S.No. 7333 **RNENS 3**

(For candidates admitted from 2006–2007 onwards)

M.Sc. DEGREE EXAMINATION, NOVEMBER 2023.

Mathematics

COMPLEX ANALYSIS

Time : Three hours Maximum : 100 marks

Answer ALL questions.

Subdivision (a), (b) and (c) in each question carry 4, 6 and 10 marks respectively.

1. (a) Prove that on a compact set every continuous function is uniformly continuous.

(b) (i) Prove that every set has a unique decomposition into components.

Or

(ii) If a linear transformation carries a circle C_1 into a circle C_2, then prove that it transforms any pair of symmetric points with respect to C_1 into a pair of symmetric points with respect to C_2.

(c) (i) Prove that the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle or on a straight line.

Or

(ii) Prove that a nonempty open set in the plane is connected if and only if any two of its points can be joined by a polygon which lies in the set.

2. (a) State and prove the maximum principle.

(b) (i) Let $f(z)$ be analytic on the set R' obtained from a rectangle R by omitting a finite number of interior points ζ_j If it is true that $\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0$ for all j then prove that $\int_{\partial R} f(z) dz = 0$.

Or

(ii) Prove that the line integral $\int_\gamma pdx + qdy$, defined in Ω depends only on the end points of γ if and only if there exists a function $U(x, y)$ in Ω with the partial derivatives $\frac{\partial U}{\partial x} = p$, $\frac{\partial U}{\partial y} = q$.

(c) (i) Suppose that $\phi(\zeta)$ is continuous on the arc γ. Then prove that the function $F_n(z) = \int_\gamma \frac{\phi(\zeta)g\zeta}{(\zeta - z)^n}$ is analytic in each of the regions determined by γ and its derivative is $F_n'(z) = nF_{n+1}(z)$.

Or

(ii) If the function $f(z)$ is analytic on R then prove that $\int_{\partial R} f(z)dz = 0$.

3. (a) Find the poles and residue of the following function: $\frac{1}{(z^2 - 1)^2}$.

(b) (i) Derive Poissons formula.

Or

(ii) Let γ be homologous to zero in Ω and such that $n(\gamma, z)$ is either 0 or 1 for any point z not on γ. Suppose that $f(z)$ and $g(z)$ are analytic in Ω, and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on γ. Then prove that $f(z)$ and $g(z)$ have the same number of zeros enclosed by γ.

(c) (i) Let Ω^+ be the part in the upper half plane of a symmetric region Ω, and let σ be the real axis in Ω. Suppose that $v(x)$ is continuous in $\Omega^+ \cup \sigma$, harmonic in Ω^+, and zero on σ. Then prove that v has a harmonic extension to Ω which satisfies the symmetry relation $v(\overline{z}) = -v(z)$. In the same situation, if v is the imaginary part of an analytic function $f(z)$ in Ω^+, then prove that $f(z)$ has an analytic extension which satisfies $f(z) = \overline{f(\overline{z})}$.

Or

(ii) Prove that the function $P_U(z)$ is harmonic for $|z| < 1$ and $\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0)$ provided that U is continuous at θ_0.

4. (a) (i) Define normal.

(ii) Define equi continuous,

(b) (i) Derive Jensen's formula.

Or

(ii) Prove that a family $\mathscr{F}$ of analytic functions is normal with respect to C if and only if the functions in $\mathscr{F}$ are uniformly bounded on every compact set.

(c) (i) Prove that the genus and the order of an entire function satisfy the double inequality $h \leqq \lambda \leqq h+1$.

Or

(ii) Derive Laurent series.

5. (a) Prove that the sum of the residues of an elliptic function is zero.

(b) (i) Prove that the zeros $a_1, \dots a_n$ and poles $b_1, \dots, b_n$ of an elliptic function satisfy $a_1 + \dots + a_n \equiv b_1 + \dots + b_n \pmod{M}$.

Or

(ii) Suppose that the boundary of a simply connected region Ω contains a line segment γ as a one-sided free boundary arc. Then prove that the function $f(z)$ which maps Ω onto the unit disk can be extended to a function which is analytic and one to one on $\Omega \cup \gamma$. Prove that the image of γ is an arc γ' on the unit circle.

(c) (i) Prove that a discrete module consists either of zero alone, of the integral multiples nw of a single complex number $w \neq 0$, or of all linear combinations $n_1 w_1 + n_2 w_2$ with integral coefficients of two numbers w_1, w_2 with nonreal ratio w_2 / w_1.

Or

(ii) Prove that every point r in the upper half plane is equivalent under the congruence subgroup mod 2 to exactly one point in $\overline{\Omega} \cup \Omega'$.
