EULER METHOD

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Basic Idea of Numerical Integration

The basic idea of any numerical method of solving the initial value problem is the following. For simplicity consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = f(x,t). \tag{1}$$

Suppose it is desired to find the solution of Eq.(1) for $t \in [1, b]$ with $x(a) = x_0$.. The interval $[a, b]$ is divided into m subintervals by the points $t_0 = a < t_1 = a + h < t_2 = a + 2h < \cdots < t_m = b$ where $h = (b - a)/m$. We can write $t_n = a + nh$, $n = 0, 1, \cdots, m$. The points t_n are called grid points and h is called a step-size. The exact solution $x(t_n)$ is approximated by a number and is denoted as x_n. The sequency $x_0, x_1, \cdots .x_m$ is called a numerical solution.
In general, the numerical solution x_{n+1} at time t_{n+1} is obtained from the formula

$$x_{n+1} = x_n + h\phi(x_n, x_{n-1}, \cdots, x_{n-k}, t_n, t_{n-1}, \cdots, t_{n-k}). \tag{2}$$

When $k = 0$

$$x_{n+1} = x_n + h\phi(x_n, t_n). \tag{3}$$

The value of x_{n+1} is calculated using (x_n, t_n) only. In this case the method is called a single-step method. When $k > 0$ the method is a multi-step method. If the error in a method is (h^{N+1}) then it is called Nth order method.

Euler Method for First-Order Differential Equations

Some of the single-step methods are Euler method and Runge–Kutta method. Consider the first-order equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = f(x,t). \tag{4}$$

Assume that the value of $x(t)$ at $t = t_0$ is given and is denoted as x_0. Suppose $f(x,t)$ remains same during a time interval t_0 to $t_1 = t_0 + h$. Then integration of Eq.(4) from t_0 to t_1 gives

$$x(t)\big|_{t_0}^{t_1} = f(x_0,t_0)\int_{t_0}^{t_1} \mathrm{d}t.$$

That is,

$$x(t_1) - x(t_0) = f(x_0,t_0)(t_1 - t_0).$$

With $h = t_1 - t_0$ we can rewrite the above equation as

$$x_1 = x_0 + hf(x_0,t_0), \tag{5a}$$

$$t_1 = t_0 + h. \tag{5b}$$

Euler Method for First-Order Differential Equations

If $f(x,t)$ changes with time then

$$\begin{aligned} x_1 &= x_0 + hf(x_0,t_0) + \frac{1}{2}h^2 f'(c), \quad c \in [t_0,t_1] \\ &= x_0 + hf(x_0,t_0) + \frac{1}{2}h^2 x''(c). \end{aligned} \tag{6}$$

In general

$$x_{n+1} = x_n + hf(x_n,t_n), \tag{7a}$$

$$t_{n+1} = t_n + h, \quad n = 0,1,\cdots . \tag{7b}$$

The above algorithm of determining $x(t)$ from $x(t_0)$ is called Euler method.

For the equation $\mathrm{d}y/\mathrm{d}x = f(y,x)$ introduce the change of variables $x \to t$ and $y \to x$ which gives $\mathrm{d}x/\mathrm{d}t = f(x,t)$ for which the Euler formula is given by Eqs.(7).

Euler Method for Second-Order Differential Equations

For a second-order equation of the form

$$\begin{aligned} \dot{x} &= f(x, y, t), \quad &(8a) \\ \dot{y} &= g(x, y, t) \quad &(8b) \end{aligned}$$

the Euler formula is

$$\begin{aligned} x_{n+1} &= x_n + hf(x_n, y_n, t_n), \quad &(9a) \\ y_{n+1} &= y_n + hg(x_n, y_n, t_n), \quad &(9b) \\ t_{n+1} &= t_n + h, \quad n = 0, 1, \cdots . \quad &(9c) \end{aligned}$$

For the equation

$$\ddot{x} + G(x, \dot{x}, t) = 0 \quad (10)$$

we write it as

$$\begin{aligned} \dot{x} &= y = f(x, y, t), \quad &(11a) \\ \dot{y} &= \ddot{x} = -G(x, y, t) = g(x, y, t). \quad &(11b) \end{aligned}$$

Then the corresponding Euler formula is given by Eqs.(9).

Error Analysis

The error in the Euler formula in obtaining the solution from t_n to t_{n+1} refering to Eq.(6)is

$$E_n = \frac{1}{2}h^2 x''(c_n). \tag{12}$$

This is the local truncation error. The global error in obtining the solution from $t_0 = a$ to $t_m = b$ is

$$\begin{aligned} E &= \sum_{n=1}^{m} E_n \\ &= \frac{1}{2}h^2 \sum_{n=1}^{m} x''(c_n) \\ &= \frac{1}{2}h\frac{(b-a)}{m} \sum_{n=1}^{m} x''(c_n) \\ &= \frac{1}{2}h\frac{(b-a)}{m} m x''(c) \\ &= \frac{1}{2}h(b-a)x''(c) = O(h). \end{aligned} \tag{13}$$

Examples

Consider the first-order equation $\dot{x} = -x + \sin t$, $x(0) = 1$. Calculate $x(0.1)$ and $x(0.2)$ by the Euler method with step size 0.1.

For the equation $\dot{x} = f(x,t)$ the Euler formula is

$$x_{n+1} = x_n + hf(x_n, t_n), \tag{14a}$$

$$t_{n+1} = t_n + h. \tag{14b}$$

Given: $f(x,t) = -x + \sin t, \quad t_n = 0, \quad x_n = x(0) = 1, \quad h = 0.1$

$t_{n+1} = t_n + h = 0.1, \ \ x_{n+1} = ?$

We find:

$$x(0.1) = x_{n+1} = 1 + 0.1 \times f(x_n = 1, t_n = 0) = 1 + 0.1 \times -1 = 0.9.$$

Next, determination of $x(0.2)$ using $x(0.1)$.

Given: $f(x,t) = -x + \sin t, \quad t_n = 0.1, \quad x_n = x(0.1) = 0.9, \quad h = 0.1$

$t_{n+1} = t_n + h = 0.2, \ \ x_{n+1} = ?$

Example

We find:

$$\begin{aligned} x(0.2) &= x_{n+1} \\ &= 0.9 + 0.1 \times f(x_n = 0.9, t_n = 0.1) \\ &= 0.9 + 0.1 \times -0.8001665 \\ &= 0.8199833. \end{aligned}$$