Edward Samuel [24] added simple rule to fuzzy zero point method for crisp and fuzzy transportation problems and it was named as improved zero point method (IZPM). The added simple rule helped to improve zero point method. As a result of the simple rule always brings an optimal solution.

Amit Kumar and Amarpreet Kaur [48] pointed out some shortcomings to find the optimal solution for fuzzy transportation problems in existing methods. To overcome the shortcomings, two new methods have been proposed based on classical methods to solve unbalanced fuzzy transportation problems with all parameters are trapezoidal fuzzy numbers.

Edward Samuel and Venkatachalapathy [25] investigated some new methodologies for solving a special type of fuzzy transportation problem by assuming that uncertain about the precise values of transportation cost only but no certainty about the supply and demand of the product. In this method, transportation costs are represented as generalized trapezoidal fuzzy numbers. Edward Samuel and Venkatachalapathy [26] solved unbalanced fuzzy transportation problem, in which all parameters are represented as triangular fuzzy numbers. Solaiappan and Jeyaraman [82] investigated zero termination method to solve fuzzy transportation problem using Robust Ranking method to arrange the fuzzy number in specific interval. Edward Samuel and Raja [27, 28] adopted some new methodologies for solving an unbalanced transportation problem in fuzzy environment without converting into a balanced one, which gives an optimal solution.

Nirbhay Mathur et al., [57] established an algorithm which is quicker in terms of runtime compared to fuzzy VAM in [44, 59]. Muruganandam and Srinivasan [60] proposed an alternative method for transportation problem in which the objective function is ranked with the help of α-solution. Rajesh Kumar Saini et al., [78] pointed out some shortcoming in existing method and proposed ranking method for solving unbalanced fuzzy transportation problem. Dhanasekar et al., [20] proposed a new algorithm named as fuzzy Vogel Approximation Method (VAM) for solving FAP in which the parameters are triangular fuzzy numbers. Many researchers [22, 39, 40, 41, 69, 72] were interested to solve fuzzy transportation problems with various techniques.

1.7.4 Assignment Problem

Assignment Problem deals with assigning a task or job from a number of origins to equal number of destinations with minimum cost or maximum profit. Kuhn [47] explored the talented work of Hungarian mathematician. Lin Chi-Jen and Ue-Pyng Wen [50] handled a labeling algorithm in efficient way for solving assignment problem in fuzzy environment.

The assignment cost is usually deterministic in nature. But, Nagarajan and Solairaju [63] expressed the assignment cost in more realistic nature. i.e., the cost has been expressed as triangular or trapezoidal fuzzy numbers. Pandian and Kavitha [73] derived two theorems; one is used for ensuring optimal solution to the fuzzy assignment problem and another is used to improve the solution from the current solution to the FAP.

Hadi Basirzadeh [6] offered ones assignment method, which reduce the matrix with '1' in each row and column. To obtain an optimal solution for assignment probem '1' has been assigned to each and every column. Nagoor Gani and Mohamed [68] transformed the fuzzy assignment problem into crisp assignment problem using new ranking in LPP form. To obtain the solution, LINGO 9.0 has been used.

Srinivasan and Geetharamani [83] formulated the fuzzy assignment problem into crisp assignment problem in LPP using Robust's ranking method and the problems are solved by Ones Assignment Method. Thorani and Ravi Shankar [94] developed the classical algorithm using fundamental theorems of fuzzy assignment problem and variations of fuzzy assignment problem. Ahmed and Afaq Ahmad [3] proposed a method for finding an optimal solution of a wide range of assignment problems, directly.

Kalaiarasi [42] applied Robust's ranking indices on triangular fuzzy numbers and used ones assignment to find an optimal solution for fuzzy assignment problem. Nirmala and Anju [71] preferred the fuzzy quantifier and ranking method for triangular and trapezoidal fuzzy numbers for FAP. This method requires less number of iterations to obtain the optimal solution.

Sofiene Abidia et al., [2] have intended hybrid genetic algorithm for parking slots assignment problem for a group of drivers. Dutta and Pal [21] modified some steps in Hungarian method to find out optimal solution of assignment problem, which reduces the computation cost. Thiruppathi and Iranian [93] proposed an innovative method to solve assignment problem in single stage.

Ashwini and Srinivasan [5] formulated the fuzzy assignment problem into crisp assignment problem in LLP form using Robust's ranking method then it is solved by Ones Assignment method. Muruganandam and Hema [61] used graded mean integration representation method and converted the crisp assignment problem in LPP which is called as Fourier Elimination method. Queen Mary and Selvi [56] defuzzified the fuzzy assignment problem into crisp values using Centroid Ranking method, then Hungarian method is used to find optimal solution. Bhavika and Mitali [8] developed a modified approach of zero suffix method to solve assignment problem. Srinivasan et al., [84] studied the assignment problem in fuzzy environment with various ranking.

1.7.5 Travelling Salesman Problem

Travelling salesman problem stated as A salesman starting his journey form one city, planning to visit the rest of the specified cities only once and return to the start. Little John et al., [52] presented a branch and bound algorithm for travelling salesman problem in 1963. In this approach the set of all tours is broken into smaller subsets and the best lower bound on the cost length is calculated.

Bellmore and Nemhauser [7] recommended dynamic programming over branch and bound method to solve smaller problems in TSP, although the expected computer processing time is greater, they assured that the maximum time is very small. Held and Karp [36] explored new approaches to solve symmetric travelling salesman problem with slight variant in spanning trees which plays an essential role. Lin and Kernighan [51] discussed a highly effective heuristic and widely applicable procedure for generating optimum and nearer to optimum solutions for symmetric travelling salesman problem. This approach is also having ability to solve combinatorial optimization problems.

Richard M Karp [43] suggested hybrid schemes to get near optimum solutions to TSP, which are having thousands of cities, in which partitioning is used in conjunction with existing heuristic algorithms. Chandrasekaran et al., [12] proposed a method which solves the travelling salesman problems with transitive fuzzy numbers. Dhanasekar et al., [19] introduced a classical based assignment approach to solve FTSP using Yager's ranking [97] with some modifications in order to satisfy the route conditions by considering element wise subtraction. Hadi Basirzadeh [34] slightly modified the working procedure to solve travelling salesman problem by defining the distance matrix and then using matrix determinant. Shweta Rana [77] solved TSP using improved genetic algorithm.

Mythili et al., [62] applied dynamic programming technique on FTSP and implemented through MATLAB program. Prabakaran et al., [74] surveyed meta-heuristic algorithms like Ant Colony Optimization to arrive the solution but returns the best optimal route when compared to others. In order to decrease the complexity and to obtain a satisfactory solution in TSP. Hanif Halim and Ismail [35] used Meta-heuristic algorithms. Dafne Lagos et al.,[49] assessed the performance of assigning service squads and incorporated the variability of service times. The initial problem was formulated as a Travelling Salesman Problem (TSP), whose solution was obtained using ant colony algorithm.

Amit Kumar Rana [76] studied a solution for FTSP using transitive fuzzy numbers, in which travelling costs are expressed as fuzzy numbers due to road and traffic conditions. Gharehchopogh and Benyamin [32] presented a new metaheuristic algorithm named as Harris Hawk Optimization (HHO) algorithm which helps to maintain the main capabilities of the HHO algorithm, on one hand and to take advantage of the capabilities of active mechanisms in the continuous-valued problem space on the other hand.

Hiplito Hermandez-Perez and Juan–Jose Salazar-Gonzalez [37] proposed a new branch-and-cut algorithm to solve travelling salesman problem. It is spitted into two parts. The first part is master problem, which allows all feasible solutions and some invalid ones like mixed integer problem. The second part is sub–problem which checks the feasibility of the master solutions and generates valid cuts which are not feasible. This concept is applied to solve split-delivery one-commodity pickup and delivery travelling salesman problem. The fundamental concepts of linear programming, transportation problem, assignment problem and travelling salesman problem are overviewed from the standard books [33, 80, 88, 89].

1.8 Scope and Motivation of the Thesis

Fuzzy set theory is not a philosopher's stone, which solves all the problems that confront us today. But it has considerable potential for practical and for mathematical applications. As basic building stones of this thesis, the primary focus is on finding some new ranking procedures on fuzzy numbers to apply them in the fuzzy numbers involved in real-life applications. It is also briefly overviewed its background, main problems, methodologies, and recent developments. A special mention is also on the main literature, professional journals, technical newsletters, professional organizations, and other relevant information.

- The first phase of this research is focused on reviewing some notions of fuzzy numbers and to propose a new fuzzy number named as Generalized Quadrilateral Fuzzy Numbers, graphical representation and its arithmetic operations. It is very useful to apply on some fuzzy optimization problems and other problems.

- The second phase of this research is applied the Generalized Quadrilateral Fuzzy Numbers on transportation problem, assignment problem and travelling salesman problem with existing algorithms and newly proposed algorithms. To recognize the significance of the proposed algorithms and approaches, these results of proposed algorithms and approaches are compared with existing method.

1.9 Organization of the Thesis

This thesis entitled **"A Study on Fuzzy Optimization Problems"** consists of seven chapters.

Chapter 1 is introductory in nature which explicates the fundamental theoretical background of this thesis. Moreover, it also records the chronological details of literature survey on the topic of the thesis. This part also explains briefly about the motivation and scope of the research work.

Chapter 2 deals with some relevant definitions and fundamentals of fuzzy numbers such as definition of fuzzy sets, fuzzy numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers and etc., and their graphical representations.

Chapter 3 presents, a new fuzzy number named as Generalized Quadrilateral Fuzzy Numbers, membership function, arithmetic operations and its graphical representation. Moreover, it also presents the working procedure for fully fuzzy transportation problem and working procedure is applied on some numerical example. The content of this chapter has been published in the following international Journals:

- **Malaya Journal of Mathematik.**
- **American International Journal of Research in Science, Technology, Engineering and Mathematics.**

Chapter 4 deals with a new algorithm named as Stephen's Algorithm is proposed for assignment problem and a distinct method is presented to solve fuzzy assignment problem. The efficiency of these methods is examined and the results are compared with existing algorithm. The content of this chapter has been published in the following international Journals:

- **Advances and Applications in Mathematical Sciences.**
- **Mathematical Analysis and Computing.**

Chapter 5 gives working procedure of Dynamic Programming Algorithm to solve both travelling salesman problem and fuzzy travelling salesman problem with some additional steps. The content of this chapter has been published in the following international Journal:

- **Malaya Journal of Mathematik**

Chapter 6 provides a new proposed approach named as DSD Approach to solve both travelling salesman problem and travelling salesman problem in fuzzy environment with four cities. The procedure of the new approach is applied on some travelling and fuzzy travelling salesman problem and the results are compared with Dynamic Programming Algorithm. The content of this chapter has been communicated in the following international Journal:

- **South East Asian Journal of Mathematics and Mathematical Sciences.**

Chapter 7 deals with a new proposed approach named as BCR Approach to solve travelling salesman problem and the travelling salesman problem in fuzzy environment with illustrations for five cities. At the end of this chapter comprise the overall summary and its result of thesis. The content of this chapter is communicated in International Journal in the following title:

- **Solving Travelling Salesman Problem by BCR Approach in Fuzzy Environment.**

Chapter 2

Preliminaries

Abstract

In this chapter, some preliminaries which are required for the present work are presented. In Section 2.1, the basic definitions of Fuzzy Numbers such as the definition of Fuzzy Set, Convex Fuzzy Set, Fuzzy Number, Triangular Fuzzy Numbers, Trapezoidal Fuzzy Numbers and Generalized Trapezoidal Fuzzy Numbers and their graphical representations are discussed. In Section 2.2, the definitions related to fuzzy linear programming problem and some of its sub-classes such as Fuzzy Transportation Problem, Fuzzy Assignment Problem and Fuzzy Travelling Salesman Problem and its important concepts are discussed.

2.1 Preliminaries

In this section, some basic notions of fuzzy numbers are discussed.

Definition 2.1.1: (**Fuzzy Set**)

A fuzzy set $\tilde{A} = (x, \mu_{\tilde{A}}(x)), x \in A, \mu_{\tilde{A}}(x) \in [0,1]$. In this pair $(x, \mu_{\tilde{A}}(x))$, the first element x belongs to the classical set A and the second element $\mu_{\tilde{A}}(x)$ belongs to the interval $[0,1]$, called membership function.

Definition 2.1.2: (**Convex fuzzy set**)

A fuzzy set $\tilde{A}$ is convex if $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda x_2)) \geq \min\,(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(x)), x_1$, $x_2 \in X$ and $\lambda \in [0,1]$. Alternatively, a fuzzy set is convex, if all α - level sets are convex.

Definition 2.1.3: (**Fuzzy Number**)

A fuzzy set $\tilde{A}$ on $\Re$ must possess at least the following three properties to qualify a fuzzy number:

(i) $\tilde{A}$ must be a normal fuzzy set;

(ii) $\tilde{A}$ must be a convex fuzzy set;

(iii) $\tilde{A}$ must be closed and bounded.

Definition 2.1.4: (Triangular Fuzzy Number)

A fuzzy number represented with three points as follows $\tilde{A} = (a_l, a_m, a_r)$ is called triangular fuzzy number. This representation is interpreted as the following membership function and graphically represented in fig 2.1

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a_l \\ \dfrac{x - a_l}{a_m - a_l}, & a_l < x \leq a_m \\ 1, & x = a_m \\ \dfrac{a_r - x}{a_r - a_m}, & a_m < x \leq a_r \\ 0, & x \geq a_r \end{cases}$$

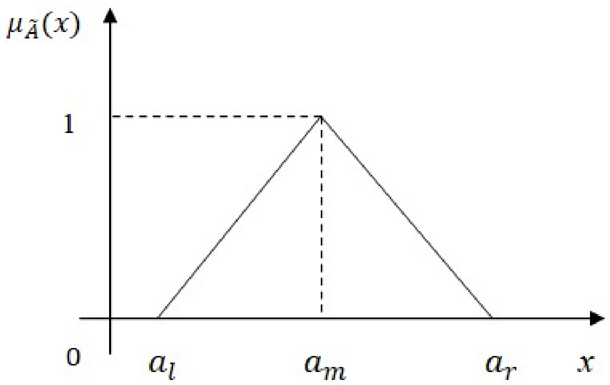

Figure 2.1: Graphical Representation of Triangular Fuzzy Numbers

Definition 2.1.5: (Trapezoidal Fuzzy Numbers)

A fuzzy number represented with four points as follows $\tilde{A} = (a, b, c, d)$ is called trapezoidal fuzzy number. This representation is interpreted as the following membership function and graphically represented in fig 2.2

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a < x \leq b \\ 1, & b \leq x \leq c \\ \dfrac{c-x}{d-c}, & c < x \leq d \\ 0, & x \geq d \end{cases}$$

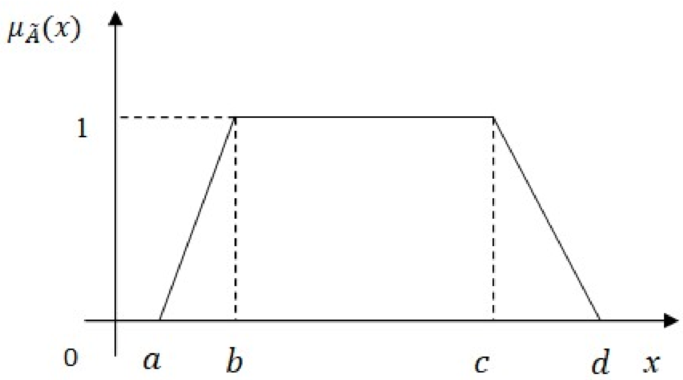

Figure 2.2: Graphical Representaion of Trapezoidal Fuzzy Numbers

Definition 2.1.6: (**Generalized Trapezoidal Fuzzy Numbers**)

A fuzzy number $\tilde{A}$ =(a,b,c,d; ω) is said to be generalized trapezoidal fuzzy numbers if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega\dfrac{(x-a)}{(b-a)}, & a \leq x < b \\ \omega, & b \leq x < c \\ \omega\dfrac{(x-d)}{(c-d)}, & c \leq x < d \end{cases}$$

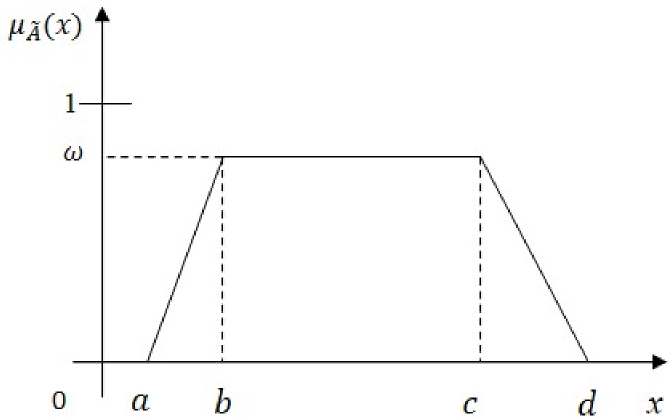

Figure 2.3: Graphical Representation of Generalized Trapezoidal Fuzzy Numbers

2.2 Some Basic Concepts of Fuzzy Linear Programming Problem and Some Subclasses

Definition 2.2.1: (**Fully Fuzzy Linear Programming Problem**)

In Linear Programming Problems, if the objective function and the constraints are expressed as fuzzy numbers, then the Linear Programming Problem is called Fully Fuzzy Linear Programming Problem.

Note: In some situations, sometimes left side or right side of the constraints may be expressed as precise value in fuzzy linear programming problems.

Definition 2.2.2: (**Fuzzy Transportation Problem**)

Fuzzy Transportation Problem is a Transportation Problem in which the parameters are expressed as fuzzy numbers. Fuzzy Transportation Problem has three parameters, fuzzy supply, fuzzy demand and fuzzy transportation costs. There are three types of transportation problem in fuzzy environment.

(i) Supply and Demand are expressed as fuzzy numbers and transportation costs are precise values.

(ii) Supply and Demand are precise values and transportation costs are expressed as fuzzy numbers.

(iii) All parameters are fuzzy numbers.

Definition 2.2.3: (**Fully Fuzzy Transportation Problem**)

In Transportation Problem, if all the parameters are expressed as fuzzy numbers then it is called as Fully Fuzzy Transportation Problem.

Definition 2.2.4: (**Fuzzy Assignment Problem**)

In Assignment Problem, if the assignment costs are expressed as fuzzy numbers, then the Assignment Problem is called Fuzzy Assignment Problem.

Definition 2.2.5: (**Fuzzy Travelling Salesman Problem**)

In Travelling Salesman Problem, if the travelling costs are expressed as fuzzy numbers then the Travelling Salesman Problem is called Fuzzy Travelling Salesman Problem.

Chapter 3

On Fuzzy Transportation Problem with Generalized Quadrilateral Fuzzy Numbers

Abstract

Aim of this chapter is to introduce Generalized Quadrilateral Fuzzy Numbers (GQFNs), arithmetic operations and graphical representation of GQFNs and to solve Fully Fuzzy Transportation problem(FFTP) by using classical methods with some necessary modifications. In these methods the supply, demand and transportation costs are represented by GQFNs. To understand and apply these ideas, relevant numerical examples are illustrated.

The part of the content of this chapter form the material of the papers published in

1. **Malaya Journal of Mathematik, 2019, S(1): 24-27**
2. **American International Journal of Research in Science, Technology, Engineering and Mathematics, 2019, 199-204.**

3.1 Generalized Quadrilateral Fuzzy Numbers (GQFNs)

In this section, a new fuzzy number named as Generalized Quadrilateral Fuzzy Number has been defined and its membership function and graphical representation are presented.

Definition 3.1.1: (**Generalized Quadrilateral Fuzzy Numbers**)

A new fuzzy number $\tilde{A} = (a, b, c, d; \omega_1, \omega_2)$ is defined as Generalized Quadrilateral Fuzzy Number, if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega_1 \dfrac{(x-a)}{(b-a)}, & a \leq x < b \\ \dfrac{(x-b)\omega_2 + (c-x)\omega_1}{(c-b)}, & b \leq x < c \\ \omega_2 \dfrac{(x-d)}{(c-d)}, & c \leq x < d \\ 0, & otherwise \\ \text{where} \quad 0 \leq \omega_1, \omega_2 \leq 1 & \end{cases}$$

3.1.1 Graphical Representation of Generalized Quadrilateral Fuzzy Numbers

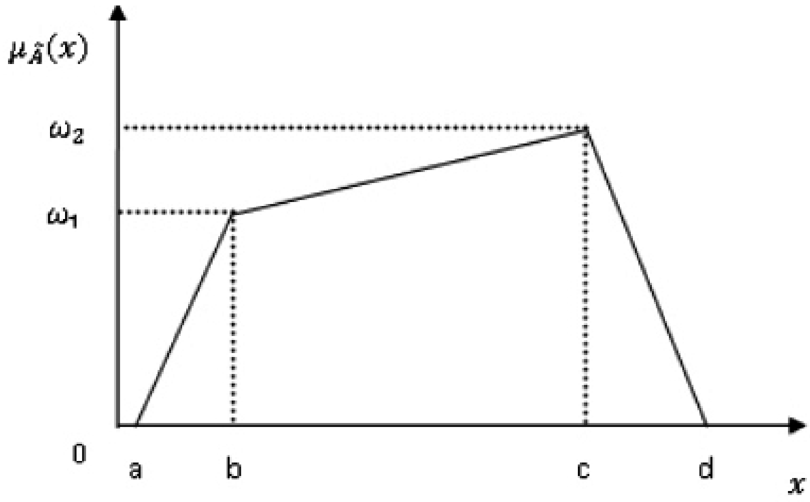

Figure 3.1: Generalized Quadrilateral Fuzzy Numbers.

3.2 Arithmetic Operations on GQFNs

In this section, the arithmetic operations between two generalized quadrilateral fuzzy numbers are defined on the universal set of real numbers $\Re$.

Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1; \omega_{A_1^1}, \omega_{A_1^2})$ and $\tilde{A}_2 = (a_2, b_2, c_2, d_2; \omega_{A_2^1}, \omega_{A_2^2})$

(i) Addition for GQFN:

$$\begin{aligned}\tilde{A}_1 \oplus \tilde{A}_2 &= (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; \\ &\quad min(\omega_{A_1^1}, \omega_{A_2^1}), min(\omega_{A_1^2}, \omega_{A_2^2})).\end{aligned}$$

(ii) Subtraction for GQFN:

$$\tilde{A}_1 \ominus \tilde{A}_2 = (a_1 - d_2, b_1 - c_2, c_1 - c_2, d_1 - b_2; \\ min(\omega_{A_1^1}, \omega_{A_2^1}), min(\omega_{A_1^2}, \omega_{A_2^2})).$$

(iii) Scalar Multiplication for GQFN:

$$\lambda\tilde{A}_1 = \begin{cases} \lambda a_1, \lambda b_1 \lambda c_1 \lambda d_1; (\omega_{A_1^1}, \omega_{A_1^2}) & \lambda \geq 0 \\ \lambda d_1, \lambda c_1 \lambda b_1 \lambda a_1; (\omega_{A_1^1}, \omega_{A_1^2}) & \lambda < 0. \end{cases}$$

3.2.1 Ranking function for GQFN

A new ranking function $\Re : F(\Re) \to \Re$, is prposed, which maps each fuzzy number into the real number. Let $\tilde{A}$=$(a, b, c, d; \omega_1, \omega_2)$, then

$$\Re(\tilde{A}) = \sum_{i=1}^{4} a_i \sum_{j=1}^{2} \omega_j$$

3.3 Fully Fuzzy Transportation Problem

More number of decision makers are interested to solve fuzzy transportation problems, where all parameters are represented as generalized quadrilateral fuzzy numbers.

Fuzzy transportation problem is formulated as

$$Min\ z = \sum_{i=1}^{m} \sum_{j=1}^{n} \widetilde{c_{ij}} \otimes \widetilde{x_{ij}}$$

Subject to

$$\sum_{j=1}^{n} \widetilde{x_{ij}} \leq \widetilde{a}_i, \qquad i = 1, 2, 3, ..., m$$

$$\sum_{i=1}^{m} \widetilde{x_{ij}} \leq \widetilde{b}_j, \qquad j = 1, 2, 3, ..., n$$

$$\tilde{x_{ij}} \geq 0, \forall i, j$$

3.4 Solving Fuzzy Transportation Problem using Unique Approach

In this section, an algorithm for solving fully fuzzy transportation problem is given with suitable numerical illustration.

3.4.1 Algorithm for Solving Fully Fuzzy Transportation Problem

The step by step procedure to solve the fully fuzzy transportation problem is given, in which all parameters are represented as generalized quadrilateral fuzzy numbers.

Step 1: Check whether the given problem is balanced transportation problem i.e $\sum_{i=1}^{m} \tilde{a}_i = \sum_{j=1}^{n} \tilde{b}_j$, if not convert it into balanced one by adding dummy row or dummy column.

Step 2: Convert the supply and demand into crisp number using ranking function in rows and columns.

Step 3: Find the minimum value either in supply or demand.

Find the minimum rank of $(\tilde{a}_i, \tilde{b}_j)$

Case(i): If minimum rank of $(\tilde{a}_i, \tilde{b}_j) = \tilde{a}_i$ then allocate $x_{ij} = \tilde{a}_i$ in $(i, j)^{th}$ cell. Ignore i^{th} row replace $\tilde{b}_j$ by $\tilde{b}_j – \tilde{a}_i$.

Case(ii): If minimum rank of $(\tilde{a}_i, \tilde{b}_j) = \tilde{b}_j$ then allocate $x_{ij} = \tilde{b}_j$ in $(i, j)^{th}$ cell. Ignore j^{th} row replace $\tilde{a}_i$ by $\tilde{a}_i – \tilde{b}_j$.

Step 4: Repeat the step 3 until $\tilde{a}_i$ and $\tilde{b}_j$ are zero.

3.4.2 Numerical Illustration

Consider the following fully fuzzy transportation problem. Here, the fuzzy supply $[\widetilde{a}_i]$ and fuzzy demand $[\widetilde{b}_j]$ are expressed as generalized quadrilateral fuzzy numbers. Determine the basic feasible solution.

Table 3.1: Fuzzy Transportation Problem

	D_1	D_2	D_3	D_4	$\widetilde{a}_i$
S_1	(1,2,3,4; 0.6,0.8)	(1,3,4,6; 0.6,0.8)	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	(1,6,7,12; 0.6,0.8)
S_2	(0,1,2,4; 0.6,0.8)	(-1,0,1,2; 0.6,0.8)	(5,6,7,8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	(0,1,2,3; 0.6,0.8)
S_3	(3,5,6,8; 0.6,0.8	(5,8,9,12; 0.6,0.8)	(12,15,16,19; 0.6,0.8)	(7,9,10,12; 0.6,0.8)	(5,10,12,15; 0.6,0.8)
$\widetilde{b}_j$	(5,7,8,10; 0.6,0.8)	(-1,5,6,10; 0.6,0.8)	(1,3,4,6; 0.6,0.8)	(1,2,3,4; 0.6,0.8)	(6,17,21,30; 0.6,0.8)

Solution

Step 1: Since $\sum_{i=1}^{m} \tilde{a}_i = \sum_{j=1}^{n} \tilde{b}_j$ the given problem is balanced fuzzy transportation problem.

step 2: Converting supply and demand into crisp number using ranking function in rows and columns.

Table 3.2: Converting the supply and demand into crisp number using ranking function

	D_1	D_2	D_3	D_4	$\Re(\widetilde{a}_i)$
S_1	(1,2,3,4; 0.6,0.8)	(1,3,4,6; 0.6,0.8)	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	4.55
S_2	(0,1,2,4; 0.6,0.8)	(-1,0,1,2; 0.6,0.8)	(5,6,7,8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	1.05
S_3	(3,5,6,8; 0.6,0.8	(5,8,9,12; 0.6,0.8)	(12,15,16,19; 0.6,0.8)	(7,9,10,12; 0.6,0.8)	7.35
$\Re(\widetilde{b_j})$	5.25	3.5	2.45	1.75	12.95

Step 3: Finding the minimum value either in supply or demand. Find minimum of $(\tilde{a}_i, \tilde{b_j})= \{5.25, 4.55\}$ $= 4.55$.

Minimum value is allocated in (S_1, D_1) cell and the minimum value is subtracted from 5.25 and the remaining values are written in the corresponding places.

Step 4: Repeat the step 3 until $\tilde{a}_i$ and $\tilde{b_j}$ are zero.

Table 3.3: First Allocation

	D_1	D_2	D_3	D_4	$\Re(\widetilde{a}_i)$
S_1	**(4.55)** (1,2,3,4; 0.6,0.8)	(1,3,4,6; 0.6,0.8)	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	0
S_2	(0,1,2,4; 0.6,0.8)	(-1,0,1,2; 0.6,0.8)	(5,6,7,8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	1.05
S_3	(3,5,6,8; 0.6,0.8)	(5,8,9,12; 0.6,0.8)	(12,15,16,19; 0.6,0.8)	(7,9,10,12;0.6,0.8)	7.35
$\Re(\widetilde{b}_j)$	0.7	3.5	2.45	1.75	8.4

Table 3.4: Second Allocaton

	D_1	D_2	D_3	D_4	$\Re(\widetilde{a}_i)$
S_1	**(4.55)** (1,2,3,4; 0.6,0.8)	(1,3,4,6; 0.6,0.8)	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	0
S_2	**(0.7)** (0,1,2,4; 0.6,0.8)	(-1,0,1,2; 0.6,0.8)	(5,6,7,8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	0.35
S_3	(3,5,6,8; 0.6,0.8)	(5,8,9,12; 0.6,0.8)	(12,15,16,19; 0.6,0.8)	(7,9,10,12; 0.6,0.8)	7.35
$\Re(\widetilde{b}_j)$	0	3.5	2.45	1.75	7.7

Table 3.5: Third allocation

	D_1	D_2	D_3	D_4	$\Re(\widetilde{a}_i)$
S_1	**(4.55)** (1,2,3,4; 0.6,0.8)	(1,3,4,6; 0.6,0.8)	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	0
S_2	**(0.7)** (0,1,2,4; 0.6,0.8)	**(0.35)** (-1,0,1,2; 0.6,0.8)	(5,6,7,8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	0
S_3	(3,5,6,8; 0.6,0.8)	(5,8,9,12; 0.6,0.8)	(12,15,16,19; 0.6,0.8)	(7,9,10,12; 0.6,0.8)	7.35
$\Re(\widetilde{b}_j)$	0	3.15	2.45	1.75	7.35

Table 3.6: Fourth Allocation

	D_1	D_2	D_3	D_4	$\Re(\widetilde{a}_i)$
S_1	**(4.55)** (1,2,3,4; 0.6,0.8)	(1,3,4,6; 0.6,0.8)	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	0
S_2	**(0.7)** (0,1,2,4; 0.6,0.8)	**(0.35)** (-1,0,1,2; 0.6,0.8)	(5,6,7,8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	0
S_3	(3,5,6,8; 0.6,0.8)	**(3.15)** (5,8,9,12; 0.6,0.8)	(12,15,16,19; 0.6,0.8)	(7,9,10,12; 0.6,0.8)	4.2
$\Re(\widetilde{b}_j)$	0	0	2.45	1.75	4.2

Table 3.7: Fifth Allocation

	D_1	D_2	D_3	D_4	$\Re(\widetilde{a}_i)$
S_1	**(4.55)** (1,2,3,4; 0.6,0.8)	(1,3,4,6; 0.6,0.8)	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	0
S_2	**(0.7)** (0,1,2,4; 0.6,0.8)	**(0.35)** (-1,0,1,2; 0.6,0.8)	(5,6,7,8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	0
S_3	(3,5,6,8; 0.6,0.8)	**(3.15)** (5,8,9,12; 0.6,0.8)	**(2.45)** (12,15,16,19; 0.6,0.8)	(7,9,10,12; 0.6,0.8)	1.75
$\Re(\widetilde{b}_j)$	0	0	0	1.75	1.75

Table 3.8: Basic Feasible Solution

	D_1	D_2	D_3	D_4	$\Re(\widetilde{a}_i)$
S_1	**(4.55)** (1,2,3,4; 0.6,0.8)	(1,3,4,6; 0.6,0.8)	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	0
S_2	**(0.7)** (0,1,2,4; 0.6,0.8)	**(0.35)** (-1,0,1,2; 0.6,0.8)	(5,6,7,8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	0
S_3	(3,5,6,8; 0.6,0.8)	**(3.15)** (5,8,9,12; 0.6,0.8)	**(2.45)** (12,15,16,19; 0.6,0.8)	**(1.75)** (7,9,10,12;0.6,0.8)	0
$\Re(\widetilde{b}_j)$	0	0	0	0	0

Here, all demand and supply are exhausted.

Therefore the initial basic feasible solution is

$$
\begin{aligned}
Minimum\widetilde{z} &= (4.55) \times (1, 2, 3, 4; 0.6, 0.8) \\
&\quad + (0.7) \times (0, 1, 2, 4; 0.6, 0.8) \\
&\quad + (0.35) \times (-1, 0, 1, 2; 0.6, 0.8) \\
&\quad + (3.15) \times (5, 8, 9, 12; 0.6, 0.8) \\
&\quad + (2.45) \times (12, 15, 16, 19; 0.6, 0.8) \\
&\quad + (1.75) \times (7, 9, 10, 12; 0.6, 0.8) \\
&= (61.6, 87.5, 100.45, 127.05; 0.6, 0.8).
\end{aligned}
$$

$$Min\ \Re(z) = 65.905.$$

3.5 Solving Fuzzy Transportation Problem using Novel Approach

In this section, to solve Fully Fuzzy Transportation Problem, the algorithm for novel approach is presented. To understand the procedure suitable numerical example is illustrated.

3.5.1 Algorithm for Solving Fuzzy Transportation Problem using Novel Approach

Step 1: Check whether the given problem is balanced transportation problem i.e $\sum_{i=1}^{m} \widetilde{a_i} = \sum_{j=1}^{n} \widetilde{b_j}$, if not convert it into balanced one by adding dummy row or dummy column.

Step 2: Identify the maximum cost in the demand and find minimum transportation cost in the corresponding column using ranking function.

Step 3: Find minimum of $(\widetilde{a_i}, \widetilde{b_j})$, Allocate $x_{ij} = min(\widetilde{a_i}, \widetilde{b_j})$

Case(i): If minimum $(\widetilde{a_i}, \widetilde{b_j}) = \widetilde{a_i}$ then allocate $x_{ij} = \widetilde{a_i}$ in $(i, j)^{th}$ cell. Ignore i^{th} row replace $\widetilde{b_j}$ by $\widetilde{b_j} - \widetilde{a_i}$.

Case(ii): If minimum $(\widetilde{a_i}, \widetilde{b_j}) = b_j$ then allocate $x_{ij} = \widetilde{b_j}$ in $(i, j)^{th}$ cell. Ignore j^{th} row replace $\widetilde{a_i}$ by $\widetilde{a_i} - \widetilde{b_j}$.

Case(iii): If there is a tie in minimum of $(\widetilde{a_i}, \widetilde{b_j})$, find the next minimum in the same column.

Step 4: Identify the next maximum cost in the demand and find the minimum transportation cost in the corresponding column.

Step 5: Repeat steps 3 and 4 until ($\widetilde{a}_i$ and $\widetilde{b}_j$) are zero.

3.5.2 Numerical Illustration

Example 3.5.1: Consider the following fully fuzzy transportation problem. Here all parameters are generalized quadrilateral fuzzy numbers. Determine the basic feasible solution.

Table 3.9: Fuzzy Transportation Problem

	D_1	D_2	D_3	D_4	$\widetilde{a}_i$
S_1	(1,2,3,4; 0.6,0.8)	(1,3,4, 6; 0.6,0.8)	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	(1,6,7,12; 0.6,0.8)
S_2	(0,1,2,4; 0.6,0.8)	(-1, 0,1,2; 0.6,0.8)	(5, 6,7, 8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	(0,1,2, 3; 0.6,0.8)
S_3	(3,5,6,8; 0.6,0.8)	(5,8,9,12; 0.6,0.8)	(12,15,16,19; 0.6,0.8)	(7,9,10,12; 0.6,0.8)	(5,10,12,15; 0.6,0.8)
$\widetilde{b}_j$	(5,7,8,10; 0.6,0.8)	(-1,5,6,10; 0.6,0.8)	(1,3,4,6; 0.6, 0.8)	(1,2,3,4; 0.6, 0.8)	

Solution:

Table 3.10: Fuzzy Transportation Problem

	D_1	D_2	D_3	D_4	$\widetilde{a}_i$
S_1	(1,2,3,4; 0.6,0.8)	(1,3,4,6; 0.6,0.8)	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	(1,6,7,12; 0.6,0.8)
S_2	(0,1,2,4; 0.6,0.8)	(-1,0,1,2; 0.6,0.8)	(5, 6,7, 8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	(0,1,2, 3; 0.6,0.8)
S_3	(3,5,6,8; 0.6,0.8)	(5,8,9,12; 0.6,0.8)	(12,15,16,19; 0.6,0.8)	(7,9,10,12; 0.6,0.8)	(5,10,12,15; 0.6,0.8)
$\widetilde{b}_j$	(5,7,8,10; 0.6,0.8)	(-1,5,6,10; 0.6,0.8)	(1,3,4,6; 0.6, 0.8)	(1,2,3,4; 0.6, 0.8)	(6,17,21,30;0.6,0.8)

Since $\sum_{i=1}^{m} \widetilde{a}_i = \sum_{j=1}^{n} \widetilde{b}_j$ the given problem is balanced fuzzy transportation problem.

Table 3.11: First Allocation

	D_1	D_2	D_3	D_4	$\widetilde{a}_i$
S_1	(1,2,3,4; 0.6,0.8)	(1,3,4, 6; 0.6,0.8)	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	(1,6,7,12; 0.6,0.8)
S_2	(0,1,2,4; 0.6,0.8) **(0,1,2,4; 0.6,0.8)**	(-1,0,1,2; 0.6,0.8)	(5, 6,7, 8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	(0,1,2, 3; 0.6,0.8)
S_3	(3,5,6,8; 0.6,0.8)	(5,8,9,12; 0.6,0.8)	(12,15,16,19; 0.6,0.8)	(7,9,10,12; 0.6,0.8)	(5,10,12,15; 0.6,0.8)
$\widetilde{b}_j$	(2,5,7,10; 0.6,0.8)	(-1,5,6,10; 0.6,0.8)	(1,3,4,6; 0.6, 0.8)	(1,2,3,4; 0.6, 0.8)	

Table 3.12: Second Allocation

	D_1	D_2	D_3	D_4	$\widetilde{a}_i$
S_1	(1,2,3,4; 0.6,0.8) **(2,5,7,10; 0.6,0.8)**	(1,3,4, 6; 0.6,0.8)	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	(-9,-1,2,10; 0.6,0.8)
S_2	(0,1,2,4; 0.6,0.8) **(0,1,2,4; 0.6,0.8)**	(-1, 0,1,2; 0.6,0.8)	(5, 6,7, 8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	(0,1,2, 3; 0.6,0.8)
S_3	(3,5,6,8; 0.6,0.8)	(5,8,9,12; 0.6,0.8)	(12,15,16,19; 0.6,0.8)	(7,9,10,12; 0.6,0.8)	(5,10,12,15; 0.6,0.8)
$\widetilde{b}_j$	(2,5,7,10; 0.6,0.8)	(-1,5,6,10; 0.6,0.8)	(1,3,4,6; 0.6, 0.8)	(1,2,3,4; 0.6, 0.8)	

Table 3.13: Third Allocation

	D_1	D_2	D_3	D_4	$\widetilde{a}_i$
S_1	(1,2,3,4; 0.6,0.8) **(2,5,7,10; 0.6,0.8)**	(1,3,4, 6; 0.6,0.8) **(-9,-1,2,10; 0.6,0.8)**	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	(-9,-1,2,10; 0.6,0.8)
S_2	(0,1,2,4; 0.6,0.8) **(0,1,2,4; 0.6,0.8)**	(-1, 0,1,2; 0.6,0.8)	(5, 6,7, 8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	(0,1,2, 3; 0.6,0.8)
S_3	(3,5,6,8; 0.6,0.8)	(5,8,9,12; 0.6,0.8)	(12,15,16,19; 0.6,0.8)	(7,9,10,12; 0.6,0.8)	(5,10,12,15; 0.6,0.8)
$\widetilde{b}_j$	(2,5,7,10; 0.6,0.8)	(-11,3,7,19;0.6,0.8)	(1,3,4,6; 0.6, 0.8)	(1,2,3,4; 0.6, 0.8)	

Table 3.14: Fourth Allocation

	D_1	D_2	D_3	D_4	$\widetilde{a}_i$
S_1	(1,2,3,4; 0.6,0.8) **(2,5,7,10; 0.6,0.8)**	(1,3,4, 6; 0.6,0.8) **(-9,-1,2,10; 0.6,0.8)**	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	(-9,-1,2,10; 0.6,0.8)
S_2	(0,1,2,4; 0.6,0.8) **(0,1,2,4; 0.6,0.8)**	(-1, 0,1,2; 0.6,0.8)	(5, 6,7, 8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	(0,1,2, 3; 0.6,0.8)
S_3	(3,5,6,8; 0.6,0.8)	(5,8,9,12; 0.6,0.8) **(-11,3,7,19;0.6,0.8)**	(12,15,16,19; 0.6,0.8)	(7,9,10,12; 0.6,0.8)	(-14,3,9,26; 0.6,0.8)
$\widetilde{b}_j$	(2,5,7,10; 0.6,0.8)	(-11,3,7,19;0.6,0.8)	(1,3,4,6; 0.6, 0.8)	(1,2,3,4; 0.6, 0.8)	

Table 3.15: Fifth Allocation

	D_1	D_2	D_3	D_4	$\widetilde{a}_i$
S_1	(1,2,3,4; 0.6,0.8) **(2,5,7,10; 0.6,0.8)**	(1,3,4, 6; 0.6,0.8) **(-9,-1,2,10; 0.6,0.8)**	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	(3,11,14,22; 0.6,0.8)
S_2	(0,1,2,4; 0.6,0.8) **(0,1,2,4; 0.6,0.8)**	(-1, 0,1,2; 0.6,0.8)	(5, 6,7, 8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	(0,1,2, 3; 0.6,0.8)
S_3	(3,5,6,8; 0.6,0.8)	(5,8,9,12; 0.6,0.8) **(-11,3,7,19;0.6,0.8)**	(12,15,16,19; 0.6,0.8) **(1,3,4,6; 0.6, 0.8)**	(7,9,10,12; 0.6,0.8)	(-20,-1,6,25; 0.6,0.8)
$\widetilde{b}_j$	(2,5,7,10; 0.6,0.8)	(-11,3,7,19;0.6,0.8)	(1,3,4,6; 0.6, 0.8)	(1,2,3,4; 0.6, 0.8)	

Table 3.16: Final Allocation

	D_1	D_2	D_3	D_4	$\widetilde{a}_i$
S_1	(1,2,3,4; 0.6,0.8) **(2,5,7,10; 0.6,0.8)**	(1,3,4, 6; 0.6,0.8) **(-9,-1,2,10; 0.6,0.8)**	(9,11,12,14; 0.6,0.8)	(5,7,8,11; 0.6,0.8)	(-9,-1,2,10; 0.6,0.8)
S_2	(0,1,2,4; 0.6,0.8) **(0,1,2,4; 0.6,0.8)**	(-1, 0,1,2; 0.6,0.8)	(5, 6,7, 8; 0.6,0.8)	(0,1,2,3; 0.6,0.8)	(0,1,2, 3; 0.6,0.8)
S_3	(3,5,6,8; 0.6,0.8)	(5,8,9,12; 0.6,0.8) **(-11,3,7,19;0.6,0.8)**	(12,15,16,19; 0.6,0.8) **(1,3,4,6; 0.6,0.8)**	(7,9,10,12; 0.6,0.8) **(1,2,3,4; 0.6,0.8)**	(-20,-1,6,25; 0.6,0.8)
$\widetilde{b}_j$	(2,5,7,10; 0.6,0.8)	(-11,3,7,19;0.6,0.8)	(1,3,4,6; 0.6, 0.8)	(1,2,3,4; 0.6, 0.8)	(-24,-4,4,24; 0.6,0.8)

Therefore the initial basic feasible solution is

$$
\begin{aligned}
Minimum\ \widetilde{z} &= [(2,5,7,10;0.6,0.8)(1,2,3,4;0.6,0.8) \\
&\quad +(0,1,2,3;0.6,0.8)(0,1,2,4;0.6,0.8) \\
&\quad +(-9,-1,2,10;0.6,0.8)(1,3,4,6;0.6,0.8) \\
&\quad +(-11,3,7,19;0.6,0.8)(5,8,9,12;0.6,0.8) \\
&\quad +(1,3,4,6;0.6,0.8)(12,15,16,19;0.6,0.8) \\
&\quad +(1,2,3,4;0.6,0.8)(7,9,10,12;0.6,0.8)] \\
&= (-66.5,71.225,124.6,250.425;0.6,0.8). \\
Min\ \Re(z) &= 66.45.
\end{aligned}
$$

Chapter 4

On Fuzzy Assignment Problem with Generalized Quadrilateral Fuzzy Numbers

Abstract

Aim of this chapter is to present the efficient algorithm named as Stephen's Algorithm and a distinct method to find the optimal solution for Fuzzy Assignment Problem. These algorithms provide less number of iterations to reach the optimality. To validate these algorithms, numerical examples are given and the results are compared with Hungarian Algorithm.

The part of the content of this chapter form the material of the papers published in

1. **Advances and Applications in Mathematical Sciences, 2021, 20(5): 887-894**
2. **Mathematical Analysis and Computing, 2021, 344, 221-229.**

4.1 Fuzzy Assignment Problem

In this section, matrix form of Assignment Problem and its mathematical formulation are given.

4.1.1 Fuzzy Assignment problem in the Matrix form

Table 4.1: Matrix form of Fuzzy Assignment Problem

Resources	Activity				
	A_1	A_2	A_3	...	A_n
Worker 1	$\widetilde{c_{11}}$	$\widetilde{c_{12}}$	$\widetilde{c_{13}}$	$\dots$	$\widetilde{c_{1n}}$
Worker 2	$\widetilde{c_{21}}$	$\widetilde{c_{22}}$	$\widetilde{c_{23}}$	$\dots$	$\widetilde{c_{2n}}$
Worker 3	$\widetilde{c_{31}}$	$\widetilde{c_{32}}$	$\widetilde{c_{33}}$	$\dots$	$\widetilde{c_{3n}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Worker n	$\widetilde{c_{n1}}$	$\widetilde{c_{n2}}$	$\widetilde{c_{n3}}$	$\dots$	$\widetilde{c_{nn}}$

4.1.2 The Mathematical formulation of Fuzzy Assignment Problem

$$Min\ \tilde{z} = \sum_{i=1}^{n}\sum_{j=1}^{n}\widetilde{c_{ij}} \otimes x_{ij}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, 2, 3, \dots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, 2, 3, \ldots, n$$

$$\text{where } x_{ij} = \begin{cases} 1 & \text{if } i^{th} \text{ person is assigned to } j^{th} \text{ job} \\ 0 & \text{otherwise} \end{cases}$$

$\widetilde{c_{ij}}$ = the fuzzy cost associated with assigning i^{th} resource to j^{th} activity.

4.2 Solving Assignment Problem using Stephen's Approach

In this section, a new algorithm named Stephen's Algorithm is proposed. To compare the result and the efficiency of this algorithm, Hungarian algorithm is presented.

4.2.1 Hungarian Algorithm (HA)

Step 1: Check whether, the given problem is balanced or not. If the problem is not balanced, balance it by adding dummy rows or columns.

Step 2: Locate the smallest element from each row and subtract the same from all other elements of the same row.

Step 3: Locate the smallest element from each column and subtract the same from all other elements of the same column.

Step 4: (a) Examine rows successively until a row with exactly single zero is found. Make an assignment to this zero by square bracket [ ] and cross off (×) all other zeros in the corresponding column. Continue the procedure until all rows have exactly one unmarked zero.

(b) Examine columns successively until a column with exactly single zero is found. Make an assignment to this zero by square bracket [ ] and cross off (×) all other zeros in the corresponding row. Continue the procedure, until all columns have exactly one unmarked zero.

Step 5: If each row and column contain exactly one assignment, then the current assignment will be optimal solution. If any one of the row or column is without assignment, the current assignment is not optimal. Go to step 6.

Step 6: Draw the minimum number of horizontal and /or vertical lines to cover all zeros using the procedure given below:

(a) Mark ($\surd$) the rows which are not having assignments.

(b) Mark ($\surd$) the columns (not already marked) which are having zeros in the marked row.

(c) Mark ($\surd$) the rows (not already marked) which are having assignments in marked column.

Repeat (b) and (c) until marking is not required.

(d) Draw minimum number of lines through all unmarked rows and marked columns to cover all these zeros.

(e) If the number of lines is equal to number of rows or number of columns in the table, then it is an optimum solution, otherwise not. If the solution is not optimal, go to step 7.

Step 7: Select the smallest uncovered element, subtract it from all other uncovered elements and add the same to all elements which are lying at the intersection of two lines except the elements covered by the lines.

Step 8: Go to step 5 and repeat the process until an optimum solution is attained.

4.2.2 Stephen's Algorithm

Step 1: Construct the assignment problem.

Step 2: Examine the smallest assignment cost of each column and then subtract the same cost from each cost in the corresponding column.

Step 3: If each column and each row has only one zero, the solution is optimal. Make the assignment, otherwise go to step 4.

Step 4: If any row has more than one zeros, find the penalty i.e., determine the difference between the smallest and the next-to-smallest. Display them alongside of the assignment table by enclosing them in parenthesis against the respective rows. Similarly compute the penalty for each column and display them below the assignment table by enclosing them in parenthesis against the respective column.

Step 5: Identify the row or column with largest penalty among all rows and columns. If a tie occurs, find the next penalty i.e., difference between second smallest cost and next to the second smallest. Repeat the process until break the tie.

Step 6: From the identified row or column, find the smallest assignment cost and make the assignment, then cross off the corresponding row and column values, except assignment cost.

Step 7: Go to step 4 and repeat the procedure until an optimum is reached.

4.2.3 Numerical Illustrations

Example 4.2.1: Consider the Assignment Problem of assigning four jobs to four persons. The assignment costs are given as follows:

Table 4.2: Assignment Problem 1

Persons	Jobs			
	A	B	C	D
I	1	4	6	3
II	9	7	10	9
III	4	5	11	7
IV	8	7	8	5

Determine the optimum assignment schedule and minimum total assignment cost.

Solution:

(a) By Hungarian Algorithm

Since the number of rows is equal to the number of columns, the given problem is a balanced Fuzzy Assignment Problem. Locate the smallest cost in each row and subtract the same cost from each cost in the corresponding row.

Table 4.3: Reduced Cost Matrix with Zero in every row

	A	B	C	D
I	0	3	5	2
II	2	0	3	2
III	0	1	7	3
IV	3	2	3	0

Locate the smallest cost in each column and subtract the same cost from each cost in the corresponding column.

Table 4.4: Reduced Cost Matrix with Zero in every row and column

	A	B	C	D
I	0	3	2	2
II	2	0	0	2
III	0	1	4	3
IV	3	2	0	0

Table 4.5: Making Assignment

	A	B	C	D
I	[0]	3	2	2
II	2	[0]	~~0~~	2
III	~~0~~	1	4	3
IV	3	2	[0]	~~0~~

Since third row has no assignment, the solution is not optimal. Therefore follow step 6 and 7 to get the following solution.

Table 4.6: Making Assignment

	A	B	C	D
I	[0]	2	1	1
II	3	~~0~~	[0]	2
III	~~0~~	[0]	3	2
IV	3	2	~~0~~	[0]

Since each row and each column has one assignment, the problem has an optimal solution. The Optimal Assignment is given below:

Table 4.7: Optimal Assignment Table

Person	Assigned to	Job	Assignment Cost
I	→	A	1
II	→	C	10
III	→	B	5
IV	→	D	5
	Total Assignment cost		21

(b) By Stephen's Algorithm

Since the number of rows is equal to the number of columns, the given problem is a balanced Fuzzy Assignment Problem. Locate the smallest cost in each column and subtract the same cost from each cost in the corresponding column.

Table 4.8: Reduced Cost Matrix with Zero in every column

	A	B	C	D
I	0	0	0	0
II	8	3	4	6
III	3	1	5	4
IV	7	3	2	2

Since the first row has four zeros, second, third and fourth row have no zeros. Therefore, the solution is not optimal.

Find the penalty using step 4.

Table 4.9: Finding Penalties 1

	A	B	C	D	Penalty
I	0	0	0	0	[0]
II	8	3	4	6	[1]
III	3	1	5	4	[2]
IV	7	3	2	2	[0]
Penalty	[3]	[1]	[2]	[2]	

Since the largest penalty is 3, choose the corresponding column. Here, the smallest assignment cost is 0. Therefore, make the assignment.

Table 4.10: Making Assignment 1

	A	B	C	D
I	[0]	*	*	*
II	*	3	4	6
III	*	1	5	4
IV	*	3	2	2

Table 4.11: Finding Penalties 2

	A	B	C	D	Penalty
I	[0]	*	*	*	[−]
II	*	3	4	6	[1]
III	*	1	5	4	[3]
IV	*	3	2	2	[0]
Penalty	[−]	[2]	[2]	[2]	

Since the largest penalty is 3, choose the corresponding column. Here, the smallest assignment cost is 1. Therefore, make the assignment.

Table 4.12: Making Assignment 2

	A	B	C	D
I	[0]	*	*	*
II	*	*	4	6
III	*	[1]	*	*
IV	*	*	2	2

Table 4.13: Finding Penalties 3

	A	B	C	D	Penalty
I	[0]	*	*	*	[−]
II	*	*	4	6	[2]
III	*	[1]	*	*	[−]
IV	*	*	2	2	[0]
Penalty	[−]	[−]	[2]	[4]	

Since the largest penalty is 4, choose the corresponding column. Here, the smallest assignment cost is 2. Therefore, make the assignment.

Table 4.14: Making Assignment 3

	A	B	C	D
I	[0]	*	*	*
II	*	*	4	*
III	*	[1]	*	*
IV	*	*	*	[2]

There is only one assignment in the table. Make the assignment.

Table 4.15: Final Assignment

	A	B	C	D
I	[0]	*	*	*
II	*	*	[4]	*
III	*	[1]	*	*
IV	*	*	*	[2]

Since each row and column has only one assignment, the solution is optimal.

The Optimal Assignment is given below:

Table 4.16: Optimal Assignment Table

Assignment	Assignment Cost
I → A	1
II → C	10
III → B	5
IV → D	5
Total Assignment cost	21

Example 4.2.2: Consider the Assignment Problem of assigning four salesmen to four areas. The assignment costs are given as follows:

Table 4.17: Assignment Problem 2

Salesman	Area			
	W	X	Y	Z
A	11	17	8	16
B	9	7	12	6
C	13	16	15	12
D	14	10	12	11

Solution:

(a) By Hungarian algorithm

Since, the number of rows is equal to the number of columns, the problem is a balanced Fuzzy Assignment Problem. Locate the smallest cost in each row and subtract the same cost from each cost in the corresponding row.

Table 4.18: Reduced cost matrix with zero in every row

	W	X	Y	Z
A	3	9	0	8
B	3	1	6	0
C	1	4	3	0
D	4	0	2	1

Locate the smallest cost in each column and subtract the same cost from each cost in the corresponding column.

Table 4.19: Reduced cost matrix with zero in every row and column

	W	X	Y	Z
A	2	9	0	8
B	2	1	6	0
C	0	4	3	0
D	3	0	1	1

Table 4.20: Making Assignment

	W	X	Y	Z
A	2	9	[0]	8
B	2	1	6	[0]
C	[0]	4	3	X
D	3	[0]	1	1

Since each row and each column has only one assignment, the problem has an optimal solution. The Optimal Assignment is given below:

Table 4.21: Optimal Assignment Table

Salesman	Assigned to	Area	Assignment Cost
A	→	Y	8
B	→	Z	6
C	→	W	13
D	→	X	10
	Total Assignment cost		37

(b) By Stephen's Algorithm

Since the number of rows is equal to the number of columns, the given problem is a balanced Assignment Problem. Locate the smallest cost in each column and subtract the same cost from each cost in the corresponding column.

Table 4.22: Reduced Cost Matrix with Zero in each column

	W	X	Y	Z
A	2	10	0	10
B	0	0	4	0
C	4	9	7	6
D	5	3	4	5

Since third and fourth row do not have zeros, the solution is not optimal.

Table 4.23: Finding Penalties 1

	W	X	Y	Z	Penalty
A	2	10	0	10	[2]
B	0	0	4	0	[0]
C	4	9	7	6	[2]
D	5	3	4	5	[1]
Penalty	[2]	[3]	[4]	[5]	

Since the largest penalty is 5, choose the corresponding column. Here, the smallest assignment cost is 0. Therefore, make the assignment.

Table 4.24: Making Assignment 1

	W	X	Y	Z
A	2	10	0	*
B	*	*	*	[0]
C	4	9	7	*
D	5	3	4	*

Table 4.25: Finding Penalties 2

	W	X	Y	Z	Penalty
A	2	10	0	*	[2]
B	*	*	*	[0]	[−]
C	4	9	7	*	[3]
D	5	3	4	*	[1]
Penalty	[2]	[6]	[4]	[−]	

Since the largest penalty is 6, choose the corresponding column. Here, the smallest assignment cost is 3. Therefore, make the assignment.

Table 4.26: Making Assignment 2

	W	X	Y	Z
A	2	*	0	*
B	*	*	*	[0]
C	4	*	7	*
D	*	[3]	*	*

Table 4.27: Finding Penalties 3

	W	X	Y	Z	Penalty
A	2	*	0	*	[2]
B	*	*	*	[0]	[−]
C	4	*	7	*	[3]
D	*	[3]	*	*	[-]
Penalty	[2]	[−]	[7]	[−]	

Since the largest penalty is 7, choose the corresponding column. Here, the smallest assignment cost is 0. Therefore, make the assignment.

Table 4.28: Making Assignment 3

	W	X	Y	Z
A	*	*	[0]	*
B	*	*	*	[0]
C	4	*	*	*
D	*	[3]	*	*

There is only one assignment in the table. Therefore make the assignment.

Table 4.29: Final Assignment

	W	X	Y	Z
A	*	*	[0]	*
B	*	*	*	[0]
C	[4]	*	*	*
D	*	[3]	*	*

Since each row and each column has only one assignment, the problem has an optimal solution.

The optimal Assignment is given below:

Table 4.30: Optimal Assignment Table

Salesman	assigned to	Area	Assignment Cost
A	→	Y	8
B	→	Z	6
C	→	W	13
D	→	X	10
	Total Assignment cost		37

4.3 Solving Fuzzy Assignment Problem using Distinct Approach

In this section, a computational procedure is formulated to solve the Fuzzy Assignment Problem with GQFNs.

4.3.1 Computational procedure

Step 1: Check whether, the number of sources is equal to the number of destinations; if it is not equal, add a dummy row or dummy column with zeros in assignment matrix.

Step 2: Convert the fuzzy assignment cost into crisp values.

Step 3: Locate the smallest element in each column.

Step 4: If each column and each row has only one assignment, optimal solution is reached. Otherwise go to step 5.

Step 5: If any row has more than one assignment, find the penalty.

Step 6: Identify the maximum penalty, from the identified row or column choose the smallest element and make the assignment then omit the corresponding row and column values except the assignment value. If there is a tie in the maximum penalty, find difference between the next smallest elements.

Step 7: Continue the process until each column and each row has only one assignment.

4.3.2 Numerical Illustration

In this section, the Assignment Problem is solved with Generalized Quadrilateral Fuzzy Numbers.

Example 4.3.1: Consider the fuzzy assignment problem of assigning four workers to four jobs. Assigning costs are represented by Generalized Quadrilateral Fuzzy Numbers and given in matrix form in table 4.31. Find the fuzzy optimal assignment.

Solution:

Since, the number of rows is equal to the number of columns, the given problem is a balanced Fuzzy Assignment Problem.

Table 4.31: Fuzzy Assignment Problem

Workers	Jobs			
	J1	J2	J3	J4
A	$(3, 5, 6, 7; 0.2, 0.4)$	$(5, 8, 11, 12; 0.3, 0.5)$	$(9, 10, 11, 15; 0.2, 0.4)$	$(5, 8, 10, 11; 0.3, 0.5)$
B	$(7, 8, 10, 11; 0.3, 0.6)$	$(3, 5, 6, 7; 0.2, 0.4)$	$(6, 8, 10, 12; 0.3, 0.6)$	$(5, 8, 9, 10; 0.2, 0.4)$
C	$(2, 4, 5, 6; 0.2, 0.4)$	$(5, 7, 10, 11; 0.3, 0.6)$	$(8, 11, 13, 15; 0.2, 0.4)$	$(4, 6, 7, 10; 0.3, 0.6)$
D	$(6, 8, 10, 12; 0.3, 0.6)$	$(2, 5, 6, 7; 0.2, 0.5)$	$(5, 7, 10, 11; 0.3, 0.5)$	$(2, 4, 5, 7; 0.2, 0.4)$

Table 4.32: Converted Fuzzy Assignment Cost into Crisp Values

Workers	Jobs			
	J1	J2	J3	J4
A	1.575	3.6	3.375	3.4
B	4.05	1.575	4.05	2.4
C	1.275	3.7125	3.525	3.0375
D	4.05	1.75	3.3	1.35

Table 4.33: Locating the Smallest element in each column

Workers	Jobs			
	J1	J2	J3	J4
A	(3, 5, 6, 7; 0.2, 0.4)	(5, 8, 11, 12; 0.3, 0.5)	(9, 10, 11, 15; 0.2, 0.4)	(5, 8, 10, 11; 0.3, 0.5)
B	(7, 8, 10, 11; 0.3, 0.6)	**(3, 5, 6, 7; 0.2, 0.4)**	(6, 8, 10, 12; 0.3, 0.6)	(5, 8, 9, 10; 0.2, 0.4)
C	**(2, 4, 5, 6; 0.2, 0.4)**	(5, 7, 10, 11; 0.3, 0.6)	(8, 11, 13, 15; 0.2, 0.4)	(4, 6, 7, 10; 0.3, 0.6)
D	(6, 8, 10, 12; 0.3, 0.6)	(2, 5, 6, 7; 0.2, 0.5)	**(5, 7, 10, 11; 0.3, 0.5)**	**(2, 4, 5, 7; 0.2, 0.4)**

Since 4^{th} row has more than one assignment, the solution is not optimal. Find the largest penalty. The largest penalty is in the 4^{th} column. The smallest value is (2, 4, 5, 7; 0.2, 0.4) and make the assignment in (D, J4), then omit the other values.

Table 4.34: Making the Assignment and deleting all other values in the corresponding rows and columns

Workers	Jobs			
	J1	J2	J3	J4
A	$(3, 5, 6, 7; 0.2, 0.4)$	$(5, 8, 11, 12; 0.3, 0.5)$	$(9, 10, 11, 15; 0.2, 0.4)$	$*$
B	$(7, 8, 10, 11; 0.3, 0.6)$	$(3, 5, 6, 7; 0.2, 0.4)$	$(6, 8, 10, 12; 0.3, 0.6)$	$*$
C	$(2, 4, 5, 6; 0.2, 0.4)$	$(5, 7, 10, 11; 0.3, 0.6)$	$(8, 11, 13, 15; 0.2, 0.4)$	$*$
D	$*$	$*$	$*$	$(\mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{7}; \mathbf{0.2}, \mathbf{0.4})$

Table 4.35: Again locating the smallest element in each column

Workers	Jobs			
	J1	J2	J3	J4
A	$(3, 5, 6, 7; 0.2, 0.4)$	$(5, 8, 11, 12; 0.3, 0.5)$	$(\mathbf{9}, \mathbf{10}, \mathbf{11}, \mathbf{15}; \mathbf{0.2}, \mathbf{0.4})$	$*$
B	$(7, 8, 10, 11; 0.3, 0.6)$	$(\mathbf{3}, \mathbf{5}, \mathbf{6}, \mathbf{7}; \mathbf{0.2}, \mathbf{0.4})$	$(6, 8, 10, 12; 0.3, 0.6)$	$*$
C	$(\mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{6}; \mathbf{0.2}, \mathbf{0.4})$	$(5, 7, 10, 11; 0.3, 0.6)$	$(8, 11, 13, 15; 0.2, 0.4)$	$*$
D	$*$	$*$	$*$	$(\mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{7}; \mathbf{0.2}, \mathbf{0.4})$

Now, each row and each column has only one assignment. Therefore optimal assignment has been reached.

Table 4.36: Delete all values in the rows and columns except assignment values

Workers	Jobs			
	J1	J2	J3	J4
A	*	*	$(\mathbf{9}, \mathbf{10}, \mathbf{11}, \mathbf{15}; \mathbf{0.2}, \mathbf{0.4})$	*
B	*	$(\mathbf{3}, \mathbf{5}, \mathbf{6}, \mathbf{7}; \mathbf{0.2}, \mathbf{0.4})$	*	*
C	$(\mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{6}; \mathbf{0.2}, \mathbf{0.4})$	*	*	*
D	*	*	*	$(\mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{7}; \mathbf{0.2}, \mathbf{0.4})$

The optimal assignment and the corresponding assignment costs are given below

Table 4.37: Optimal Assignment Table

Worker	Job	Assignment cost
A	3	$(\mathbf{9}, \mathbf{10}, \mathbf{11}, \mathbf{15}; \mathbf{0.2}, \mathbf{0.4})$
B	2	$(\mathbf{3}, \mathbf{5}, \mathbf{6}, \mathbf{7}; \mathbf{0.2}, \mathbf{0.4})$
C	1	$(\mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{6}; \mathbf{0.2}, \mathbf{0.4})$
D	4	$(\mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{7}; \mathbf{0.2}, \mathbf{0.4})$

$$\text{Minimum Assignment cost} = (9, 10, 11, 15; 0.2, 0.4) + (3, 5, 6, 7; 0.2, 0.4) + (2, 4, 5, 6; 0.2, 0.4) + (2, 4, 5, 7; 0.2, 0.4)$$

$$\text{Minimum Assignment cost} = (16, 23, 27, 35; 0.2, 0.4)$$

Rank of minimum Assignment cost $= 7.575$.

Chapter 5

On Fuzzy Travelling Salesman Problem with Generalized Quadrilateral Fuzzy Numbers

Abstract

In travelling salesman problem, travelling costs are represented as crisp values. But in practice, it is impossible to express travelling costs as crisp values. In this chapter, travelling costs are expressed as Generalized Quadrilateral Fuzzy Numbers. Since Dynamic Programming is a mathematical tool for solving complex problems, Dynamic Programming algorithms are applied to solve TSP and FTSP. To validate this technique, few numerical examples are illustrated.

The part of the content of this chapter form the material of the paper published in
Malaya Journal of Mathematik, 2021, S(1): 448-450.

5.1 Travelling Salesman Problem in Fuzzy Environment

In this section, a matrix form of fuzzy travelling salesman problem and its mathematical formulation have been presented.

5.1.1 Matrix form of Travelling Salesman Problem with Fuzzy Parameters

Table 5.1: Fuzzy Travelling Salesman Problem

From	To				
	City 1	City 2	City 3	...	City n
City 1	∞	$\widetilde{c_{12}}$	$\widetilde{c_{13}}$	...	$\widetilde{c_{1n}}$
City 2	$\widetilde{c_{21}}$	∞	$\widetilde{c_{23}}$	...	$\widetilde{c_{2n}}$
City 3	$\widetilde{c_{31}}$	$\widetilde{c_{32}}$	∞	...	$\widetilde{c_{3n}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
City n	$\widetilde{c_{n1}}$	$\widetilde{c_{n2}}$	$\widetilde{c_{n3}}$	...	∞

5.1.2 The Mathematical Formulation of Fuzzy Travelling Salesman Problem

$$\tilde{z} = \sum_{i=1}^{n}\sum_{j=1}^{n} \widetilde{c_{ij}} \otimes x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, 2, 3, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, 2, 3, \ldots, n$$

where $x_{ij} = \begin{cases} 1 & \text{if the salesman travelling from } i^{th} \text{ city to } j^{th} \text{ city} \\ 0 & \text{otherwise} \end{cases}$

$\widetilde{c_{ij}}$ = fuzzy travelling cost from i^{th} city to j^{th} city.

5.2 Solution for Travelling Salesman Problem using Dynamic Programming Algorithm

In this section, Dynamic Programming Algorithm is applied to solve travelling salesman problem. The given procedure is applicable for travelling salesman problem limited with four cities. The procedure is extendable for travelling salesman problem with n-cities according to our purpose.

5.2.1 Dynamic Programming Algorithm (DPA) for Travelling Salesman Problem with four cities

Stage 1:

$$\phi(B, A) = C_{BA}$$

$$\phi(C, A) = C_{CA}$$

$$\phi(D, A) = C_{DA}$$

Stage 2:

$$\phi\left(C,[B]\right) = C_{CB} + C_{BA}$$

$$\phi\left(D,[B]\right) = C_{DB} + C_{BA}$$

$$\phi\left(B,[C]\right) = C_{BC} + C_{CA}$$

$$\phi\left(D,[C]\right) = C_{DC} + C_{CA}$$

$$\phi\left(B,[D]\right) = C_{BD} + C_{DA}$$

$$\phi\left(C,[D]\right) = C_{CD} + C_{DA}$$

Stage 3:

$$\begin{aligned}
\phi\left(D,[B,C]\right) &= min\left\{C_{DB} + \phi\left(B,[C]\right), C_{DC} + \tilde{\phi}\left(C,[B]\right)\right\} \\
\phi\left(C,[B,D]\right) &= min\left\{C_{CB} + \phi\left(B,[D]\right), C_{CD} + \tilde{\phi}\left(D,[B]\right)\right\} \\
\phi\left(B,[C,D]\right) &= min\left\{C_{BC} + \phi\left(C,[D]\right), C_{BD} + \tilde{\phi}\left(D,[C]\right)\right\}
\end{aligned}$$

Stage 4:

$$\begin{aligned}
F &= \phi(A,[B,C,D]) \\
&= min\left\{C_{AB} + \phi(B,[C,D]), C_{AC} + \phi(C,[B,D]),\right. \\
&\qquad \left. C_{AD} + \phi(D,[B,C])\right\}
\end{aligned}$$

The minimum value is optimal travelling cost. The corresponding route is the optimal travelling route.

5.2.2 Numerical Illustrations

Example 5.2.1: Consider a travelling salesman problem: A salesman is planned to visit 4 cities. He would like to start his journey from a particular city, visits each city only once and returns to the home city. The travelling costs are given in table 5.2. Find the least cost route.

Table 5.2: Travelling Salesman Problem 1

From	To			
	City A	City B	City C	City D
City A	∞	4	7	3
City B	4	∞	6	3
City C	7	6	∞	7
City D	3	3	7	∞

Solution:

Stage 1:

$$\phi(B, A) = C_{BA} = 4$$
$$\phi(C, A) = C_{CA} = 7$$
$$\phi(D, A) = C_{DA} = 3$$

Stage 2:

$$\phi(C, [B]) = C_{CB} + C_{BA} = 6 + 4 = 10$$
$$\phi(D, [B]) = C_{DB} + C_{BA} = 3 + 4 = 7$$
$$\phi(B, [C]) = C_{BC} + C_{CA} = 6 + 7 = 13$$

$$\begin{aligned}
\phi(D,[C]) &= C_{DC}+C_{CA}=7+7=14\\
\phi(B,[D]) &= C_{BD}+C_{DA}=3+3=6\\
\phi(C,[D]) &= C_{CD}+C_{DA}=7+3=10
\end{aligned}$$

Stage 3:

$$\begin{aligned}
\phi(D,[B,C]) &= min\left\{C_{DB}+\phi(B,[C]),C_{DC}+\tilde{\phi}(C,[B])\right\}\\
&= min\{3+13,7+10\}\\
&= min\{16,17\}\\
&= 16\\
\phi(C,[B,D]) &= min\left\{C_{CB}+\phi(B,[D]),C_{CD}+\tilde{\phi}(D,[B])\right\}\\
&= min\{6+6,7+7\}\\
&= min\{12,14\}\\
&= 12\\
\phi(B,[C,D]) &= min\left\{C_{BC}+\phi(C,[D]),C_{BD}+\tilde{\phi}(D,[C])\right\}\\
&= min\{6+10,3+14\}\\
&= min\{16,17\}\\
&= 16
\end{aligned}$$

Stage 4:

$$\begin{aligned}
F &= \phi(A,[B,C,D])\\
&= min\{C_{AB}+\phi(B,[C,D]),C_{AC}+\phi(C,[B,D]),\\
&\qquad C_{AD}+\phi(D,[B,C])\}\\
&= min\{4+16,7+12,3+16\}\\
&= min\{20,19,19\}\\
&= 19.
\end{aligned}$$

The optimal routes are

$A \to C \to B \to D \to A$ and

$A \to D \to B \to C \to A.$

The optimal travelling cost is 19.

Example 5.2.2: Consider a travelling salesman problem: A salesman has planned to visit 4 cities. He would like to start his journey from a particular city, visits each city only once and returns to the home city. The travelling costs are given in table 5.3. Find the least cost route.

Table 5.3: Travelling Salesman Problem 2

From	To			
	City A	City B	City C	City D
City A	∞	2	5	7
City B	6	∞	3	8
City C	8	7	∞	4
City D	12	4	6	∞

Solution:

Stage 1:

$$\phi(B, A) = C_{BA} = 6$$

$$\phi(C, A) = C_{CA} = 8$$

$$\phi(D, A) = C_{DA} = 12$$

Stage 2:

$$\begin{aligned}
\phi(C,[B]) &= C_{CB}+C_{BA}=7+6=13\\
\phi(D,[B]) &= C_{DB}+C_{BA}=4+6=10\\
\phi(B,[C]) &= C_{BC}+C_{CA}=3+8=11\\
\phi(D,[C]) &= C_{DC}+C_{CA}=6+8=14\\
\phi(B,[D]) &= C_{BD}+C_{DA}=8+12=20\\
\phi(C,[D]) &= C_{CD}+C_{DA}=4+12=16
\end{aligned}$$

Stage 3:

$$\begin{aligned}
\phi(D,[B,C]) &= min\left\{C_{DB}+\phi(B,[C]),C_{DC}+\tilde{\phi}(C,[B])\right\}\\
&= min\{4+11,6+13\}\\
&= min\{15,19\}\\
&= 15\\
\phi(C,[B,D]) &= min\left\{C_{CB}+\phi(B,[D]),C_{CD}+\tilde{\phi}(D,[B])\right\}\\
&= min\{7+20,4+10\}\\
&= min\{27,14\}\\
&= 14\\
\phi(B,[C,D]) &= min\left\{C_{BC}+\phi(C,[D]),C_{BD}+\tilde{\phi}(D,[C])\right\}\\
&= min\{3+16,8+14\}\\
&= min\{19,22\}\\
&= 19
\end{aligned}$$

Stage 4:

$$\begin{aligned} F &= \phi\left(A, [B, C, D]\right) \\ &= min\left\{C_{AB} + \phi\left(B, [C, D]\right), C_{AC} + \phi\left(C, [B, D]\right), \right. \\ &\qquad \left. C_{AD} + \phi\left(D, [B, C]\right)\right\} \\ &= min\left\{2 + 19, 5 + 14, 7 + 15\right\} \\ &= min\left\{21, 19, 22\right\} \\ &= 19. \end{aligned}$$

The optimal route is

$A \to C \to D \to B \to A$

The optimal travelling cost is 19.

5.3 Solution for Fuzzy Travelling Salesman Problem using Dynamic Programming Algorithm

In this section, Dynamic Programming Algorithm is applied to solve travelling salesman problem with Generalized Quadrilateral Fuzzy Numbers. The given procedure is applicable for travelling salesman problem limited with four cities. But, the procedure is extendable for travelling salesman problem with n-cities according to our purpose.

5.3.1 Dynamic Programming Algorithm (DPA) for Travelling Salesman Problem with Four Cities in Fuzzy Environment

Stage 1:

$$\begin{aligned}\tilde{\phi}(B,A) &= \tilde{\phi}_{BA}\\ \tilde{\phi}(C,A) &= \tilde{\phi}_{CA}\\ \tilde{\phi}(D,A) &= \tilde{\phi}_{DA}\end{aligned}$$

Stage 2:

$$\tilde{\phi}(C,[B]) = \tilde{\phi}_{CB} + \tilde{\phi}_{BA}$$

$$\tilde{\phi}(D,[B]) = \tilde{\phi}_{DB} + \tilde{\phi}_{BA}$$

$$\tilde{\phi}(B,[C]) = \tilde{\phi}_{BC} + \tilde{\phi}_{CA}$$

$$\tilde{\phi}(D,[C]) = \tilde{\phi}_{DC} + \tilde{\phi}_{CA}$$

$$\tilde{\phi}(B,[D]) = \tilde{\phi}_{BD} + \tilde{\phi}_{DA}$$

$$\tilde{\phi}(C,[D]) = \tilde{\phi}_{CD} + \tilde{\phi}_{DA}$$

Stage 3:

$$\begin{aligned}\tilde{\phi}(D,[B,C]) &= min\left\{\tilde{\phi}_{DB} + \tilde{\phi}(B,[C]), \tilde{\phi}_{DC} + \tilde{\phi}(C,[B])\right\}\\ \tilde{\phi}(C,[B,D]) &= min\left\{\tilde{\phi}_{CB} + \tilde{\phi}(B,[D]), \tilde{\phi}_{CD} + \tilde{\phi}(D,[B])\right\}\\ \tilde{\phi}(B,[C,D]) &= min\left\{\tilde{\phi}_{BC} + \tilde{\phi}(C,[D]), \tilde{\phi}_{BD} + \tilde{\phi}(D,[C])\right\}\end{aligned}$$

Stage 4:

$$\begin{aligned}\tilde{F} &= \tilde{\phi}\,(A,[B,C,D]) \\ &= min\{\tilde{\phi}_{AB} + \tilde{\phi}(B,[C,D]), \\ &\qquad \tilde{\phi}_{AC} + \tilde{\phi}(C,[B,D]), \tilde{\phi}_{AD} + \tilde{\phi}(D,[B,C])\}\end{aligned}$$

The minimum value is fuzzy optimal travelling cost. The corresponding route is the optimal travelling route.

5.3.2 Numerical Illustrations

Example 5.3.1: Consider a Fuzzy Travelling Salesman Problem: A salesman has planned to visit 4 cities. He would like to start his journey from a particular city, visits each city only once and returns to the starting city. The travelling costs are given in table 5.4. Find the least cost route for Fuzzy Travelling Salesman Problem.

Table 5.4: Fuzzy Travelling Salesman Problem 1

From	To			
	A	B	C	D
A	∞	$(4, 6, 8, 10; 0.4, 0.6)$	$(5, 7, 9, 11; 0.3, 0.5)$	$(6, 8, 10, 12; 0.2, 0.6)$
B	$(4, 6, 8, 10; 0.4, 0.6)$	∞	$(2, 4, 6, 8; 0.5, 0.8)$	$(1, 3, 5, 7; 0.5, 0.7)$
C	$(5, 7, 9, 11; 0.3, 0.5)$	$(2, 4, 6, 8; 0.5, 0.8)$	∞	$(3, 5, 7, 9; 0.4, 0.7)$
D	$(6, 8, 10, 12; 0.2, 0.6)$	$(1, 3, 5, 7; 0.5, 0.7)$	$(3, 5, 7, 9; 0.4, 0.7)$	∞

Solution:

Stage 1:

$$\begin{aligned}
\tilde{\phi}(B, A) &= \tilde{\phi}_{BA} = (4, 6, 8, 10; 0.4, 0.6) \\
\tilde{\phi}(C, A) &= \tilde{\phi}_{CA} = (5, 7, 9, 11; 0.3, 0.5) \\
\tilde{\phi}(D, A) &= \tilde{\phi}_{DA} = (6, 8, 10, 12; 0.2, 0.6)
\end{aligned}$$

Stage 2:

$$\begin{aligned}
\tilde{\phi}(C, [B]) = \tilde{\phi}_{CB} + \tilde{\phi}_{BA} &= \{(2, 4, 6, 8; 0.5, 0.8) + (4, 6, 8, 10; 0.4, 0.6)\} \\
&= (6, 10, 14, 18; 0.4, 0.6) \\
\tilde{\phi}(D, [B]) = \tilde{\phi}_{DB} + \tilde{\phi}_{BA} &= \{(1, 3, 5, 7; 0.5, 0.7) + (4, 6, 8, 10; 0.4, 0.6)\} \\
&= (5, 9, 13, 17; 0.4, 0.6) \\
\tilde{\phi}(B, [C]) = \tilde{\phi}_{BC} + \tilde{\phi}_{CA} &= \{(2, 4, 6, 8; 0.5, 0.8) + (5, 7, 9, 11; 0.3, 0.5)\} \\
&= (7, 11, 15, 19; 0.3, 0.5) \\
\tilde{\phi}(D, [C]) = \tilde{\phi}_{DC} + \tilde{\phi}_{CA} &= \{(3, 5, 7, 9; 0.4, 0.7) + (5, 7, 9, 11; 0.3, 0.5)\} \\
&= (8, 12, 16, 20; 0.3, 0.5) \\
\tilde{\phi}(B, [D]) = \tilde{\phi}_{BD} + \tilde{\phi}_{DA} &= \{(1, 3, 5, 7; 0.5, 0.7) + (6, 8, 10, 12; 0.2, 0.6)\} \\
&= (7, 11, 15, 19; 0.2, 0.6) \\
\tilde{\phi}(C, [D]) = \tilde{\phi}_{CD} + \tilde{\phi}_{DA} &= \{(3, 5, 7, 9; 0.4, 0.7) + (6, 8, 10, 12; 0.2, 0.6)\} \\
&= (9, 13, 17, 21; 0.2, 0.6).
\end{aligned}$$

Stage 3:

$$\begin{aligned}\tilde{\phi}\left(D,[B,C]\right) &= min\left\{\tilde{\phi}_{DB}+\tilde{\phi}\left(B,[C]\right),\tilde{\phi}_{DC}+\tilde{\phi}\left(C,[B]\right)\right\}\\ &= min\left\{(1,3,5,7;0.5,0,7)+(7,11,15,19;0.3,0.5),\right.\\ &\qquad\left.(3,5,7,9;0.4,0.7)+(6,10,14,18;0.4,0.6)\right\}\\ &= min\{(8,14,20,26;0.3,0.5),(9,15,21,27;0.4,0.6)\}\\ &= (8,14,20,26;0.3,0.5)\end{aligned}$$

Similarly,

$$\begin{aligned}\tilde{\phi}\left(C,[B,D]\right) &= min\left\{\tilde{\phi}_{CB}+\tilde{\phi}\left(B,[D]\right),\tilde{\phi}_{CD}+\tilde{\phi}\left(D,[B]\right)\right\}\\ &= min\left\{(7,10,11,15;0.4,0.5)+(9,13,15,23;0.2,0.3),\right.\\ &\qquad\left.(5,7,8,11;0.3,0.4)+(13,18,20,29;0.3,0.4)\right\}\\ &= min\left\{(16,23,26,38;0.3,0.4),(18,25,28,40;0.3,0.4)\right\}\\ &= (9,15,21,27;0.2,0.6)\end{aligned}$$

$$\begin{aligned}\tilde{\phi}\left(B,[C,D]\right) &= min\left\{\tilde{\phi}_{BC}+\tilde{\phi}\left(C,[D]\right),\tilde{\phi}_{BD}+\tilde{\phi}\left(D,[C]\right)\right\}\\ &= min\left\{(7,1011,14;0.4,0.5)+(11,16,18,26;0.3,0.4),\right.\\ &\qquad\left.(3,4,5,8;0.2,0.3)+(11,13,16,24;0.3,0.4)\right\}\\ &= min\left\{(18,26,29,40;0.3,0.4),(14,17,21,32;0.2,0.3)\right\}\\ &= (9,15,21,27;0.3,0.5)\end{aligned}$$

Stage 4:

$$\begin{aligned}
\tilde{F} &= \tilde{\phi}\,(A, [B, C, D]) \\
&= min\{\tilde{\phi}_{AB} + \tilde{\phi}(B, [C, D]), \\
&\qquad \tilde{\phi}_{AC} + \tilde{\phi}(C, [B, D]), \tilde{\phi}_{AD} + \tilde{\phi}(D, [B, C])\} \\
&= min\{(4, 6, 8, 10; 0.4, 0.6) + (9, 15, 21, 27; 0.3, 0.5), \\
&\qquad (5, 7, 9, 11; 0.3, 0.5) + (9, 15, 21, 27; 0.2, 0.6), \\
&\qquad (6, 8, 10, 12; 0.2, 0.6) + (8, 14, 20, 26; 0.3, 0.5)\} \\
&= min\{(13, 21, 29, 37; 0.3, 0.5), (14, 22, 30, 38; 0.2, 0.5)\} \\
&= (14, 22, 30, 38; 0.2, 0.5).
\end{aligned}$$

The optimal routes are

$A \to C \to B \to D \to A$ and

$A \to D \to B \to C \to A$

The fuzzy optimal travelling cost is $(14, 22, 30, 38; 0.2, 0.5)$.

Example 5.3.2: Consider a Fuzzy Travelling Salesman Problem: A salesman has planned to visit 4 cities. He would like to start his journey from a particular city, visits each city only once and returns to the starting city. The travelling costs are given in table 5.5. Find the least cost route for Fuzzy Travelling Salesman Problem.

Table 5.5: Fuzzy Travelling Salesman Problem 2

From	To			
	A	B	C	D
A	∞	$(5, 8, 11, 12; 0.4, 0.6)$	$(9, 10, 11, 15; 0.2, 0.4)$	$(5, 8, 10, 11; 0.4, 0.6)$
B	$(7, 8, 10, 11; 0.4, 0.6)$	∞	$(6, 8, 10, 12; 0.4, 0.6)$	$(5, 8, 9, 10; 0.3, 0.5)$
C	$(2, 4, 5, 6; 0.2, 0.4)$	$(5, 7, 10, 11; 0.4, 0.6)$	∞	$(4, 6, 7, 10; 0.2, 0.5)$
D	$(6, 8, 10, 12; 0.4, 0.6)$	$(2, 5, 6, 7; 0.3, 0.5)$	$(5, 7, 10, 11; 0.4, 0.6)$	∞

Solution:

Stage 1:

$$\tilde{\phi}(B, A) = \tilde{\phi}_{BA} = (7, 8, 10, 11; 0.4, 0.6)$$

$$\tilde{\phi}(C, A) = \tilde{\phi}_{CA} = (2, 4, 5, 6; 0.2, 0.4)$$

$$\tilde{\phi}(D, A) = \tilde{\phi}_{DA} = (6, 8, 10, 12; 0.4, 0.6)$$

Stage 2:

$$\begin{aligned}
\tilde{\phi}(C, [B]) = \tilde{\phi}_{CB} + \tilde{\phi}_{BA} &= \{(5, 7, 10, 11; 0.4, 0.6) + (7, 8, 10, 11; 0.4, 0.6)\} \\
&= (12, 15, 20, 22; 0.4, 0.6) \\
\tilde{\phi}(D, [B]) = \tilde{\phi}_{DB} + \tilde{\phi}_{BA} &= \{(2, 5, 6, 7; 0.3, 0.5) + (7, 8, 10, 11; 0.4, 0.6)\} \\
&= (9, 13, 16, 18; 0.3, 0.5) \\
\tilde{\phi}(B, [C]) = \tilde{\phi}_{BC} + \tilde{\phi}_{CA} &= \{(6, 8, 10, 12; 0.4, 0.6) + (2, 4, 5, 6; 0.2, 0.4)\} \\
&= (8, 12, 15, 18; 0.2, 0.4) \\
\tilde{\phi}(D, [C]) = \tilde{\phi}_{DC} + \tilde{\phi}_{CA} &= \{(5, 7, 10, 11; 0.4, 0.6) + (2, 4, 5, 6; 0.2, 0.4)\} \\
&= (7, 11, 15, 17; 0.2, 0.4) \\
\tilde{\phi}(B, [D]) = \tilde{\phi}_{BD} + \tilde{\phi}_{DA} &= \{(5, 8, 9, 10; 0.3, 0.5) + (6, 8, 10, 12; 0.4, 0.6)\} \\
&= (11, 16, 19, 22; 0.3, 0.5) \\
\tilde{\phi}(C, [D]) = \tilde{\phi}_{CD} + \tilde{\phi}_{DA} &= \{(4, 6, 7, 10; 0.2, 0.5) + (6, 8, 10, 12; 0.4, 0.6)\} \\
&= (10, 14, 17, 22; 0.3, 0.5).
\end{aligned}$$

Stage 3:

$$\begin{aligned}
\tilde{\phi}(D,[B,C]) &= min\left\{\tilde{\phi}_{DB}+\tilde{\phi}(B,[C]),\tilde{\phi}_{DC}+\tilde{\phi}(C,[B])\right\}\\
&= min\{(2,5,6,7;0.3,0.5)+(8,12,15,18;0.2,0.4),\\
&\qquad (5,7,10,11;0.4,0.6)+(12,15,20,22;0.4,0.6)\}\\
&= min\{(10,17,21,25;0.2,0.4),(17,22,30,33;0.4,0.6)\}\\
&= (10,17,21,25;0.2,0.4)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tilde{\phi}(C,[B,D]) &= min\left\{\tilde{\phi}_{CB}+\tilde{\phi}(B,[D]),\tilde{\phi}_{CD}+\tilde{\phi}(D,[B])\right\}\\
&= min\{(5,7,10,11;0.4,0.6)+(11,16,19,22;0.3,0.5),\\
&\qquad (4,6,7,10;0.2,0.5)+(9,13,16,18;0.3,0.5)\}\\
&= min\{(16,23,29,33;0.3,0.5),(13,19,23,28;0.3,0.5)\}\\
&= (13,19,23,28;0.3,0.5)
\end{aligned}$$

$$\begin{aligned}
\tilde{\phi}(B,[C,D]) &= min\left\{\tilde{\phi}_{BC}+\tilde{\phi}(C,[D]),\tilde{\phi}_{BD}+\tilde{\phi}(D,[C])\right\}\\
&= min\{(6,8,10,12;0.4,0.6)+(10,14,17,22;0.3,0.5),\\
&\qquad (5,8,9,10;0.3,0.5)+(7,11,15,17;0.2,0.4)\}\\
&= min\{(16,22,27,34;0.3,0.5),(12,19,24,27;0.2,0.4)\}\\
&= (12,19,24,27;0.2,0.4)
\end{aligned}$$

Stage 4:

$$\begin{aligned}\tilde{F} &= \tilde{\phi}\,(A, [B, C, D]) \\ &= min\{\tilde{\phi}_{AB} + \tilde{\phi}(B, [C, D]), \\ &\quad \tilde{\phi}_{AC} + \tilde{\phi}(C, [B, D]), \tilde{\phi}_{AD} + \tilde{\phi}(D, [B, C])\} \\ &= min\{(17, 27, 35, 39; 0.2, 0.4), (22, 29, 34, 43; 0.2, 0.4), \\ &\quad (15, 25, 31, 36; 0.2, 0.4)\} \\ &= (15, 25, 31, 36; 0.2, 0.4).\end{aligned}$$

The optimal travelling route is $A \to D \to B \to C \to A$

The optimal travelling cost is $(15, 25, 31, 36; 0.2, 0.4)$.

5.3.3 Results

In this section, the results of TSP and FTSP are given in table 5.6.

Table 5.6: Optimal Solution

Examples	**Optimal Sequence**	**Total Travelling Cost**
Example 1	$A \to C \to B \to D \to A$ $A \to D \to B \to C \to A$	19
Example 2	$A \to C \to D \to B \to A$	19
Example 3	$A \to C \to B \to D \to A$ $A \to D \to B \to C \to A.$	(14,22,30,38; 0.2,0.5)
Example 4	$A \to D \to B \to C \to A$	$(15, 25, 31, 36; 0.2, 0.4)$

Chapter 6

On Travelling Salesman Problem with DSD Approach

Abstract

This chapter aims to describe a new and efficient approach named as DSD approach to solve Travelling Salesman Problem and it is validated through illustrations. To prove the significance of the new method, the result of the Dynamic Programming Algorithm is compared with DSD approach. DSD approach is extended to solve Fuzzy Travelling Salesman Problem. The advantage of this approach is very easy to understand and number of steps is very less and fairly short in computational time.

The part of the content of this chapter form the material of the paper will be appeared in **South East Asian Journal of Mathematics and Mathematical Sciences.**

6.1 Solving Travelling Salesman Problem using DSD Approach

In this section, Dynamic Programming procedure and DSD approach are reviewed.

6.1.1 Dynamic Programming Algorithm (DPA) for Four Cities

Since Dynamic Programming Algorithm was discussed in chapter 5, the algorithm has been left in this Chapter.

6.1.2 DSD Approach for Solving Travelling Salesman Problem

Phase I

Step 1: Find the lowest travelling cost from three cities to first city. Fix the lowest travelling cost. The cities are denoted by circles(nodes) and the cities are connected by arrow.

Step 2: Find the remaining two possibilities from the latest city.

Step 3: Find the rest of the final possibilities.

Step 4: Go to the starting node.

Step 5: Find the two possible routes.

Phase II

Step 1: Find the lowest travelling cost from first city to other three cities. Fix the lowest travelling cost. The cities are denoted by circles(nodes) and the cities are connected by arrow.

Step 2: Find the remaining two possibilities from the latest city.

Step 3: Find the rest of the final possibilities.

Step 4: Go to the starting node.

Step 5: Find the two possible routes.

Finally, Find the least possible route cost among the two phases. The least cost is called optimal travelling cost. The corresponding route is the optimal route.

Note: If there is a tie in step 1, take the two possible routes and do the same process for it.

6.1.3 Numerical Illustrations

To illustrate the effectiveness of DSD method, TSP is solved by Dynamic Programming Algorithm and DSD Approach.

Example 6.1.1: Solve the following travelling salesman problem with four cities and find the least cost route for TSP.

Table 6.1: Travelling Salesman Problem 1

From	To			
	City A	City B	City C	City D
City A	∞	3	6	2
City B	3	∞	5	2
City C	6	5	∞	6
City D	2	2	6	∞

Solution:

a) Solution by Dynamic Programming Algorithm

Stage 1:

$$\begin{aligned} \phi(B, A) &= C_{BA} = 3 \\ \phi(C, A) &= C_{CA} = 6 \\ \phi(D, A) &= C_{DA} = 2 \end{aligned}$$

Stage 2:

$$\begin{aligned} \phi(C, [B]) &= C_{CB} + C_{BA} = 5 + 3 = 8 \\ \phi(D, [B]) &= C_{DB} + C_{BA} = 2 + 3 = 5 \\ \phi(B, [C]) &= C_{BC} + C_{CA} = 5 + 6 = 11 \\ \phi(D, [C]) &= C_{DC} + C_{CA} = 6 + 6 = 12 \end{aligned}$$

$$\phi\left(B,[D]\right) = C_{BD}+C_{DA}=2+2=4$$
$$\phi\left(C,[D]\right) = C_{CD}+C_{DA}=6+2=8$$

Stage 3:

$$\begin{aligned}\phi\left(D,[B,C]\right) &= min\left\{C_{DB}+\phi\left(B,[C]\right),C_{DC}+\phi\left(C,[B]\right)\right\}\\ &= min\left\{2+11,6+8\right\}\\ &= min\left\{13,14\right\}\\ &= 13\end{aligned}$$

$$\begin{aligned}\phi\left(C,[B,D]\right) &= min\left\{C_{CB}+\phi\left(B,[D]\right),C_{CD}+\phi\left(D,[B]\right)\right\}\\ &= min\left\{5+4,6+5\right\}\\ &= min\left\{9,11\right\}\\ &= 9\end{aligned}$$

$$\begin{aligned}\phi\left(B,[C,D]\right) &= min\left\{C_{BC}+\phi\left(C,[D]\right),C_{BD}+\phi\left(D,[C]\right)\right\}\\ &= min\left\{5+8,2+12\right\}\\ &= min\left\{13,14\right\}\\ &= 13\end{aligned}$$

Stage 4:

$$\begin{aligned}F &= \phi(A,[B,C,D])=min\left\{\phi(A,B)+\phi(B,[C,D]),\phi(A,C)\right.\\ &\qquad \left.+\phi(C,[B,D]),+\phi(A,D)+\phi(D,[B,C])\right\}\\ &= min\left\{3+13,6+9,2+13\right\}\\ &= min\left\{16,15,15\right\}\\ &= 15.\end{aligned}$$

The optimal routes are

$A \to C \to B \to D \to A$ and

$A \to D \to B \to C \to A$

The optimal travelling cost is 15.

The problem has multiple solutions.

b) Solution by DSD Approach

Phase I

Step 1: Finding the minimum fuzzy value from three cities to city A:

$B \to A = 3$

$C \to A = 6$

$D \to A = 2$

Minimum travelling cost is $D \to A$.

Fix the assignment.

Step 2: The remaining two possibilities are $A \to B$ and $A \to C$.

Step 3: The final possibilities are $B \to C$ and $C \to B$.

Step 4: Go to the starting node i.e., $C \to D$ and $B \to D$.

Step 5: The two possible routes:

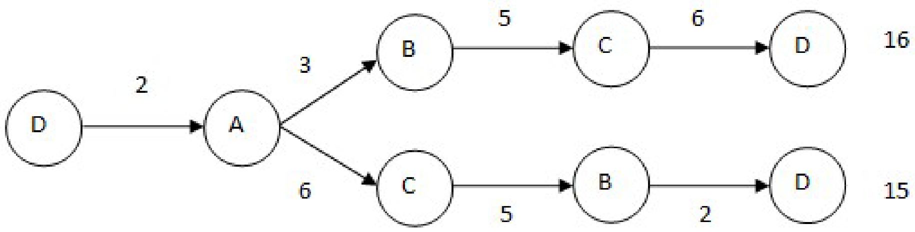

Figure 6.1: Problem 1: Phase I: Possible Sequence

Table 6.2: Phase I: Possible sequence and Travelling cost

Possible Sequence	Travelling Cost
$A \to B \to C \to D \to A$	16
$A \to C \to B \to D \to A$	15

Phase II

Step 1 Find minimum fuzzy value from city A to rest of the 3 cities:

$A \to B = 3$

$A \to C = 6$

$A \to D = 2$

Minimum travelling cost is $A \to D$.

Fix the assignment.

Step 2: The remaining two possibilities are $D \to B$ and $D \to C$.

Step 3: The final possibilities $B \to C$ and $C \to B$.

Step 4: Go to the starting node i.e., $C \to A$ and $B \to A$.

Step 5: The two possible routes:

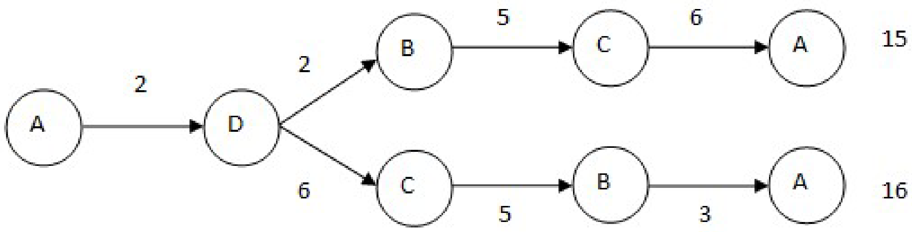

Figure 6.2: Problem 1: Phase II: Possible Sequence

Table 6.3: Phase II: Possible sequence and Travelling cost

Possible Sequence	Travelling Cost
$A \to D \to B \to C \to A$	15
$A \to D \to C \to B \to A$	16

Table 6.4: Results of Phase I and Phase II

Phases	Possible Sequence	Travelling Cost
Phase I	$A \to B \to C \to D \to A$	16
	$A \to C \to B \to D \to A$	15
Phase II	$A \to D \to B \to C \to A$	15
	$A \to D \to C \to B \to A$	16

Finally, finding the least possible route cost among the two phases. The least cost is called optimal travelling cost. The corresponding route is the optimal route.

The minimum total travelling cost is 15.

Therefore, the optimal routes are

$A \to C \to B \to D \to A$ and

$A \to D \to B \to C \to A$.

The optimal travelling cost is 15.

Note: The problem has multiple solutions.

Example 6.1.2: Consider the fuzzy travelling salesman problem: A salesman has planned to visit four cities. He would like to start his journey from a particular city, visits each city only once and return to the starting city. The travelling costs are given in the table below. Find the least cost route for the TSP.

Table 6.5: Travelling Salesman Problem 2

From	To			
	City A	City B	City C	City D
City A	∞	46	16	40
City B	41	∞	50	40
City C	82	32	∞	60
City D	40	40	36	∞

Solution:

a) Solution by Dynamic Programming Algorithm

Stage 1:

$$\begin{aligned}\phi(B,A) &= C_{BA}=41\\ \phi(C,A) &= C_{CA}=82\\ \phi(D,A) &= C_{DA}=40\end{aligned}$$

Stage 2:

$$\begin{aligned}\phi(C,[B]) &= C_{CB}+C_{BA}=32+41=73\\ \phi(D,[B]) &= C_{DB}+C_{BA}=40+41=81\\ \phi(B,[C]) &= C_{BC}+C_{CA}=50+82=132\\ \phi(D,[C]) &= C_{DC}+C_{CA}=36+82=118\\ \phi(B,[D]) &= C_{BD}+C_{DA}=40+40=80\\ \phi(C,[D]) &= C_{CD}+C_{DA}=60+40=100\end{aligned}$$

Stage 3:

$$\begin{aligned}\phi(D,[B,C]) &= min\left\{C_{DB}+\phi(B,[C]),C_{DC}+\tilde{\phi}(C,[B])\right\}\\ &= min\{40+132,36+73\}\\ &= min\{172,109\}\\ &= 109\end{aligned}$$

$$
\begin{aligned}
\phi\left(C,[B,D]\right) &= min\left\{C_{CB}+\phi\left(B,[D]\right),C_{CD}+\tilde{\phi}\left(D,[B]\right)\right\}\\
&= min\left\{32+80,60+81\right\}\\
&= min\left\{112,141\right\}\\
&= 112
\end{aligned}
$$

$$
\begin{aligned}
\phi\left(B,[C,D]\right) &= min\left\{C_{BC}+\phi\left(C,[D]\right),C_{BD}+\tilde{\phi}\left(D,[C]\right)\right\}\\
&= min\left\{50+100,40+118\right\}\\
&= min\left\{150,158\right\}\\
&= 150
\end{aligned}
$$

Stage 4:

$$
\begin{aligned}
F &= \phi(A,[B,C,D]) = min\left\{\phi(A,B)+\phi(B,[C,D]),\phi(A,C)\right.\\
&\qquad \left.+\phi(C,[B,D]),\phi(A,D)+\phi(D,[B,C])\right\}\\
&= min\{46+150,16+112,40+109\}\\
&= min\{196,128,149\}\\
&= 128.
\end{aligned}
$$

The optimal route is $A\rightarrow C\rightarrow B\rightarrow D\rightarrow A$

The optimal travelling cost is 128.

The problem has no multiple solutions.

b) Solution by DSD Approach

Phase I

Step 1 Finding the minimum fuzzy value from three cities to city A:

$B \to A = 41$

$C \to A = 82$

$D \to A = 40$

Minimum travelling cost is $D \to A$.

Fix the assignment.

Step 2: The remaining two possibilities are $A \to B$ and $A \to C$.

Step 3: The final possibilities are $B \to C$ and $C \to B$.

Step 4: Go to the starting node i.e., $C \to D$ and $B \to D$.

Step 5: The two possible routes.

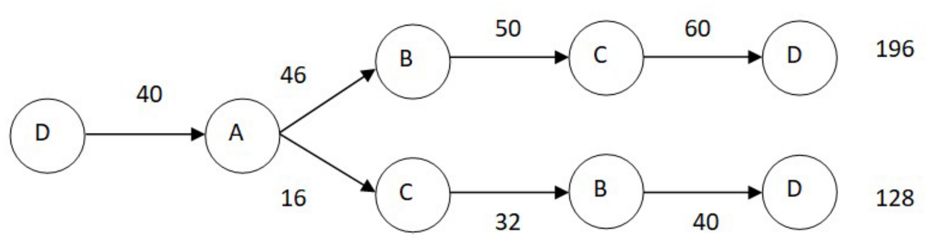

Figure 6.3: Problem 2: Possible Route 1

Table 6.6: Phase I: Possible Sequence and Travelling cost

Possible Sequence	**Travelling Cost**
$D \to A \to B \to C \to D$	196
$D \to A \to C \to B \to D$	128

Phase II

Step 1 Finding the minimum fuzzy value from city A to rest of the 3 cities.

$A \to B = 46$

$A \to C = 16$

$A \to D = 40$

Minimum travelling cost is $A \to C$.

Fix the assignment.

Step 2: The remaining two possibilities are $C \to B$ and $C \to D$.

Step 3: The final possibilities are $B \to D$ and $D \to B$.

Step 4: Go to the starting node i.e., $D \to A$ and $B \to A$.

Step 5: The two possible routes.

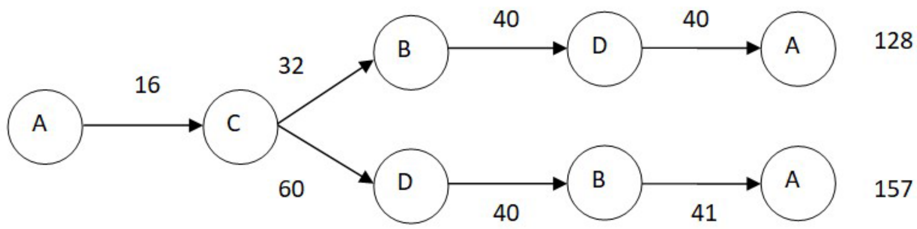

Figure 6.4: Problem 2: Possible Route 2

Table 6.7: Phase II: Possible sequence and Travelling cost

Possible Sequence	Travelling Cost
$A \to C \to B \to D \to A$	128
$A \to C \to D \to B \to A$	157

Table 6.8: Results of Phase I and Phase II

Phases	Possible Sequence	Travelling Cost
Phase I	$D \to A \to B \to C \to D$	196
	$D \to A \to C \to B \to D$	128
Phase II	$A \to C \to B \to D \to A$	128
	$A \to C \to D \to B \to A$	157

Finally, finding the least possible route cost among the two phases. The least cost is called optimal travelling cost. The corresponding route is the optimal route.

The minimum travelling cost is 128.

Therefore, the optimal route is $A \to C \to B \to D \to A$.

The optimal travelling cost is 128.

Note: The problem has no multiple solutions.

6.2 Solving Travelling Salesman Problem using DSD Approach in Fuzzy Environment

In this section, DSD Approach is applied on Fuzzy Travelling Salesman Problem.

Phase I

Step 1: Find the lowest fuzzy travelling cost from 3 cities to first city. Fix the lowest fuzzy travelling cost. The cities are denoted by circles (node) and travelling costs are denoted by arrows.

Step 2: Find the remaining two possibilities from the latest city.

Step 3: Find the rest of the final possibilities.

Step 4: Go to the starting node.

Step 5: Find the two possible routes.

Phase II

Step 1: Find the lowest fuzzy travelling cost from city A to three cities. Fix the lowest fuzzy travelling cost. The cities are denoted by circles (node) and travelling costs are denoted by arrows.

Step 2: Find the remaining two possibilities from the latest city.

Step 3: Find the rest of the final possibilities.

Step 4: Go to the starting node.

Step 5: Find the two possible routes.

Finally, Find the least possible fuzzy route cost among the two phases. The least fuzzy cost is called optimal travelling cost. The corresponding route is the optimal route.

Note: If there is a tie in step 1, take the two possible routes and do the same process for it.

6.2.1 Numerical Illustrations

Example 6.2.1: Consider a fuzzy travelling salesman problem: A salesman has planned to visit four cities. He would like to start his journey from a particular city, visits each city only once and return to the starting city. The travelling costs are given in the table below. Find the least cost route for FTSP.

Table 6.9: Fuzzy Travelling Salesman Problem 1

From	To			
	A	B	C	D
A	∞	$(8, 9, 10, 13; 0.5, 0.6)$	$(3, 6, 8, 13; 0.2, 0.3)$	$(7, 8, 9, 12; 0.4, 0.5)$
B	$(7, 9, 10, 14; 0.4, 0.5)$	∞	$(7, 10, 11, 14; 0.4, 0.5)$	$(3, 4, 5, 8; 0.2, 0.3)$
C	$(6, 7, 8, 11; 0.3, 0.4)$	$(7, 10, 11, 15; 0.4, 0.5)$	∞	$(5, 7, 8, 11; 0.3, 0.4)$
D	$(6, 9, 10, 15; 0.3, 0.4)$	$(6, 9, 11, 15; 0.3, 0.4)$	$(5, 6, 8, 13; 0.3, 0.4)$	∞

Solution: a) Solution by Dynamic Programming Algorithm

Stage 1:

$$\begin{aligned}
\tilde{\phi}(B,A) = \tilde{\phi}_{BA} &= (7,9,10,14;0.4,0.5) \\
\tilde{\phi}(C,A) = \tilde{\phi}_{CA} &= (6,7,8,11;0.3,0.4) \\
\tilde{\phi}(D,A) = \tilde{\phi}_{DA} &= (6,9,10,15;0.3,0.4)
\end{aligned}$$

Stage 2:

$$\begin{aligned}
\tilde{\phi}(C,[B]) = \tilde{\phi}_{CB} + \tilde{\phi}_{BA} &= \{(7,10,11,15;0.4,0.5) \\
&\quad +(7,9,10,14;0.4,0.5)\} \\
&= (14,19,21,29;0.4,0.5) \\
\tilde{\phi}(D,[B]) = \tilde{\phi}_{DB} + \tilde{\phi}_{BA} &= \{(6,9,11,15;0.3,0.4) \\
&\quad +(7,9,10,14;0.4,0.5)\} \\
&= (13,18,21,29;0.3,0.4) \\
\tilde{\phi}(B,[C]) = \tilde{\phi}_{BC} + \tilde{\phi}_{CA} &= \{(7,10,11,14;0.4,0.5) \\
&\quad +(6,7,8,11;0.3,0.4)\} \\
&= (13,17,19,25;0.3,0.4) \\
\tilde{\phi}(D,[C]) = \tilde{\phi}_{DC} + \tilde{\phi}_{CA} &= \{(5,6,8,13;0.3,0.4) \\
&\quad +(6,7,8,11;0.3,0.4)\} \\
&= (11,13,16,24;0.3,0.4) \\
\tilde{\phi}(B,[D]) = \tilde{\phi}_{BD} + \tilde{\phi}_{DA} &= \{(3,4,5,8;0.2,0.3) \\
&\quad +(6,9,10,15;0.3,0.4)\} \\
&= (9,13,15,23;0.2,0.3)
\end{aligned}$$

$$\begin{aligned}\tilde{\phi}\,(C,[D]) = \tilde{\phi}_{CD} + \tilde{\phi}_{DA} &= \{(5,7,8,11;0.3,0.4)\\ &\quad +(6,9,10,15;0.3,0.4)\}\\ &= (11,16,18,26;0.3,0.4)\end{aligned}$$

Stage 3:

$$\begin{aligned}\tilde{\phi}\,(D,[B,C]) &= min\left\{\tilde{\phi}_{DB} + \tilde{\phi}(B,[C]), \tilde{\phi}_{DC} + \tilde{\phi}(C,[B])\right\}\\ &= min\,\{(6,9,11,15;0.3,0.4) + (13,17,19,25;0.3,0.4)\\ &\qquad (5,6,8,13;0.3,0.4) + (14,19,21,26;0.4,0.5)\}\\ &= min\,\{(19,26,30,40;0.3,0.4),(19,25,29,39;0.3,0.4)\}\\ &= (19,25,29,39;0.3,0.4)\end{aligned}$$

Similarly,

$$\begin{aligned}\tilde{\phi}\,(C,[B,D]) &= min\left\{\tilde{\phi}_{CB} + \tilde{\phi}(B,[D]), \tilde{\phi}_{CD} + \tilde{\phi}(D,[B])\right\}\\ &= min\,\{(7,10,11,15;0.4,0.5) + (9,13,15,23;0.2,0.3),\\ &\qquad (5,7,8,11;0.3,0.4) + (13,18,20,29;0.3,0.4)\}\\ &= min\,\{(16,23,26,38;0.3,0.4),(18,25,28,40;0.3,0.4)\}\\ &= (16,23,26,38;0.3,0.4)\end{aligned}$$

$$\begin{aligned}\tilde{\phi}\,(B,[C,D]) &= min\left\{\tilde{\phi}_{BC} + \tilde{\phi}(C,[D]), \tilde{\phi}_{BD} + \tilde{\phi}(D,[C])\right\}\\ &= min\,\{(7,10,11,14;0.4,0.5) + (11,16,18,26;0.3,0.4),\\ &\qquad (3,4,5,8;0.2,0.3) + (11,13,16,24;0.3,0.4)\}\\ &= min\,\{(18,26,29,40;0.3,0.4),(14,17,21,32;0.2,0.3)\}\\ &= (14,17,21,32;0.2,0.3)\end{aligned}$$

Stage 4:

$$\begin{aligned}
\tilde{F} &= \tilde{\phi}\,(A, [B, C, D]) \\
&= min\left\{\tilde{\phi}(A, B) + \tilde{(}B, [C, D]), \tilde{\phi}(A, C)\right. \\
&\qquad \left. + \tilde{\phi}(C, [B, D]), \tilde{\phi}(A, D) + \tilde{\phi}(D, [B, C])\right\} \\
&= min\,\{(8, 9, 10, 13; 0.5, 0.6) + (14, 17, 21, 32; 0.2, 0.3), \\
&\qquad (3, 6, 8, 13; 0.2, 0.3) + (16, 23, 26, 38; 0.3, 0.4), \\
&\qquad (7, 8, 9, 12; 0.4, 0.5) + (19, 25, 29, 39; 0.3, 0.4)\} \\
&= min\,\{(22, 26, 31, 45; 0.2, 0.3), (19, 29, 34, 51; 0.2, 0.3), \\
&\qquad (26, 33, 38, 53; 0.3, 0.4)\} \\
&= (19, 29, 34, 51; 0.2, 0.3).
\end{aligned}$$

The optimal travelling route is $A \to C \to B \to D \to A$

The optimal fuzzy travelling cost is $(19, 29, 34, 51; 0.2, 0.3)$.

b) Solution by DSD Approach

Phase I

Step 1: Finding the minimum fuzzy value from three cities to city A using ranking function

$B \to A = (7, 9, 10, 14; 0.4, 0.5)$

$C \to A = (6, 7, 8, 11; 0.3, 0.4)$

$D \to A = (6, 9, 10, 15; 0.3, 0.4)$

Minimum Travelling cost is $C \to A$.

Fix the assignment.

Step 2: The remaining two possibilities are $A \to B$ and $A \to D$.

Step 3: The final possibilities are $B \to D$ and $D \to B$.

Step 4: Go to the starting node i.e., $D \to C$ and $B \to C$.

Step 5: The two possible routes:

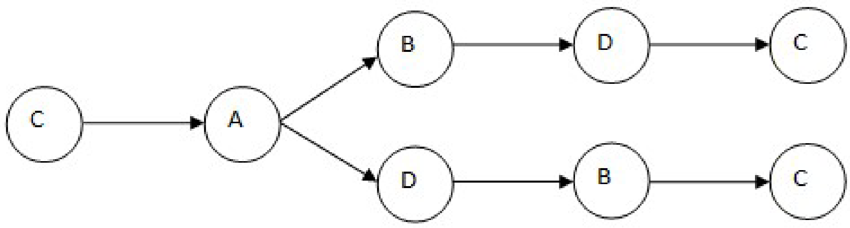

Figure 6.5: FTSP 1: Possible Route 1

Table 6.10: Phase I: Possible Sequence and Fuzzy Travelling Cost

Possible Sequence	**Fuzzy Travelling Cost**
$C \to A \to B \to D \to C$	(22, 26, 31, 45; 0.2, 0.3)
$C \to A \to D \to B \to C$	(26, 34, 39, 52; 0.3, 0.4).

Phase II

Step 1: Finding the minimum fuzzy value from city A to rest of the 3 cities.

$A \to B = (8, 9, 10, 13; 0.5, 0.6)$

$A \to C = (3, 6, 8, 13; 0.2, 0.3)$

$A \to D = (7, 8, 9, 12; 0.4, 0.5)$

Minimum travellig cost is $A \to C$.

Fix the assignment.

Step 2: The remaining two possibilities are $C \to B$ and $C \to D$.

Step 3: The final possibilities are $B \to D$ and $D \to B$.

Step 4: Go to the starting node i.e., $D \to A$ and $B \to A$.

Step 5: The two possible routes:

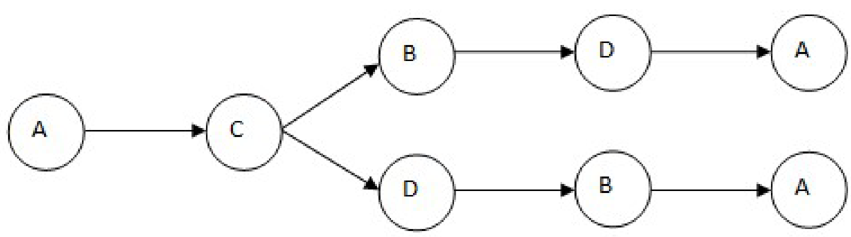

Figure 6.6: FTSP 1: Possible Route 2

Table 6.11: Phase II: Possible Sequence and Fuzzy Travelling Cost

Possible Sequence	Fuzzy Travelling Cost
$A \to C \to B \to D \to A$	(19, 29, 34, 51; 0.2, 0.3)
$A \to C \to D \to B \to A$	(21, 31, 37, 53; 0.2, 0.3)

Table 6.12: Results of Phase I and Phase II

Phases	Possible Sequence	Travelling Cost
Phase I	$C \to A \to B \to D \to C$	(22, 26, 31, 45; 0.2, 0.3)
	$C \to A \to D \to B \to C$	(26, 34, 39, 52; 0.3, 0.4)
Phase II	$A \to C \to B \to D \to A$	(19, 29, 34, 51; 0.2, 0.3)
	$A \to C \to D \to B \to A$	(21, 31, 37, 53; 0.2, 0.3)

Finally, finding the least possible route cost among the two phases. The least cost is called optimal travelling cost. The corresponding route is the optimal route.

Therefore, the optimal route is $A \to C \to B \to D \to A$.

The optimal fuzzy travelling cost is $(19, 29, 34, 51; 0.2, 0.3)$.

Note: The problem does not have multiple solutions.

Example 6.2.2: Consider the fuzzy travelling salesman problem: A salesman has planned to visit four cities. He would like to start his journey from a particular city, visits each city only once and return to the starting city. The travelling costs are given in table 6.13. Find the least cost route for FTSP.

Table 6.13: Fuzzy Travelling Salesman Problem 2

From	To			
	A	B	C	D
A	∞	$(8, 9, 10, 19; 0.4, 0.6)$	$(3, 6, 8, 14; 0.2, 0.4)$	$(7, 8, 9, 17; 0.4, 0.6)$
B	$(7, 9, 10, 19; 0.4, 0.6)$	∞	$(7, 10, 11, 21; 0.4, 0.6)$	$((3, 4, 5, 9; 0.3, 0.5)$
C	$(6, 7, 8, 15; 0.2, 0.4)$	$(7, 10, 11, 13; 0.4, 0.6)$	∞	$(5, 7, 8, 10; 0.2, 0.5)$
D	$(6, 9, 10, 19; 0.4, 0.6)$	$(6, 9, 10, 19; 0.3, 0.5)$	$(5, 6, 8, 14; 0.4, 0.6)$	∞

Solution: b) Solution by Dynamic Programming Algorithm

Stage 1:

$$\begin{aligned}
\tilde{\phi}(B,A) = \tilde{\phi}_{BA} &= (7,9,10,19;0.4,0.6)\\
\tilde{\phi}(C,A) = \tilde{\phi}_{CA} &= (6,7,8,15;0.2,0.4)\\
\tilde{\phi}(D,A) = \tilde{\phi}_{DA} &= (6,9,10,19;0.4,0.6)
\end{aligned}$$

Stage 2:

$$\begin{aligned}
\tilde{\phi}(C,[B]) = \tilde{\phi}_{CB} + \tilde{\phi}_{BA} &= \{(7,10,11,13;0.4,0.6)\\
&\quad +(7,9,10,19;0.4,0.6)\}\\
&= (14,19,21,32;0.4,0.6)\\
\tilde{\phi}(D,[B]) = \tilde{\phi}_{DB} + \tilde{\phi}_{BA} &= \{(6,9,10,19;0.3,0.5)\\
&\quad +(7,9,10,19;0.4,0.6)\}\\
&= (13,18,21,39;0.3,0.5)\\
\tilde{\phi}(B,[C]) = \tilde{\phi}_{BC} + \tilde{\phi}_{CA} &= \{(7,10,11,21;0.4,0.6)\\
&\quad +((6,7,8,15;0.2,0.4)\}\\
&= (13,17,19,36;0.2,0.4)\\
\tilde{\phi}(D,[C]) = \tilde{\phi}_{DC} + \tilde{\phi}_{CA} &= \{(5,6,8,14;0.4,0.6)\\
&\quad +(6,7,8,15;0.2,0.4)\}\\
&= (11,13,16,29;0.2,0.4)\\
\tilde{\phi}(B,[D]) = \tilde{\phi}_{BD} + \tilde{\phi}_{DA} &= \{(3,4,5,9;0.3,0.5)\\
&\quad +(6,9,10,19;0.4,0.6)\}\\
&= (9,13,15,28;0.3,0.5)
\end{aligned}$$

$$\begin{aligned}\tilde{\phi}(C,[D]) = \tilde{\phi}_{CD} + \tilde{\phi}_{DA} &= \{(5,7,8,10;0.2,0.5)\\ &\quad +(6,9,10,19;0.4,0.6)\}\\ &= (11,16,18,29;0.3,0.5).\end{aligned}$$

Stage 3:

$$\begin{aligned}\tilde{\phi}(D,[B,C]) &= min\left\{\tilde{\phi}_{DB} + \tilde{\phi}(B,[C]), \tilde{\phi}_{DC} + \tilde{\phi}(C,[B])\right\}\\ &= min\{(6,9,10,19;0.3,0.5) + (13,17,19,36;0.2,0.4),\\ &\qquad (5,6,8,14;0.4,0.6) + (14,19,21,32;0.4,0.6)\}\\ &= min\{(19,26,30,56;0.2,0.4),(19,25,29,46;0.4,0.6)\}\\ &= (19,26,30,56;0.2,0.4)\end{aligned}$$

Similarly,

$$\begin{aligned}\tilde{\phi}(C,[B,D]) &= min\left\{\tilde{\phi}_{CB} + \tilde{\phi}(B,[D]), \tilde{\phi}_{CD} + \tilde{\phi}(D,[B])\right\}\\ &= min\{(7,10,11,13;0.4,0.6) + (9,13,15,28;0.3,0.5),\\ &\qquad (5,7,8,10;0.2,0.5) + (13,18,21,39;0.3,0.5)\}\\ &= min\{(16,23,26,41;0.3,0.5),(18,25,29,49;0.3,0.5)\}\\ &= (16,23,26,41;0.3,0.5)\end{aligned}$$

$$\begin{aligned}\tilde{\phi}(B,[C,D]) &= min\left\{\tilde{\phi}_{BC} + \tilde{\phi}(C,[D]), \tilde{\phi}_{BD} + \tilde{\phi}(D,[C])\right\}\\ &= min\{(7,10,11,21;0.4,0.6) + (11,16,18,29;0.3,0.5),\\ &\qquad (3,4,5,9;0.3,0.5) + (11,13,16,29;0.2,0.4)\}\\ &= min\{(18,26,29,50;0.3,0.5),(14,17,21,38;0.2,0.4)\}\\ &= (14,17,21,38;0.2,0.4)\end{aligned}$$

Stage 4:

$$\begin{aligned}
\tilde{F} &= \tilde{\phi}\left(A,[B,C,D]\right) \\
&= min\left\{\tilde{\phi}(A,B)+\tilde{(}B,[C,D]),\tilde{\phi}(A,C)\right. \\
&\qquad \left. +\,\tilde{\phi}(C,[B,D]),\tilde{\phi}(A,D)+\tilde{\phi}(D,[B,C])\right\} \\
&= min\{(8,9,10,19;0.4,0.6)+(14,17,21,38;0.2,0.4), \\
&\qquad (3,6,8,14;0.2,0.4)+(16,23,26,41;0.3,0.5), \\
&\qquad (7,8,9,17;0.4,0.6)+(19,26,30,56;0.2,0.4)\} \\
&= min\{(22,26,31,57;0.2,0.4),(19,29,34,55;0.2,0.4), \\
&\qquad (26,34,39,73;0.2,0.4)\} \\
&= (22,26,31,57;0.2,0.4).
\end{aligned}$$

The optimal travelling route is $A \rightarrow B \rightarrow D \rightarrow C \rightarrow A$.

The minimum value optimal travelling cost is $(22,26,31,57;0.2,0.4)$.

b)Solution by DSD Approach

Phase I

Step 1: Finding the minimum fuzzy value from three cities to city A using ranking function

$B \rightarrow A = (7,9,10,19;0.4,0.6)$

$C \rightarrow A = (6,7,8,15;0.2,0.4)$

$D \rightarrow A = (6,9,10,19;0.4,0.6)$

Minimum travelling cost is $C \to A$.

Fix the assignment.

Step 2: The remaining two possibilities are $A \to B$ and $A \to D$.

Step 3: The final possibilities are $B \to D$ and $D \to B$.

Step 4: Go to the starting node i.e., $D \to C$ and $B \to C$.

Step 5: The two possible routes:

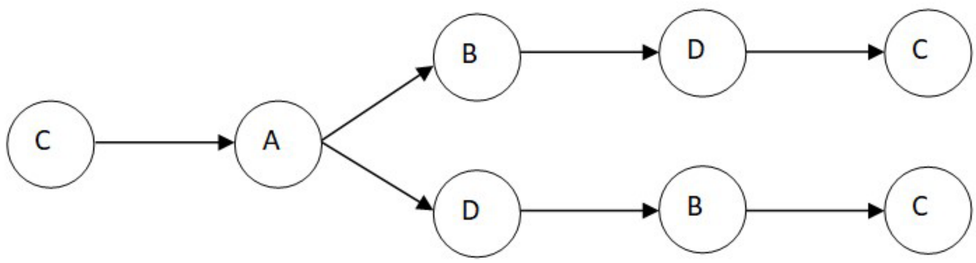

Figure 6.7: FTSP 2: Possible Route 1

Table 6.14: Phase I: Possible sequence and Fuzzy Travelling cost

Possible Sequence	Fuzzy Travelling Cost
$C \to A \to B \to D \to C$	(22,26,31,57;0.2,0.4)
$C \to A \to D \to B \to C$	(26,34,39,73;0.2,0.4)

Phase II

Step 1: Finding the minimum fuzzy value from city A to rest of the three cities.

$A \to B = (8, 9, 10, 13; 0.5, 0.6)$

$A \to C = (3, 6, 8, 13; 0.2, 0.3)$

$A \to D = (7, 8, 9, 12; 0.2, 0.4)$

Minimum travelling cost is $A \to C$.

Fix the assignment.

Step 2: The remaining two possibilities are $C \to B$ and $C \to D$.

Step 3: The final possibilities are $B \to D$ and $D \to B$.

Step 4: Go to the starting node i.e., $D \to A$ and $B \to A$.

Step 5: Finding the two possible routes:

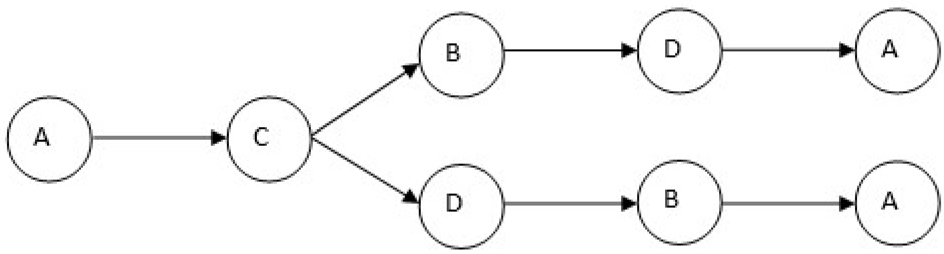

Figure 6.8: FTSP 2: Possible Route 2

Table 6.15: Phase II: Possible Sequence and Fuzzy Travelling Cost

Possible Sequence	**Fuzzy Travelling Cost**
$A \to C \to B \to D \to A$	(26,34,39,73;0.2,0.4)
$A \to C \to D \to B \to A$	(26,33,38,63;0.2,0.4)

Table 6.16: Results of Phase I and Phase II

Phases	**Possible Sequence**	**Travelling Cost**
Phase I	$C \to A \to B \to D \to C$	(22,26,31,57;0.2,0.4)
	$C \to A \to D \to B \to C$	(26,34,39,73;0.2,0.4)
Phase II	$A \to C \to B \to D \to A$	(26,34,39,73;0.2,0.4)
	$A \to C \to D \to B \to A$	(26,33,38,63;0.2,0.4)

Finally, finding the least possible route cost among the two phases. The least cost is called optimal travelling cost. The corresponding route is the optimal route.

Therefore, the optimal route is $A \to B \to D \to C \to A$.

The optimal fuzzy travelling cost is $(22, 26, 31, 57; 0.2, 0.4)$.

Note: Therefore, the problem does not have multiple solutions.

Chapter 7

On Fuzzy Travelling Salesman Problem with BCR Approach

Abstract

Travelling Salesman Problem (TSP) is one of the most significant optimization problems and brings remarkable advances in recent years. The focus of this chapter is to present a new approach named as BCR approach to solve both travelling salesman problem and fuzzy travelling salesman problem. The BCR approach is more understandable and applicable in real life situations. The main contribution of this work is to reduce the computational time, which is illustrated through an example.

The part of the content of this chapter form the material of the following paper will be appeared in International Journal.

Solving Travelling Salesman Problem by BCR Approach in Fuzzy Environment.

7.1 Solving Travelling Salesman Problem using BCR Approach

In this section, BCR approach is proposed to solve Travelling Salesman Problem for five cities. To prove the efficiency of BCR approach, few problems are solved by Hungarian algorithm and BCR approach.

7.1.1 BCR Approach for Solving Travelling Salesman Problem

Let the cities are represented by circles and the distances or costs are represented by arrows

Phase I

Step 1: Find the minimum value from four cities to first city. Fix the assignment.

Step 2: Find the next minimum value, which is continuation of connected city in step 1.

Step 3: Find the remaining two possibilities from step 2.

Step 4: Connect the rest of the final possibilities.

Step 5: Go to the starting city.

Step 6: Find the two possible routes.

Phase II

Step 1: Find the minimum value from first city to rest of the four cities. Fix the assignment.

Step 2: Find the next minimum value, which is continuation of connected city in step 1.

Step 3: Find the remaining two possibilities from step 2.

Step 4: Connect the rest of the final possibilities.

Step 5: Go to the starting city.

Step 6: Find the two possible routes.

Finally, find the minimum travelling cost between two phases. The minimum travelling cost is called optimal travelling cost. The corresponding sequence is called optimal sequence.

Note: If there is a tie, take the two possibilities and do the same process for it.

7.1.2 Numerical Illustrations

To illustrate the proficiency of the proposed method, few travelling salesman problems are solved by BCR approach.

Example 7.1.1: Consider the travelling salesman problem: A salesman has planned to visit five cities. He would like to start his journey from a particular city, visits each city only once and returns to his own city. The travelling cost is given in the table 7.1. Find the least cost route.

Table 7.1: Travelling Salesman Problem with 5 cities

From	To				
	A	B	C	D	E
A	∞	3	6	2	3
B	3	∞	5	2	3
C	6	5	∞	6	4
D	2	2	6	∞	6
E	3	3	4	6	∞

Solution:

a) Hungarian Method:

Since the number of rows is equal to number of columns, the problem is a balanced assignment problem.

Finding the smallest travelling cost from each row and subtract the same from all other travelling costs in the same row.

Table 7.2: Reduced cost matrix with zero in every row

	A	B	C	D	E
A	∞	1	4	0	1
B	1	∞	3	0	1
C	2	1	∞	2	0
D	0	0	4	∞	4
E	0	0	1	3	∞

Finding the smallest travelling cost from each column and subtract the same from all other travelling costs in the same column.

Table 7.3: Reduced cost matrix with zero in every row and column

	A	B	C	D	E
A	∞	1	3	0	1
B	1	∞	2	0	1
C	2	1	∞	2	0
D	0	0	4	∞	4
E	0	0	0	3	∞

Making the assignment

Table 7.4: Making Assignment

	A	B	C	D	E
A	∞	1	3	[0]	1
B	1	∞	2	~~0~~	1
C	2	1	∞	2	[0]
D	[0]	~~0~~	4	∞	4
E	~~0~~	~~0~~	[0]	3	∞

Since the number of assignment is less than 5, the assignment is not optimal.

Using the step 6 and step 7 of Hungarian Algorithm which is discussed in Chapter 3, we get the following solution.

Table 7.5: Again Making Assignment

	A	B	C	D	E
A	∞	1	2	[0]	X
B	[0]	∞	1	X	1
C	2	1	∞	3	[0]
D	X	[0]	4	∞	1
E	X	X	[0]	3	∞

This table provides an optimal solution to the assignment problem, but not solution to travelling salesman problem. The salesman starting his journey from city 'A' and returning to his home city without visiting cities C and E. i.e., The salesman has visited the following cities. $A \to D \to B \to A$.

Therefore, the solution is not a solution to travelling salesman problem. In order to find the optimal solution to TSP, the next non-zero element in the table is to be found. The next non-zero element in the table is '1'. Since the element '1' occurs in three different cells, the following cases araise.

Case (i): Choosing the cell (B, C) instead of zero assignment in the cell (B, A)

Table 7.6: Case (i): Making Assignment

	A	B	C	D	E
A	∞	1	2	[0]	~~0~~
B	~~0~~	∞	[1]	~~0~~	~~0~~
C	2	1	∞	3	[0]
D	~~0~~	[0]	4	∞	4
E	[0]	~~0~~	~~0~~	3	∞

The optimal assignment is $A \to D, D \to B, B \to C, C \to E, E \to A$.

Case (ii): Choosing the cell (C, B) instead of zero assignment in the cell (C, E)

Table 7.7: Case (ii): Making Assignment

	A	B	C	D	E
A	∞	1	2	~~0~~	[0]
B	~~0~~	∞	1	[0]	~~0~~
C	2	[1]	∞	3	~~0~~
D	[0]	~~0~~	4	∞	4
E	~~0~~	~~0~~	[0]	3	∞

The optimal assignment $A \to E, E \to C, C \to B, B \to D, D \to A$.

The two cases provide the optimal solution to travelling salesman problem.

Table 7.8: The Optimal Solution for Problem 1

Optimal Sequence	**Travelling Cost**
$A \to D \to B \to C \to E \to A$	16
$A \to E \to C \to B \to D \to A$	16

The problem has multiple solutions.

The optimal travelling cost = 16.

b) Solution by BCR Approach:

Phase I

Step 1: Finding the minimum value from four cities to city A.

$B \to A = 3$

$C \to A = 6$

$D \to A = 2$

$E \to A = 3$

The minimum travelling cost is $D \to A$.

Step 2: Finding the next minimum value, which is continuation of connected city in step 1.

$A \to B = 3$

$A \to C = 6$

$A \to E = 3$

There are two minimum travelling costs in this step.

$A \to B = 3$ and $A \to E$.

Since there is a tie, there are two cases arise.

Case (i) If the minimum travelling cost is $A \to B$, then the rest of the steps are as follows:

Step 3: The remaining two possibilities from step 2 are

$B \to C$ and $B \to E$.

Step 4: Connecting the rest of the final possibilities:

$C \to E$ and $E \to C$.

Step 5: Go to the starting node i.e., $E \to D$ and $C \to D$.

Step 6: Finding the two possible routes.

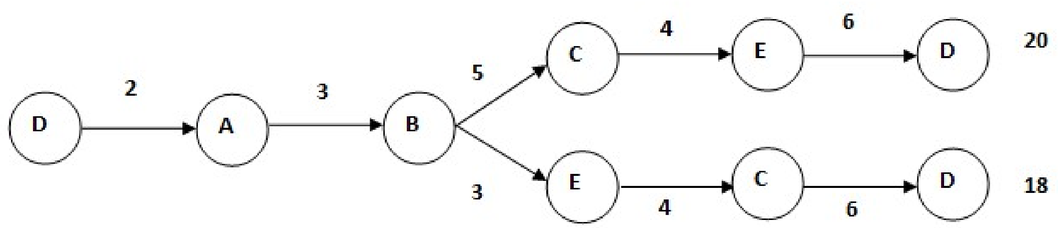

Figure 7.1: Problem 1: Phase I: Case (i): Possible Route 1

Table 7.9: Problem 1: Phase I: Case(i): Possible sequence and Travelling cost

Possible Sequence	**Travelling Cost**
$A \to B \to C \to E \to D \to A$	20
$A \to B \to E \to C \to D \to A$	18

Case (ii) If the minimum travelling cost is $A \to E$, then the rest of the steps are as follows:

Step 3: The remaining two possibilities from step 2 are $E \to B$ and $E \to C$.

Step 4: Connecting the rest of the final possibilities: $B \to C$ and $C \to B$.

Step 5: Go to the starting node i.e., $C \to D$ and $B \to D$.

Step 6: Finding the two possible routes:

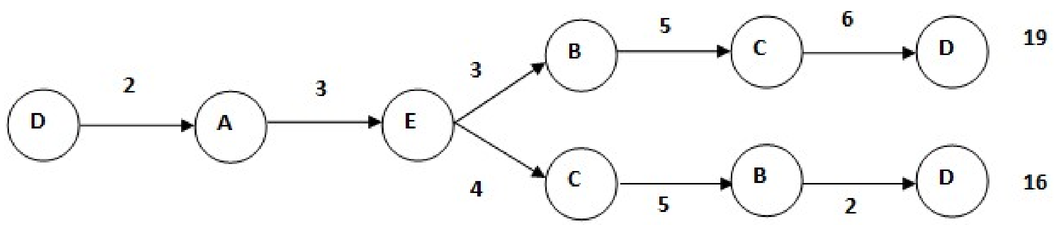

Figure 7.2: Problem 1: Phase I: Case (ii): Possible Route 2

Table 7.10: Problem 1: Phase I: Case(ii): Possible Sequence and Travelling Cost

Possible Sequence	**Travelling Cost**
$A \to E \to B \to C \to D \to A$	19
$A \to E \to C \to B \to D \to A$	16

Phase II

Step 1: Finding the minimum value from city A to rest of the four cities.

$A \to B = 3$

$A \to C = 6$

$A \to D = 2$

$A \to E = 3$

The minimum travelling cost is $A \to D$.

Step 2: Finding the next minimum value, which is continuation of connected city in step 1.

$D \to B = 2$

$D \to C = 6$

$D \to E = 6$

The minimum travelling cost is $D \to B$.

Step 3: The remaining two possibilities from step 2.

$B \to C$ and $B \to E$.

Step 4: Connecting the rest of the final possibilities:

$C \to E$ and $E \to C$.

Step 5: Go to the starting node i.e., $E \to A$ and $C \to A$.

Step 6: Finding the two possible routes:

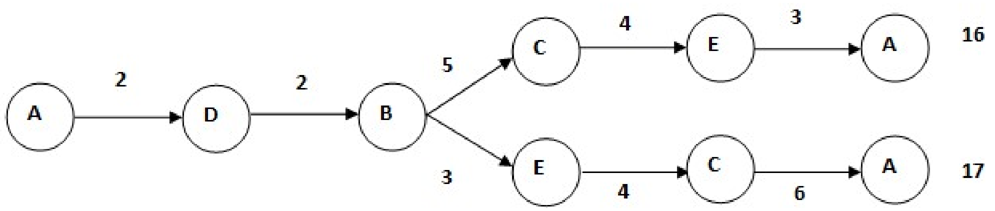

Figure 7.3: Problem 1: Phase II: Possible Route 1

Table 7.11: Problem 1: Phase II: Possible Sequence and Travelling Cost

Possible Sequence	**Travelling Cost**
$A \to D \to B \to C \to E \to A$	16
$A \to D \to B \to E \to C \to A$	17

All possible solutions are given below:

Table 7.12: Results of Two Phases

Phases	**Possible Sequence**	**Travelling Cost**
Phase I (Case (i))	$A \to B \to C \to E \to D \to A$	20
	$A \to B \to E \to C \to D \to A$	18
Phase I (Case (ii))	$A \to E \to B \to C \to D \to A$	19
	$A \to E \to C \to B \to D \to A$	16
Phase II	$A \to D \to B \to C \to E \to A$	16
	$A \to D \to B \to E \to C \to A$	17

Finally, finding the least possible route cost among the two phases. The least cost is called optimal travelling cost. The corresponding route is the optimal route.

Table 7.13: Optimal Solution

Optimal Sequence	**Travelling Cost**
$A \to E \to C \to B \to D \to A$	16
$A \to D \to B \to C \to E \to A$	16

The minimum travelling cost is 16.

Note: The problem has multiple solutions.

Example 7.1.2: Consider the following travelling salesman problem so as to minimize the cost per cycle.

Table 7.14: Travelling Salesman Problem with 5 cities

From	To				
	A	B	C	D	E
A	∞	2	5	7	1
B	6	∞	3	8	2
C	8	7	∞	4	7
D	12	4	6	∞	5
E	1	3	2	8	∞

Solution:

a) Hungarian Algorithm:

Since the number of rows is equal to number of columns, the problem is a balanced assignment problem. Finding the smallest travelling cost from each row and subtract the same from all other travelling cost in the same row.

Table 7.15: Reduced cost matrix with zero in every row

	A	B	C	D	E
A	∞	1	4	6	0
B	4	∞	1	6	0
C	4	3	∞	0	3
D	8	0	2	∞	1
E	0	2	1	7	∞

Finding the smallest travelling cost from each column and subtract the same from all other travelling cost in the same column.

Table 7.16: Reduced cost matrix with zero in every row and column

	A	B	C	D	E
A	∞	1	3	6	0
B	4	∞	0	6	0
C	4	3	∞	0	3
D	8	0	1	∞	1
E	0	2	0	7	∞

Table 7.17: Making Assignment

	A	B	C	D	E
A	∞	1	3	6	[0]
B	4	∞	[0]	6	X
C	4	3	∞	[0]	3
D	8	[0]	1	∞	1
E	[0]	2	X	7	∞

The optimal assignment for assignment problem is

$A \to E, B \to C, C \to D, D \to E, E \to A.$

This table provides an optimal solution to the assignment problem, but not solution to travelling salesman problem. The salesman is starting his journey from city 'A' and returning to his home city without visiting cities B, C and D. i.e., $A \to E \to A$. Therefore, this is not a solution to travelling salesman problem.

In order to find the optimal solution to TSP, the next non-zero element has to be chosen from the table. The next non-zero element in the table is '1'. Since, the element '1' occurs in three different cells, three cases arise.

Case (i): Choosing the cell (A, B) instead of zero assignment in the cell (A,E)

Table 7.18: Case (i): Making Assignment

	A	B	C	D	E
A	∞	[1]	3	6	~~0~~
B	4	∞	[0]	6	~~0~~
C	4	3	∞	[0]	3
D	8	~~0~~	1	∞	[1]
E	[0]	2	~~0~~	7	∞

The resultant solution is $A \to B, B \to C, C \to D, D \to E, E \to A$.

The resultant solution to travelling salesman problem is

$A \to B \to C \to D \to E \to A$.

The travelling cost = 15.

Case (ii) Choosing the cell (D, C), instead of zero assignment in the cell (D, B).

Table 7.19: Case (ii): Making Assignment

	A	B	C	D	E
A	∞	[1]	3	6	~~0~~
B	4	∞	~~0~~	6	~~0~~
C	4	3	∞	[0]	3
D	8	~~0~~	[1]	∞	1
E	[0]	2	~~0~~	7	∞

The resultant solution is $A \to B, B \to E, C \to D, D \to C, E \to A$.

Case (iii) Choosing the cell (D, E) instead of zero assignment in the cell (D, B).

Table 7.20: Case (iii): Making Assignment

	A	B	C	D	E
A	∞	[1]	3	6	~~0~~
B	4	∞	[0]	6	~~0~~
C	4	3	∞	[0]	3
D	8	~~0~~	1	∞	[1]
E	[0]	2	~~0~~	7	∞

The resultant solution is $A \to B, B \to C, C \to D, D \to E, E \to A$.

The resultant solution to travelling salesman problem is

$$A \to B \to C \to D \to E \to A.$$

The travelling cost is Rs.15.

Table 7.21: The Possible Solution for Problem 2

Cases	**Possible**	**Total Travelling Cost**
Case (i)	$A \to B \to C \to D \to E \to A$	15
Case (ii)	$A \to B \to E \to A$	–
Case (iii)	$A \to B \to C \to D \to E \to A$	15

It is observed that case(ii) is not satisfying the travelling salesman route condition. Therefore case(ii) is not optimal solution, case (i) and (iii) have same schedule and same travelling cost. Therefore the optimal travelling schedule is $A \to B \to C \to D \to E \to A$.

The corresponding optimal travelling cost is Rs.15.

b) Solution by BCR Approach:

Phase I

Step 1: Finding the minimum value from four cities to city A.

$B \to A = 6$

$C \to A = 8$

$D \to A = 12$

$E \to A = 1$

The minimum travelling cost is $E \to A$.

Step 2: Finding the next minimum value, which is continuation of connected city in step 1.

$A \to B = 2$

$A \to C = 5$

$A \to D = 7$

The minimum travelling cost is $A \to B$.

Step 3: The remaining two possibilities from step 2 are

$B \to C$ and $B \to D$.

Step 4: Connect the rest of the final possibilities:

$C \to D$ and $D \to C$.

Step 5: Go to the starting node i.e., $D \to E$ and $C \to E$.

Step 6: Finding the two possible routes:

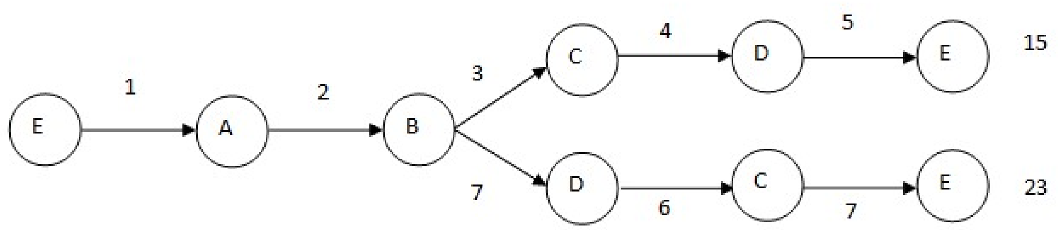

Figure 7.4: Possible Route 1

Table 7.22: Possible Sequence and Travelling Cost in Phase I

Possible Sequence	Travelling Cost
$A \to B \to C \to D \to E \to A$	15
$A \to B \to D \to C \to E \to A$	23

Phase II

Step 1: Finding the minimum value from city A to rest of the four cities.

$A \to B = 2$

$A \to C = 5$

$A \to D = 7$

$A \to E = 1$

The minimum travelling cost is $A \to E$.

Step 2: Finding the next minimum value, which is continuation of connected city in step 1.

$E \to B = 3$

$E \to C = 2$

$E \to D = 8$

The minimum travelling cost is $E \to C$.

Step 3: The remaining two possibilities from step 2 are $C \to B$ and $C \to D$.

Step 4: Connect the rest of the final possibilities:

$B \to D$ and $D \to B$.

Step 5: Go to the starting node i.e.,$D \to A$ and $B \to A$.

Step 6: Finding the two possible routes:

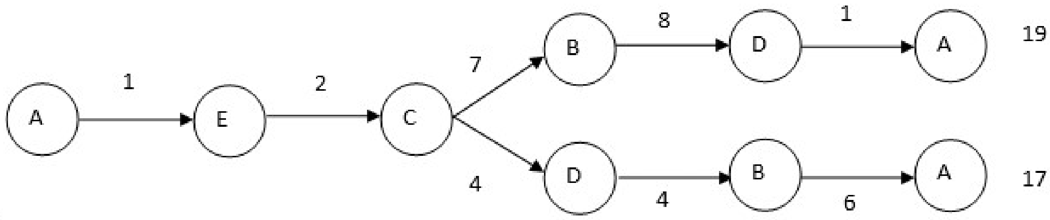

Figure 7.5: Possible Route 2

Table 7.23: Possible Sequence and Travelling cost in Phase II

Possible Sequence	**Travelling Cost**
$A \to E \to C \to B \to D \to A$	19
$A \to E \to C \to D \to B \to A$	17

All possible sequences and travelling costs are given below:

Finally, finding the least possible route cost among the two phases. The least cost is called optimal travelling cost. The corresponding route is the optimal route. The least travelling cost is 15.

Therefore, the optimal route is $A \to B \to C \to D \to E \to A$

The optimal travelling cost is 15.

Note: The problem does not have multiple solutions.

Table 7.24: All Possible Sequences and Travelling costs in Phase I and Phase II

Phases	**Possible Sequence**	**Travelling Cost**
Phase I	$A \to B \to C \to D \to E \to A$	15
	$A \to B \to D \to C \to E \to A$	23
Phase II	$A \to E \to C \to B \to D \to A$	19
	$A \to E \to C \to D \to B \to A$	17

7.1.3 Comparison Analysis

Table 7.25: Comparison Table

Examples	**Hungarian Algortihm**	**BCR Approach**
Example 1	$A \to E \to C \to B \to D \to A$	16
	$A \to D \to B \to C \to E \to A$	16
Example 2	$A \to B \to C \to D \to E \to A$	15

7.2 Solving Travelling Salesman Problem using BCR Approach in Fuzzy Environment

In this section, BCR approach is proposed to solve fuzzy travelling salesman problem for five cities. Suitable illustrations are given to prove the effectiveness of BCR approach.

7.2.1 BCR Approach for Solving Fuzzy Travelling Salesman Problem

Let the cities are represented by circles and the distances or costs are represented by arrow.

Phase I

Step 1: Find minimum fuzzy travelling cost from four cities to first city. Fix the assignment.

Step 2: Find the next minimum fuzzy travelling cost, which is continuation of connected city in step 1.

Step 3: Find the remaining two possibilities from step 2.

Step 4: Connect the rest of the final possibilities.

Step 5: Go to the starting city.

Step 6: Find the two possible routes.

Phase II

Step 1: Find minimum fuzzy travelling cost from first city to rest of the four cities. Fix the assignment.

Step 2: Find the next minimum fuzzy travelling cost, which is continuation of connected city in step 1.

Step 3: Find the remaining two possibilities from step 2.

Step 4: Connect the rest of the final possibilities.

Step 5: Go to the starting city.

Step 6: Find the two possible routes.

Finally, find the minimum fuzzy travelling cost among two phases. The minimum travelling cost is called optimal travelling cost. The corresponding sequence is called optimal sequence.

Note: If there is a tie, take the two possibilities and do the same process for it.

7.2.2 Numerical Illustrations

In order to apply BCR approach on travelling salesman problem in fuzzy environment, few numerical examples are illustrated.

Example 7.2.1: Consider the fuzzy travelling salesman problem: A salesman has planned to visit five cities. He would like to start his journey from a particular city, visits each city only once and return to the starting city. The travelling costs are given in the table 7.26 below. Find the least cost route for FTSP.

Table 7.26: Fuzzy Travelling Salesman Problem 1

From	To				
	A	B	C	D	E
A	∞	$(5,6,8,9;0.4,0.6)$	$(3,4,6,7;0.3,0.5)$	$(0,2,4,6;0.2,0.4)$	$(3,5,7,8;0.3,0.5)$
B	$(5,6,8,9;0.4,0.6)$	∞	$(5,6,10,11;0.4,0.6)$	$(2,3,4,6;0.3,0.5)$	$(0,2,4,6;0.2,0.4)$
C	$(3,4,6,7;0.3,0.5)$	$(5,6,10,11;0.4,0.6)$	∞	$(4,5,7,8;0.3,0.5)$	$(0,1,3,4;0.2,0.4)$
D	$(0,2,4,6;0.2,0.4)$	$(2,3,5,6;0.3,0.5)$	$(4,5,7,8;0.3,0.5)$	∞	$(0,1,3,4;0.2,0.4)$
E	$(3,5,7,8;0.3,0.5)$	$(0,2,4,6;0.2,0.4)$	$(0,1,3,4;0.2,0.5)$	$(0,1,3,4;0.2,0.5)$	∞

Solution by BCR Approach:

Phase I

Step 1: Finding the minimum value from four cities to city A

$B \to A = (5, 6, 8, 9; 0.4, 0.6)$

$C \to A = (3, 4, 6, 7; 0.3, 0.5)$

$D \to A = (0, 2, 4, 6; 0.2, 0.4)$

$E \to A = (3, 5, 7, 8; 0.3, 0.5)$

The minimum travelling cost is $D \to A$.

Step 2: Finding the next minimum value, which is continuation of connected city in step 1.

$A \to B = (5, 6, 8, 9; 0.4, 0.6)$

$A \to C = (3, 4, 6, 7; 0.3, 0.5)$

$A \to E = (3, 5, 7, 8; 0.3, 0.5)$

The minimum travelling cost is $A \to C$.

Step 3: The remaining two possibilities from step 2 are

$C \to B$ and $C \to E$.

Step 4: Connecting the rest of the final possibilities:

$B \to E$ and $E \to B$.

Step 5: Go to the starting node i.e., $E \to D$ and $B \to D$.

Step 6: Finding the two possible routes:

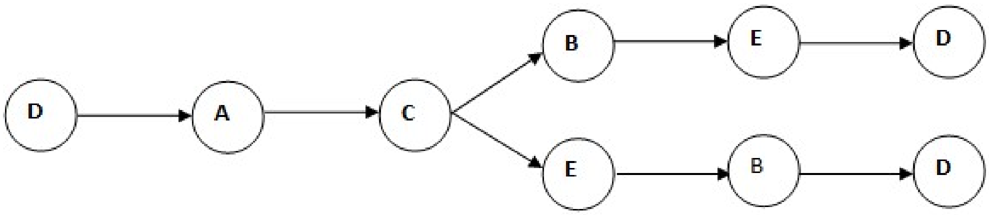

Figure 7.6: Possible Route 1

Table 7.27: Possible sequence and Travelling cost in Phase I

Possible Sequence	Travelling Cost
$A \to C \to B \to E \to D \to A$	(8, 15, 27, 34; 0.2, 0.4)
$A \to C \to E \to B \to D \to A$	(5, 12, 21, 29; 0.2, 0.4)

Phase II

Step 1: Finding the minimum value from city A to rest of the four cities.

$A \to B = (5, 6, 8, 9; 0.4, 0.6)$

$A \to C = (3, 4, 6, 7; 0.3, 0.5)$

$A \to D = (0, 2, 4, 6; 0.2, 0.4)$

$A \to E = (3, 5, 7, 8; 0.3, 0.5)$

The minimum travelling cost is $A \to D$.

Step 2: Finding the next minimum value, which is continuation of connected city in step 1.

$D \to B = (2, 3, 5, 6; 0.3, 0.5)$

$D \to C = (2, 3, 5, 6; 0.3, 0.5)$

$D \to E = (0, 1, 3, 4; 0.2, 0.4)$

The minimum travelling cost is $D \rightarrow E$.

Step 3: The remaining two possibilities from step 2 are

$E \rightarrow B$ and $E \rightarrow C$.

Step 4: Connecting the rest of the final possibilities:

$B \rightarrow C$ and $C \rightarrow B$.

Step 5: Go to the starting node i.e., $C \rightarrow A$ and $B \rightarrow A$.

Step 6: Finding the two possible routes:

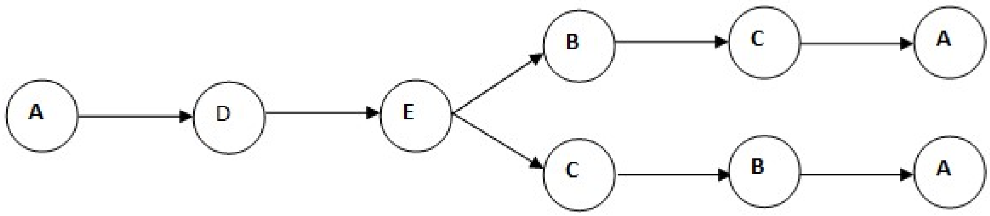

Figure 7.7: Possible Route 2

Table 7.28: Possible Sequence and Travelling cost in Phase II

Possible Sequence	Travelling Cost
$A \rightarrow D \rightarrow E \rightarrow B \rightarrow C \rightarrow A$	(8, 15, 27, 34; 0.2, 0.4)
$A \rightarrow D \rightarrow E \rightarrow C \rightarrow B \rightarrow A$	(10, 16, 28, 34; 0.2, 0.4)

Table 7.29: Results of Phase I and Phase II

Phase	**Possible Sequence**	**Travelling Cost**
Phase I	$A \to C \to B \to E \to D \to A$	(8, 15, 27, 34; 0.2, 0.4)
	$A \to C \to E \to B \to D \to A$	(5, 12, 21, 29; 0.2, 0.4)
Phase II	$A \to D \to E \to B \to C \to A$	(8, 15, 27, 34; 0.2, 0.4)
	$A \to D \to E \to C \to B \to A$	(10, 16, 28, 34; 0.2, 0.4)

Finally, find the least possible route cost among the two phases. The least cost is called optimal travelling cost. The corresponding route is the optimal route.

The minimum travelling cost is (5, 12, 21, 29; 0.2, 0.4).

Therefore, the optimal route is $A \to C \to E \to B \to D \to A$

The optimal travelling cost is $(5, 12, 21, 29; 0.2, 0.4)$

Note: Therefore, the problem does not have multiple solutions.

Example 7.2.2: Consider the fuzzy travelling salesman problem: A salesman has planned to visit 5 cities. He would like to start his journey from a particular city, visits each city only once and return to the starting city. The travelling costs are given in the table 7.30. Find the least cost route for FTSP.

Table 7.30: Fuzzy Travelling Salesman Problem 2

From	To				
	A	B	C	D	E
A	∞	$(1, 2, 4, 5; 0.3, 0.6)$	$(3, 7, 10, 14; 0.2, 0.5)$	$(2, 3, 5, 6; 0.3, 0.6)$	$(0, 1, 4, 5; 0.4, 0.6)$
B	$(0, 2, 4, 6; 0.3, 0.6)$	∞	$(1, 4, 6, 9; 0.3, 0.5)$	$(1, 2, 3, 4; 0.4, 0.6)$	$(1, 7, 15, 21; 0.2, 0.4)$
C	$(3, 7, 10, 14; 0.2, 0.5)$	$(3, 4, 6, 7; 0.3, 0.5)$	∞	$(2, 5, 15, 18; 0.2, 0.4)$	$(1, 2, 5, 6; 0.3, 0.5)$
D	$(1, 3, 5, 7; 0.3, 0.6)$	$(0, 2, 3, 5; 0.4, 0.6)$	$(2, 5, 15, 18; 0.2, 0.4)$	∞	$(0, 2, 16, 18; 0.2, 0.5)$
E	$(0, 1, 4, 5; 0.4, 0.6)$	$(1, 7, 15, 21; 0.2, 0.4)$	$(2, 5, 15, 18; 0.2, 0.4)$	$(0, 2, 16, 18; 0.2, 0.5)$	∞

Solution by BCR Approach:

Phase I

Step 1: Find minimum value from city A to rest of the four cities.

$B \to A = (0, 2, 4, 6; 0.3, 0.6)$

$C \to A = (3, 7, 10, 14; 0.2, 0.5)$

$D \to A = (1, 3, 5, 7; 0.3, 0.6)$

$E \to A = (0, 1, 4, 5; 0.4, 0.6)$

The minimum travelling cost is $E \to A$.

Step 2: Find the next minimum value, which is continuation of connected city in step 1.

$A \to B = (1, 2, 4, 5; 0.3, 0.6)$

$A \to C = (3, 7, 10, 14; 0.2, 0.5)$

$A \to D = (2, 3, 5, 6; 0.3, 0.6)$

The minimum travelling cost is $A \to B$.

Step 3: The remaining two possibilities from step 2 are

$B \to C$ and $B \to D$.

Step 4: Connecting the rest of the final possibilities:

$C \to D$ and $D \to C$.

Step 5: Go to the starting node i.e., $D \to E$ and $C \to E$.

Step 6: Finding the two possible routes:

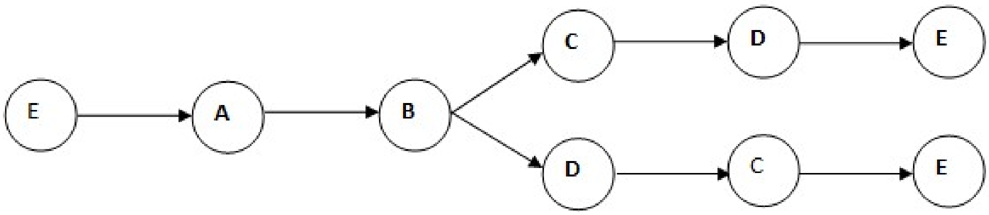

Figure 7.8: Possible Route 1

Table 7.31: Possible sequence and Travelling Cost in Phase I

Possible Sequence	Travelling Cost
$A \to B \to C \to D \to E \to A$	$(4, 14, 45, 55; 0.2, 0.4)$
$A \to B \to D \to C \to E \to A$	$(5, 12, 31, 38; 0.2, 0.4)$

Phase II

Step 1: Finding the minimum value from city A to rest of the three cities.

$A \to B = (1, 2, 4, 5; 0.3, 0.6)$

$A \to C = (3, 7, 10, 14; 0.2, 0.5)$

$A \to D = (2, 3, 5, 6; 0.3, 0.6)$

$A \to E = (0, 1, 4, 5; 0.4, 0.6)$

The minimum travelling cost is $A \to E$.

Step 2: Finding the next minimum value, which is continuation of connected city in step 1.

$E \to B = (1, 7, 15, 21; 0.2, 0.4)$

$E \to C = (2, 3, 4, 5; 0.3, 0.5)$

$E \to D = (0, 2, 16, 18; 0.2, 0.5)$

The minimum travelling cost is $E \to C$.

Step 3: The remaining two possibilities from step 2 are $C \to B$ and $C \to D$.

Step 4: Connecting the rest of the final possibilities $B \to D$ and $D \to B$.

Step 5: Go to the starting node i.e., $D \to A$ and $B \to A$.

Step 6: Finding the two possible routes:

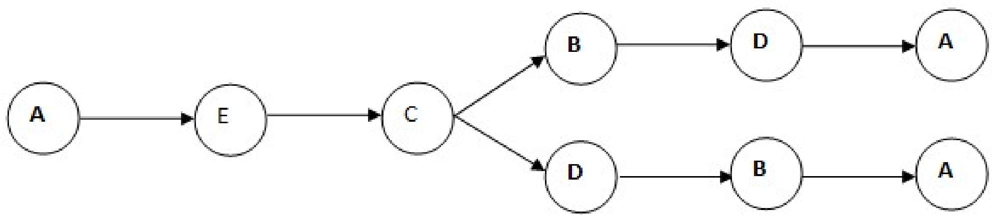

Figure 7.9: Possible Route 2

Table 7.32: Possible Sequence and Travelling Cost in Phase II

Possible Sequence	**Travelling Cost**
$A \rightarrow E \rightarrow C \rightarrow B \rightarrow D \rightarrow A$	(7, 15, 33, 41; 0.2, 0.4)
$A \rightarrow E \rightarrow C \rightarrow D \rightarrow B \rightarrow A$	(4, 15, 41, 52; 0.2, 0.4)

All possible sequences and travelling costs are given below:

Table 7.33: All Possible Sequences and Travelling Costs in Phase I and Phase II

Phase	**Possible Sequence**	**Travelling Cost**
Phase I	$A \to B \to C \to D \to E \to A$	(4, 14, 45, 55; 0.2, 0.4)
	$A \to B \to D \to C \to E \to A$	(5, 12, 31, 38; 0.2, 0.4)
Phase II	$A \to E \to C \to B \to D \to A$	(7, 15, 33, 41; 0.2, 0.4)
	$A \to E \to C \to D \to B \to A$	(4, 15, 41, 52; 0.2, 0.4)

Finally, finding the least possible route cost among the two phases. The least cost is called optimal travelling cost. The corresponding route is the optimal route.

The optimal route is $A \to B \to D \to C \to E \to A$.

The optimal travelling cost is $(5, 12, 31, 38; 0.2, 0.4)$.

Note: Therefore, the problem does not have multiple solutions.

7.3 Conclusion and Future Outlook

The concluding chapter puts forth the findings and scope of the thesis. They are summarized as follows: In Chapter 1 the overall view of optimization theory and its applications, fuzzy set theory and how fuzzy theory is applied in optimization, linear programming problems and subclasses of linear programming problems are briefly summarized. Literature of journals, professional organizations and other relevant information are also presented.

In Chapter 2, some rudimentary definitions and important results which are indispensable to the thesis work such as optimization, fuzzy sets, fuzzy numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, fuzzy optimization, etc., are presented.

In Chapter 3, a new fuzzy number named as Generalized Quadrilateral Fuzzy Numbers and its graphical representation, arithmetic operations of the Generalized Quadrilateral Fuzzy Numbers have been introduced. To solve Fuzzy Transportation problem in fuzzy environment existing methods with some required steps were used. These methods bring Initial basic feasible solution to the fuzzy transportation problem.

In Chapter 4, a new algorithm named as Stephen's Algorithm and a distinct working procedure for fuzzy assignment problem have been proposed. Hungarian Algorithm is summarized to validate Stephen's Algorithm with suitable illustrations.

In Chapter 5, working procedure of Dynamic Programming Algorithm is summarized and also applied on both travelling salesman problem and fuzzy travelling salesman problem. Suitable illustrations were given to understand and to follow the working procedure. This algorithm always provides an optimal solution to both travelling salesman problem and fuzzy travelling salesman problem.

In Chapter 6, ground breaking approach named as DSD approach has been proposed to solve both travelling salesman problem and fuzzy travelling salesman problem with four cities. Since Dynamic programming algorithm provides the optimal solution to both travelling salesman problem and fuzzy travelling salesman problem, the result of the DSD approach is compared with Dynamic programming algorithm results. DSD approach provides optimal solution with limited number of steps. The significance of the approach is almost same structure to solve all travelling salesman problem with four cities.

In Chapter 7, an innovative approach named as BCR approach has been introduced to solve both travelling salesman problem and fuzzy travelling salesman problem with five cities. The working procedure of Hungarian algorithm and BCR approach have been presented. Few numerical illustrations have been provided to examine the efficiency of BCR. It is found that the BCR approach reduces the computational time and minimizes the number of steps to reach optimal solution.

Future work

Generalized Quadrilateral Fuzzy Numbers have been applied to solve transportation problem, assignment problem and travelling salesman problem with limited algorithms. It may be extended to solve all subclasses of the linear programming problems and all other algorithms.

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