(6 pages)

S.No. 7331 **RNENS 1**

(For candidates admitted from 2006–2007 onwards)

M.Sc. DEGREE EXAMINATION, NOVEMBER 2023.

Mathematics

ALGEBRA

Time : Three hours Maximum : 100 marks

Answer ALL questions

Subdivsions a,b and c in each question carry 4,6 and 10 marks respectively

1. (a) Define the following:

(i) Integer monic.

(ii) Normalizer.

(b) (i) Let G be a group and suppose that G is the internal direct product of $N_1,\cdots,N_n$. Let $N_1 \times N_2 \times \cdots \times N_n$. Then prove that G and T are isomorphic.

Or

(ii) State and prove Gauss Lemma.

(c) (i) If p is a prime number and $p \mid o(G)$, then prove that G has an element of order p.

Or

(ii) Show that if R is unique factorization domain, then so is $R[x]$.

2. (a) Define the following:

(i) Algebraic extension.

(ii) Algebraic of degree n.

(b) (i) Prove that K is a normal extension of F if and only if K is the splitting field of some polynomial over F.

Or

(ii) Let C be the field of complex numbers and suppose that the division ring D is algebraic over C. then prove that $D = C$.

(c) (i) Prove that a polynomial of degree n over a field can have at most n roots in any extension field.

Or

(ii) State and prove Wedderburn theorem.

3. (a) If f and g are linear functionals on a vector space V. then g is a scalar multiple of f if and only if the null space of g contains the null space of f, that is. if and only if $f(\alpha)=0$ implies $g(\alpha)=0$.

(b) (i) Let T be a linear transformation from V into W. Then prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

Or

(ii) Let V be a finite-dimensional vector space over the field F. For each vector α in V define $L_{\alpha}(f)=f(\alpha), f$ in V^{*}. Then Prove that the mapping $\alpha \rightarrow L_{\alpha}$ is isomorphism of V onto V^{**}.

(c) (i) Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose that V is finite dimensional. Then prove that $rank(T)+nullity(T)=\dim V$.

Or

(ii) Let V be a finite-dimensional vector space over the field F, and let W be a subspace of V then prove that

$\dim W+\dim W^{\circ}=\dim V$.

4. (a) Prove that a linear combination of n-linear functions is n-linear.

(b) (i) Let p, f and g be polynomials over the field F. Suppose that p is a prime polynomial and that p divides the product f g. Then prove that either p divides f or p divides g.

Or

(ii) If f, d are polynomials over a field F and d is different from 0 then there exist polynomials q, r in $F[x]$ such that

(1) $f=dq+r$.

(2) either $r=0$ or deg r < deg d.

Then prove that the polynomials q, r satisfying (1) and (2) are unique.

(c) (i) Prove that if F is a field, a non-scalar monic polynomial in F[x] can be factored as a product of monic primes in F[x] in one and, except for order, only one way.

Or

(ii) Let D be an n-linear function on $n \times n$ matrices over K. Suppose D has the property that $D(A)=0$ whenever two adjacent rows of A are equal. Then prove that D is alternating.

5. (a) Let W be an invariant subspace for T. The characteristic polynomial for the restriction operator T_W divides the characteristic polynomial for T. Then prove that the minimal polynomial for T_W divides the minimal polynomial for T.

(b) (i) Let V be finite dimensional vector space over the field F and let T be a linear operator on V. Then prove that T is diagonalizable if and only if the minimal polynomial for T has the form $p=(x-c_1)\cdots(x-c_k)$ where $c_1,\cdots,c_k$ are distinct elements of F.

Or

(ii) Let T be a linear operator on an n-dimensional vector space V [or, let A be an $n \times n$ matrix]. Then prove that the characteristic and minimal polynomials for T [for A] have the same roots, except for multiplicities.

(c) (i) Let $\mathcal{F}$ be a commuting family of trainagulable linear operators on V. Let W be a proper subspace of V which is invariant under $\mathcal{F}$. Then prove that there exist a vector α in V such that

(1) α is not in W;

(2) for each T in $\mathcal{F}$, the vecotr $T\alpha$ is in the subspace spanned by α and W.

Or

(ii) Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix $A=\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ prove that T is diagonalizable.