Mathematical Physics - II

Fourier Transform

Prof. K. THAMILMARAN

Department of Nonlinear Dynamics
School of Physics, Bharathidasan University
Tiruchirapalli-620 024

maran.cnld@gamil.com

Jean Baptiste Joseph Fourier (1768-1830)

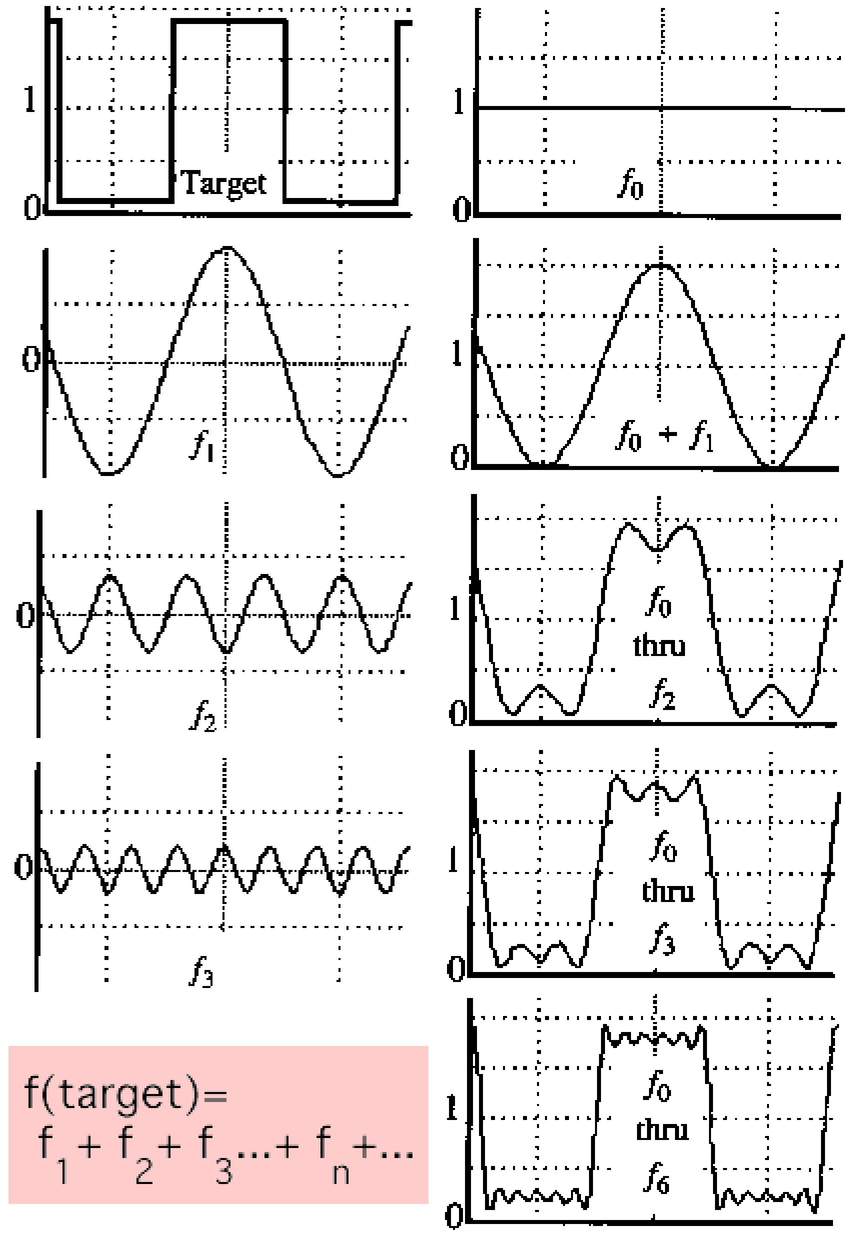
- Had crazy idea (1807):
- **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don’t believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it’s true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

A Sum of Sinusoids

- Our building block:

$$A\sin(\omega x + \phi)$$

- Add enough of them to get any signal *f(x)* you want!
- How many degrees of freedom?
- What does each control?
- Which one encodes the coarse vs. fine structure of the signal?



f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

Fourier Transform

- We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of x:

f(x) → **Fourier Transform** → ***F(ω)***

- **For every ω from 0 to inf, *F(ω)* holds the amplitude *A* and phase ϕ of the corresponding sine**

$$A\sin(\omega x + \phi)$$

  - **How can *F* hold both? Complex number trick!**

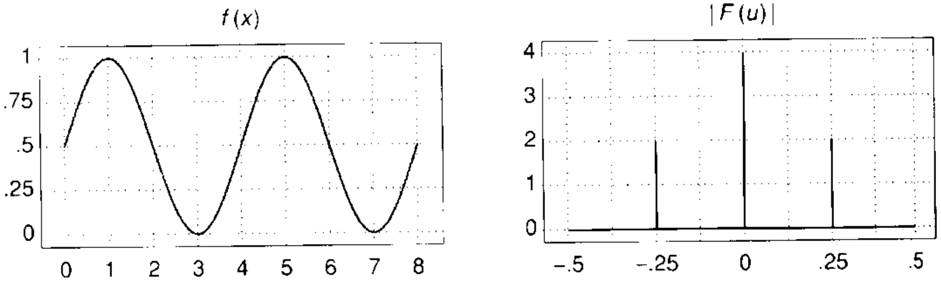
$$F(\omega) = R(\omega) + iI(\omega)$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$$

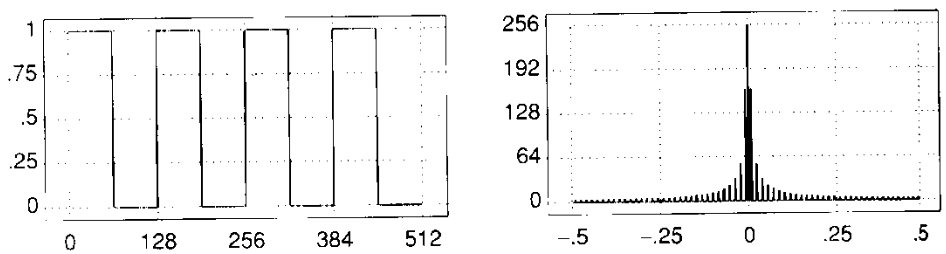
$$\phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

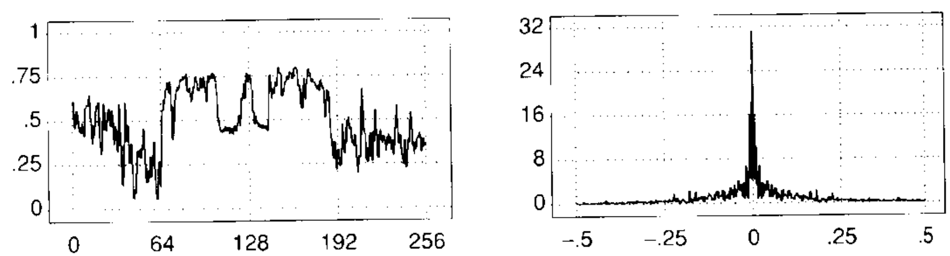
F(ω) → **Inverse Fourier Transform** → ***f(x)***

Frequency Spectra



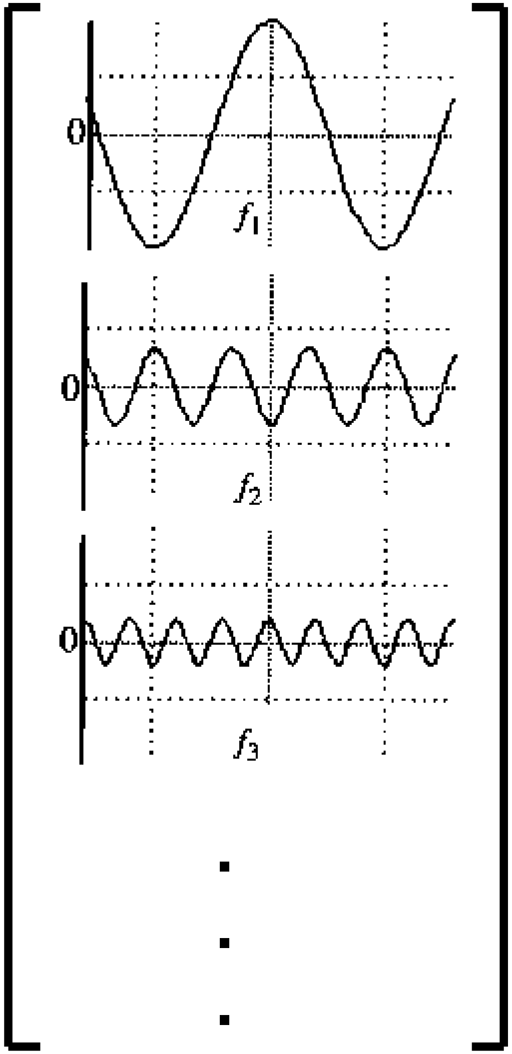
(a)


(b)


(c)

FT: Just a change of basis

$$\mathbf{M} * f(x) = F(\omega)$$



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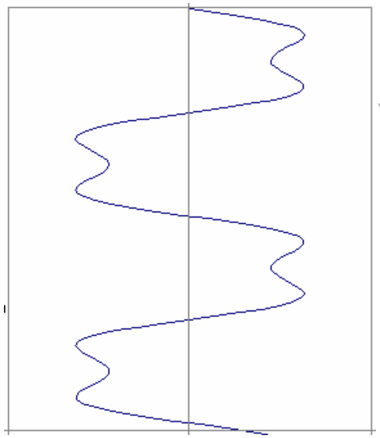

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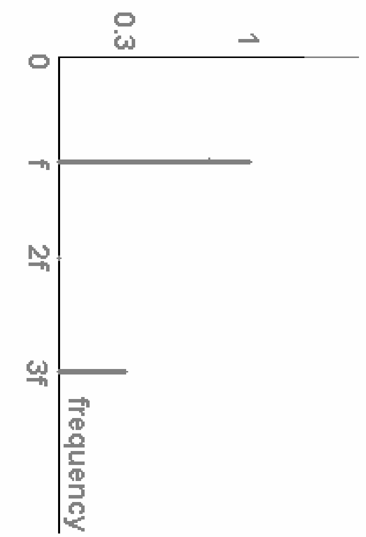
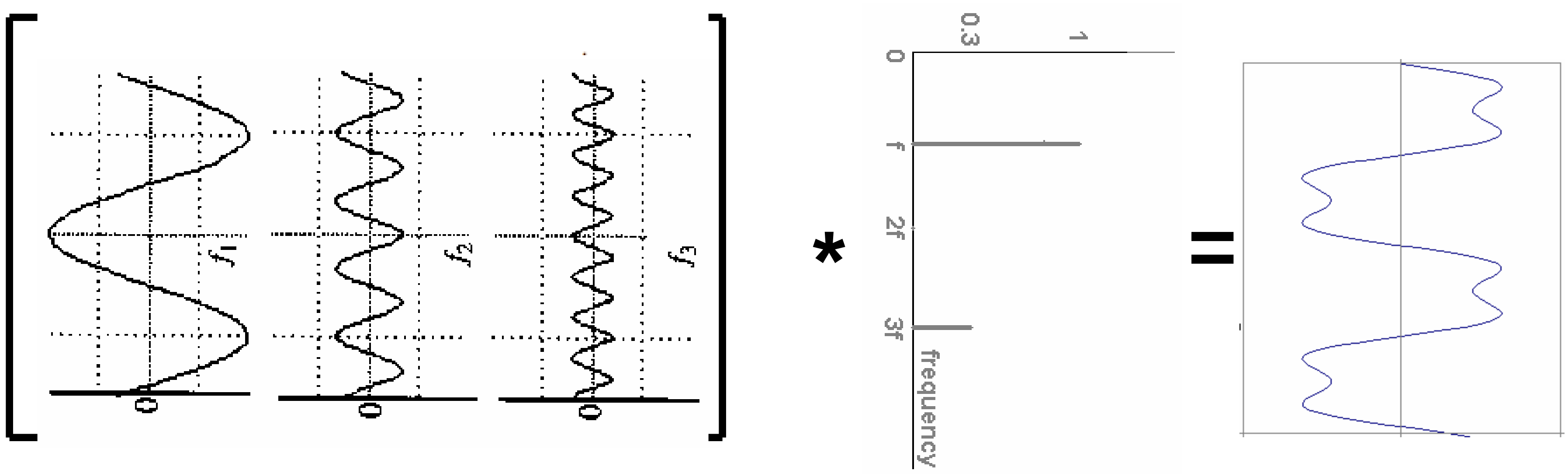


IFT: Just a change of basis

$$\mathbf{M}^{-1} * F(\omega) = f(x)$$



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Fourier Transform

- **Also, defined as:**

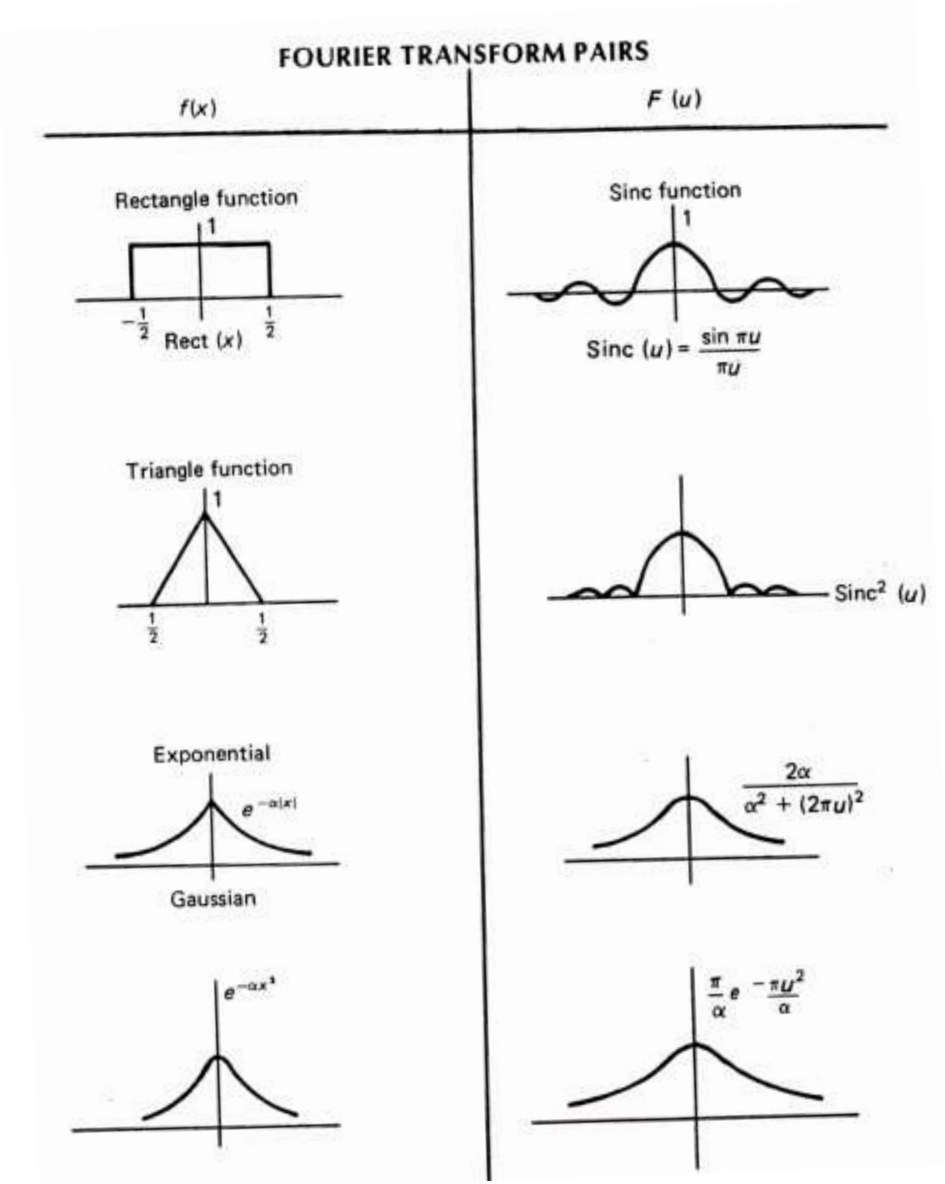
$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} dx$$

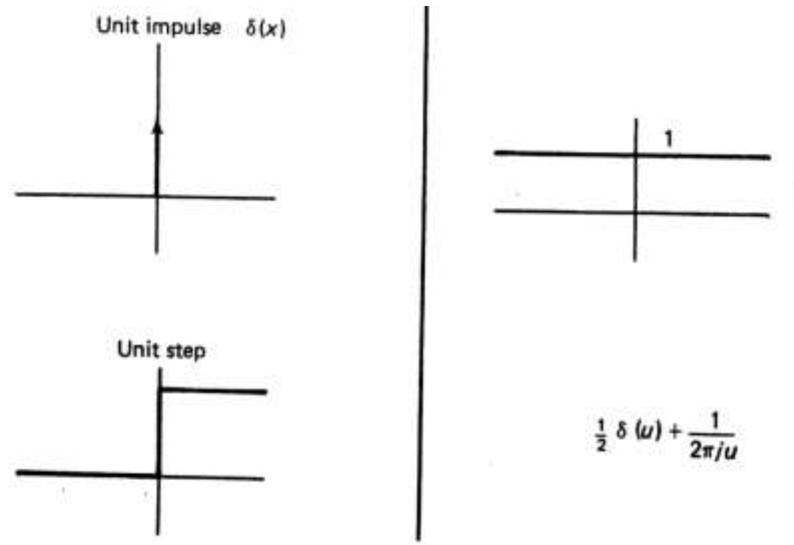
Note: $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

- **Inverse Fourier Transform (IFT)**

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} dx$$

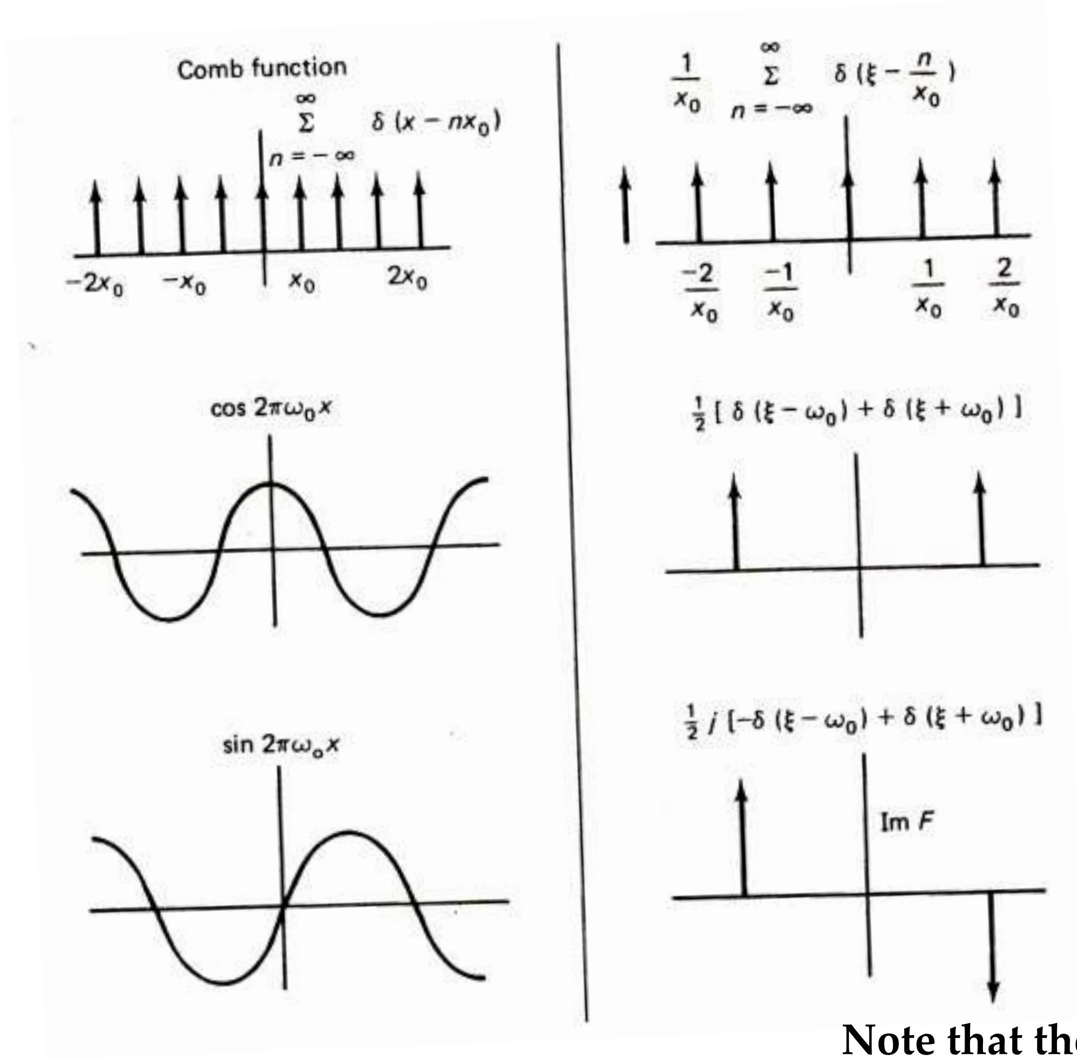
Fourier Transform Pairs (I)





Note that these are derived using angular frequency (e^{-iux})

Fourier Transform Pairs (I)



Note that these are derived using angular frequency (e^{-iux})

Fourier Transform and Convolution

Let $g = f * h$

Then $G(u) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux} dx$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau) h(x-\tau) e^{-i2\pi ux} d\tau dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \left[h(x-\tau) e^{-i2\pi u(x-\tau)} dx \right]$$

$$= \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \int_{-\infty}^{\infty} \left[h(x') e^{-i2\pi ux'} dx' \right]$$

$$= F(u) H(u)$$

Convolution in spatial domain

$\Longleftrightarrow$ **Multiplication in frequency domain**

Fourier Transform and Convolution

Spatial Domain *(x)* | **Frequency Domain *(u)***

$$g = f * h \quad \longleftrightarrow \quad G = FH$$

$$g = fh \quad \longleftrightarrow \quad G = F * H$$

So, we can find *g(x)* by Fourier transform

$$\begin{tikzcd}
g & = & f \arrow[d, "\boxed{\mathbf{FT}}"] & * & h \arrow[d, "\boxed{\mathbf{FT}}"] \\
G \arrow[u, "\boxed{\mathbf{IFT}}"] & = & F & \times & H
\end{tikzcd}$$

Properties of Fourier Transform

	Spatial Domain *(x)*	**Frequency Domain *(u)***
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Note that these are derived using frequency ($e^{-i2\pi u x}$)

Properties of Fourier Transform

Parseval's theorem:

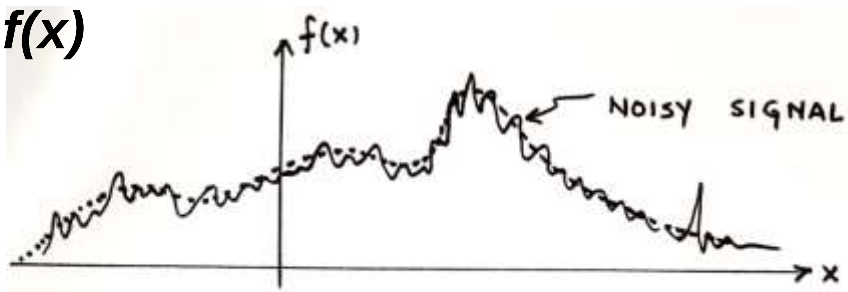
$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\xi)|^2 d\xi$$

$$\int_{-\infty}^{\infty} f(x) g^*(x)\ dx = \int_{-\infty}^{\infty} F(\xi) G^*(\xi)\ d\xi$$

$f(x)$	$F(\xi)$
Real(R)	Real part even (RE) Imaginary part odd (IO)
Imaginary (I)	RO,IE
RE,IO	R
RE,IE	I
RE	RE
RO	IO
IE	IE
IO	RO
Complex even (CE)	CE
CO	CO

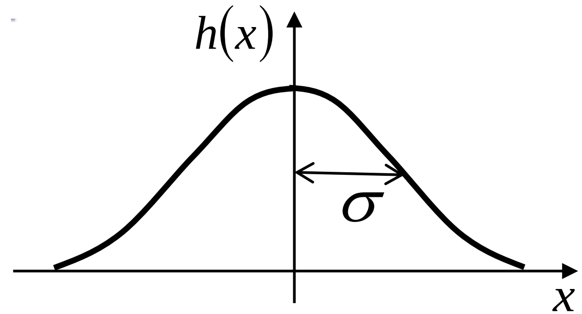
Example use: Smoothing/Blurring

- **We want a smoothed function of *f(x)***



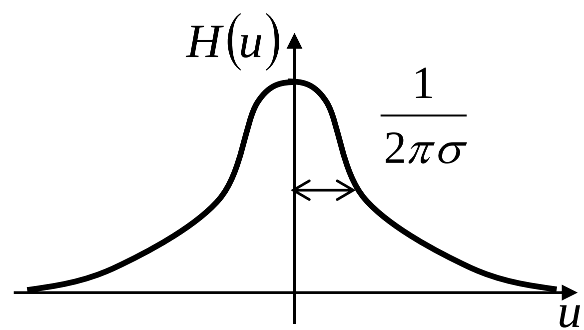
$$g(x) = f(x) * h(x)$$

- **Let us use a Gaussian kernel**



$$h(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \frac{x^2}{\sigma^2} \right]$$

- **Then**



$$H(u) = \exp\left[-\frac{1}{2} (2\pi u)^2 \sigma^2 \right]$$

$$G(u) = F(u) H(u)$$

***H(u)* attenuates high frequencies in *F(u)* (Low-pass Filter)!**