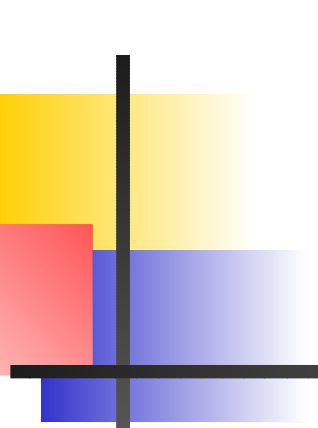




EULER METHOD

...

S. Rajasekar

School of Physics

Bharathidasan University

Tiruchirapalli - 620 024

email: rajasekar@cnld.bdu.ac.in
srj.bdu@gmail.com

Basic Idea of Numerical Integration

The basic idea of any numerical method of solving the initial value problem is the following. For simplicity consider

$$\frac{dx}{dt} = \dot{x} = f(x, t). \quad (1)$$

Suppose it is desired to find the solution of Eq.(1) for $t \in [a, b]$ with $x(a) = x_0$. The interval $[a, b]$ is divided into m subintervals by the points $t_0 = a < t_1 = a + h < t_2 = a + 2h < \dots < t_m = b$ where $h = (b - a)/m$. We can write $t_n = a + nh$, $n = 0, 1, \dots, m$. The points t_n are called **grid points** and h is called a **step-size**. The exact solution $x(t_n)$ is approximated by a number and is denoted as x_n . The sequence $x_0, x_1, \dots, x_m$ is called a **numerical solution**.

In general, the numerical solution x_{n+1} at time t_{n+1} is obtained from the formula

$$x_{n+1} = x_n + h\phi(x_n, x_{n-1}, \dots, x_{n-k}, t_n, t_{n-1}, \dots, t_{n-k}). \quad (2)$$

When $k = 0$

$$x_{n+1} = x_n + h\phi(x_n, t_n). \quad (3)$$

The value of x_{n+1} is calculated using (x_n, t_n) only. In this case the method is called a **single-step method**. When $k > 0$ the method is a **multi-step method**. If the error in a method is (h^{N+1}) then it is called **Nth order method**.

Euler Method for First-Order Differential Equations

Some of the single-step methods are Euler method and Runge–Kutta method. Consider the first-order equation of the form

$$\frac{dx}{dt} = \dot{x} = f(x, t). \quad (4)$$

Assume that the value of $x(t)$ at $t = t_0$ is given and is denoted as x_0 . Suppose $f(x, t)$ remains same during a time interval t_0 to $t_1 = t_0 + h$. Then integration of Eq.(4) from t_0 to t_1 gives

$$x(t)|_{t_0}^{t_1} = f(x_0, t_0) \int_{t_0}^{t_1} dt.$$

That is,

$$x(t_1) - x(t_0) = f(x_0, t_0)(t_1 - t_0).$$

With $h = t_1 - t_0$ we can rewrite the above equation as

$$x_1 = x_0 + hf(x_0, t_0), \quad (5a)$$

$$t_1 = t_0 + h. \quad (5b)$$

Euler Method for First-Order Differential Equations

If $f(x, t)$ changes with time then

$$\begin{aligned}x_1 &= x_0 + hf(x_0, t_0) + \frac{1}{2}h^2 f'(c), \quad c \in [t_0, t_1] \\&= x_0 + hf(x_0, t_0) + \frac{1}{2}h^2 x''(c).\end{aligned}\tag{6}$$

In general

$$x_{n+1} = x_n + hf(x_n, t_n),\tag{7a}$$

$$t_{n+1} = t_n + h, \quad n = 0, 1, \dots.\tag{7b}$$

The above algorithm of determining $x(t)$ from $x(t_0)$ is called **Euler method**.

For the equation $dy/dx = f(y, x)$ introduce the change of variables $x \rightarrow t$ and $y \rightarrow x$ which gives $dx/dt = f(x, t)$ for which the Euler formula is given by Eqs.(7).

Euler Method for Second-Order Differential Equations

For a second-order equation of the form

$$\dot{x} = f(x, y, t), \quad (8a)$$

$$\dot{y} = g(x, y, t) \quad (8b)$$

the Euler formula is

$$x_{n+1} = x_n + hf(x_n, y_n, t_n), \quad (9a)$$

$$y_{n+1} = y_n + hg(x_n, y_n, t_n), \quad (9b)$$

$$t_{n+1} = t_n + h, \quad n = 0, 1, \dots. \quad (9c)$$

For the equation

$$\ddot{x} + G(x, \dot{x}, t) = 0 \quad (10)$$

we write it as

$$\dot{x} = y = f(x, y, t), \quad (11a)$$

$$\dot{y} = \ddot{x} = -G(x, y, t) = g(x, y, t). \quad (11b)$$

Then the corresponding Euler formula is given by Eqs.(9).

Error Analysis

The error in the Euler formula in obtaining the solution from t_n to t_{n+1} referring to Eq.(6) is

$$E_n = \frac{1}{2}h^2 x''(c_n). \quad (12)$$

This is the local truncation error. The global error in obtaining the solution from $t_0 = a$ to $t_m = b$ is

$$\begin{aligned} E &= \sum_{n=1}^m E_n \\ &= \frac{1}{2}h^2 \sum_{n=1}^m x''(c_n) \\ &= \frac{1}{2}h \frac{(b-a)}{m} \sum_{n=1}^m x''(c_n) \\ &= \frac{1}{2}h \frac{(b-a)}{m} m x''(c) \\ &= \frac{1}{2}h(b-a)x''(c) = O(h). \end{aligned} \quad (13)$$

Examples

Consider the first-order equation $\dot{x} = -x + \sin t$, $x(0) = 1$. Calculate $x(0.1)$ and $x(0.2)$ by the Euler method with step size 0.1.

For the equation $\dot{x} = f(x, t)$ the Euler formula is

$$x_{n+1} = x_n + hf(x_n, t_n), \quad (14a)$$

$$t_{n+1} = t_n + h. \quad (14b)$$

Given: $f(x, t) = -x + \sin t$, $t_n = 0$, $x_n = x(0) = 1$, $h = 0.1$

$$t_{n+1} = t_n + h = 0.1, \quad x_{n+1} = ?$$

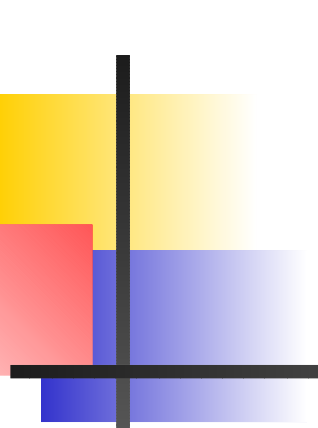
We find:

$$x(0.1) = x_{n+1} = 1 + 0.1 \times f(x_n = 1, t_n = 0) = 1 + 0.1 \times -1 = 0.9.$$

Next, determination of $x(0.2)$ using $x(0.1)$.

Given: $f(x, t) = -x + \sin t$, $t_n = 0.1$, $x_n = x(0.1) = 0.9$, $h = 0.1$

$$t_{n+1} = t_n + h = 0.2, \quad x_{n+1} = ?$$



Example

We find:

$$\begin{aligned}x(0.2) &= x_{n+1} \\&= 0.9 + 0.1 \times f(x_n = 0.9, t_n = 0.1) \\&= 0.9 + 0.1 \times -0.8001665 \\&= 0.8199833.\end{aligned}$$