Mathematical Physics - II

Complex Analysis

K. THAMILMARAN

Department of Nonlinear Dynamics
School of Physics, Bharathidasan University
Tiruchirapalli-620 024

maran.cnld@gamil.com

Complex Variables & Functions

Complex numbers : $£ = \{ z = x + i y \; ; \; \forall \, x, y \in ¡ \; \}$ (Ordered pair of real numbers)

Complex conjugate : $z^* = x - i y$

$r = \sqrt{x^2 + y^2}$ modulus

Polar representation : $z = r\, e^{i\theta} = r\, e^{i(\theta + 2\pi n)}$

$\theta = \tan^{-1} \frac{y}{x}$ argument

$\rightarrow \quad e^{i\theta} = \cos\theta + i \sin\theta$

Multi-valued function $\rightarrow$ single-valued in each branch

E.g., $z^{1/m} = r^{1/m}\, e^{i(\theta + 2\pi n)/m}$ **has *m* branches.**

$\ln z = \ln r + i(\theta + 2\pi n)$ **has an infinite number of branches.**

Cauchy Reimann Conditions

$\boxed{z = x + iy}$

Derivative : $$\frac{d\,f(z)}{dz} = f'(z) \quad = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} \quad = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

where limit is independent of path of $\delta z \to 0$.

Let $$f(z) = u(z) + i\,v(z)$$

$$\to \quad \delta z = \delta x + i\,\delta y \qquad \delta f = \delta u + i\,\delta v \quad \to \quad \frac{\delta f}{\delta z} = \frac{\delta u + i\,\delta v}{\delta x + i\,\delta y}$$

$$\delta z = \delta x \quad \to \quad \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$\delta z = \delta y \quad \to \quad \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta y \to 0} \left(-i \frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \right) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$\therefore$ **f' exists** $\quad \to \quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ **&** $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ **Cauchy- Reimann Conditions**

$z = x + iy$ $\quad\quad$ $f(z) = u(z) + i\,v(z)$

f' **exists** $\rightarrow$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ **&** $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ **Cauchy- Reimann Conditions**

$$f(z) = f(x, y) \quad \rightarrow \quad \delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

If the CRCs are satisfied, $\delta f = \left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right)(\delta x + i\,\delta y)$

$\rightarrow$ $\dfrac{\delta f}{\delta z} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$ **is independent of path of** $\delta z \rightarrow 0$.

i.e., f' exists $\leftrightarrow$ CRCs satisfied.

Analytic Functions

$f(z)$ is analytic in $R \subseteq C$ $\leftrightarrow$ f' exists & single-valued in R.

Note: Multi-valued functions can be analytic within each branch.

$f(z)$ is an entire function if it is analytic $\forall\, z \in C \setminus \{\infty\}$.

z_0 is a singular point of $f(z)$ if $f'(z)$ doesn't exist at $z = z_0$.

Example z^2 is Analytic

$$\boxed{z = x + i\,y}$$

$$f(z) = z^2 = x^2 - y^2 + 2i\,xy = u + i\,v$$

$$\rightarrow \quad \begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \end{aligned} \quad \rightarrow \quad \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

$\therefore$ ***f*′ exists & single-valued $\forall$ finite *z*.**

i.e., z^2 is an entire function.

$$z = x + i\,y$$

Example z^* is Not Analytic

$$f(z) = z^* = x - i\,y \;= u + i\,v$$

$$\rightarrow \quad \begin{aligned} u &= x \\ v &= -y \end{aligned} \quad \rightarrow \quad \frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$$

$\therefore f'$ **doesn't exist** $\forall z$, **even though it is continuous every where.**

i.e., z^2 **is nowhere analytic.**

Harmonic Functions

CRCs $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \qquad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

By definition, derivatives of a real function *f* depend only on the local behavior of *f*.

But derivatives of a complex function *f* depend on the global behavior of *f*.

Let $\psi(z) = u + iv$

ψ **is analytic** $\rightarrow$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \qquad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\rightarrow$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

i.e., The real & imaginary parts of ψ must each satisfy a 2-D Laplace equation.

(*u* & *v* are harmonic functions)

CRCs $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Contours of *u* & *v* are given by $u(x,y) = c \qquad v(x,y) = c'$

$$\rightarrow \qquad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \qquad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$

Thus, the slopes at each point of these contours are

$$m_u = \left(\frac{dy}{dx}\right)_u = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \qquad m_v = \left(\frac{dy}{dx}\right)_v = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

CRCs $\rightarrow$ $m_u m_v = -1$

at the intersections of these 2 sets of contours

i.e., these 2 sets of contours are orthogonal to each other.
(*u* & *v* are complementary)

Derivatives of Analytic Functions

$\boxed{z = x + iy}$

Let *f* (*z*) be analytic around *z*, then

$$\frac{d\,f(x)}{dx} = g(x) \quad \to \quad \frac{d\,f(z)}{dz} = g(z)$$

Proof

:

***f* (*z*) analytic** $\to$

$$f'(z) = \frac{\partial f(x+iy)}{\partial x} = \left.\frac{d\,f(x)}{dx}\right|_{x=z} = g(z)$$

E.g. $\dfrac{d\,x^n}{dx} = n\,x^{n-1} \quad \to \quad \dfrac{d\,z^n}{dz} = n\,z^{n-1}$

$\therefore$ **Analytic functions can be defined by Taylor series of the same coefficients as their real counterparts.**

Example
Derivative of Logarithm

CRCs $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1} \dfrac{y}{x}$

$$\frac{d \ln z}{dz} = \frac{1}{z}$$ **for z within each branch.**

Proof : $\ln z = \ln r + i\left(\theta + 2\pi n\right) = u + iv \quad \rightarrow \quad u = \ln r, \quad v = \theta + 2\pi n$

$$\frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{x}{r^2} = \frac{\partial v}{\partial y} \qquad \frac{\partial \theta}{\partial y} = \left(1 + \frac{y^2}{x^2}\right)^{-1} \frac{1}{x} = \frac{x}{r^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{r}\frac{\partial r}{\partial y} = \frac{y}{r^2} = -\frac{\partial v}{\partial x} \qquad \frac{\partial \theta}{\partial x} = \left(1 + \frac{y^2}{x^2}\right)^{-1} \left(-\frac{y}{x^2}\right) = -\frac{y}{r^2}$$

$\rightarrow$ **ln z is analytic within each branch.**

$$\therefore \quad \frac{d \ln z}{dz} = \frac{\partial \ln z}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{r^2} - i\frac{y}{r^2} = \frac{1}{x + iy} = \frac{1}{z}$$ **QED** $r^2 = z z^*$

Cauchy's Integral Theorem

Contour = curve in *z*-plane

$$C: \quad z(t) = x(t) + i\, y(t)$$

$$f(z) = u(z) + i\, v(z)$$

Contour integrals :

$$\int_C dz\ f(z) = \int_{t_0}^{t_1} dt \left[x(t) + i\, y(t) \right] \left[u(t) + i\, v(t) \right]$$

$$= \int_{t_0}^{t_1} dt \left[xu - yv + i\left(xv + yu \right) \right]$$

The *t* -integrals are just Reimann integrals

A contour is closed if $z(t_0) = z(t_1)$

Closed contour integral : $\oint_C dz\ f(z)$

(positive sense = counter-clockwise)

Statement of Theorem

A region is simply connected if every closed curve in it can be shrunk continuously to a point.

Let C be a closed contour inside a simply connected region $R \subseteq \mathbb{C}$.

If $f(z)$ is analytic in R, then

$$\oint_C dz\, f(z) = 0$$

Cauchy's integral theorem

Example z^n on Circular Contour

$z = r\,e^{i\theta} \quad \rightarrow \quad dz = i\,r\,e^{i\theta}\,d\theta$ **on a circular contour**

$$\therefore \quad \oint_C dz\, z^n = i\, r^{n+1} \int_0^{2\pi} d\theta\, e^{i(n+1)\theta}$$

$$= \begin{cases} \dfrac{r^{n+1}}{n+1} e^{i(n+1)\theta}\Big|_0^{2\pi} & n = \text{integers} \;\&\; n \neq -1 \\ i \displaystyle\int_0^{2\pi} d\theta & n = -1 \end{cases}$$

$$= \begin{cases} 0 & n = \text{integers} \;\&\; n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

Example z^n on Square Contour

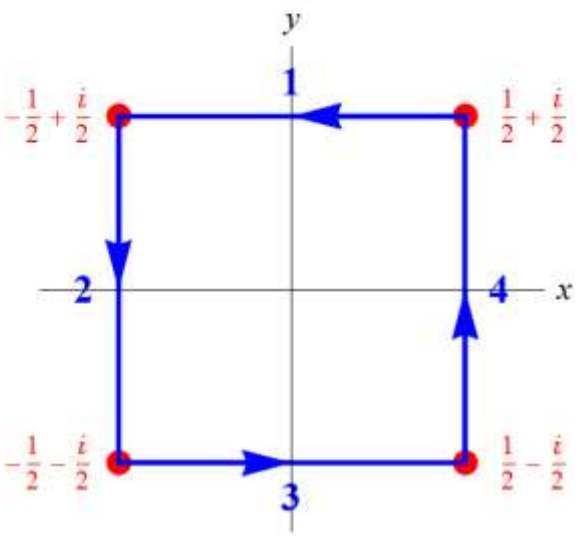
Contour integral

from $z = z_0$ to $z = z_1$ along a straight line:

$$z(t) = z_0 + (z_1 - z_0)\, t \qquad t:\ 0 \to 1$$

$$\to \qquad dz = (z_1 - z_0)\, dt$$

$$\int_{z_0}^{z_1} dz\, f(z) = \int_0^1 dt\, (z_1 - z_0)\, f\left[z_0 + (z_1 - z_0)\, t\right]$$



Mathematica

For $n = -1$, each line segment integrates to $i\pi/2$.

For other integer n, the segments cancel out in pairs.

$$\to \qquad \oint_C dz\, z^n = \begin{cases} 2\pi i & n = -1 \\ 0 & n = \text{integers} \ \&\ n \neq -1 \end{cases}$$

Proof of Theorem

$$\oint_C dz\, f(z) = 0$$

$$z = x + iy \qquad f = u + iv \quad \rightarrow \quad \oint_C dz\, f(z) = \oint_C (u\,dx - v\,dy) + i \oint_C (v\,dx + u\,dy)$$

Stokes theorem : $\oint_{\partial S} d\mathbf{r} \cdot \mathbf{V} = \int_S d\boldsymbol{\sigma} \cdot \nabla \times \mathbf{V}$ ***S* simply-connected**

For *S* in *x*-*y* plane : $\oint_{\partial S} \left(V_x\,dx + V_y\,dy \right) = \int_S dx\,dy \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$

$$\rightarrow \quad \oint_C dz\, f(z) = \int_S dx\,dy \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + i \int_S dx\,dy \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)$$

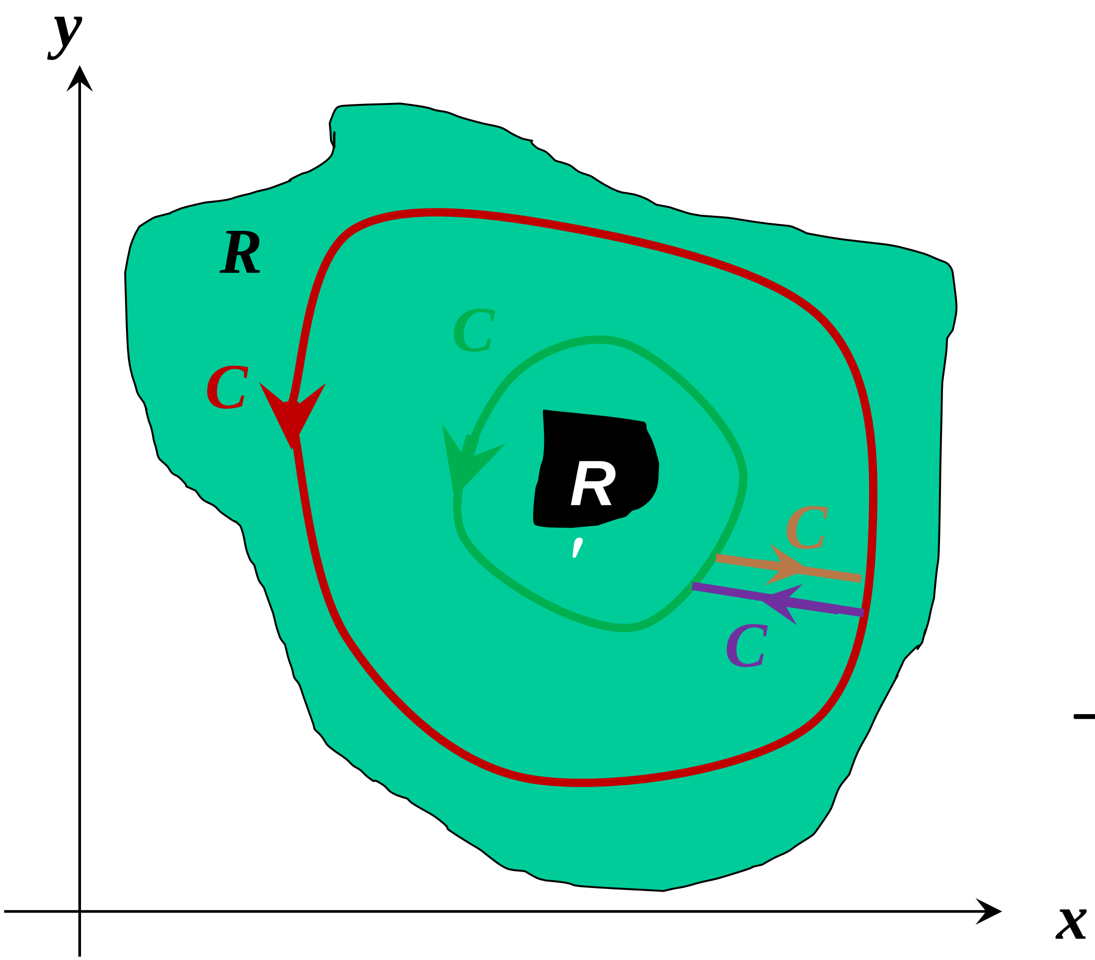
$$= 0$$ **QED**

***f* analytic in *S* → CRCs**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Note: The above (Cauchy's) proof requires $\partial_x u$, etc, be continuous.

Multiply Connected Regions



$$\left[\int_C + \int_C - \int_C + \int_C\right] dz\ f(z) = 0$$

$$\left[\int_C + \int_C\right] dz\ f(z) = 0$$

$$\rightarrow \qquad \int_C dz\ f(z) = \int_C dz\ f(z)$$

Value of integral is unchanged for any continuous deformation of *C* inside a region in which *f* is analytic.

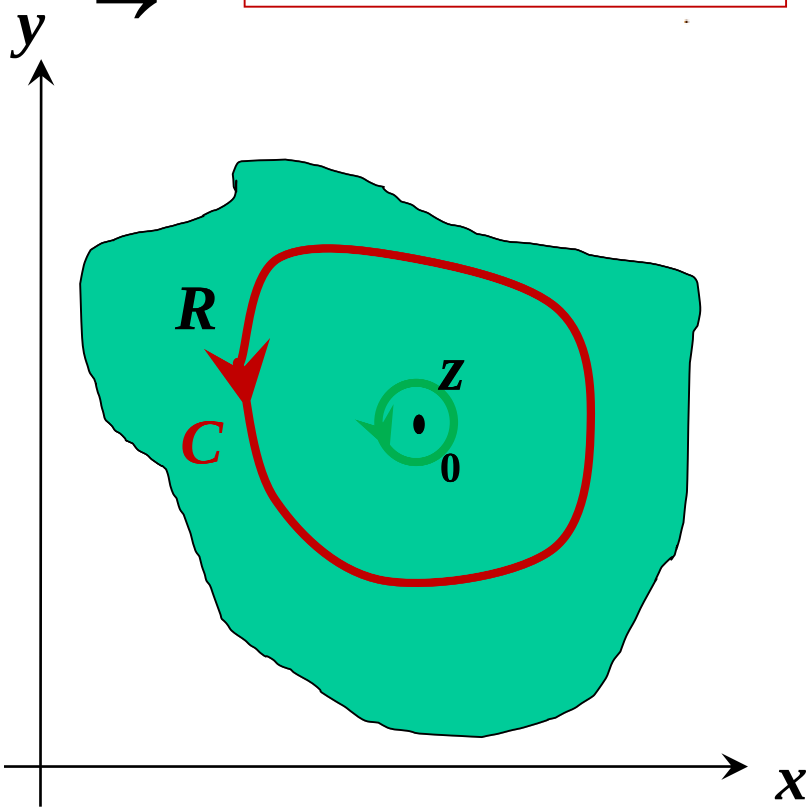
Cauchy's Integral Formula

Let *f* be analytic in *R* & $C \subset R$.

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0}}$$

$\forall$ **z_0 inside region bounded by *C*.**

$\rightarrow$ **Cauchy's Integral Formula**



$$\oint_C dz \, \frac{f(z)}{z - z_0} = \oint_C dz \, \frac{f(z)}{z - z_0}$$

On *C* : $z - z_0 = r\, e^{i\theta}$ $\qquad dz = i\, r\, e^{i\theta}\, d\theta$

$$\lim_{r \to 0} \oint_C dz \, \frac{f(z)}{z - z_0} = \int_0^{2\pi} d\theta \, i\, r\, e^{i\theta} \frac{f(z_0)}{r\, e^{i\theta}}$$

$$= 2\pi i\, f(z_0)$$

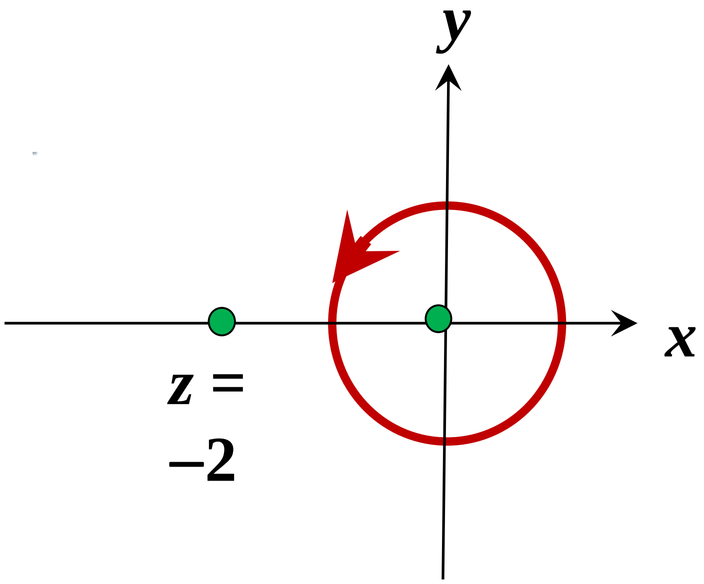
QED

Example An Integral

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0}$$

$$I = \oint_C dz \, \frac{1}{z(z+2)}$$

C = CCW over unit circle centered at origin.



$(z+2)^{-1}$ is analytic inside C.

$$\rightarrow \qquad I = 2\pi i \left. \frac{1}{z+2} \right|_{z=0} = \pi i$$

Alternatively

$$\frac{1}{z(z+2)} = \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z+2} \right)$$

$$\rightarrow \qquad I = \frac{1}{2}\left[\oint_C dz \, \frac{1}{z} - \oint_C dz \, \frac{1}{z+2} \right] = \frac{1}{2}(2\pi i - 0) = \pi i$$

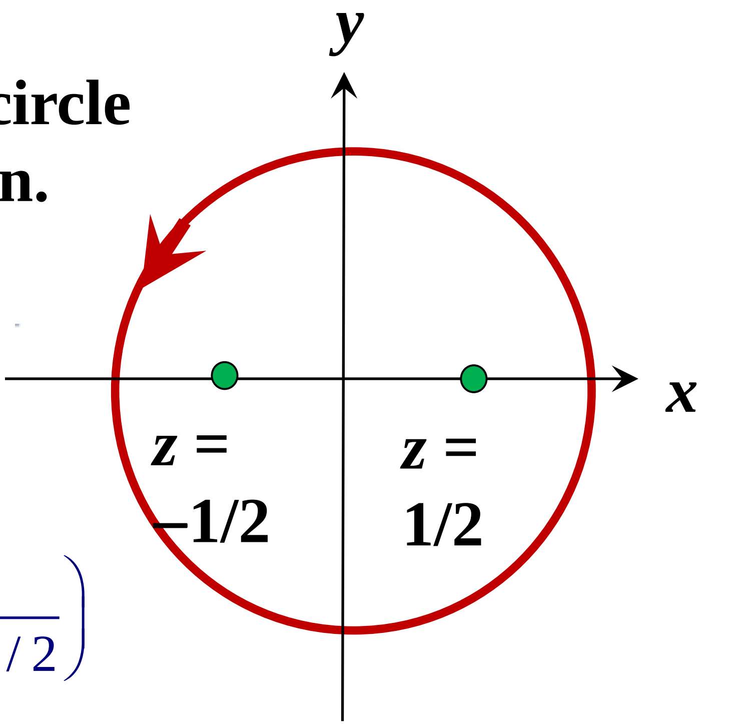
Example

Integral with 2 Singular Factors

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0}$$

$$I = \oint_C dz \, \frac{1}{4z^2 - 1}$$

C = CCW over unit circle centered at origin.



$$\frac{1}{4z^2 - 1} = \frac{1}{(2z+1)(2z-1)}$$

$$= \frac{1}{2}\left(\frac{1}{2z-1} - \frac{1}{2z+1} \right) = \frac{1}{4}\left(\frac{1}{z-1/2} - \frac{1}{z+1/2} \right)$$

$$\therefore \qquad I = 2\pi i \, \frac{1}{4}(1-1) = 0$$

Derivatives

$$f'(z_0) = \frac{1}{2\pi i}\oint_C dz\,\frac{\partial}{\partial z_0}\left[\frac{f(z)}{z-z_0}\right] \quad = \frac{1}{2\pi i}\oint_C dz\,\frac{f(z)}{(z-z_0)^2}$$

$$f''(z_0) = \frac{2}{2\pi i}\oint_C dz\,\frac{f(z)}{(z-z_0)^3}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C dz\,\frac{f(z)}{(z-z_0)^{n+1}}$$

$$\boxed{f(z_0) = \frac{1}{2\pi i}\oint_C dz\,\frac{f(z)}{z-z_0}}$$

f **analytic in** $\boldsymbol{R \supset C}$**.**

$\boldsymbol{f^{(n)}}$ **analytic inside** ***C*****.**

Example

Use of Derivative Formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C dz\, \frac{f(z)}{(z - z_0)^{n+1}}$$

f analytic in $R \supset C$.

$$I = \oint_C dz\, \frac{\sin^2 z}{(z-a)^4}$$

C = CCW over circle centered at a.

Let $f(z) = \sin^2 z$

$\rightarrow$

$$f^{(3)}(a) = \frac{3!}{2\pi i} \oint_C dz\, \frac{\sin^2 z}{(z-a)^4}$$

$$f'(z) = 2\sin z \cos z$$

$$f''(z) = 2\left(\cos^2 z - \sin^2 z\right)$$

$\rightarrow$ $$I = \frac{2\pi i}{3!} f^{(3)}(a) = -\frac{8\pi i}{3} \sin a \cos a$$

$$f^{(3)}(z) = -8 \sin z \cos z$$

Laurent Expansion

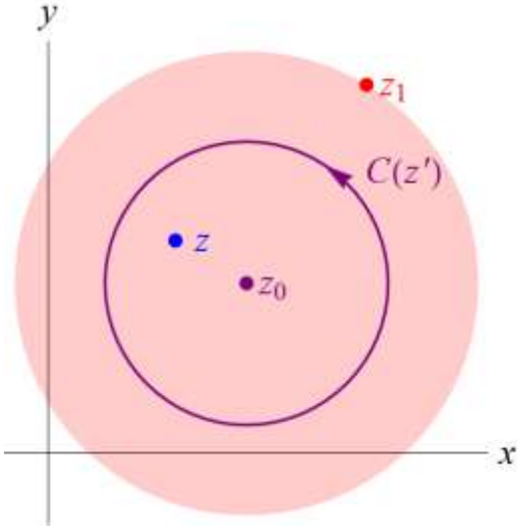
$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C dz\, \frac{f(z)}{(z-z_0)^{n+1}}$$

$$f(z_0) = \frac{1}{2\pi i}\oint_C dz\, \frac{f(z)}{z-z_0}$$

$$f(z) = \frac{1}{2\pi i}\oint_C dz'\, \frac{f(z')}{z'-z}$$

$$\frac{1}{z'-z} = \frac{1}{z'-z_0-(z-z_0)} = \frac{1}{z'-z_0}\left(1-\frac{z-z_0}{z'-z_0}\right)^{-1} = \frac{1}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n$$

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_C dz'\, \frac{f(z')}{(z'-z)^{n+1}}$$



$$\to f(z) = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{n!} f^{(n)}(z_0)$$ **Taylor series**

(f analytic in $R \supset C$)

Let z_1 be the closest singularity from z_0 , then the radius of convergence is $|z_1 - z_0|$.i.e., series converges for

$$|z-z_0| < |z_1 - z_0|$$

Laurent Series

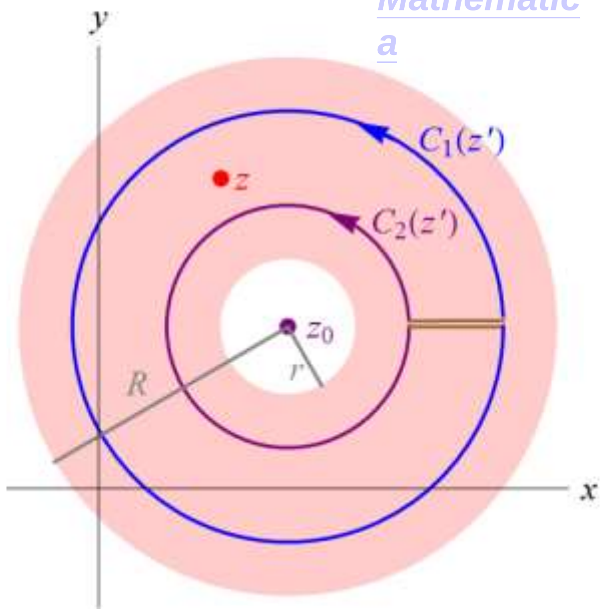
$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C dz \, \frac{f(z)}{(z - z_0)^{n+1}}$$

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0}$$

Let f be analytic within an annular region $r \le |z - z_0| \le R$

$$\rightarrow \qquad f(z) = \frac{1}{2\pi i}\left[\oint_{C_1} - \oint_{C_2}\right] dz' \, \frac{f(z')}{z' - z}$$

Mathematica



$$\mathbf{C_1}: \quad \frac{1}{z' - z} = \frac{1}{z' - z_0 - (z - z_0)} = \frac{1}{z' - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0}\right)^n$$

$$\mathbf{C_2}: \quad \frac{1}{z' - z} = -\frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{z' - z_0}{z - z_0}\right)^n$$

$\rightarrow$

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} dz' \, \frac{f(z')}{(z' - z_0)^{n+1}} + \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{(z - z_0)^{n+1}} \oint_{C_2} dz' \, (z' - z_0)^n f(z')$$

$$f(z)=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_{C_1} dz'\,\frac{f(z')}{(z'-z_0)^{n+1}}+\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{(z-z_0)^{n+1}}\oint_{C_2} dz'\,(z'-z_0)^n f(z')$$

$$\sum_{n=0}^{\infty}\frac{1}{(z-z_0)^{n+1}}\oint_{C_2} dz'\,(z'-z_0)^n f(z')=\sum_{n=-\infty}^{-1}(z-z_0)^n \oint_{C_2} dz'\,\frac{f(z')}{(z'-z_0)^{n+1}}$$

$\rightarrow$

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$$

Laurent series

$$a_n=\frac{1}{2\pi i}\oint_C dz'\,\frac{f(z')}{(z'-z_0)^{n+1}}$$

C **within *f*'s region of analyticity**

Example

Laurent Expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{(z' - z_0)^{n+1}}$$

$$f(z) = \frac{1}{z(z-1)}$$

Consider expansion about $z_0 = 0$ $\rightarrow$ ***f*** **is analytic for** $0 < |z| < 1$

Expansion via binomial theorem :

$$f(z) = -\left(\frac{1}{z} + \frac{1}{1-z} \right) = -\left(\frac{1}{z} + \sum_{n=0}^{\infty} z^n \right)$$

Laurent series :

$$a_n = \frac{1}{2\pi i} \oint_C dz \frac{1}{z^{n+1}\, z(z-1)} = -\frac{1}{2\pi i} \sum_{k=0}^{\infty} \oint_C dz \frac{z^k}{z^{n+2}} = \begin{cases} -1 & n \geq -1 \\ 0 & otherwise \end{cases}$$

$$\rightarrow \qquad f(z) = -\sum_{n=-1}^{\infty} z^n$$

Calculus of Residues

$$\oint_C dz\,(z-z_0)^n = 2\pi i\,\delta_{n,-1}$$

$(C$ circles z_0 once$)$

Residue Theorem :

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n} \quad \to \quad \oint_C dz\, f(z) = 2\pi i\, a_{-1}$$

a_{-1} = **residue of f at z_0 .**

For a meromorphic function f with n isolated singularities, one can deform C to circumvent them all so that

$$\oint_C dz\, f(z) - \sum_{i=1}^{n} \oint_{C_i} dz\, f(z) = 0$$

$$\to \quad \boxed{\oint_C dz\, f(z) = 2\pi i \sum_{i=1}^{n} a_{-1,i}}$$

Residue Theorem

$a_{-1,i}$ = residue at singularity i.

Computing Residues

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

If f has a simple pole at z_0 , $f(z) = \sum_{n=-1}^{\infty} a_n (z - z_0)^n$

$$\rightarrow \quad (z - z_0) f(z) = a_{-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+1}$$

$$\therefore \quad \boxed{a_{-1} = \lim_{z \to z_0} (z - z_0) f(z)}$$

simple poles

If f has a pole of order m at z_0 , $f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n$

$$\rightarrow \quad (z - z_0)^m f(z) = \sum_{k=0}^{m-2} a_{-(m-k)} (z - z_0)^k + a_{-1} (z - z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m}$$

$$\therefore \quad \boxed{a_{-1} = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{d z^{m-1}} \left[(z - z_0)^m f(z) \right]}$$

poles of order m

Example Computing Residues

Some examples :

$$\mathrm{Res}\left(\frac{1}{4z+1}\,;\,z=-\frac{1}{4}\right)=\lim_{z\to -1/4}\left(z+\frac{1}{4}\right)\frac{1}{4z+1}=\frac{1}{4}$$

$$\mathrm{Res}\left(\frac{1}{\sin z}\,;\,z=0\right)=\lim_{z\to 0} z\,\frac{1}{\sin z}=1$$

$$\mathrm{Res}\left(\frac{\ln z}{z^2+4}\,;\,z=2e^{i\pi/2}\right)=\lim_{z\to 2e^{i\pi/2}}(z-2i)\,\frac{\ln z}{z^2+4}=\lim_{z\to 2e^{i\pi/2}}\frac{\ln z}{z+2i}=\frac{\ln 2+i\pi/2}{4i}$$

$$\mathrm{Res}\left(\frac{z}{\sin^2 z}\,;\,z=\pi\right)=\lim_{z\to\pi}\frac{d}{dz}\left[(z-\pi)^2\frac{z}{\sin^2 z}\right]$$

$$=\lim_{z\to\pi}\left[(-\pi+z)\big(-\pi+3z+2(\pi-z)\,z\cot z\big)\csc^2 z\right]=1$$

$$\cot x = \frac{1}{x} - \frac{x}{3} + O\left(x^3\right)$$

$$\frac{1}{z+2} = \frac{1}{2}\left(1 - \frac{z}{2} + O\left(z^2\right)\right)$$

$$\operatorname{Res}\left(\frac{\cot \pi z}{z(z+2)} ; z=0\right)$$

$$= a_{-1} \text{ of } \left[\frac{1}{z} \cdot \frac{1}{2}\left(1 - \frac{z}{2} + O\left(z^2\right)\right)\left[\frac{1}{\pi z} - \frac{\pi z}{3} + O\left(z^3\right)\right]\right] \quad = -\frac{1}{4\pi}$$

Alternatively, $z = 0$ is a pole of 2nd order :

$$\operatorname{Res}\left(\frac{\cot \pi z}{z(z+2)} ; z=0\right) = \lim_{z \to 0} \frac{d}{dz}\left(z^2 \frac{\cot \pi z}{z(z+2)}\right)$$

$$= \lim_{z \to 0}\left[-\frac{z \cot \pi z}{(2+z)^2} + \frac{\cot \pi z}{2+z} - \frac{\pi z \csc^2 \pi z}{2+z}\right] \quad = -\frac{1}{4\pi}$$

$$\operatorname{Res}\left(e^{-1/z} ; z=0\right) = a_{-1} \text{ of } \left[1 - \frac{1}{z} + O\left(z^{-2}\right)\right] \quad = -1$$