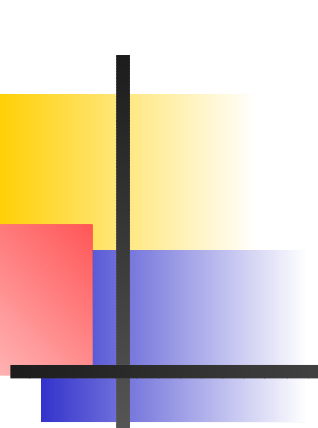




ZITTERBEWEGUNG – JITTERY MOTION OF A FREE PARTICLE

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Equations of motion of $\vec{X}$ & $\vec{\alpha}$

The velocity of a free particle is

$$\begin{aligned}
 \frac{d\vec{X}}{dt} &= \frac{1}{i\hbar} [\vec{X}, H] = \frac{1}{i\hbar} [\vec{X}, c\vec{\alpha} \cdot \vec{p} + \beta mc^2] \\
 &= \frac{1}{i\hbar} (\vec{X}c\vec{\alpha} \cdot \vec{p} - c\vec{\alpha} \cdot \vec{p}\vec{X} + \vec{X}\beta mc^2 - \beta mc^2 \vec{X}) \\
 &= -\frac{c}{i\hbar} \vec{\alpha} \cdot \vec{p} \vec{X} \\
 &= -\frac{c}{i\hbar} \vec{\alpha} \cdot (-i\hbar) \nabla \vec{X} \\
 &= c \vec{\alpha} .
 \end{aligned} \tag{1}$$

Next, the time evolution of $\vec{\alpha}$ is found to be

$$\begin{aligned}
 \frac{d\vec{\alpha}}{dt} &= \frac{1}{i\hbar} [\vec{\alpha}, H] \\
 &= \frac{1}{i\hbar} (\alpha H - H \alpha) \\
 &= \frac{1}{i\hbar} (\vec{\alpha} (c \vec{\alpha} \cdot \vec{p} + \beta mc^2) - H \vec{\alpha}) \\
 &= \frac{1}{i\hbar} (c \vec{p} + \vec{\alpha} \beta mc^2 - H \vec{\alpha}) .
 \end{aligned} \tag{2}$$

Determination of $\vec{\alpha}$

Adding and subtracting $H\vec{\alpha}$ to the above equation we get

$$\begin{aligned}\frac{d\vec{\alpha}}{dt} &= \frac{1}{i\hbar} (c\vec{p} + \vec{\alpha}\beta mc^2 - H\vec{\alpha} + H\vec{\alpha} - H\vec{\alpha}) \\ &= \frac{1}{i\hbar} (c\vec{p} + \vec{\alpha}\beta mc^2 + c\vec{p} + \beta\vec{\alpha}mc^2 - 2H\vec{\alpha}) \\ &= \frac{2}{i\hbar} (c\vec{p} - H\vec{\alpha}) .\end{aligned}\tag{3}$$

Since H and $\vec{p}$ are time-independent the Eq. (3) is written as

$$\frac{d\vec{\alpha}}{dt} = \frac{2iH}{\hbar} \left(-\frac{c\vec{p}}{H} + \vec{\alpha} \right) .\tag{4}$$

Defining $\vec{\alpha}' = \vec{\alpha} - c\vec{p}/H$ the above equation becomes

$$d\vec{\alpha}'/dt = (2iH/\hbar)\vec{\alpha}'\tag{5}$$

whose solution is

$$\vec{\alpha}'(t) = \vec{\alpha}'(0)e^{2iHt/\hbar} .\tag{6}$$

Determination of $\vec{X}$

Then $\vec{\alpha}(t)$ is obtained as

$$\vec{\alpha}(t) = \frac{c\vec{p}}{H} + e^{2iHt/\hbar} \left(\vec{\alpha}(0) - \frac{c\vec{p}}{H} \right) . \quad (7)$$

Now, Eq. (1) gives

$$\frac{d\vec{X}}{dt} = \frac{c^2\vec{p}}{H} + c e^{2iHt/\hbar} \left(\vec{\alpha}(0) - \frac{c\vec{p}}{H} \right) . \quad (8)$$

Although the momentum $\vec{p}$ is a constant of motion, that is independent of time, the velocity ($\vec{v} = d\vec{X}/dt$) is not. The solution of Eq. (8) is

$$\vec{X}(t) = \vec{X}(0) + \frac{c^2\vec{p}t}{H} + \frac{c\hbar}{2iH} \left(e^{2iHt/\hbar} - 1 \right) \left(\vec{\alpha}(0) - \frac{c\vec{p}}{H} \right) . \quad (9)$$

In Eq. (9) the first two terms of the right-side describe the uniform motion and it corresponds to the classical dynamics of a free relativistic particle. The last term is a feature of relativistic quantum mechanics. This term oscillates extremely rapidly with high frequency, $\omega \sim mc^2/\hbar$, (or period $\sim \hbar/(mc^2)$) and with amplitude $\hbar/(mc)$ (*how?*). This high frequency oscillatory or jittery or trembling motion is called *zitterbewegung*. The frequency ω is so high that the deviation from the classical mechanics term $c^2\vec{p}t/H \approx \vec{v}t$ is undetectable.