Poynting's Theorem in Electrodynamics

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Poynting's Theorem

The work necessary to assemble a static charge distribution

$$W_e = \frac{\varepsilon_0}{2}\int E^2 \, d\tau \quad ; \; E \text{ - Electric field}$$

The work required to get current going is

$$W_m = \frac{1}{2\mu_0}\int B^2 \, d\tau \quad ; \; B \text{ - Magnetic field}$$

The total energy stored in electromagnetic fields is

$$U_{em} = \frac{1}{2}\int \left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau$$

According to Lorentz force law, the work done on a charge q is

$$F \cdot dl = q(E + v \times B) \cdot v \, dt = qE \cdot v \, dt$$

Here $q = \rho \, d\tau$ and $\rho v = J$ and so the rate at which work is done on all charges in a volume is

$$\frac{dW}{dt} = \int_V (E \cdot J) \, d\tau$$

From the equation of continuity

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$$

Also it is known that

$$\nabla \cdot E = 4\pi \rho$$

$$\rho = \frac{1}{4\pi} (\nabla \cdot E)$$

$$\frac{\partial \rho}{\partial t} = \frac{1}{4\pi} \nabla \cdot \frac{\partial E}{\partial t}$$

$$\therefore \nabla \cdot J + \frac{1}{4\pi} \nabla \cdot \frac{\partial E}{\partial t} = 0$$

$$\Rightarrow \nabla \cdot \left[J + \frac{1}{4\pi} \frac{\partial E}{\partial t} \right] = 0$$

Hence

$$\nabla \times B = \frac{4\pi}{c} \left[J + \frac{1}{4\pi} \cdot \frac{\partial E}{\partial t} \right]$$

$$\therefore \nabla \times B = \frac{4\pi}{c} J + \frac{1}{c} \frac{\partial E}{\partial t}$$ (this is Corrected Ampere's law)

$$J = \frac{c}{4\pi}\left(\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t}\right)$$

$$J = \frac{1}{4\pi}\left(c\,\nabla \times B - \frac{\partial E}{\partial t}\right)$$

The total rate of doing work by the field is

$$W = \int J \cdot E \, d^3x$$

$$= \int \frac{1}{4\pi}\left(c\,\nabla \times B - \frac{\partial E}{\partial t}\right) \cdot E \, d^3x$$

$$= \frac{1}{4\pi}\int\left[c\; E \cdot (\nabla \times B) - E \cdot \frac{\partial E}{\partial t}\right] d^3x$$

Since $E \cdot (\nabla \times B) = B \cdot (\nabla \times E) - \nabla \cdot (E \times B)$

$$W = \frac{1}{4\pi}\int\left[c\left(B \cdot (\nabla \times E)\right) - c\;\nabla \cdot (E \times B) - E \cdot \frac{\partial E}{\partial t}\right] d^3x$$

$$\therefore W = \frac{1}{4\pi} \int \left(-B \cdot \frac{\partial B}{\partial t} - c\, \nabla \cdot (E \times B) - E \cdot \frac{\partial E}{\partial t} \right) d^3x$$

Meanwhile

$$B \cdot \frac{\partial B}{\partial t} = \frac{1}{2} \cdot \frac{\partial}{\partial t}\left(B^2\right) \quad \text{and} \quad E \cdot \frac{\partial E}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t}\left(E^2\right)$$

$$\therefore W = \frac{1}{4\pi} \int \left(-\frac{1}{2} \cdot \frac{\partial}{\partial t}(B^2) - \frac{1}{2} \frac{\partial}{\partial t}(E^2) \right) d^3x - \frac{1}{4\pi} \int c\, \nabla \cdot (E \times B)\, d^3x$$

Hence the rate at which work is done on charges is

$$\frac{dW}{dt} = -\frac{d}{dt} \int_V \frac{1}{2} \left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau - \frac{1}{\mu_0} \oint_S (E \times B) \cdot da$$

where S is the surface bounding V. This is Poynting's theorem or 'work-energy theorem of electrodynamics'.

The first integral is the total energy stored in the field, U_{em} and the second term is the rate at which energy is carried out of V, across its boundary surface, by the em fields.

The energy per unit time per unit area transported by the fields is called Poynting's vector

$$S = \frac{1}{\mu_0}(E \times B)$$

$\therefore$ Poynting's theorem can be compactly written as

$$\frac{dw}{dt} = -\frac{dU_{em}}{dt} - \oint_S S \cdot da$$

Hence Poynting's theorem states that the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface. Poynting's theorem is the conservation of energy in electromagnetics.

$$\therefore \quad \frac{\partial U}{\partial t} + \nabla \cdot S = -J \cdot E$$

Change in energy of electromagnetic fied	+	energy flowing off	=	Total work done

is the representation of conservation of energy.