Maxwell's Equation
Scalar and Vector Potentials

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Maxwell's Equation

Writing the fields (E and B) on the left and the sources (ρ and J) on right. (9)

(i) $\nabla . E = \frac{\rho}{\varepsilon_0}$; flux of E thro' Closed surface $\Big\} = \frac{\text{charge inside}}{\varepsilon_0}$

(ii) $\nabla \cdot B = 0$; flux of B thro' closed surface = 0

(iii) $\nabla \times E + \frac{\partial B}{\partial t} = 0$; line integral of E around loop $\Big\} +$ rate of change of flux of B thro' loop $\Big\} = 0$

(iv) $\nabla \times B - \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} = \mu_0 J$; integral of B $- \mu_0 \varepsilon_0$ (rate of change of flux of E thro' loop) = μ_0 (current thro' loop)

This form emphasizes that all electromagnetic fields are attributable to charges and currents.

"Charges produce field" and "field affects charges"

The simple approach to understand Maxwell's current role can be discussed as follows

$$\nabla \cdot J = \nabla \cdot (\nabla \times B) = 0$$ is valid for steady-current.

By equation of continuity

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = \nabla \cdot \left(J + \frac{\partial E}{\partial t} \right) = 0$$; applying coulomb's law.

Maxwell replaced J by $\left(J + \frac{\partial E}{\partial t} \right)$ - Maxwell's current for time dependent fields. Its presence means that a changing electric field causes a magnetic field. This addition is important for rapidly flucating fields and in particular with electromagnetic radiation like light.

Vector and Scalar potentials.

The Maxwell equations consist of a set of coupled first-order partial differential equations relating the various components of electric and magnetic fields. Although it can be solved as such, it is convinent to introduce potentials, obtaining a smaller number of second-order equations.

Since $\nabla \cdot B = 0$, one can define B in terms of a vector potential

$$B = \nabla \times A$$

Then the other homogenous equation of Faraday's law can be written as

$$\nabla \times \left(E + \frac{\partial A}{\partial t} \right) = 0$$

This means that the quantity with vanishing curl in above equation can be written as the gradient of some scalar potential, ϕ

$$E + \frac{\partial A}{\partial t} = -\nabla\phi$$

$$E = -\nabla\phi - \frac{\partial A}{\partial t}$$

The definition of B and E in terms of potentials A and ϕ Satisfies identically the two homogeneous Maxwell equations. The dynamic behavior of A and ϕ will be determined by the two inhomogenous Maxwell equations.

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla . A) = -\rho/\varepsilon_0$$

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \left(\nabla . A + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\mu_0 J$$

The set of four Maxwell equations were now reduced into two equations. But they are still coupled equations. To uncouple the equations, the gradient of some scalar function Λ can be added to the vector potential, A

$$A \rightarrow A' = A + \nabla\Lambda$$

For electric field to be unchanged, scalar potential must be

Simultaneously transformed

$$\phi \rightarrow \phi' = \phi - \frac{\partial \Lambda}{\partial t}$$ (The above two are known as Gauge transformations.)

Thus one can choose a set of potentials (A, ϕ) such that

$$\nabla \cdot A + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

Using this the pair of equations can be uncoupled as

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\rho/\varepsilon_0$$

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 J$$

Thus two inhomogenous independent equations are formed.

The above three equations are equivalent in all respects to Maxwell equations.

Magnetic charges

There is a symmetry in Maxwell's equations, Particularly in free space

$$\nabla \cdot E = 0 \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

If one replace E by B and B by $-\mu_0\varepsilon_0 E$, the first pair of equations turns into second and vice versa. This symmetry between E and B is spoiled, by the charge term in Gauss's law and current term in Ampere's law. If

$$\nabla \cdot E = \frac{1}{\varepsilon_0} \rho_e \qquad \nabla \times E = -\mu_0 J_m - \frac{\partial B}{\partial t}$$

$$\nabla \cdot B = \mu_0 \rho_m \qquad \nabla \times B = \mu_0 J_e + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

Then ρ_m is density of magnetic charge, ρ_e is density of electric charge, J_m is current of magnetic charge, J_e is current of electric charge. Both charges would be conserved

$$\nabla \cdot J_m = -\frac{\partial \rho_m}{\partial t} \quad ; \quad \nabla \cdot J_e = -\frac{\partial \rho_e}{\partial t}$$

But in nature, magnetic charge does not exist.