

(6 pages)

S.No. 7336

RNENS 6

(For candidates admitted from 2006–2007 onwards)

M.Sc. DEGREE EXAMINATION, NOVEMBER 2023.

Mathematics

TOPOLOGY AND FUNCTIONAL ANALYSIS

Time : Three hours

Maximum : 100 marks

Subdivisions (a), (b) and (c) in each question carry 4, 6 and 10 marks respectively.

1. (a) Let X be a set; let $\mathcal{B}$ be a basis for a topology τ on X . Then show that τ equals the collection of all unions of elements of $\mathcal{B}$.
- (b) (i) Let Y be a subspace of X . Then prove that a set is closed in Y if and only if it equals the intersection of a closed set of X with Y .

Or

- (ii) Let X and Y be topological spaces; let $f : X \rightarrow Y$. Then show the following are equivalent:

- (1) f is continuous
- (2) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

- (c) (i) Prove that the topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Or

- (ii) (1) Prove that the union of a collection of connected subspaces of X that have a point in common is connected
- (2) Prove that the image of a connected space under a continuous map is connected.

2. (a) Prove that every closed subspace of a compact space is compact.

(b) (i) State and prove the tube lemma.

Or

(ii) Show that a subspace of a completely regular space is completely regular.

(c) (i) Prove that a subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the euclidean metric d or the square metric ρ .

Or

(ii) Suppose that X has a countable basis. Then prove that.

(1) Every open covering of X contains a countable subcollection covering X .

(2) There exists a countable subset of X that is dense in X .

3. (a) Let X be a set; let $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then if a subset of X that intersects every element of $\mathcal{D}$, then show that A is an element of $\mathcal{D}$.

(b) (i) Let X be a completely regular space. Then show that there exists a compactification Y of X having the property that every bounded continuous map $f: X \rightarrow \mathbb{R}$ extends uniquely to a continuous map of Y into $\mathbb{R}$.

Or

(ii) Let (X, d) be a metric space. Then show that there is an isometric imbedding of X into a complete metric space.

(c) (i) State and prove Tychonoff theorem.

Or

(ii) State and prove classical version of Ascoli's theorem.

4. (a) (i) Define a normed linear space.
 (ii) Give any five examples of a Banach space.
 (b) (i) If P is a projection on a Banach space B , and if M and N are its range and null space, then show that M and N are closed linear subspaces of B such that $B = M \oplus N$.

Or

- (ii) If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M , then show that there exists a functional f_0 in N^* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$.
 (c) (i) Suppose T is a continuous linear transformation from the Banach space B onto the Banach space B' . Then prove that the image of each open sphere centered on the origin in B contains an open sphere centered on the origin in B' .

Or

- (ii) If N is a normed linear space, then show that the closed unit sphere S^* in N^* is a compact Hausdorff space in the weak* topology.

5. (a) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.
 (b) (i) Show that every non-zero Hilbert space contains a complete orthonormal set.

Or

- (ii) If T is normal, then prove that an eigenvector x of T with eigenvalue λ if and only if x is an eigenvector of T^* with eigenvalue $\bar{\lambda}$.
 (c) (i) If P is a projection on H with range M and null space N , then show that $M \perp N \Leftrightarrow P$ is self-adjoint; and in this case, $N = M^\perp$.

Or

- (ii) Show that if T is normal, then the M_i 's are pairwise orthogonal and each M_i reduces T .