Mathematical Physics - II

Laplace Transform

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The Laplace Transform of a function, f(t), is defined as;

$$L[f(t)] = F(s) = \int_{0}^{\infty} f(t)e^{-st}dt$$

The Inverse Laplace Transform is defined by

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{ts}ds$$

Laplace Transform of the unit step.

$$L[u(t)] = \int_0^\infty 1 e^{-st} dt = \frac{-1}{s} e^{-st} \Big|_0^\infty$$

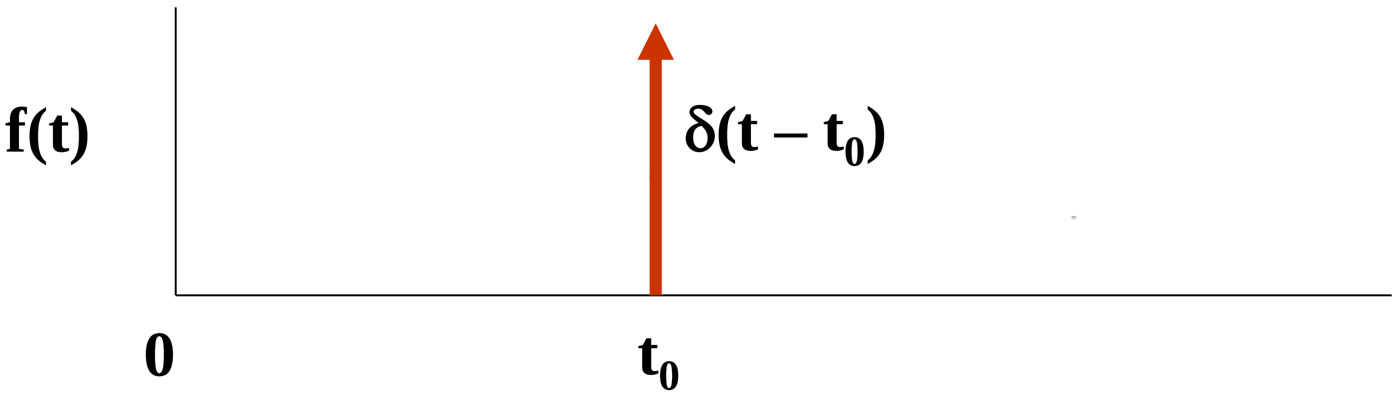
$$L[u(t)] = \frac{1}{s}$$

The Laplace Transform of a unit step is:

$$\boxed{\frac{1}{s}}$$

The Laplace transform of a unit impulse:

Pictorially, the unit impulse appears as follows:



Mathematically:

$$\delta(t - t_0) = 0 \quad t \neq 0$$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0)\,dt = 1 \qquad \varepsilon > 0$$

The Laplace transform of a unit impulse:

An important property of the unit impulse is a sifting or sampling property. The following is an important.

$$\int_{t_1}^{t_2} f(t)\delta(t-t_0)dt = \begin{cases} f(t_0) & t_1 < t_0 < t_2 \\ 0 & t_0 < t_1, t_0 > t_2 \end{cases}$$

The Laplace transform of a unit impulse:

In particular, if we let f(t) = δ(t) and take the Laplace

$$L[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = e^{-0s} = 1$$

An important point to remember:

$$f(t) \Leftrightarrow F(s)$$

The above is a statement that f(t) and F(s) are transform pairs. What this means is that for each f(t) there is a unique F(s) and for each F(s) there is a unique f(t). If we can remember the Pair relationships between approximately 10 of the Laplace transform pairs we can *go a long way.*

Building transform pairs:

$$L[e^{-at}u(t)] = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty e} e^{-(s+a)t} dt$$

$L(e$

$$L[e^{-at}u(t)] = \frac{-e^{-st}}{(s+a)}\Big|_0^{\infty} = \frac{1}{s+a}$$

A transform pair $\quad e^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{s+a}$

Building transform pairs:

$$L[tu(t)] = \int_0^\infty t e^{-st} dt$$

$$\int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

u = t

dv = e^-st^dt

$$tu(t) \Leftrightarrow \frac{1}{s^2}$$

A transform pair

Building transform pairs:

$$L[\cos(wt)] = \int_0^\infty \frac{(e^{jwt} + e^{-jwt})}{2} e^{-st} dt$$

$$= \frac{1}{2}\left[\frac{1}{s - jw} - \frac{1}{s + jw}\right]$$

$$= \frac{s}{s^2 + w^2}$$

$$\cos(wt)u(t) \quad \Leftrightarrow \quad \frac{s}{s^2 + w^2}$$

A transform pair

Time Shift

$$L[f(t-a)u(t-a)] = \int_a^{\infty} f(t-a)e^{-st}$$

$$Let\ x = t-a, then\ dx = dt\ and\ t = x+a$$

$$As\ t \to a,\ x \to 0\ and\ as\ t \to \infty, x \to \infty.\ So,$$

$$\int_0^{\infty} f(x)e^{-s(x+a)}dx = e^{-as}\int_0^{\infty} f(x)e^{-sx}dx$$

$$L[f(t-a)u(t-a)] = e^{-as}F(s)$$

Frequency Shift

$$L[e^{-at}f(t)] = \int_0^\infty [e^{-at}f(t)]e^{-st}dt$$

$$= \int_0^\infty f(t)e^{-(s+a)t}dt = F(s+a)$$

$$L[e^{-at}f(t)] = F(s+a)$$

Example: Using Frequency Shift

Find the $L[e^{-at}\cos(wt)]$

In this case, f(t) = cos(wt) so,

$$F(s)=\frac{s}{s^2+w^2}$$

$$and\ \ F(s+a)=\frac{(s+a)}{(s+a)^2+w^2}$$

$$L[e^{-at}\cos(wt)]=\frac{(s+a)}{(s+a)^2+(w)^2}$$

Time Integration:

The property is:

$$L\left[\int_0^{\infty} f(t)dt\right]=\int_0^{\infty}\left[\int_0^{t} f(x)dx\right]e^{-st}dt$$

Integrate by parts:

$$Let\ u=\int_0^{t} f(x)dx,\quad du=f(t)dt$$

and

$$dv=e^{-st}dt,\quad v=-\frac{1}{s}e^{-st}$$

Time Integration:

Making these substitutions and carrying out The integration shows that

$$L\left[\int_0^{\infty} f(t)dt\right] = \frac{1}{s}\int_0^{\infty} f(t)e^{-st}dt$$

$$= \frac{1}{s}F(s)$$

Time Differentiation:

If the L[f(t)] = F(s), we want to show:

$$L[\frac{df(t)}{dt}] = sF(s) - f(0)$$

Integrate by parts:

$$u = e^{-st},\ du = -se^{-st}dt\ and$$

$$dv = \frac{df(t)}{dt}dt = df(t),\ so\ v = f(t)$$

Time Differentiation:

Making the previous substitutions gives,

$$L\left[\frac{df}{dt}\right] = f(t)e^{-st}\Big|_0^{\infty} - \int_0^{\infty} f(t)\left[-se^{-st}\right]dt$$

$$= 0 - f(0) + s\int_0^{\infty} f(t)e^{-st}dt$$

So we have shown:

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

Time Differentiation:

We can extend the previous to show;

$$L\left[\frac{df(t)^2}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$$

$$L\left[\frac{df(t)^3}{dt^3}\right] = s^3F(s) - s^2 f(0) - sf'(0) - f''(0)$$

general case

$$L\left[\frac{df(t)^n}{dt^n}\right] = s^nF(s) - s^{n-1} f(0) - s^{n-2} f'(0)$$

$$-\ldots - f^{(n-1)}(0)$$

Transform Pairs:

f(t)	F(s)
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-st}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$

Transform Pairs:

f(t)	F(s)
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(wt)$	$\frac{w}{s^2+w^2}$
$\cos(wt)$	$\frac{s}{s^2+w^2}$

Transform Pairs:

f(t)	F(s)
$e^{-at}\sin(wt)$	$\dfrac{w}{(s+a)^2+w^2}$
$e^{-at}\cos(wt)$	$\dfrac{s+a}{(s+a)^2+w^2}$
$\sin(wt+\theta)$	$\dfrac{s\sin\theta+w\cos\theta}{s^2+w^2}$
$\cos(wt+\theta)$	$\dfrac{s\cos\theta-w\sin\theta}{s^2+w^2}$

Common Transform Properties:

f(t)	F(s)
$f(t-t_0)u(t-t_0), t_0 \geq 0$	$e^{-t_o s}F(s)$
$f(t)u(t-t_0), t \geq 0$	$e^{-t_o s}L[f(t+t_0)$
$e^{-at}f(t)$	$F(s+a)$
$\dfrac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - s^0 f^{n-1}f(0)$
$tf(t)$	$-\dfrac{dF(s)}{ds}$
$\int_0^t f(\lambda)d\lambda$	$\dfrac{1}{s}F(s)$

Theorem: Initial Value Theorem:

If the function f(t) and its first derivative are Laplace transformable and f(t) Has the Laplace transform F(s), and the $\lim_{s \to \infty} sF(s)$ exists, then

$$\lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t) = f(0)$$

Initial Value Theorem

The utility of this theorem lies in not having to take the inverse of F(s) in order to find out the initial condition in the time domain. This is particularly useful in circuits and systems.

Example: **Initial Value Theorem:**

Given;

$$F(s)=\frac{(s+2)}{(s+1)^2+5^2}$$

Find f(0)

$$f(0)=\lim_{s\to\infty} sF(s)=\lim_{s\to\infty} s\frac{(s+2)}{(s+1)^2+5^2}=\lim_{s\to\infty}\left[\frac{s^2+2s}{s^2+2s+1+25}\right]$$

$$=\lim_{s\to\infty}\frac{s^2/s^2+2s/s^2}{s^2/s^2+2s/s^2+(26/s^2)}=1$$

Theorem: Final Value Theorem:

If the function f(t) and its first derivative are Laplace transformable and f(t) has the Laplace transform F(s), and the $\lim_{s \to \infty} sF(s)$ exists, then

$$\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t) = f(\infty)$$

Final Value Theorem

Again, the utility of this theorem lies in not having to take the inverse of F(s) in order to find out the final value of f(t) in the time domain. This is particularly useful in circuits and systems.

Example: **Final Value Theorem:**

Given:

$$F(s) = \frac{(s+2)^2 - 3^2}{\left[(s+2)^2 + 3^2\right]} \qquad \textit{note} \;\; F^{-1}(s) = te^{-2t}\cos 3t$$

Find $f(\infty)$.

$$f(\infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} s \frac{(s+2)^2 - 3^2}{\left[(s+2)^2 + 3^2\right]} = 0$$