Mathematical Physics - II

Green’s Function

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What is a Green's Function?

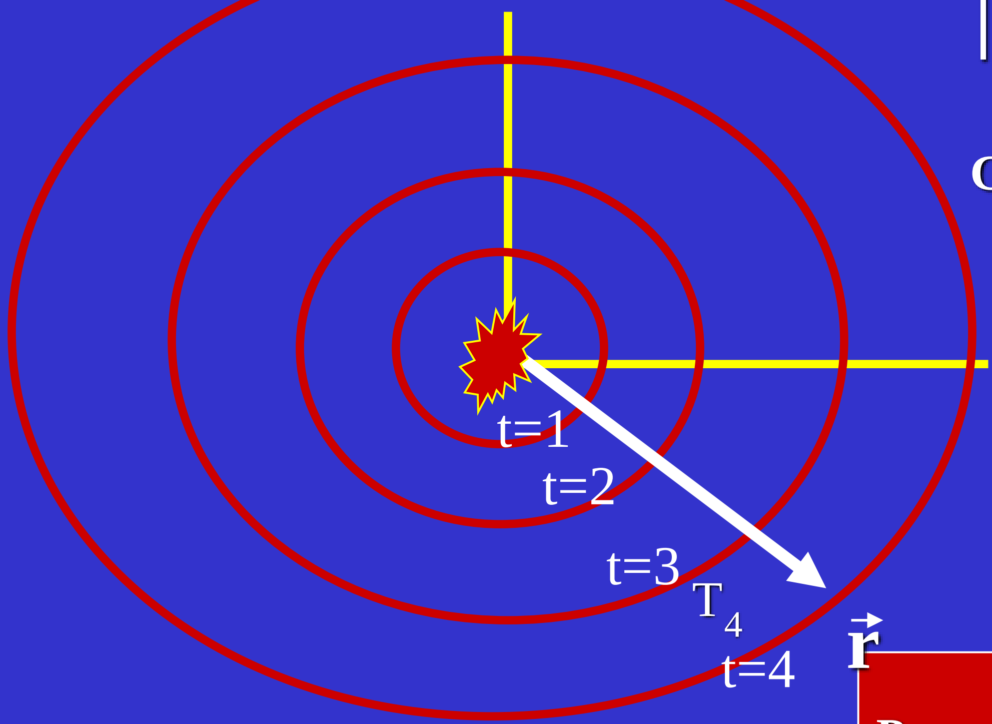
Answer: Impulse-Point Src. Response of Medium

Listening & source coords.

Listening time

$$G(x,t|x_s,t_s) = \frac{\delta(t-|\vec{r}|/c)}{|\vec{r}|} = \begin{cases} 1 & t=|\vec{r}|/c \\ 0 & \text{ottherwise} \end{cases}$$

Geometrical Spreading



Propagation time $= \dfrac{|\vec{r}|}{c} = \dfrac{\sqrt{x^2 + y^2 + z^2}}{c}$

What is a Green's Function?

Answer: Impulse-Point Src. Response of Medium

$$G(x,t|x_s\ ,0\ ) = \frac{\delta(t - |\vec{r}|/c)}{|\vec{r}|} = \begin{cases} 1 & t\ =|\vec{r}|/c \\ 0 & \text{ottherwise} \end{cases}$$

Fourier

$$\tilde{G}(x|x_s) = \frac{e^{iwr/c}}{|\vec{r}|}$$

derivative

$$\hat{n} \cdot \vec{\nabla} G(x|x_s) = \frac{iw/c\ e^{iwr/c}}{|\vec{r}|}\ \hat{n} \cdot \hat{r} + O(1/r^2\ )$$

Wave in Heterogeneous Medium

$$G = Ae^{i(w\tau - wt)} \quad \text{satisfies} \quad \ddot{G} = c^2 \nabla^2 G$$

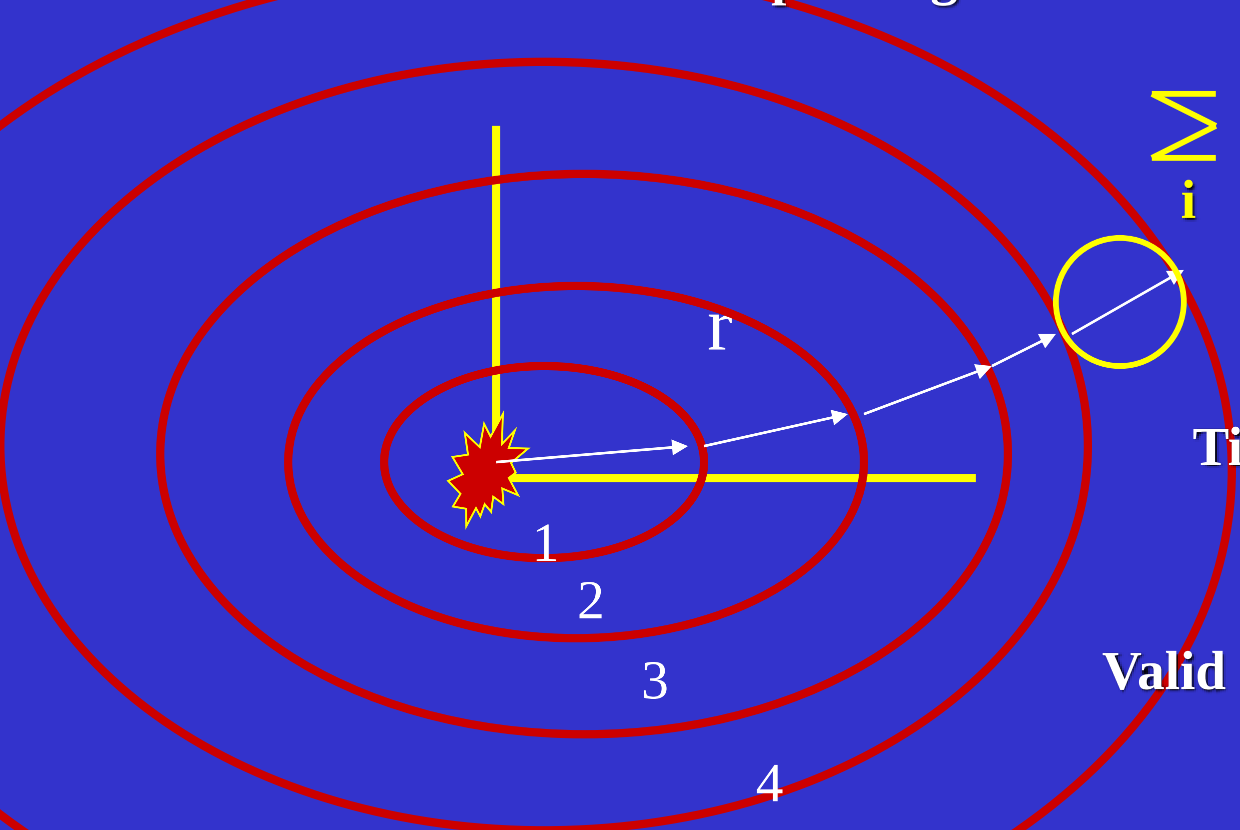
Geometrical speading

except at origin

$$\sum_i kr_i = \sum_i (\overbrace{kc_i}^{w})\, r_i / c_i = w\tau$$

Time taken along ray segment

Valid at high w and smooth media

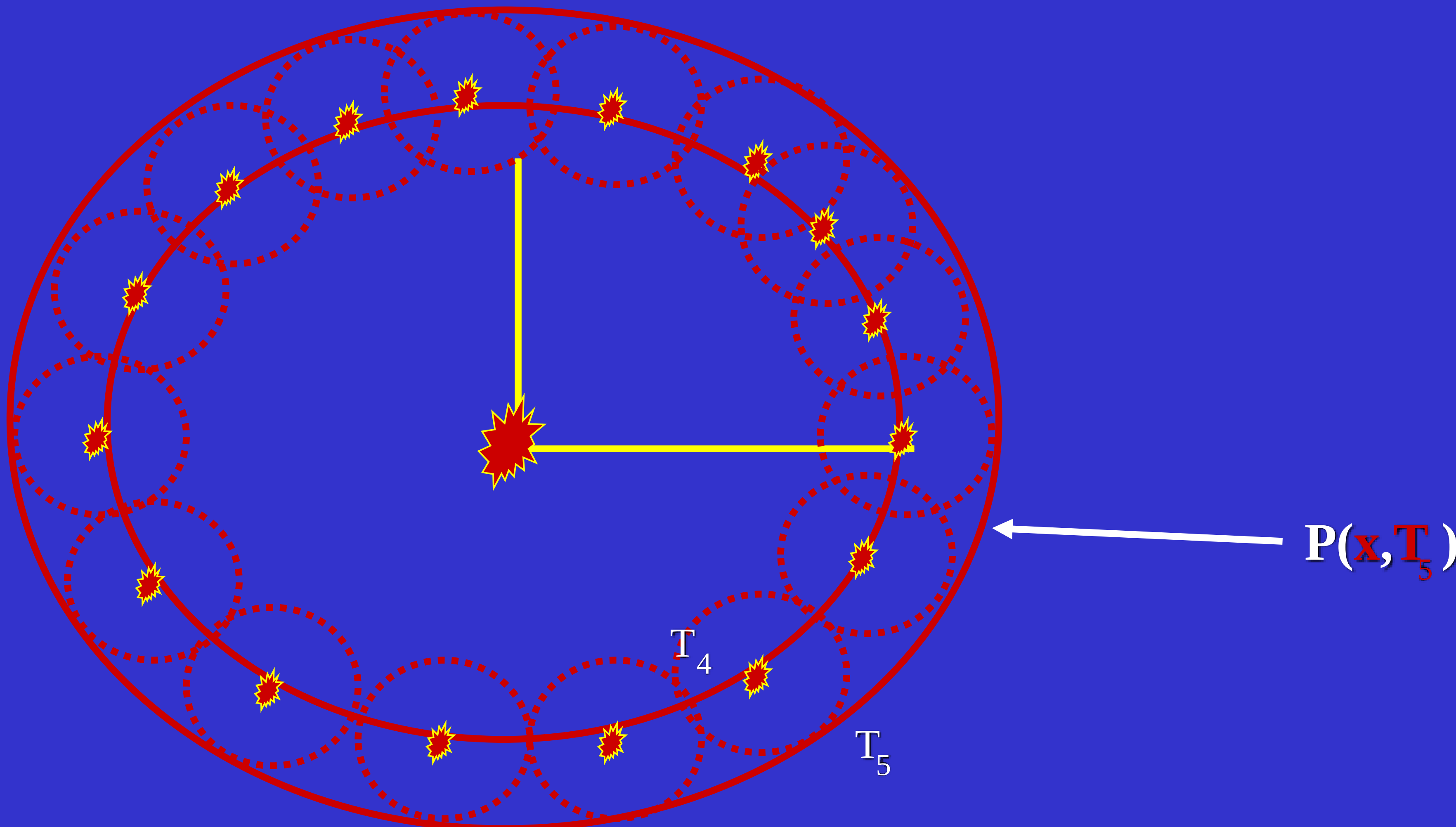


Outline

- Green’s Functions
- Forward Modeling
- SRME
- Acausal GF, Stationarity, Reciprocity
- Interferometry
- Migration

What is Huygen's Principle?

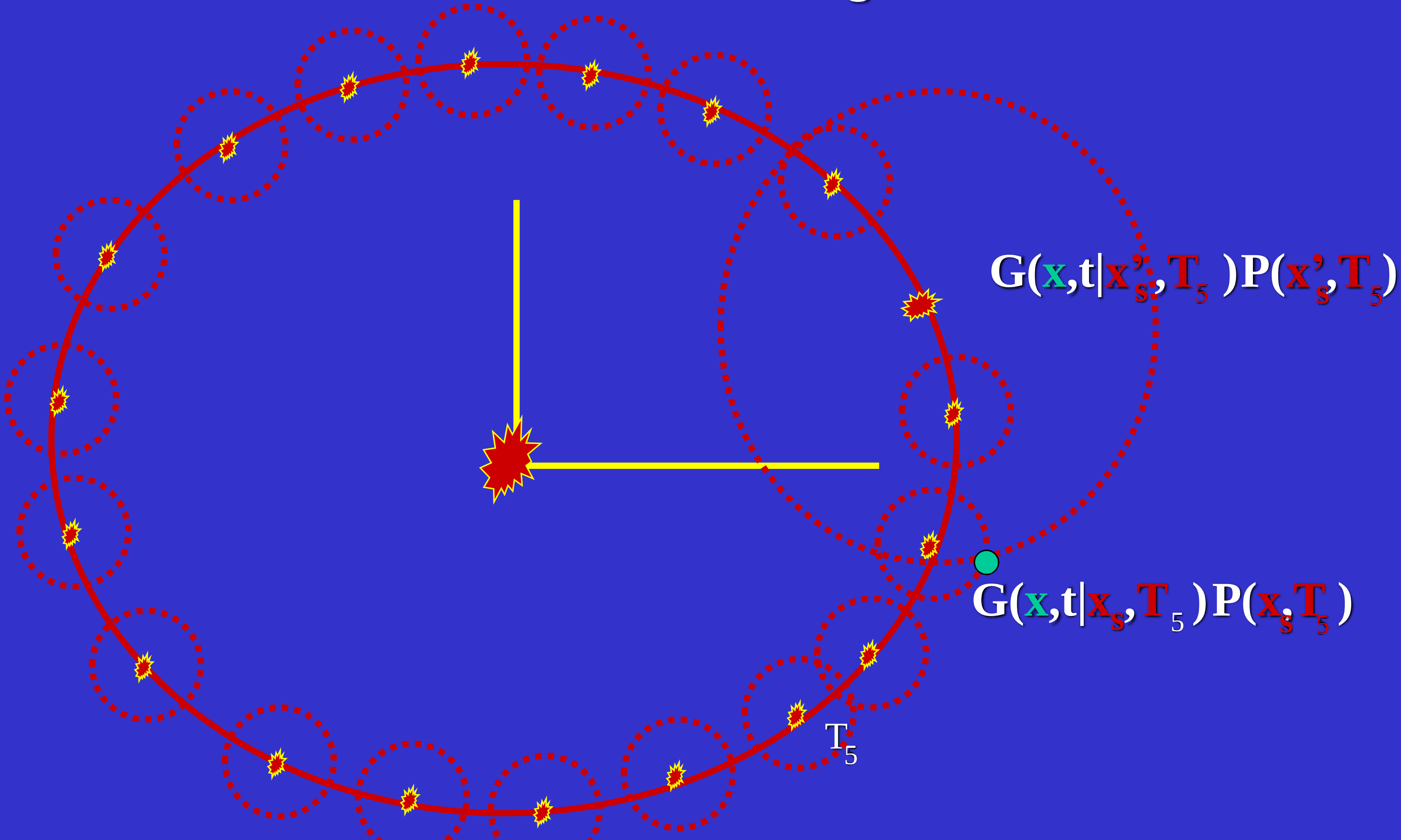
Answer: Every pt. on a wavefront is a secondary src. pt.
Common tangent kinematically defines next wavefront.



What is Huygen-Fresnel Principle?

Answer: Every pt. on a wavefront is a secondary src. pt.

Next wavefront is sum of weighted Green's functions.



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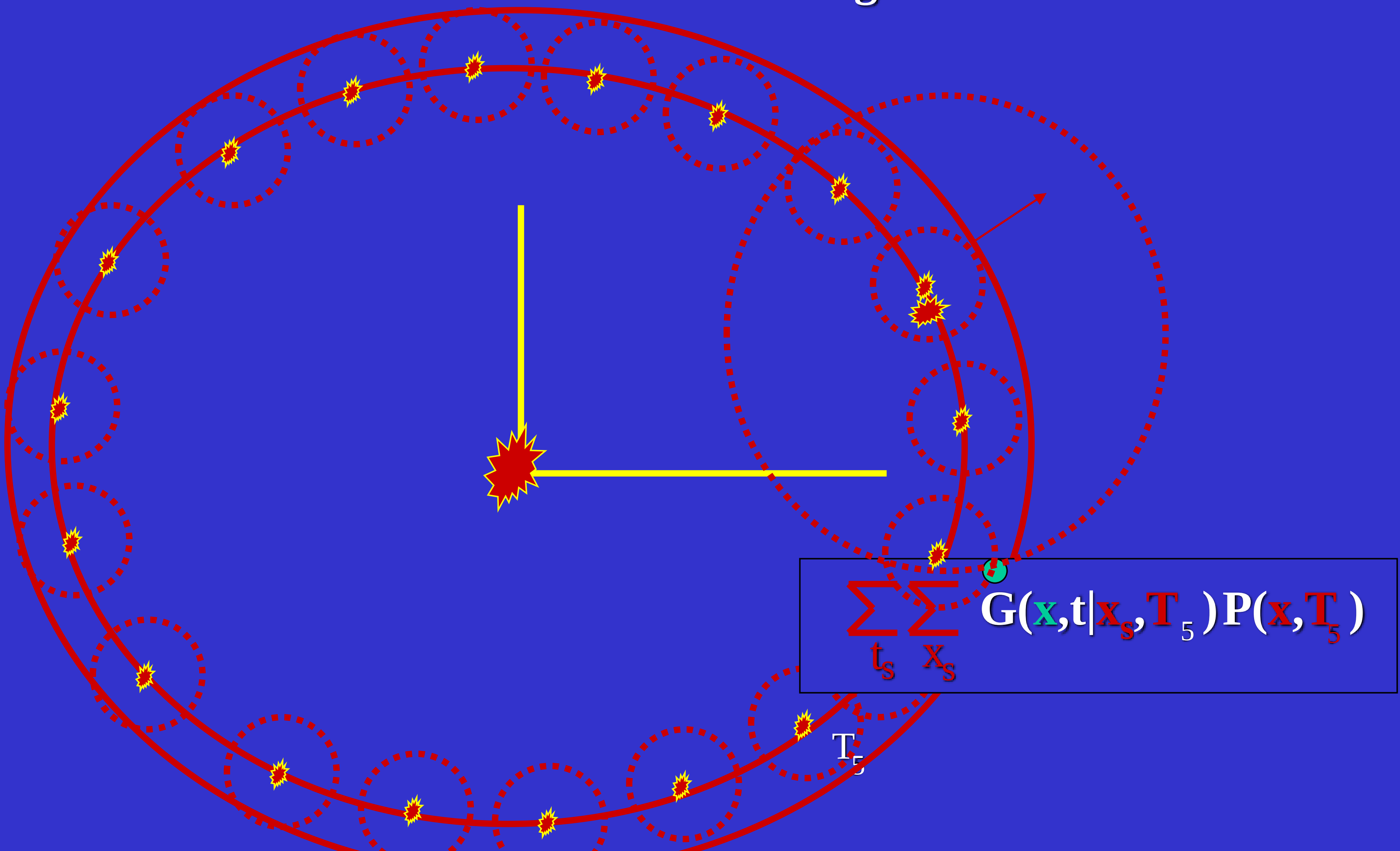
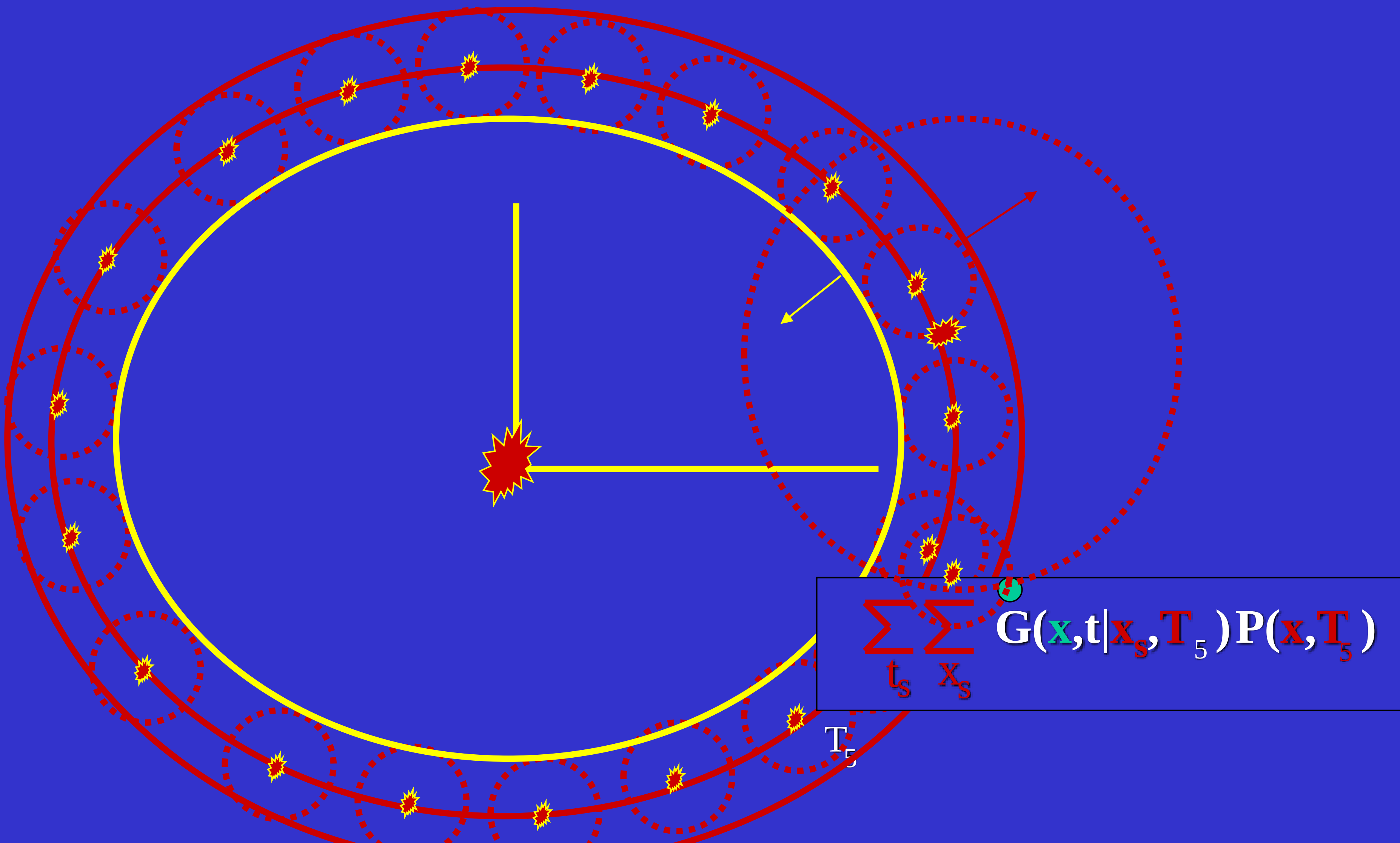


Problem with Huygen's Principle?



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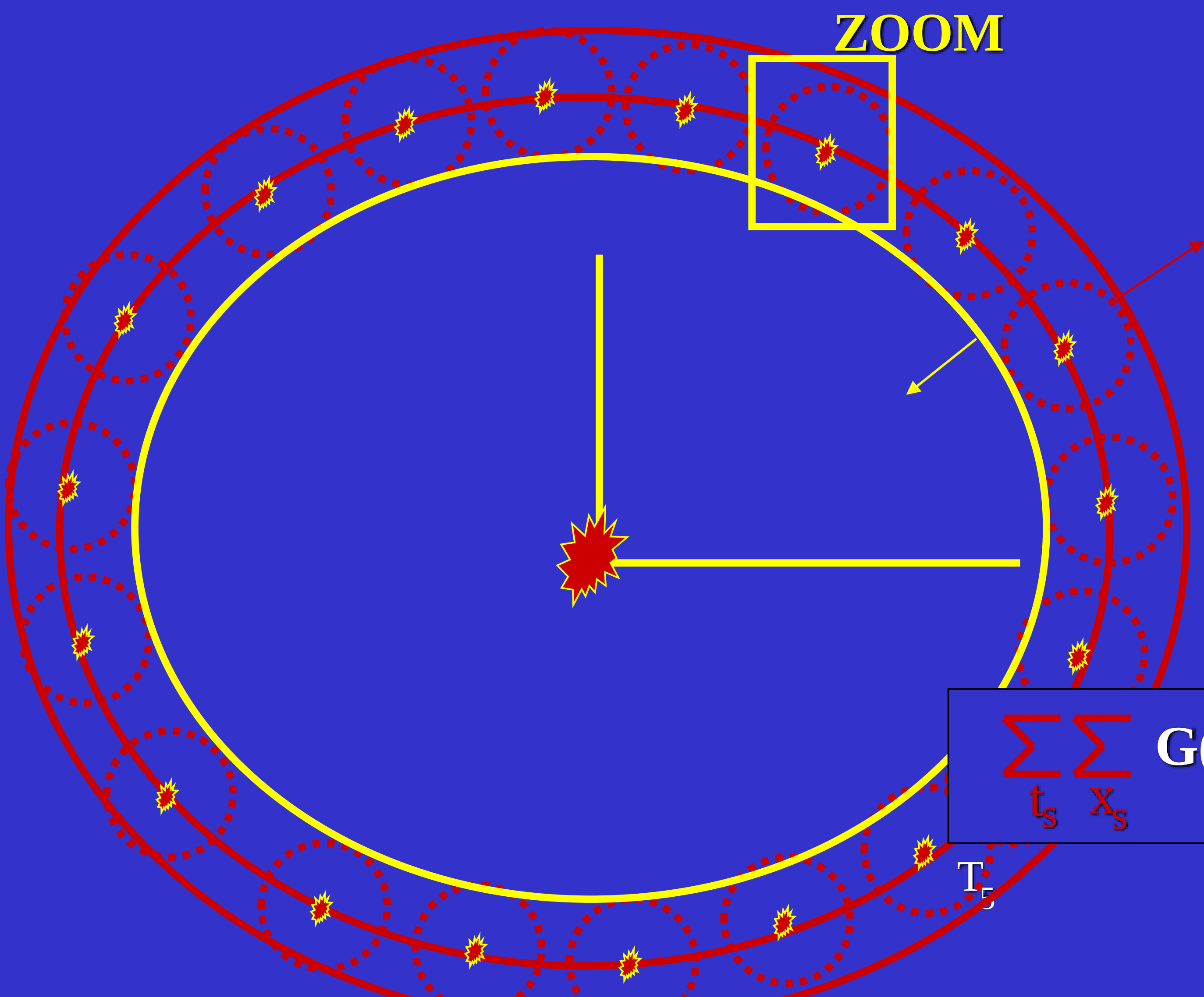
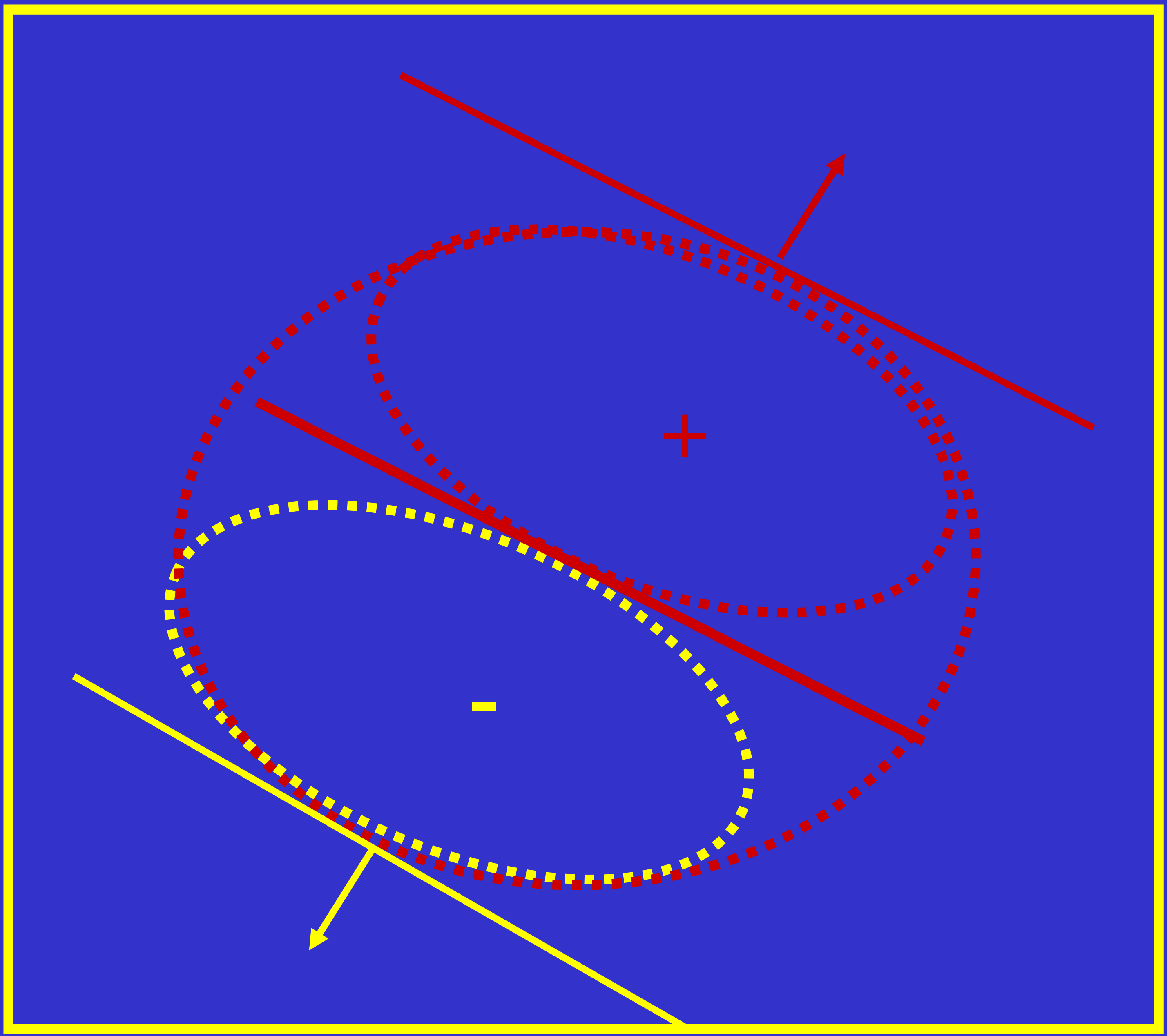


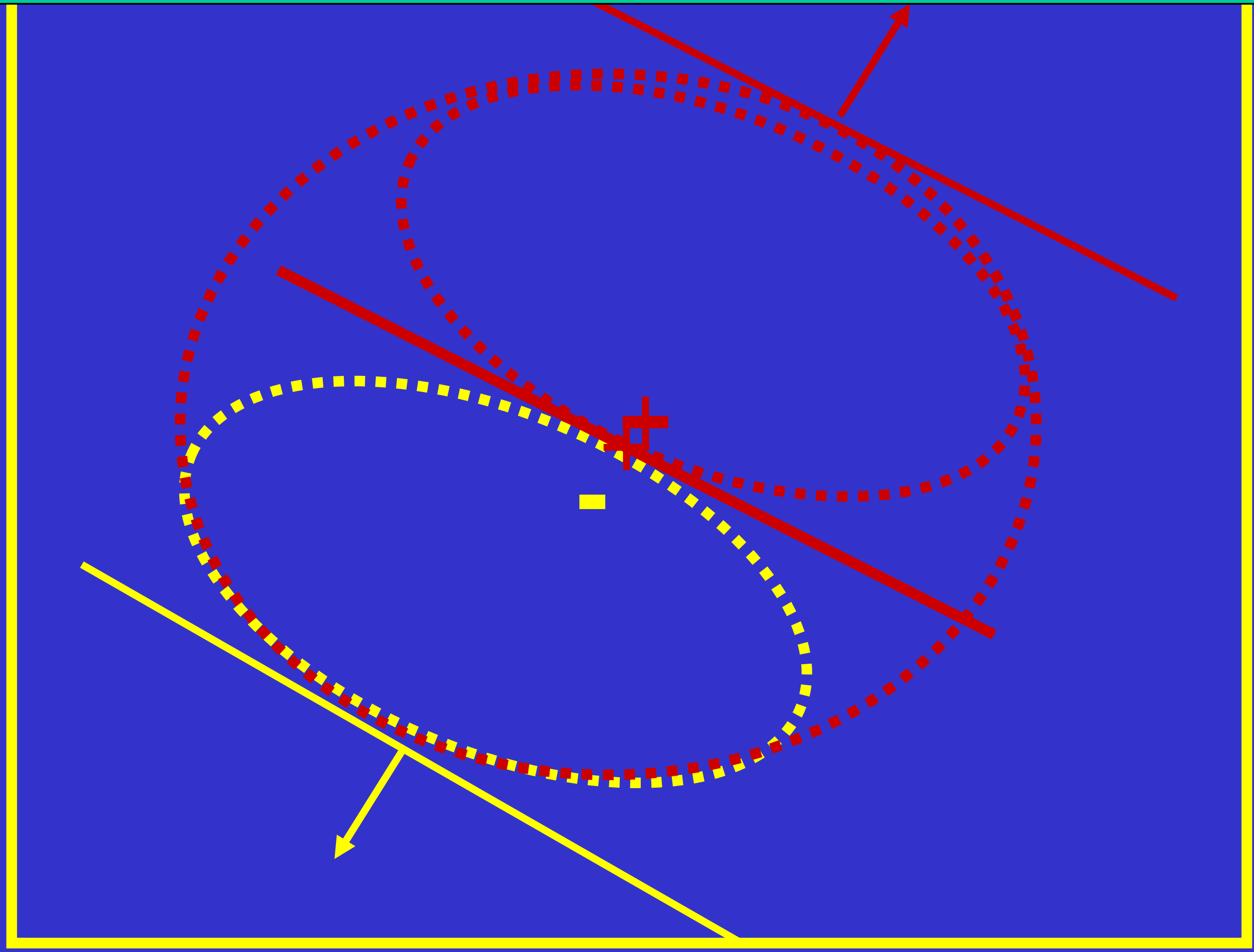
$$\sum_{t_s}\sum_{x_s} G(\mathbf{x},t|\mathbf{x}_s,\mathbf{T}_5)\,P(\mathbf{x},\mathbf{T}_5)$$

Problem with Huygen's Principle?



Problem with Huygen's Principle?

$$\frac{G(x,t|x'_s,T_5)P(x',T_5) \;-G(x,t|x'_s,T_5)P(x',T_5)}{dn} = PdG/dn$$



Green's Theorem Forward Modeling

$$P(x) = \oint \left\{ G(x|\mathbf{x}) \frac{d}{dn} P(\mathbf{x}) - P(\mathbf{x}) \frac{d}{dn} G(x|\mathbf{x}) \right\} d^2\mathbf{x}$$

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- Green's Functions
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- Acausal GF, Stationarity, Reciprocity
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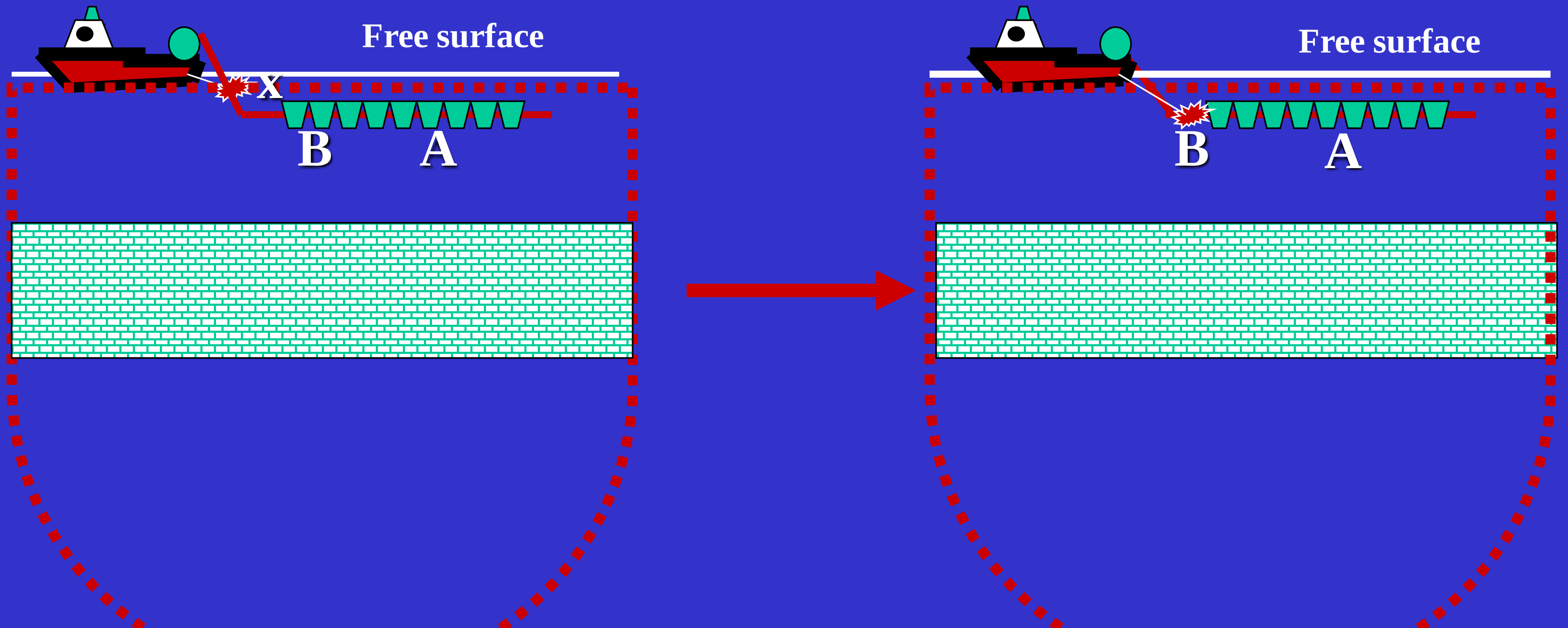
Reciprocity Eqn. of Convolution Type

0. Define Problem

Given: G(A|x)
G(B|x)

Find: G(A|B)



Reciprocity Eqn. of Convolution Type

1. Helmholtz Eqns:

$$[\nabla^2 + k^2]\, G(A|x) = -\delta(x-A);$$

$$[\nabla^2 + k^2]\, P(B|x)^* = -\delta(x-B)$$

2. Multiply by G(A|x) and P(B|x) and subtract*

$$P(B|x)^*[\nabla^2 + k^2]\, G(A|x) = -\delta(x-A)\; P(B|x)^*$$

$$G(A|x)\;[\nabla^2 + k^2]\, P(B|x)^* = -\delta(x-B)\; G(A|x)^*$$

$$P(B|x)^* \nabla^2 G(A|x) - G(A|x) \nabla^2 P(B|x)^* = \delta(B-x)G(A|x) - \delta(A-x)P(B|x)^*$$

$$\nabla\cdot\{P(B|x)^* \nabla G(A|x) - G(A|x) \nabla P(B|x)^*\} = \delta(B-x)G(A|x) - \delta(A-x)P(B|x)^*$$

Reciprocity Eqn. of Convolution Type

3. Integrate over a volume

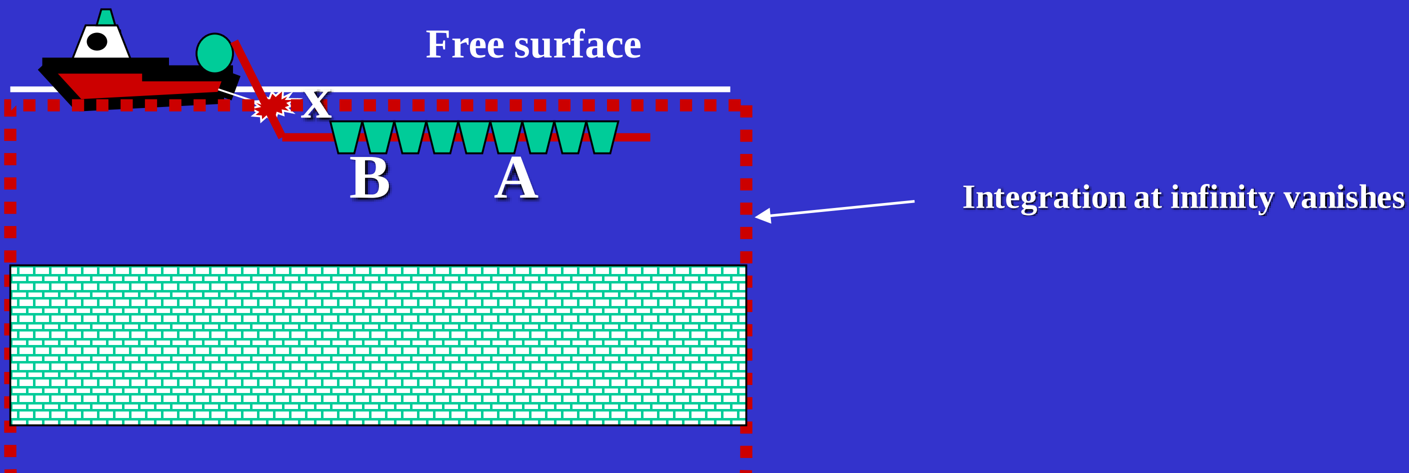
$$\int \nabla \cdot \{ P(B|x)^* \nabla G(A|x) - G(A|x) \nabla P(B|x)^* \}\, d^3x = G(A|B) - P(B|A)^*$$

4. Gauss's Theorem

$$\oint \{ P(B|x)^* \nabla G(A|x) - G(A|x) \nabla P(B|x)^* \} \cdot \vec{n}\, d^2x = G(A|B) - P(B|A)^*$$

Source line

G(A|B)



Reciprocity Eqn. of Convolution Type

3. Integrate over a volume

$$\int \nabla \cdot \{P(B|x)\, \nabla G(A|x) - G(A|x)\, \nabla P(B|x)\}\, d^3x = G(A|B) - P(B|A)$$

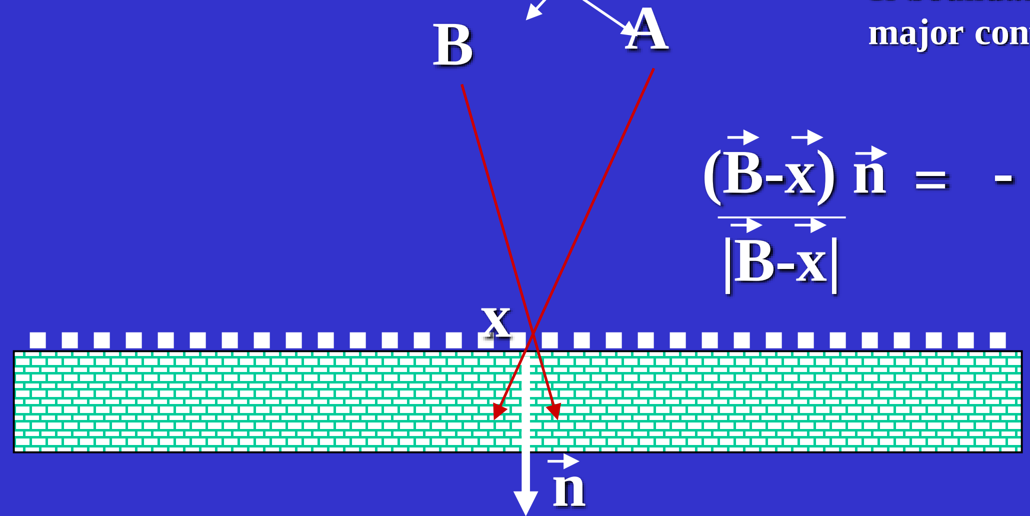
Hence, the two integrands above are opposite and equal in far-field hi-freq. approximation

4. Gauss's Theorem

$$\int_{\text{Layer interface}} \{P(B|x)\, \nabla G(A|x) - G(A|x)\, \nabla P(B|x)\} \cdot \vec{n}\, d^2x = G(A|B) - P(B|A)$$

If boundary is far away the major contribution is specular reflection

$$\frac{(\vec{B}-\vec{x})\,\vec{n}}{|\vec{B}-\vec{x}|} = -\frac{(\vec{A}-\vec{x})\,\vec{n}}{|\vec{A}-\vec{x}|}$$



Reciprocity Eqn. of Convolution Type

$$2\int_{1^{st}\ \text{interface}} P(B|x)\ \nabla G(A|x) \cdot \vec{n}\ d^2x \;=\; G(A|B) - \;\; P(B|A)$$

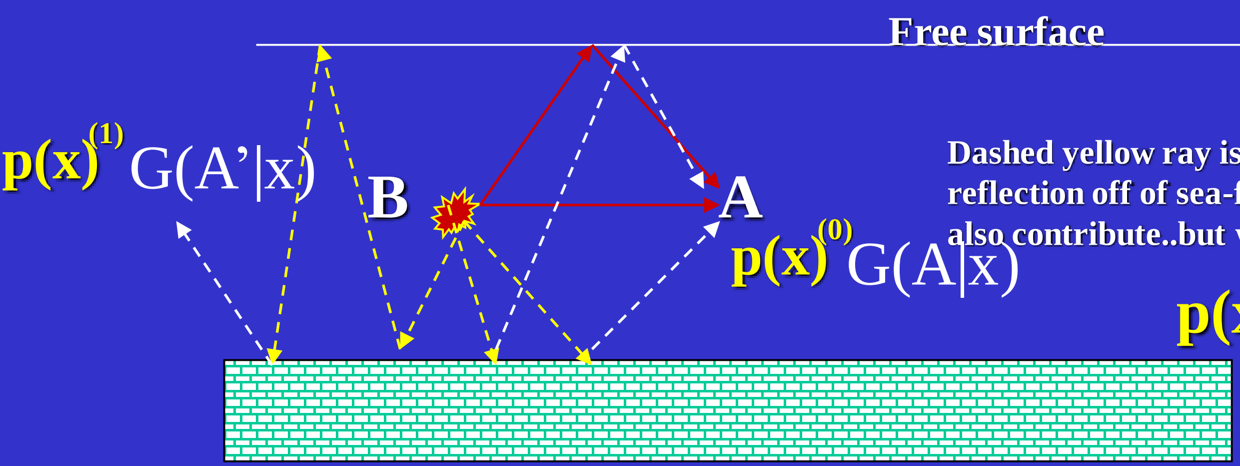
Far field $\;\vec{n}\cdot\vec{r} = 1$

$$2ik\int_{1^{st}\ \text{interface}} P(B|x)G(A|x)d^2x \;+\; P(B|A) = G(A|B)$$

Freeze B and hide it

$$2ik\int_{1^{st}\ \text{interface}} \underbrace{p(x)G(A|x)d^2x}_{\text{Reflected waves}} \;=\!- \underbrace{p(A)}_{\text{Ghost direct-Total pressure}} + g(A)$$

Remember, g(A) is a ½ space shot gather with src at B and rec at A



Dashed yellow ray is just a primary and ghost reflection off of sea-floor. .Higher-order multiples will also contribute..but we assume they are weak

$$p(x) = p(x)^{(0)} + p(x)^{(1)} + p(x)^{(2)} + \ldots$$

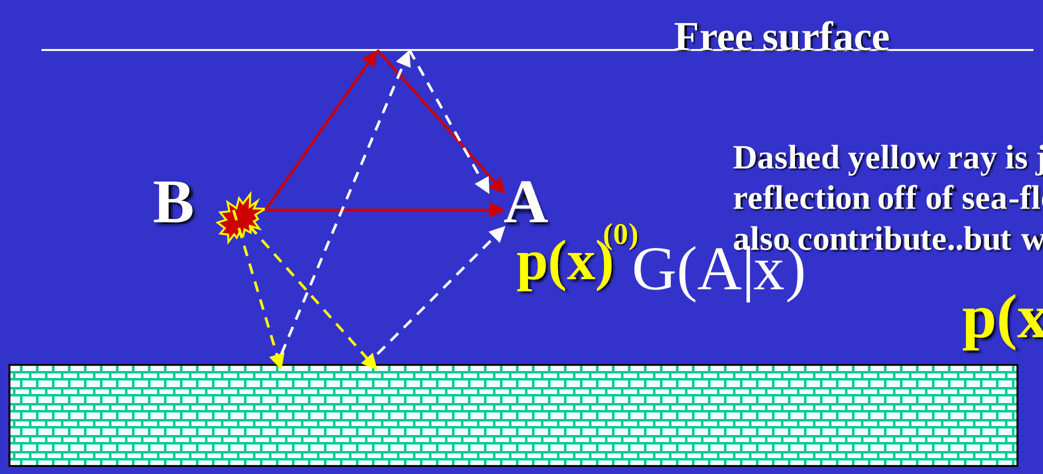
$$\cong p(x)^{(0)} \longleftarrow \textbf{Primary reflection}$$

Summary

- **Green's theorem, far-field, hi-freq. approx.**

$$2ik \int_{1^{st}\ \text{interface}} \underbrace{p(x)G(A|x)}_{\text{Reflected waves}}\, d^2x = -\underbrace{p(A)}_{\text{Scattered pressure field}} + g(A)$$

$$2ik \int p(x)G(A|x)\,\underbrace{\vec{n}\cdot\vec{r}}_{\text{obliquity}}\, d^2x = p(A)^{scatt.}$$



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- **If A=B and p(x)=cnst. Then we reduce to intuitive ZO modeling formula**