

Nonlinear Oscillators

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Why numerical solutions?

- In most of the occasions nonlinear differential equations (describing dynamical systems) – may not possess closed solutions (or) not exactly solvable.
- Local stability analysis can be done to understand the local dynamics to some extent.
- Numerical solutions becomes essential in order to study the complete dynamics (For eg. chaos and other related phenomena).

ODEs

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3, \dots, x_n, t), \quad i = 1, 2, \dots, n \quad (1)$$

Any n -th order ODE of the form

$$\frac{d^n x}{dt^n} + a_1(t) \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_{n-1}(t) \frac{dx}{dt} + a_n(t)x = 0, \quad (2)$$

can always be written in the form (1) as $x_1 = x$,

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3,$$

$$\vdots$$

$$\dot{x}_n = -a_1(t)x_{n-1} - \dots - a_{n-1}(t)x_2 - a_n(t)x_1 \quad (3)$$

Euler method

- Simple technique for handling first order initial value problems.
- Basic explicit method for solving ordinary differential equations.
- For simplicity, consider first order ODE $\dot{x} = f(x, t)$ with initial condition: $x(t_0) = x_0$

$$x_{n+1} = x_n + hf(x_n, t_n), \quad (4)$$

where $t_n = t_0 + nh$, h time step, and $x_n = x(t_n)$.

Fourth order Runge-Kutta method

- Most commonly used method for solving ordinary differential equations – often referred to as “RK4”.
- Consider the first order ODE $\dot{x} = f(x, t)$ with initial condition: $x(t_0) = x_0$

$$x_{n+1} = x_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4), \quad (5)$$
$$t_{n+1} = t_n + h,$$

where

$$k_1 = h f(x_n, t_n), \quad k_2 = h f\left(x_n + \frac{k_1}{2}, t_n + \frac{h}{2}\right),$$
$$k_3 = h f\left(x_n + \frac{k_2}{2}, t_n + \frac{h}{2}\right), \quad k_4 = h f(x_n + k_3, t_n + h). \quad (6)$$

- Round-off error
- Local truncation error
- Global error
- Euler method

$$E(x(b), h) = O(h), \quad t \in (a, b)$$

- Fourth order Runge-Kutta method

$$E(x(b), h) = O(h^4), \quad t \in (a, b)$$

Linear Oscillators

$$\frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} + \omega_0^2 x = f \sin \omega t \quad (7)$$

Free Oscillations

Free linear harmonic oscillator ($\alpha = 0$, $f = 0$)

$$x(t) = A \cos \omega_0 t. \quad (8)$$

Initial conditions: $x(0) = A$, $\dot{x}(0) = 0$

Free Oscillations

Free linear harmonic oscillator ($\alpha = 0$, $f = 0$)

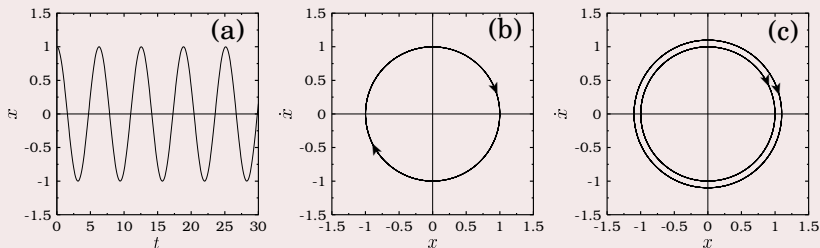


Figure: Solution curve and phase portrait of the free linear harmonic oscillator

Damped Oscillations

$\alpha \neq 0$ and $f = 0$

The explicit solution

$$x(t) = A_1 \exp(m_1 t) + A_2 \exp(m_2 t), \quad (9)$$

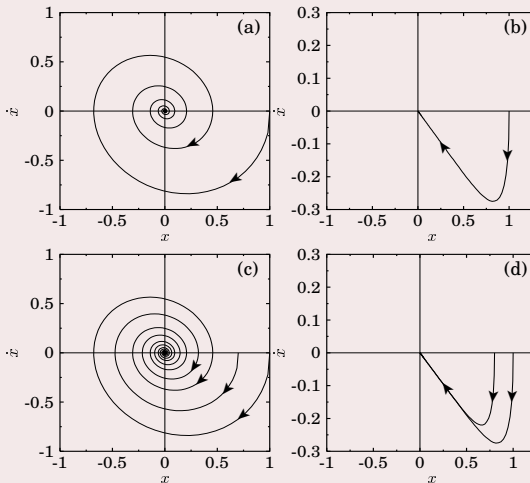
where

$$m_{1,2} = \frac{1}{2} \left[-\alpha \pm \sqrt{\alpha^2 - 4\omega_0^2} \right],$$

A_1 and A_2 are constants

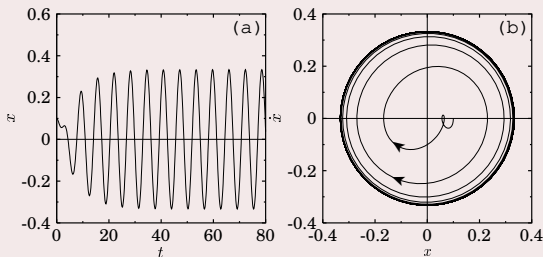
- Under damping: $0 < \alpha < 2\omega_0$
- Critical damping: $\alpha = 2\omega_0$
- Over damping : $\alpha > 2\omega_0$

Damped linear harmonic oscillator ($\alpha \neq 0, f = 0$)



Damped and Forced Oscillations

$$\ddot{x} + \alpha \dot{x} + \omega_0^2 x = f \sin \omega t. \quad (10)$$



Resonance

$$\ddot{x} + \alpha \dot{x} + \omega_0^2 x + \beta x^3 = f \sin \omega t. \quad (11)$$

Parameters : $\alpha = 0.5$, $\omega_0^2 = -1$, $\beta = 1$, and $\omega = 1$

Limit cycle

$$\ddot{x} - b(1 - x^2)\dot{x} + x = 0. \quad (12)$$

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= b(1 - x^2)y - x \end{aligned} \quad (13)$$

$$\lambda_{\pm} = \frac{1}{2} \left(b \pm \sqrt{b^2 - 4} \right) \quad (14)$$

Limit cycle

Sl. No.	Range of b	Nature of eigenvalues	Type
1.	$-\infty < b < -2$	$\lambda_{\pm} < 0, \lambda_+ \neq \lambda_-$	stable node
2.	$b = -2$	$\lambda_+ = \lambda_- < 0$	stable star
3.	$-2 < b < 0$	$\lambda_{\pm} = \alpha \pm i\beta, \alpha < 0$	stable focus
4.	$b = 0$	$\lambda_{\pm} = \pm i\beta$	center
5.	$0 < b < 2$	$\lambda_{\pm} = \alpha \pm i\beta, \alpha > 0$	unstable focus
6.	$b = 2$	$\lambda_+ = \lambda_- > 0$	unstable star
7.	$2 < b < \infty$	$\lambda_{\pm} > 0, \lambda_+ \neq \lambda_-$	unstable node