PART C — ($3 \times 10 = 30$)

Answer any THREE questions.

16. If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then prove that $W_1 + W_2$ is finite-dimensional and
$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$.

17. Let V and W be vector spaces over the field F. Let T and U be linear transformations from V into W. The function $(T + U)$ defined by -
$$(T+U)(\propto) = T \propto + U \propto$$
Is a linear transformation from V into W. If c is any element of F, the function (cT) defined by
$$(cT)(\propto) = c(T \propto)$$
Is a linear transformation from V into W. Then prove that the set of all linear transformations from V into W, together with the addition and scalar multiplication defined above, is a vector space over the field F.

18. State and prove Taylor's formula.

19. State and prove Cayley-Hamilton theorem.

20. State and prove Primary Decomposition Theorem.

(For candidates admitted from 2016–2021 batch)

M.Sc. DEGREE EXAMINATION, NOVEMBER 2023.

Mathematics

LINEAR ALGEBRA

Time : Three hours Maximum : 75 marks

PART A — ($10 \times 2 = 20$)

Answer ALL questions.

1. Find the minimal polynomial of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
2. Define linear functional.
3. Define Commutative ring.
4. Given example for direct sum.
5. Find the characteristic values of $\begin{bmatrix} 1 & -5 \\ 0 & 3 \end{bmatrix}$
6. Define a subspace of V.
7. Define an annihilator.
8. Define T – conductor.
9. What is the general form of a Vandermonde matrix?
10. When the subspaces are said to be independent?

PART B — ($5 \times 5 = 25$)

Answer ALL questions, choosing either (a) or (b).

11. (a) If A and B are row-equivalent $m \times n$ matrices, show that the homogeneous systems of linear equations $AX = 0$ and $BX = 0$ have exactly the same solutions.

Or

(b) If W is a subspace of a finite-dimensional vector space V, then prove that every linearly independent subset of W is finite and is part of a basis for W.

12. (a) If A is an $m \times n$ matrix with entries in the field F, then show that

row rank (A) = *column rank* (A).

Or

(b) Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. If T is invertible, then prove that the inverse function T^{-1} is a linear transformation from W onto V.

13. (a) Let F be a field of characteristic zero and f a polynomial over F with deg $f \leq n$. Then prove that the scalar c is a root of f of multiplicity r if and only if

$(D^* f)(c) = 0$, $0 \leq k \leq r-1$ and $(D^r f)(c) \neq 0$.

Or

(b) Let D be an n-linear function on $n \times n$ matrices over K. Suppose D has the property that $D(A) = 0$ whenever two adjacent rows of A are equal. Then prove that D is alternating.

14. (a) Let K be a commutative ring with identity, and let A and B be $n \times n$ matrices over K. Then show that $\det(AB) = (\det A)(\det B)$

Or

(b) Let T be a linear operator on an n-dimensional vector space V [or, let A be an $n \times n$ matrix]. Then prove that the characteristic and minimal polynomials for T [for A] have the same roots, except for multiplicities.

15. (a) Let W be an invariant subspace for T. The characteristic polynomial for the restriction operator T_w divides the characteristic polynomial for T. Then prove that the minimal polynomial for T_w divides the minimial polynomial for T.

Or

(b) If $V = W_1 \oplus \ldots \oplus W_k$, then prove that there exist k linear operators $E_1, \ldots, E_k$ on V such that

(i) each E_i is a projection ($E_i^2 = E_i$);

(ii) $E_i E_j = 0$, if $i \neq j$;

(iii) $I = E_1 + \ldots + E_k$;

(iv) The range of E_i is W_i.