Introduction to Nonlinear Dynamics

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Why numerical solutions?

- In most of the occasions nonlinear differential equations (describing dynamical systems) – may not possess closed solutions (or) not exactly solvable.
- Local stability analysis can be done to understand the local dynamics to some extent.
- Numerical solutions becomes essential in order to study the complete dynamics (For eg. chaos and other related phenomena).

ODEs

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3, \cdots, x_n, t), \quad i = 1, 2, \ldots, n \tag{1}$$

Any n-th order ODE of the form

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = 0, \tag{2}$$

can always be written in the form (1) as $x_1 = x$,

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
&\vdots \\
\dot{x}_n &= -a_1(t)x_{n-1} - \cdots - a_{n-1}(t)x_2 - a_n(t)x_1
\end{aligned} \tag{3}$$

Euler method

- Simple technique for handling first order initial value problems.
- Basic explicit method for solving ordinary differential equations.
- For simplicity, consider first order ODE $\dot{x} = f(x, t)$ with initial condition: $x(t_0) = x_0$

$$x_{n+1} = x_n + hf(x_n, t_n), \tag{4}$$

where $t_n = t_0 + nh$, h time step, and $x_n = x(t_n)$.

Forth order Runge-Kutta method

- Most commonly used method for solving ordinary differential equations – often referred to as "RK4".
- Consider the first order ODE $\dot{x} = f(x, t)$ with initial condition: $x(t_0) = x_0$

$$x_{n+1} = x_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \tag{5}$$
$$t_{n+1} = t_n + h,$$

where

$$k_1 = h\,f(x_n, t_n), \qquad k_2 = h\,f\left(x_n + \frac{k_1}{2}, t_n + \frac{h}{2}\right),$$
$$k_3 = h\,f\left(x_n + \frac{k_2}{2}, t_n + \frac{h}{2}\right), \qquad k_3 = h\,f\left(x_n + k_3, t_n + h\right). \tag{6}$$

- Euler method

$$E(x(b), h) = O(h), \quad t \in (a, b)$$

- Forth order Runge-Kutta method

$$E(x(b), h) = O(h^4), \quad t \in (a, b)$$

Linear Oscillators

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \omega_0^2 x = f \sin \omega t \tag{7}$$

Free Oscillations

Free linear harmonic oscillator ($\alpha = 0$, $f = 0$)

$$x(t) = A\cos\omega_0 t. \tag{8}$$

Initial conditions: $x(0) = A$, $\dot{x}(0) = 0$

Free Oscillations

Free linear harmonic oscillator ($\alpha = 0,\ f = 0$)

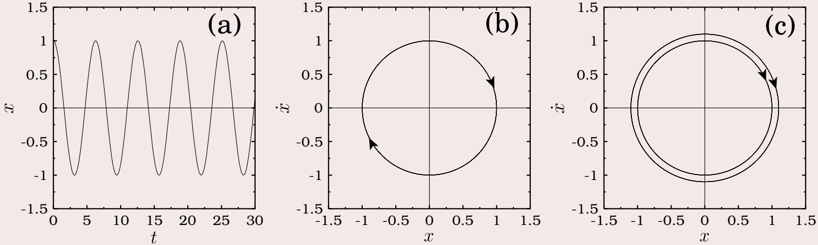


Figure: Solution curve and phase portrait of the free linear harmonic oscillator

Damped Oscillations

$\alpha \neq 0$ and $f = 0$

The explicit solution

$$x(t) = A_1 \exp\left(m_1 t\right) + A_2 \exp\left(m_2 t\right), \tag{9}$$

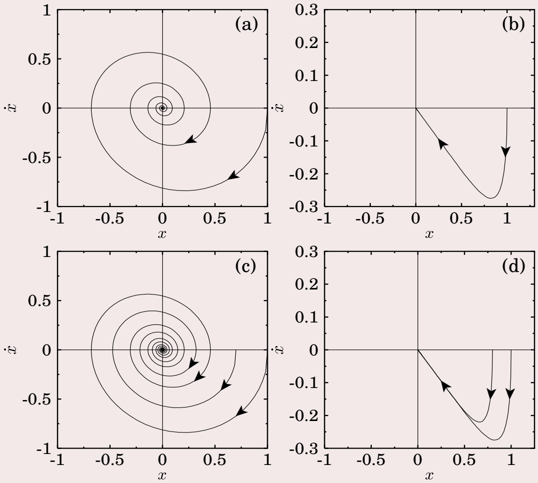
where

$$m_{1,2} = \frac{1}{2}\left[-\alpha \pm \sqrt{\alpha^2 - 4\omega_0^2}\right],$$

A_1 and A_2 are constants

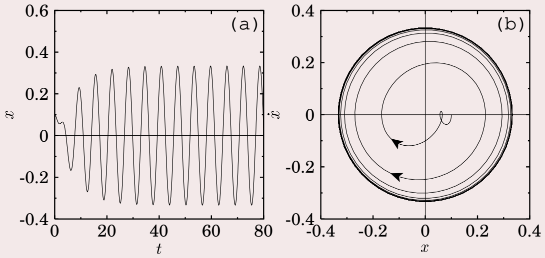
- Under damping: $0 < \alpha < 2\omega_0$
- Critical damping: $\alpha = 2\omega_0$
- Over damping : $\alpha > 2\omega_0$

Damped linear harmonic oscillator ($\alpha \neq 0$, $f = 0$)

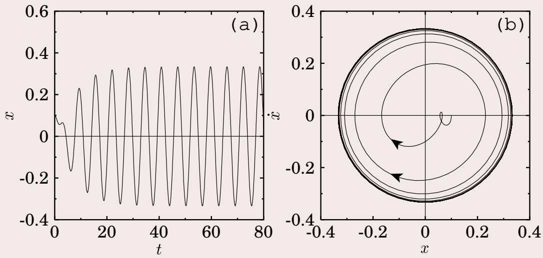


Damped and Forced Oscillations

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x = f \sin\omega t. \quad (10)$$



Resonance



Resonance