(6 pages)

S.No. 7332 **RNENS 2**

(For candidates admitted from 2006–2007 onwards)

M.Sc. DEGREE EXAMINATION, NOVEMBER 2023.

Mathematics

REAL ANALYSIS

Time : Three hours Maximum : 100 marks

Answer ALL the questions.

Subdivisions (a), (b) and (c) in each questions carries 4, 6 and 10 marks respectively.

1. (a) (i) Prove that every neighbourhood is an open set.

(ii) Prove that e is irrational.

(b) (i) Suppose $K \subset Y \subset X$. Then prove that K is compact relative to X iff K is compact relative to Y.

Or

(ii) State and prove Weierstrass theorem.

(c) (i) Prove that compact subsets of metric spaces are closed.

Or

(ii) Suppose s_n, t_n, are complex sequences and $\lim\limits_{n\to\infty} s_n = s$, $\lim\limits_{n\to\infty} t_n = t$. Then prove that

(1) $\lim\limits_{n\to\infty}(s_n + t_n) = s + t$

(2) $\lim\limits_{n\to\infty}(s_n t_n) = st$

(3) $\lim\limits_{n\to\infty}(cs_n) = cs$ for any number c.

2. (a) (i) Define a continuous function.

(ii) Define local maximum of a function.

(b) (i) Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$, $(a \le t \le b)$ then prove that h is differentiable at x, and $h'(x) = g'(f(x))f'(x)$.

Or

(ii) Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then prove that there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

(c) (i) Let f be a continuous mapping of a compact metric space X into a metric space Y. Then prove that f is uniformly continuous on a compact metric space X.

Or

(ii) State and prove Taylor's theorem.

3\. (a) (i) Write the integration by parts formula.

(ii) Define common refinement.

(b) (i) Prove that $\underline{\int_a^b} f d\alpha \le \overline{\int_a^b} f d\alpha$.

Or

(ii) State and prove the fundamental theorem of calculus.

(c) (i) Prove that $\mathscr{R}(\alpha)$ on $[a, b]$ iff for every $\in > 0$ there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \in$.

Or

(ii) Let $f \in \mathscr{R}(\alpha)$ on $[a, b]$. For $a \le x \le b$, put $F(x) = \int_a^x f(t)dt$. Then prove that F is continuous on $[a, b]$; Furthermore, if f is continuous at a point x_0 of $[a, b]$, then also prove that F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

4\. (a) (i) Define pointwise convergence with an example.

(ii) Define pointwise bounded with an example.

(b) (i) Let a be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \ldots$ and suppose $f_n \to f$ uniformly on $[a, b]$. Then prove that $f \in \mathscr{R}(\alpha)$ on $[a, b]$, and $\int_a^b f d\alpha = \lim_{n\to\infty} f_n d\alpha$.

Or

(ii) If K is a compact metric space, if $f_n \in C(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ converges uniformly on K, then prove that $\{f_n\}$ is equicontinuous on K.

(c) (i) Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E and suppose that $\lim_{t \to x} f_n(t) = A_n$, $(n = 1, 2, 3, \ldots)$. Then prove that $\{A_n\}$ converges and $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$.

Or

(ii) State and prove the Stone-Weierstrass theorem.

5. (a) (i) State Lebesgue's monotone convergence theorem.

(ii) Define outer measure.

(b) (i) Suppose $f = f_1 + f_2$, where $f_i \in \mathscr{L}(\mu)$ on $E (i = 1, 2)$. Then prove that $f \in \mathscr{L}(\mu)$ on E, and $\int_E f d\mu = \int_E f_1 d\mu + \int_E f_2 d\mu$.

Or

(ii) State and prove fatou's theorem.

(c) (i) (1) Let f and g be a measurable real-valued functions defined on X, let F be real and continuous on R^2, and put $h(x) = F(f(x), g(x))$ where $x \in X$. Then prove that h is measurable.

(2) If f is measurable then prove that $|f|$ is measurable.

Or

(ii) State and prove Lebesgue's dominated convergence theorem.