HEISENBERG UNCERTAINTY RELATION

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What is mean by uncertainty?

A striking deviations of quantum mechanics from classical mechanics is that any measurement of a microscopic system involves the interaction with the measuring instruments. Let a measurement of an observable A yields the value a and is the eigenvalue of the operator A belonging to the eigenfunction ϕ_a. The individual measurements will deviate from $\langle a \rangle$. Denote the deviation of a from $\langle a \rangle$ as $\tilde{a}$. *What would be the value of average deviation?* We obtain

$$\langle \tilde{a} \rangle = \frac{1}{N} \sum (a - \langle a \rangle) = 0 \; . \tag{1}$$

Why? $\langle \tilde{a} \rangle$ is not useful for describing how much an individual value of an observable deviates from the expected or mean value. The mean-square deviation or variance (σ^2), that is, the average of the squares of the deviations would be nonzero. The variance is

$$\sigma^2 = (\Delta a)^2 = \frac{1}{N} \sum (a - \langle a \rangle)^2 = \langle a^2 \rangle - \langle a \rangle^2 \; . \tag{2}$$

The square-root of mean-square deviation is called *standard deviation*. This quantity could be taken as a measure of the spread in the measured values. This spread, denoted as ΔA, is called the *uncertainty* in the measurement of A.

Heisenberg Uncertainty Relation

In a quantum state ψ, the uncertainty in an observable A is given by

$$(\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2 \; . \tag{3}$$

Then

$$\Delta A = \left[\langle \tilde{A}^2 \rangle \right]^{1/2} = \left[\int_{-\infty}^{\infty} \psi^* \tilde{A}^2 \psi \, \mathrm{d}\tau \right]^{1/2} = \left[\int_{-\infty}^{\infty} \psi^* \tilde{A} (\tilde{A} \psi) \, \mathrm{d}\tau \right]^{1/2} \; . \tag{4}$$

Since $\tilde{A}$ is Hermitian we have

$$\Delta A = \left[\int_{-\infty}^{\infty} (\tilde{A}\psi)^* (\tilde{A}\psi) \, \mathrm{d}\tau \right]^{1/2} = \left[\int_{-\infty}^{\infty} \psi_1^* \psi_1 \, \mathrm{d}\tau \right]^{1/2} = ||\psi_1|| \; , \tag{5}$$

where

$$\psi_1 = \tilde{A}\psi = (A - \langle A \rangle)\psi \; . \tag{6}$$

Similarly,

$$\psi_2 = \tilde{B}\psi = (B - \langle B \rangle)\psi \; . \tag{7}$$

Heisenberg Uncertainty Relation

If ψ_1 and ψ_2 are two states then from Eq. (**??**) we write

$$||\psi_1|| \cdot ||\psi_2|| \geq \left| \int_{-\infty}^{\infty} \psi_1^* \psi_2 \, \mathrm{d}\tau \right| . \tag{8}$$

This is known as *Schwarz's inequality*. We rewrite the Eq. (8) as

$$||\psi_1|| \cdot ||\psi_2|| \geq \left| \mathrm{Im} \int_{-\infty}^{\infty} \psi_1^* \psi_2 \, \mathrm{d}\tau \right| . \tag{9}$$

Suppose we write

$$\int_{-\infty}^{\infty} \psi_1^* \psi_2 \, \mathrm{d}\tau = R + \mathrm{i}I \ , \quad \int_{-\infty}^{\infty} \psi_2^* \psi_1 \, \mathrm{d}\tau = R - \mathrm{i}I \ , \tag{10}$$

where R and I are real numbers. Then I is given by

$$I = \mathrm{Im}(R + \mathrm{i}I) = \frac{1}{2\mathrm{i}} \left[(R + \mathrm{i}I) - (R + \mathrm{i}I)^* \right] \ . \tag{11}$$

In terms of the integrals in Eq. (10) the Eq. (11) is written as

$$\mathrm{Im} \int_{-\infty}^{\infty} \psi_1^* \psi_2 \, \mathrm{d}\tau = \frac{1}{2\mathrm{i}} \left[\int_{-\infty}^{\infty} \psi_1^* \psi_2 \, \mathrm{d}\tau - \int_{-\infty}^{\infty} \psi_2^* \psi_1 \, \mathrm{d}\tau \right] \ . \tag{12}$$

Heisenberg Uncertainty Relation

Then

$$||\psi_1|| \cdot ||\psi_2|| \geq \left| \frac{1}{2\mathrm{i}} \left[\int_{-\infty}^{\infty} \psi_1^* \psi_2 \, \mathrm{d}\tau - \int_{-\infty}^{\infty} \psi_2^* \psi_1 \, \mathrm{d}\tau \right] \right| . \tag{13}$$

That is,

$$\begin{aligned} \Delta A \Delta B &\geq \frac{1}{2} \left| \frac{1}{\mathrm{i}} \left[\int_{-\infty}^{\infty} (\tilde{A}\psi)^* (\tilde{B}\psi) \, \mathrm{d}\tau - \int_{-\infty}^{\infty} (\tilde{B}\psi)^* (\tilde{A}\psi) \, \mathrm{d}\tau \right] \right| \\ &\geq \frac{1}{2} \left| \frac{1}{\mathrm{i}} \left[\int_{-\infty}^{\infty} \psi^* \tilde{A}\tilde{B}\psi \, \mathrm{d}\tau - \int_{-\infty}^{\infty} \psi^* \tilde{B}\tilde{A}\psi \, \mathrm{d}\tau \right] \right| \\ &\geq \frac{1}{2} \left| \frac{1}{\mathrm{i}} \left\langle \left[\tilde{A}, \tilde{B} \right] \right\rangle \right| . \end{aligned} \tag{14}$$

The commutator $[\tilde{A}, \tilde{B}]$ is shown to be

$$\begin{aligned} \left[\tilde{A}, \tilde{B} \right] &= \tilde{A}\tilde{B} - \tilde{B}\tilde{A} \\ &= (A - \langle A \rangle)(B - \langle B \rangle) - (B - \langle B \rangle)(A - \langle A \rangle) \\ &= [A, B] . \end{aligned} \tag{15}$$

Heisenberg Uncertainty Relation

or

$$(\Delta A)^2(\Delta B)^2 \geq -\frac{1}{4}\langle[A, B]\rangle^2 \ . \tag{16}$$

Equation (16) is known as *Heisenberg uncertainty principle*. It is the uncertainty principle for any pair of observables A and B. This equation with equality sign is commonly referred as the *minimum uncertainty product*.

For a canonically conjugate pair of operators A and B we have $[A, B] = \mathrm{i}\hbar$. Hence,

$$(\Delta A)^2(\Delta B)^2 \geq \hbar^2/4 \ \text{ or } \ (\Delta A)(\Delta B) \geq \hbar/2 \ . \tag{17}$$

Thus, the statement of the uncertainty principle is the following:
The value of both the observables of a canonically conjugate pairs of a quantum mechanical system cannot be specified precisely and simultaneously.

The crucial word here is *simultaneously*. The uncertainty relation gives the lower bound on the product of the uncertainties of two pairs of variables. But it says nothing about the accuracy with which individual observables can be measured. An important point is that the uncertainty relation has nothing to do with the precision of the measuring instrument or technique.

Examples of Uncertainty Relation

Because $[x, p_x] = \mathrm{i}\hbar$, the position-momentum uncertainty principle is

$$(\Delta x)(\Delta p_x) \geq \hbar/2 \; , \tag{18}$$

where

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 \; , \quad (\Delta p_x)^2 = \langle p_x^2 \rangle - \langle p_x \rangle^2 \; . \tag{19}$$

In momentum space $p = \hbar k$ and hence the uncertainty relation is

$$(\Delta x)(\Delta k) \geq 1/2 \; . \tag{20}$$

From the equality sign we have $\Delta x = 1/(2\Delta k)$. This implies that if Δx is small then Δk must be large and vice-versa. We cannot make Δx and Δk small simultaneously (beyond the restriction $\Delta x \Delta k \geq 1/2$).