ZITTERBEWEGUNG – JITTERY MOTION OF A FREE PARTICLE

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Equations of motion of $\vec{X}$ & $\vec{\alpha}$

The velocity of a free particle is

$$
\begin{aligned}
\frac{\mathrm{d}\vec{X}}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\left[\vec{X}, H\right] \quad &= \quad \frac{1}{\mathrm{i}\hbar}\left[\vec{X},\, c\vec{\alpha}\cdot\vec{p} + \beta m c^2\right] \\
&= \quad \frac{1}{\mathrm{i}\hbar}\left(\vec{X}c\vec{\alpha}\cdot\vec{p} - c\vec{\alpha}\cdot\vec{p}\vec{X} + \vec{X}\beta m c^2 - \beta m c^2\vec{X}\right) \\
&= \quad -\frac{c}{\mathrm{i}\hbar}\vec{\alpha}\cdot\vec{p}\vec{X} \\
&= \quad -\frac{c}{\mathrm{i}\hbar}\vec{\alpha}\cdot(-\mathrm{i}\hbar)\nabla\vec{X} \\
&= \quad c\,\vec{\alpha}\ . \qquad (1)
\end{aligned}
$$

Next, the time evolution of $\vec{\alpha}$ is found to be

$$
\begin{aligned}
\frac{\mathrm{d}\vec{\alpha}}{\mathrm{d}t} \quad &= \quad \frac{1}{\mathrm{i}\hbar}\left[\vec{\alpha}, H\right] \\
&= \quad \frac{1}{\mathrm{i}\hbar}\left(\alpha H - H\alpha\right) \\
&= \quad \frac{1}{\mathrm{i}\hbar}\left(\vec{\alpha}\left(c\,\vec{\alpha}\cdot\vec{p} + \beta m c^2\right) - H\vec{\alpha}\right) \\
&= \quad \frac{1}{\mathrm{i}\hbar}\left(c\,\vec{p} + \vec{\alpha}\beta m c^2 - H\vec{\alpha}\right)\ . \qquad (2)
\end{aligned}
$$

Determination of $\vec{\alpha}$

Adding and subtracting $H\vec{\alpha}$ to the above equation we get

$$\begin{aligned}
\frac{\mathrm{d}\vec{\alpha}}{\mathrm{d}t} &= \frac{1}{\mathrm{i}\hbar}\left(c\,\vec{p} + \vec{\alpha}\beta mc^2 - H\vec{\alpha} + H\vec{\alpha} - H\vec{\alpha}\right) \\
&= \frac{1}{\mathrm{i}\hbar}\left(c\,\vec{p} + \vec{\alpha}\beta mc^2 + c\,\vec{p} + \beta\vec{\alpha}mc^2 - 2H\vec{\alpha}\right) \\
&= \frac{2}{\mathrm{i}\hbar}\left(c\,\vec{p} - H\vec{\alpha}\right) \ . \qquad (3)
\end{aligned}$$

Since H and $\vec{p}$ are time-independent the Eq. (3) is written as

$$\frac{\mathrm{d}\vec{\alpha}}{\mathrm{d}t} = \frac{2\mathrm{i}H}{\hbar}\left(-\frac{c\,\vec{p}}{H} + \vec{\alpha}\right) \ . \qquad (4)$$

Defining $\vec{\alpha'} = \vec{\alpha} - c\,\vec{p}/H$ the above equation becomes

$$\mathrm{d}\vec{\alpha'}/\mathrm{d}t = (2\mathrm{i}H/\hbar)\vec{\alpha'} \qquad (5)$$

whose solution is

$$\vec{\alpha'}(t) = \vec{\alpha'}(0)\mathrm{e}^{2\mathrm{i}Ht/\hbar}. \qquad (6)$$

Determination of $\vec{X}$

Then $\vec{\alpha}(t)$ is obtained as

$$\vec{\alpha}(t) = \frac{c\,\vec{p}}{H} + \mathrm{e}^{2\mathrm{i}Ht/\hbar}\left(\vec{\alpha}(0) - \frac{c\,\vec{p}}{H}\right) \ . \tag{7}$$

Now, Eq. (1) gives

$$\frac{\mathrm{d}\vec{X}}{\mathrm{d}t} = \frac{c^2\vec{p}}{H} + c\,\mathrm{e}^{2\mathrm{i}Ht/\hbar}\left(\vec{\alpha}(0) - \frac{c\,\vec{p}}{H}\right) \ . \tag{8}$$

Although the momentum $\vec{p}$ is a constant of motion, that is independent of time, the velocity ($\vec{v} = \mathrm{d}\vec{X}/\mathrm{d}t$) is not. The solution of Eq. (8) is

$$\vec{X}(t) = \vec{X}(0) + \frac{c^2\,\vec{p}\,t}{H} + \frac{c\hbar}{2\mathrm{i}H}\left(\mathrm{e}^{2\mathrm{i}Ht/\hbar} - 1\right)\left(\vec{\alpha}(0) - \frac{c\,\vec{p}}{H}\right) \ . \tag{9}$$

In Eq. (9) the first two terms of the right-side describe the uniform motion and it corresponds to the classical dynamics of a free relativistic particle. The last term is a feature of relativistic quantum mechanics. This term oscillates extremely rapidly with high frequency, $\omega \sim mc^2/\hbar$, (or period $\sim \hbar/(mc^2)$) and with amplitude $\hbar/(mc)$ *(how?)*. This high frequency oscillatory or jitterly or trembling motion is called *zitterbewegung*. The frequency ω is so high that the deviation from the classical mechanics term $c^2\vec{p}t/H \approx \vec{v}t$ is undetectable.