(6 pages)

S.No. 6865 **P 22 MACC 11**

(For candidates admitted from 2022–2023 onwards)

M.Sc. DEGREE EXAMINATION, NOVEMBER 2023.

Mathematics

ALGEBRA

Time : Three hours Maximum : 75 marks

SECTION A — (20 marks)

Answer ALL questions.

I. (A) Choose the correct answer. $(5 \times 1 = 5)$

1. If n is a positive integer and a is relatively prime to n, then ——————

   (a) $a^{\phi(n)} \equiv 1 \bmod n$ (b) $a^{\phi(n)} \equiv 0 \bmod n$

   (c) $a^{\phi(n)} \not\equiv 1 \bmod 1$ (d) $a^{\phi(n)} \equiv n \bmod n$

2. A mapping ϕ from a group G into a group $\vec{G}$ is said to be a homomorphism if for all $a, b \in G$ ——————.

   (a) $\phi\left(\frac{a}{b}\right) = \frac{\phi(a)}{\phi(b)}$ (b) $\phi(ab) = \phi(a)\phi(b)$

   (c) $\phi(ab) \neq \phi(a)\phi(b)$ (d) None of the above

3. If U is an ideal of a ring R and $1 \in U$, then $U =$ ——————

   (a) R (b) R/U

   (c) RU (d) U/R

4. If $f(x), g(x)$ are two non zero element of $F[x]$, then $\deg(f(x)\, g(x)) =$ ——————

   (a) $\deg f(x) - \deg g(x)$ (b) $\deg f(x). \deg g(x)$

   (c) $\deg f(x) + \deg g(x)$ (d) $\deg f(x) / \deg g(x)$

5. If $a \in K$ is algebraic of degree n over F, then ——————

   (a) $[F(a) : F] = n + 1$ (b) $F(a) = F$

   (c) $[F(a) : F] = n$ (d) $[F(a) : F] = n - 1$

(B) Fill in the blanks : $(5 \times 1 = 5)$

6. The null set is the set having ——————

7. If G is a finite group, and $H \neq G$ is a subgroup of G such that $o(G) \not\mid i(H)!$ then H must contain a —————— normal subgroup of G.

8. In the Euclidean ring R, a and b in R are said to be ____________ if their greatest common divisor is a unit of R.

9. If $a \in R$ is an ____________ and $a \mid bc$, then $a \mid b$ or $a \mid c$.

10. If $p(x) \in F[x]$, then an element a lying in some extension field of F is called a root of $p(x)$ if ____________.

II. Answer the following questions. $(5 \times 2 = 10)$

11. Define mapping.

12. Define Kernel.

13. Define homomorphism.

14. Define Primitive.

15. Define Algebraic extension.

SECTION B — $(5 \times 5 = 25)$

Answer ALL questions, choosing either (a) or (b).

16. (a) If $\sigma : S \rightarrow T$, $\tau : T \rightarrow U$, and $\mu : U \rightarrow V$, then prove that $(\sigma \circ \tau) = \sigma \circ (\tau \circ \mu)$.

Or

(b) Prove that HK is a subgroup of G if and only if $HK = KH$.

17. (a) Show that if ϕ is a homomorphism of G into $\bar{G}$ with kernel K, then K is a normal subgroup of G.

Or

(b) Prove that conjugacy is an equivalence relation on G.

18. (a) If R is a commutative ring with unit element and M is an ideal of R, then prove that M is a maximal ideal of R if and only if R/M is a field.

Or

(b) If p is a prime number of the form $4n+1$, then prove that the congruence $x^2 \equiv -1 \bmod p$.

19. (a) State and prove the division algorithm.

Or

(b) If $u, v \in V$ then prove that $|(u, v)| \leq \|u\| \|v\|$.

20. (a) If K is a finite extension of F, then prove that $G(K, F)$ is a finite group and its orders, $o(G(K F))$ satisfies $o(G(K,F)) \leq [K:F]$.

Or

(b) Let $f(x) \in F[x]$ be of degree $n \geq 1$, then prove that there is an extension E of F of degree at most $n!$ in which $f(x)$ has n roots.

SECTION C — $(3 \times 10 = 30)$

Answer any THREE questions.

21. If G is a group, then prove that

(a) The identity element of G is unique.

(b) Every $a \in G$ has a unique inverse in G.

(c) For every $a \in G$, $(a^{-1})^{-1} = a$.

(d) For all $a, b \in G, (a, b)^{-1} = b^{-1}.a^{-1}$.

22. If p is a prime number and $p \mid o(G)$, then prove that G has an element of order p.

23. State and prove Unique Factorization Theorem.

24. State and prove Eisenstein criterion.

25. Prove that the element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.