

Chaos and Strange Nonchaotic Attractors

Paulsamy Muruganandam

Department of Physics
Bharathidasan University
Tiruchirappalli – 620024, TN

E-mail: anand@bdu.ac.in

Outline

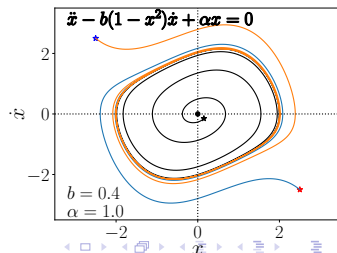
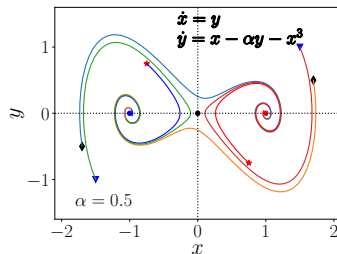
- ▶ Introduction
- ▶ Attractors
- ▶ Chaos
- ▶ Strange Nonchaotic Attractor (SNA)
- ▶ SNA: Characterisation and Routes
- ▶ Conclusions

Attractors

- ▶ A set of numerical values toward which a system tends to evolve, for a wide variety of starting conditions of the system.
- ▶ If the evolving variable is two- or three-dimensional, the attractor of the dynamic process can be represented geometrically in two or three dimensions
- ▶ An attractor can be a point, a finite set of points, a curve, a manifold, or even a complicated set with a fractal structure known as a strange attractor.
- ▶ Describing the attractors of chaotic dynamical systems has been one of the achievements of chaos theory.

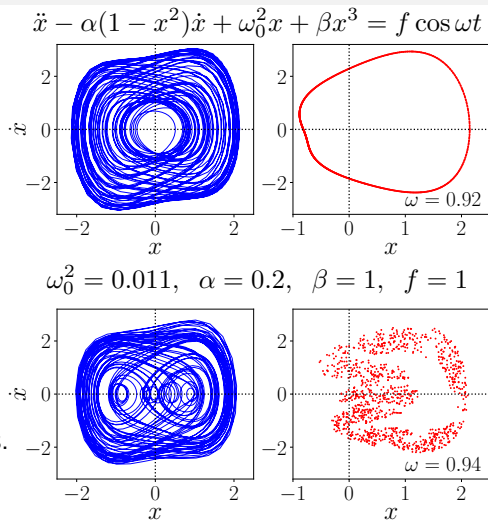
Types of attractors

- ▶ Point attractor (eg. equilibrium point)
 - ▶ An attractive fixed point is easy to find because many inputs lead to it.
 - ▶ This means that if you meant to start with a particular initial value, but messed up and started with a slightly-incorrect value, you will still probably end up with the same output.
 - ▶ This captures a stability that attractive fixed points.
- ▶ Periodic (eg. limit cycle)
 - ▶ A phenomenon, process or motion, which repeats itself after equal intervals of time is called periodic.



Types of attractors

- ▶ Quasiperiodic (eg. torus)
 - property of a system that displays irregular periodicity.
 - a pattern of recurrence with a component of unpredictability that does not lend itself to precise measurement.
- ▶ Strange (Chaos) – bounded aperiodic long-termed behavior in a deterministic nonlinear system with sensitive dependence on initial conditions.



Chaos

When the present determines the future, but the approximate present does not approximately determine the future

— Edward Lorenz

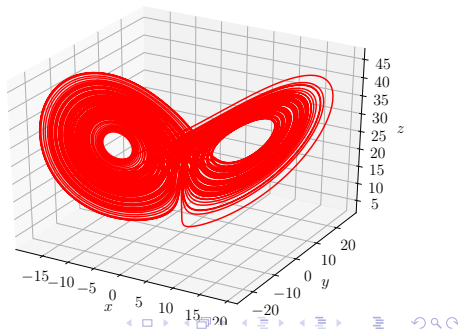
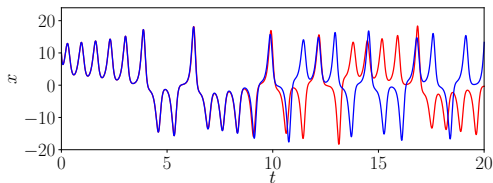
- ▶ Chaos is bounded aperiodic long-term behavior of a deterministic system that exhibits sensitive dependence on initial conditions.
- ▶ One of the first models which was shown to exhibit chaotic behaviour in numerical simulation was the fluid convection model introduced in 1963 by Lorenz.

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(\rho - z) - y$$

$$\dot{z} = xz - bz$$

$$\sigma = 10, \rho = 14, b = 8/3$$



Routes to Chaos

► Period doubling route

- A period T orbit (limit cycle) loses stability to an attracting period- $2T$ orbit as the parameter passes some critical value. At some later parameter value, the period- $2T$ orbit loses stability to a period- $4T$ orbit, etc.
- The parameter values at which these period doublings occur form an increasing sequence converging to some finite value μ_∞ , at which the original fixed point is replaced by an aperiodic attractor (which may be chaotic for $\mu > \mu_c$).
- The parameter values at which the bifurcations occur form an increasing sequence $\{\mu_k, k = 1, 2, \dots, \infty\}$ that is bounded above by a number with the property that

$$\lim_{k \rightarrow \infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} = \delta \quad (\text{Feigenbaum constant})$$

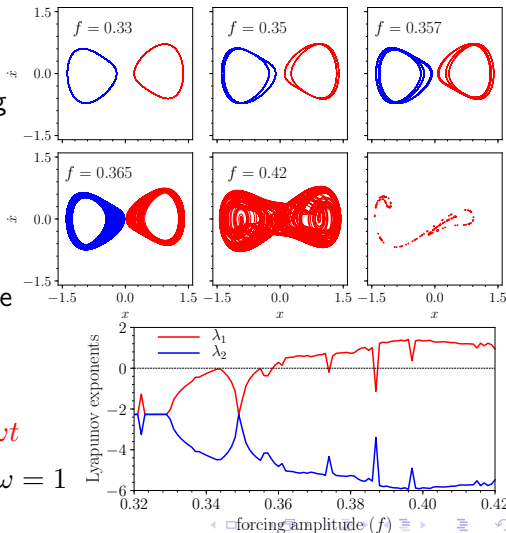
- Logistic map: $x_{n+1} = \mu x_n(1 - x_n)$, $x \in (0, 1)$ and $\mu \in (0, 4)$
 $\mu_\infty = 3.5699$ and $\delta \approx 4.6692$

Period doubling phenomenon: Duffing oscillator

- ▶ Introduced by the Dutch physicist Duffing in 1981 to describe the hardening spring effect.
- ▶ Describes the motion of a damped oscillator with a more complicated potential.
- ▶ Eg. a spring pendulum whose spring's stiffness does not exactly obey Hooke's law.

$$\ddot{x} + \alpha \dot{x} + \omega_0^2 x + \beta x^3 = f \sin \omega t$$

$$\omega_0^2 = -1, \quad \alpha = 0.2, \quad \beta = 1, \quad \omega = 1$$



Intermittency Transition

- ▶ An attracting periodic orbit for $\mu < \mu_c$ (or $\mu > \mu_c$) can disappear and be replaced by a chaotic attractor for $\mu > \mu_c$ (or $\mu < \mu_c$).
- ▶ For μ slightly larger than μ_c , initial conditions near the periodic orbit remain there for a long time.
- ▶ This regular behaviour is interrupted by a “burst” in which the trajectory moves away from a neighbourhood of the periodic orbit and exhibits possibly irregular behaviour.
- ▶ The dynamical behaviour for $\mu > \mu_c$ is characterized by an infinite sequence of intervals of nearly periodic (laminar) motion followed by bursts.
- ▶ The length of the laminar regions scales as $(\mu - \mu_c)^{-\frac{1}{2}}$ for μ near μ_c



Y. Pomeau and P. Manneville, Commun. Math. Phys. **74**, 189 (1980)

Intermittency Transition

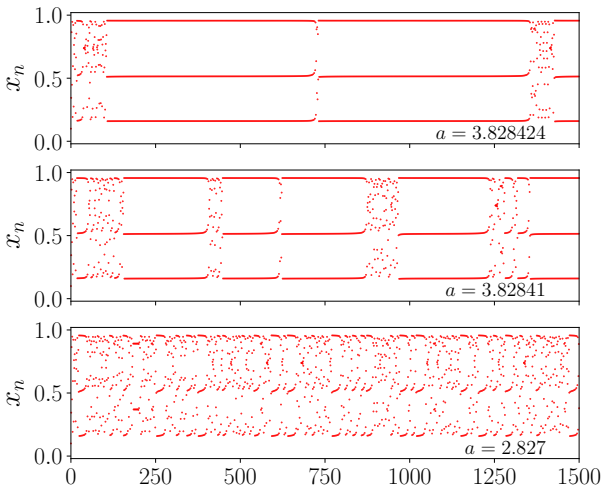
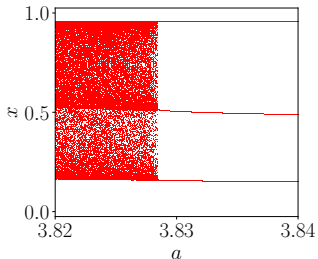
- ▶ When a motion alternates randomly between long regular or laminar phases and relatively short irregular bursts, it is said that the motion is intermittent or that there is an intermittency.
- ▶ In the intermittent route to chaos, initially the time series consists of regular laminar motion interrupted by irregular bursts. The laminar motions between two successive bursts have different durations which are randomly distributed over the time intervals.
- ▶ As a control parameter is increased, the length of the laminar region decreases and the bursts become very frequent so that at a critical value of the parameter the laminar phases disappear altogether and the motion becomes fully chaotic.
- ▶ Three bifurcations, namely saddle-node, Hopf and inverse-period doubling, and the corresponding intermittency transitions are called type-I, type-II and type-III, respectively.

Intermittency Transition

- ▶ Logistic map

$$x_{n+1} = ax_n(1 - x_n)$$

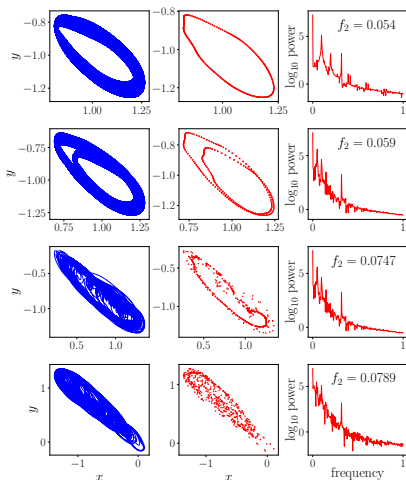
- ▶ Bifurcation diagram



Quasiperiodic Route to Chaos

- ▶ Quasiperiodicity is the property of a system that displays irregular periodicity.
- ▶ Quasiperiodic behavior is a pattern of recurrence with a component of unpredictability that does not lend itself to precise measurement.
- ▶ The system is initially in a stationary state and becomes unstable and undergoes Hopf bifurcation after a change of the control parameter.
- ▶ At the Hopf bifurcation, a limit cycle is generated around the equilibrium point the stability of which depends on the control parameter.
- ▶ If the two frequencies of this oscillation, ω_1 and ω_2 , are not commensurate (that is, ω_1/ω_2 is not a rational number), then the observed motion is not periodic but quasiperiodic.

Quasiperiodic Route to Chaos ▶ Simple nonlinear circuit model



A. Venkatesan et al, Phys. Lett. A **259**, 246 (1999)

$$\dot{x} = y - g(x)$$

$$\dot{y} = -\beta [x + (1 + \nu)y] + f_1 \sin \theta + f_2 \sin \phi$$

$$\dot{\theta} = \omega_1$$

$$\dot{\phi} = \omega_2$$

$$g(x) = bx + 0.5(a - b)(|x + 1| - |x - 1|)$$

$$a = -1.02, b = -0.55$$

▶ The parameters are

▶ $\beta = 1$

▶ $\nu = 0.015$

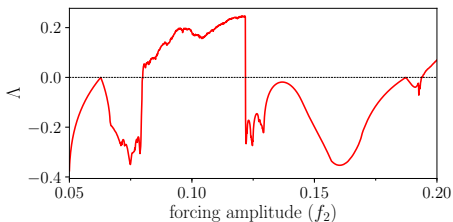
▶ $\omega_1 = 2$

▶ $\omega_2 = \frac{1}{2}(\sqrt{5} - 1)$

▶ $f_1 = 0.08$

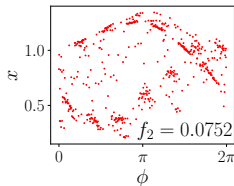
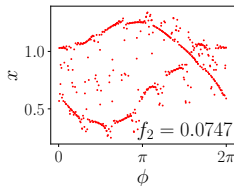
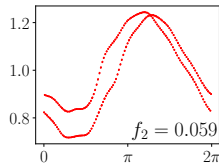
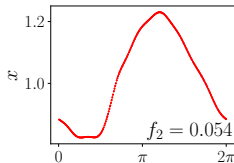
▶ $f_2 \in (0.05, 0.20)$

Quasiperiodic Route to Chaos



Maximal Lyapunov exponent Λ
versus f_2 for $f_1 = 0.08$.

Projections of attractors (Poincaré section) in the (x, ϕ) plane with ϕ modulo 2π for $f_1 = 0.08$ and several values of f_2 .



Characterization of Chaos

- ▶ Lyapunov exponents
 - ▶ Can be used to characterize the sensitive dependence of initial conditions.
 - ▶ Describes the rate of divergence (or convergence) of nearby trajectories onto the attractor in different directions (in the phase space).
 - ▶ It is sufficient to consider the largest (nonzero) Lyapunov exponent, λ_m .
 - ▶ $\lambda_m < 0$: equilibrium points and periodic (including quasi-periodic) orbits.
 - ▶ $\lambda_m > 0$: chaos
- ▶ Power spectrum
 - ▶ A simple way to check for periodicity and chaos.
 - ▶ Fourier transform of the time series data.
 - ▶ Broadband spectrum with spikes indicating predominant frequencies.
- ▶ Correlation function and dimension

Strange nonchaotic attractor

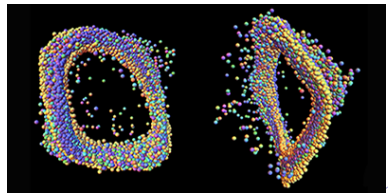
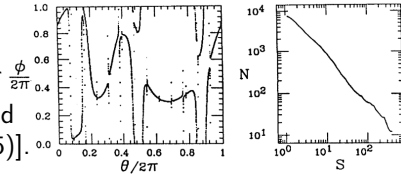
- ▶ SNAs which commonly occur in quasiperiodically forced systems were first described by Grabogi, Ott and Yorke in 1984.
- ▶ Strange nonchaotic attractors are not chaotic but aperiodic and display strange geometric properties. They are generic in the quasiperiodically forced systems and are typically found in the neighbourhood of related periodic or quasiperiodic or strange chaotic attractors in parameter space.
- ▶ In a sense they represent dynamics which is intermediate between quasiperiodic and chaotic: there is no sensitive dependence on initial conditions, similar to motion on regular (periodic or quasiperiodic) attractors, but the motion is aperiodic similar to the dynamics of chaotic attractors.

Strange nonchaotic attractor

- ▶ SNAs are not at all rare. They are typical attractors in quasiperiodically driven nonlinear system.
- ▶ They have been found in many theoretical models such as Pendulum, Duffing oscillator. Logistic Map, Henon Map, Circular Map etc.
- ▶ Number of experiments have verified the existence of SNAs
 - ▶ Magneto elastic ribbon exponent [Ditto et al. PRL 65, 533 (1990)]
 - ▶ Model of SQUID system Zhou et al. [PRA 45, 5394 (1992)]
 - ▶ Electronic circuits - Ueda's circuit, Chua's circuit, MLC circuit.
- ▶ Experiments of glow discharge in neon gas plasma in which SNA are observed [Ding et al., PRE 55, 3769 (1997)]
- ▶ Model study of a neuronal membrane system as well as EEG data in which SNAs are found [Mandell and Selz, J. Stat. Phys. 70, 355 (1993)].

Strange nonchaotic attractor

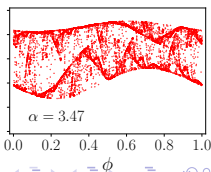
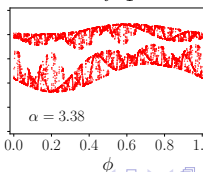
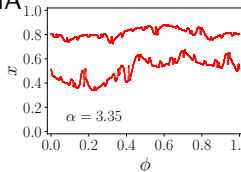
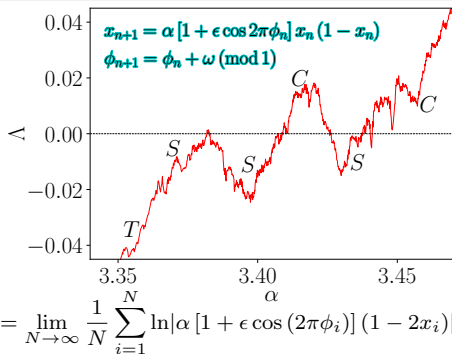
- ▶ An important area where the study of SNAs finds conceptual application is the case of quantum mechanical systems with quasiperiodic potentials.
- ▶ There is an unexpected link between wave-function localization phenomena and the related strange nonchaotic dynamics of an auxiliary variable, which was first pointed out by Bondeson et al [PRL 55, 2103 (1985)].
- ▶ For Kepler's "golden" stars, Ditto et al. [PRL 114, 054101 (2015)] have shown the evidence of the first observation of strange nonchaotic dynamics in nature outside the laboratory.
- ▶ This discovery could aid the classification and detailed modelling of variable stars.



Characterization of SNA

► Lyapunov Exponent (LE)

- The Lyapunov exponents describe essentially the rate of divergence or convergence of nearby trajectories onto the attractor in different directions in the phase space.
- At least one of the Lyapunov exponents is always positive for chaos and all are negative for Torus and SNA

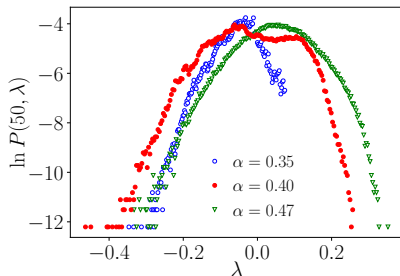


Finite time Lyapunov exponent (FTLE)

- ▶ The probability distribution of FTLE can be taken as another statistical measure for characterizing the dynamics.

$$x_{n+1} = \alpha [1 + \epsilon \cos(2\pi\phi_n)] x_n (1 - x_n)$$
$$\phi_{n+1} = \phi_n + \omega \pmod{1}$$

- ▶ The distribution takes on negative values for quasiperiodic behaviour
- ▶ It distribution takes on positive values for chaos.
- ▶ In the case of SNA, it takes on both positive and negative values with tail extending predominantly into the negative region.



$$\lambda_N = \frac{1}{N} \sum_{i=1}^N \ln |\alpha [1 + \epsilon \cos(2\pi\phi_i)] (1 - 2x_i)|$$

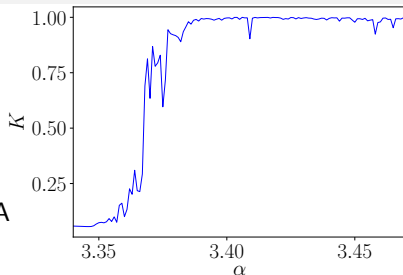
$$\epsilon = 0.05 \quad \omega = \frac{1}{2} (\sqrt{5} - 1)$$

$P(N, d\lambda)$ = Probability that λ_N lies between λ and $\lambda + d\lambda$

0-1 Test

- ▶ 0-1 test clearly helps to distinguish tori, SNA and chaos, from the time series alone.

- ▶ Gopal et al., Chaos 23, 023123 (2013)
 - ▶ value '0' for torus,
 - ▶ value in between '0' and '1' for SNA
 - ▶ value tends to '1' for Chaos



- ▶ Given an observation $x(j)$ for $j = 1, \dots, N$. Define a set of translational variables as

$$p(n) = \sum_{j=1}^n x(j) \cos cj, \quad q(n) = \sum_{j=1}^n x(j) \sin cj, \quad c \in (0, \pi)$$

- ▶ The diffusive (or non-diffusive) behaviour of p and q can be investigated by analyzing the mean square displacement M .

0-1 Test

- ▶ If the dynamics is regular then the mean square displacement is a bounded function in time, whereas the mean square displacement scales linearly with time for chaos.
- ▶ Compute the mean squared displacement as

$$M(k) = \lim_{N \rightarrow \infty} \frac{1}{N-k} \sum_{j=1}^{N-k} \left([p(j+k) - p(j)]^2 + [q(j+k) - q(j)]^2 \right)$$

$k = 1, 2, \dots, k_m$, k_m is usually taken as $N/10$.

- ▶ K is computed by a linear regression for the log-log plot of the mean square displacement as

$$K = \lim_{k \rightarrow \infty} \frac{\log M(k)}{\log k}$$

0-1 Test

- ▶ Alternatively, K can be computed by the correlation method as

$$K = \text{corr}(\xi, M) = \frac{\text{cov}(\xi, M)}{\sqrt{\text{var}(\xi)\text{var}(M)}},$$

where $\xi = \{1, 2, \dots, k_m\}$ and $M = \{M(1), M(2), \dots, M(k_m)\}$.



G. A. Gottwald, and I. Melbourne, The 0-1 Test for Chaos: A Review. In: C. Skokos, G. Gottwald and J. Laskar (eds) *Chaos Detection and Predictability. Lecture Notes in Physics*, vol 915 (Springer, Berlin, Heidelberg, 2016)



R. Gopal, A. Venkatesan, and M. Lakshmanan, *Applicability of 0-1 test for strange nonchaotic attractors*, Chaos 23, 023123 (2013);

Phase sensitivity

- ▶ Torus and SNA both show no exponential divergence to initial separation of near by orbits, yet they can be distinguished with respect to their sensitivities of the phases of the external force.
- ▶ Phase Sensitivity Measure which gives an idea of how the phase of the external force influence the state variables of the system.
 - ▶ If the attractor $x(\theta)$ is viewed as a fractal curve, then its nondifferentiability can be detected by examining the separation of points that are initially close in θ .
 - ▶ calculating the derivative $dx/d\theta$ along an orbit, and finding its maximal value. This yields the phase sensitivity function,

$$\Gamma_N = \min_{x,\theta} \left[\max_{1 \leq N} \left| \frac{dx_N}{d\theta} \right| \right],$$

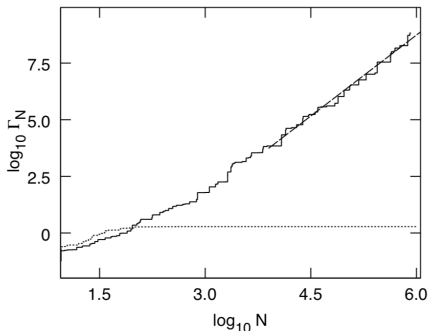
as the smallest such realization for arbitrary (x, θ) so that a bound can be set on the rate of growth of Γ over the entire attractor.

Phase sensitivity

- For a chaotic attractor, the sensitivity grows exponentially while for a SNA, Γ_N grows as power,

$$\Gamma_N \propto N^\mu$$

with $\mu > 1$



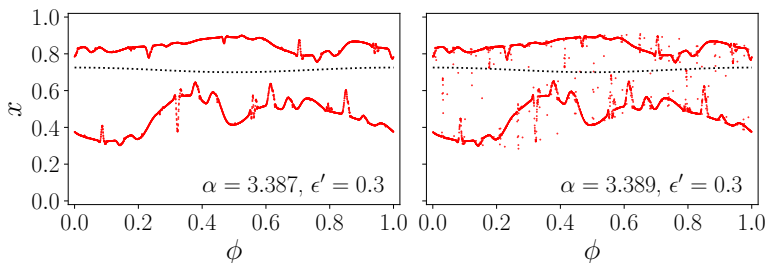
Singular Continuous Spectra

- ▶ Quantitatively confirm the strange nonchaotic nature of the dynamics, was first proposed in the investigations of a model of quasiperiodic lattices and quasiperiodically forced quantum systems.
- ▶ In general, power spectra of dissipative dynamical system can be either discrete, or continuous, or a combination of both.
- ▶ While discrete spectra are generated by regular motions such as periodic or quasiperiodic, continuous spectra are usually generated by irregular motions, either chaotic or random.
- ▶ A singular continuous spectrum is an intermediate stage between discrete and continuous spectra.

Routes to SNA

► Torus collisions

- Heagy and Hammel [Physica D 70, 140 (1994)] identified the birth of a SNA with the collision between the doubled quasiperiodic torus and its unstable parent.
- A torus collision is thus a general feature of forced systems and is a common mechanism for SNA creation.



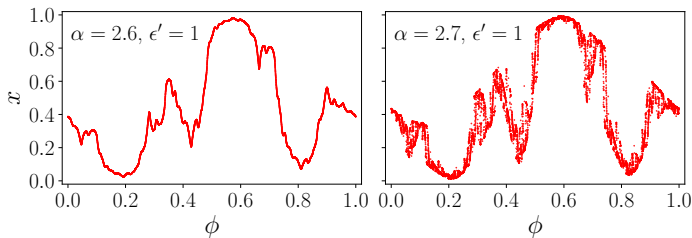
Routes to SNA

$$x_{n+1} = \alpha [1 + \epsilon \cos(2\pi\phi_n)] x_n (1 - x_n)$$

► Fractalization

$$\phi_{n+1} = \phi_n + \omega \pmod{1}$$

- “fractalization” route for the creation of SNAs [Kaneko, Prog. Theor. Phys. 71, 1112 (1984); Nishikawa & Kaneko, PRE 54, 6114 (1996)]
- A quasiperiodic torus gets increasingly wrinkled and transforms into a SNA without the apparent mediation of any nearby unstable periodic orbit.
- This route to SNA and eventually to chaos has also been observed in higher-dimensional systems [Sosnovtseva et al., PLA 218, 255 (1996)].



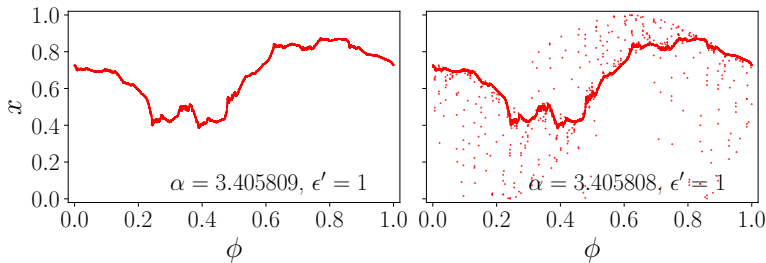
Routes to SNA

$$x_{n+1} = \alpha [1 + \epsilon \cos(2\pi\phi_n)] x_n (1 - x_n)$$

► Intermittency

$$\phi_{n+1} = \phi_n + \omega \pmod{1}$$

- Crisis-induced intermittency Collision of the wrinkled torus with the boundary resulting in a sudden widening of the attractor and birth of SNA.
- Type-III intermittency Taming of torus doublings due to subharmonic bifurcations.



Routes to SNA

- ▶ The blowout bifurcation
 - ▶ A symmetric low-dimensional invariant subspace containing a quasiperiodic torus, a blowout bifurcation [Ott & Sommerer, PLA 188, 39 (1994)] leads to the formation of a SNA [Yalçinkaya & Lai, PRL 77, 5039 (1996)].
 - ▶ Trajectories starting in the invariant subspace, S , remain in S .
 - ▶ The Lyapunov exponent Λ has two components, one of which, Λ_T , is defined for trajectories in S with respect to perturbations in a transverse subspace T .
 - ▶ A positive Λ_T indicates that trajectories in the vicinity of S are repelled away from it, and this gives rise to strangeness.
 - ▶ At the blowout bifurcation, Λ_T changes its sign, becoming positive as a system parameter varies. If, concurrently, $\Lambda < 0$, the attractor is a SNA.

Conclusion

- ▶ Strange nonchaotic attractors are an important class of dynamical attractors that are generic in quasiperiodically driven nonlinear dynamical systems, both mappings as well as flows.
- ▶ Systems where SNAs arise naturally span a wide range since the possibility of such dynamics devolves on a combination of dissipation, nonlinearity and quasiperiodic modulation.

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