Classical Mechanics: Rigid body dynamics

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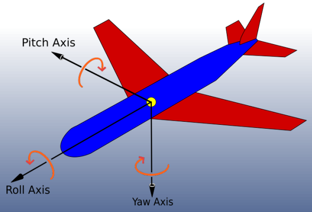
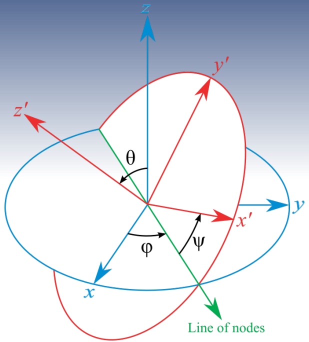


Euler (or) Eulerian angles

- It is necessary to seek three independent parameters **(or generalized coordinates)** that specify the orientation of a rigid body in such a manner that the corresponding orthogonal matrix of transformation has the determinant $+1$.
- The most common and useful parameters are the **Euler (or) Eulerian angles**



- Transformation from a given cartesian coordinate system to another by means of three successive rotations performed in a **specific sequence**.
- The Euler angles are defined as the three successive angles of rotation.

Euler (or) Eulerian angles

(1) Rotate the initial system of axes xyz, by an angle ϕ counterclockwise about z-axis — The resultant coordinate system is labelled the $\xi\eta\zeta$ axes (an intermediate coordinate set).

(2) The intermediate axes, $\xi\eta\zeta$, are rotated about the ξ axis counterclockwise by an angle θ to produce another intermediate coordinate set, $\xi'\eta'\zeta'$ axes — The ξ' axis is at the intersection of xy and $\xi'\eta'$ planes and is known as **line of nodes**.

(3) Finally the $\xi'\eta'\zeta'$ axes are rotated counterclockwise by an angle ψ about the ζ' axis to produce the desired $x'y'z'$ system of axes.

The Euler angles ϕ, θ and ψ completely specify the orientation of the $x'y'z'$ system relative to xyz — the generalized coordinates.

Euler (or) Eulerian angles

- The elements of complete transformation $\boldsymbol{A}$ – obtained as triple product of separate rotations. Initial rotation about z axis

$$\boldsymbol{\xi} = \boldsymbol{D}\boldsymbol{x}$$

- Transformation from $\xi\eta\zeta$ to $\xi'\eta'\zeta'$ — described by a matrix $\boldsymbol{C}$

$$\boldsymbol{\xi}' = \boldsymbol{C}\boldsymbol{\xi}$$

- Finally,

$$\boldsymbol{x}' = \boldsymbol{B}\boldsymbol{\xi}'$$

$$\boldsymbol{A} = \boldsymbol{BCD}$$

$$\boldsymbol{D} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Euler (or) Eulerian angles

The product $\boldsymbol{A} = \boldsymbol{BCD}$

$$\boldsymbol{A} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi - \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi - \cos\theta\cos\phi\sin\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

The inverse transformation from body coordinates to space axes

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{x}'$$

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi - \cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi - \cos\theta\cos\phi\sin\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}$$

Rate of change of a vector

The change in a time dt of the components of a general vector $\vec{G}$ as seen by an observer in the body system of axes will differ from the corresponding changes as seen by an observer in the space system.

$$\left(d\vec{G}\right)_{\text{space}} = \left(d\vec{G}\right)_{\text{body}} + \left(d\vec{G}\right)_{\text{rot.}}$$

$$\left(d\vec{G}\right)_{\text{rot.}} = d\vec{\Omega} \times \vec{G} \implies \left(d\vec{G}\right)_{\text{space}} = \left(d\vec{G}\right)_{\text{body}} + d\vec{\Omega} \times \vec{G}$$

$$\left(\frac{d\vec{G}}{dt}\right)_{\text{space}} = \left(\frac{d\vec{G}}{dt}\right)_{\text{body}} + \vec{\omega} \times \vec{G}, \quad \text{where } \vec{\omega} dt = d\vec{\Omega}$$

An operator acting on some given vector:

$$\left(\frac{d}{dt}\right)_{s} = \left(\frac{d}{dt}\right)_{r} + \vec{\omega} \times$$

Rigid body equations of motion

- Any general displacement: a **translation** plus a **rotation** (Chasles theorem)
- possible to split the problem into two separate phases:
  - one concerned solely with the translational motion of the body
  - and the other with the rotational motion
- 6 coordinates – **3 Cartesian coordinates** of a point fixed on the rigid body – **3 Euler angles** for the motion about the point
- origin of the coordinate system is chosen to be the center of mass (c.m.)
- similar division holds for the total kinetic energy T,

$$T = \frac{1}{2}Mv^2 + T'(\phi, \theta, \psi)$$

- The potential energy can also be divided into two parts and so the Lagrangian.

Rigid body equations of motion

- $\vec{R}_1$ and $\vec{R}_2$ — position vectors of the origins of two set of body coordinates (relative to a fixed set of coordinates, i.e. the space axes)
- $\vec{R}_2 = \vec{R}_1 + \vec{R}$, where $\vec{R}$ is the difference vector
- The origin of the second set of axes – considered as a point defined relative to the first set – the time derivative of $\vec{R}_2$ is then given by

$$\left(\frac{d\vec{R}_2}{dt}\right)_s = \left(\frac{d\vec{R}_1}{dt}\right)_s + \left(\frac{d\vec{R}}{dt}\right)_s \equiv \left(\frac{d\vec{R}_1}{dt}\right)_s + \vec{\omega}_1 \times \vec{R}$$

- Alternatively, the origin of the first coordinate system – considered as fixed in the second system with the position vector -$\vec{R}$ – the time derivative of $\vec{R}_1$ relative to the fixed space axes

$$\left(\frac{d\vec{R}_1}{dt}\right)_s = \left(\frac{d\vec{R}_2}{dt}\right)_s - \left(\frac{d\vec{R}}{dt}\right)_s \equiv \left(\frac{d\vec{R}_2}{dt}\right)_s - \vec{\omega}_2 \times \vec{R}$$

Rigid body equations of motion

- Comparison of the above two expressions implies $(\vec{\omega}_1 - \vec{\omega}_2) \times \vec{R} = 0$
- This means any difference in the angular velocity vectors at two arbitrary points must lie along the **line joining the two points**
- Assuming $\vec{\omega}$ continuous – only possibility for all pairs of points is that the two angular velocities must be equal, i.e., $\vec{\omega}_1 = \vec{\omega}_2$
- The angular velocity is the same for all coordinate system fixed in the rigid body.

Rigid body equations of motion

- When a rigid body moves with one point staitonary, the total angular momentum about that point: $\vec{L} = m_i \left(\vec{r}_i \times \vec{v}_i\right)$ (summation over i implied) – $\vec{r}_i$ raidus vector and $\vec{v}_i$ is the velocity of the i-th particle.
- Since $\vec{r}_i$ is a fixed vector relative to the body, the velocity $\vec{v}_i$ with respect to the space set of axes arises solely from the rotational motion of the rigid body about the fixed point– i.e., $\vec{v}_i = \vec{\omega} \times \vec{r}_i$
- The total angular momentum

$$\begin{aligned} \vec{L} &= m_i \left[\vec{r}_i \times \left(\vec{\omega} \times \vec{r}_i\right)\right], \\ &= m_i \left[\vec{\omega} r_i^2 - \vec{r}_i \left(\vec{r}_i \cdot \vec{\omega}\right)\right]. \end{aligned}$$

The x component of $\vec{L}$

- Each component of $\vec{L}$ is a linear function of all the components of the angular velocity.
- *The angular momentum vector is related to the angular velocity by a linear transformation*.

$$L_x = \omega_x m_i \left(r_i^2 - x_i^2\right) - \omega_y m_i x_i y_i - \omega_z m_i x_i z_i$$

Rigid body equations of motion

- To emphasize the similarity of L_x with the equations of a linear transformation

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z.$$

Similarly,

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z,$$
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z.$$

- I_{xx}, I_{xy}, etc. are the nine elements of the transformation matrix

$$I_{xx} = m_i\left(r_i^2 - x_i^2\right),$$
$$I_{xy} = -m_i x_i y_i.$$

- For continuous bodies the summation is replaced by a volume integration

$$I_{xx} = \int_V \rho(\vec{r})\left(r^2 - x^2\right) dV,$$

If the coordinate axes are denoted by x_j, $j = 1, 2, 3$ (x, y, z) then the matrix element I_{jk}

$$I_{jk} = \int_V \rho(\vec{r})\left(r^2\delta_{jk} - x_j x_k\right) dV.$$

The relation between $\vec{L}$ and $\vec{\omega}$

$$\vec{L} = \boldsymbol{I}\vec{\omega}.$$

Rigid body equations of motion

- The symbol $\boldsymbol{I}$ stands for the operator whose matrix elements are the inertia coefficients, and $\vec{\omega}$ and $\vec{L}$ are column matrices.
- $\boldsymbol{I}$ an operator acting upon the vector ω and not the coordinate system.
- The vectors $\vec{L}$ and $\vec{\omega}$ are two physically different vectors having different dimensions — i.e., not the merely same vector represented in two coordinate systems.
- Unlike the operator of rotation $\boldsymbol{I}$ will have dimensions (mass times length squared) – and not restricted by any orthogonality condition.
- The operator $\boldsymbol{I}$ acting upon the vector $\vec{\omega}$ results in the physically new vector $\vec{L}$

Tensors

- The quantity $\boldsymbol{I}$ — considered as defining the quotient of $\vec{L}$ and $\vec{\omega}$ for the product $\boldsymbol{I}$ and $\vec{\omega}$ given $\vec{L}$
- The quotient of two quantities often not a member of same class as the dividing factors – but they may belong to a more complicated class.
- For instance, the quotients of two integers is in general not an integer — rather a rational number.
- Similarly the quotient of two vectors cannot be defined consistently within the class of vectors
- In our case, $\boldsymbol{I}$ is a new type of quantity – a tensor of the second rank.
- Tensor of N-th rank (Cartesian coordinate) — defined as a quantity having 3^N components, $T_{ijk\ldots}$

Tensors

- They transform under an orthogonal transformation of coordinates, $\boldsymbol{A}$ according to

$$T'_{ijk\cdots}(X') = a_{il}a_{jm}a_{km}T_{lmn\cdots}$$

- A tensor of the zero rank has one component, which is invariant under orthogonal transformation – a scalar is a tensor of zero rank
- A tensor of the first rank has three components: $T'_i = a_{ij}T_j$.
- A tensor of the second rank has nine components: $T'_{ij} = a_{ik}a_{jl}T_{kl}$
- How to distinguish a second-rank tensor $\boldsymbol{T}$ and the square matrix from its components?
  - A tensor is defined only in terms of its transformation properties under orthogonal transformations.
  - A matrix is in no way restricted in the types of transformations it may undergo — indeed considered as entirely independent of its properties under some particular class of transformations.

The Inertia tensor & moment of inertia

- The quantity $\boldsymbol{I}$ is identified as a second-rank tensor and is usually called the **moment of inertia tensor** or **inertia tensor**.
- The kinetic energy of the motion about a point

$$T = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i)$$

Upon permuting the vectors in the triple do product

$$T = \frac{\vec{\omega}}{2} \cdot m_i (\vec{r}_i \times \vec{v}_i)$$

- $$T = \frac{\vec{\omega} \cdot \vec{L}}{2} = \frac{\vec{\omega} \cdot \boldsymbol{I} \cdot \vec{\omega}}{2}$$

or

$$T = \frac{\omega^2}{2} \hat{n} \cdot \boldsymbol{I} \cdot \hat{n} = \frac{1}{2} I \omega^2, \quad \vec{\omega} = \omega \hat{n}$$

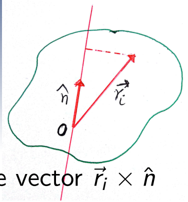
I is a scalar, defined by

$$I = \hat{n} \cdot \boldsymbol{I} \cdot \hat{n} \equiv m_i \left[r_i^2 - (\vec{r}_i \cdot \hat{n})^2 \right],$$

known as the **moment of inertia about the axis of rotation**

The Inertia tensor & moment of inertia

The moment of inertia about an axis is defined as the sum, over the particles of the body, of the product of the particle mass and the square of the perpendicular distance from the axis.



The perpendicular distance is equal to the magnitude of the vector $\vec{r}_i \times \hat{n}$

$$I = m_i \, (\vec{r}_i \times \hat{n}) \cdot (\vec{r}_i \times \hat{n})$$

$$\text{Multiply and divide by } \omega^2 \quad \Longrightarrow \quad I = \frac{m_i}{\omega^2} \, (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i)$$

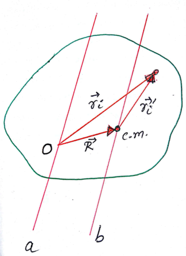
$(\vec{\omega} \times \vec{r}_i)$ is the relative velocity $\vec{v}_i$ measured in the space system of axes

$$I = \frac{2T}{\omega^2}$$

The value of moment of inertia depends upon the direction of the axis of rotation — As $\vec{\omega}$ changes its direction with respect to the body in the course of time, the moment of inertia must also be considered a function of time.

The Inertia tensor & moment of inertia

Let $\vec{R}$ be a vector from the origin 'O' to the center of mass, and let $\vec{r}_i$ and r_i' are the radii vectors from 'O' and the center of mass, respectively, to the i-th particle.



The three vectors are connected by the relation

$$\vec{r}_i = \vec{R} + \vec{r}_i'$$

The moment of inertia about the axis a is

$$I_a = m_i \left(\vec{r}_i \times \hat{n}\right)^2 = m_i \left[\left(\vec{r}_i' + \vec{R}\right) \times \hat{n}\right]^2$$
$$= M\left(\vec{R} \times \hat{n}\right)^2 + m_i \left(\vec{r}_i' \times \hat{n}\right)^2 + 2m_i \left(\vec{R} \times \hat{n}\right) \cdot \left(\vec{r}_i' \times \hat{n}\right)$$
$$= M\left(\vec{R} \times \hat{n}\right)^2 + m_i \left(\vec{r}_i' \times \hat{n}\right)^2 - 2\left(\vec{R} \times \hat{n}\right) \cdot \left(\hat{n} \times m_i \vec{r}_i'\right)$$

$$\boxed{\begin{aligned} I_a &= I_b + M\left(\vec{R} \times \hat{n}\right)^2 \\ &= I_b + MR^2 \sin^2\theta \end{aligned}}$$

by the definition of center of mass $m_i r_i' = 0$

The Inertia tensor & moment of inertia

- The inertia tensor is defined from the kinetic energy of rotation of an axis

$$T_{\text{rotation}} = \frac{1}{2} m_i \left(\vec{\omega} \times \vec{r}_i\right)^2 \equiv \frac{1}{2} \omega_\alpha \omega_\beta m_i \left(\delta_{\alpha\beta} r_i^2 - r_{i\alpha} r_{i\beta}\right).$$

- T_{rotation} is in bilinear form in the components of $\vec{\omega}$: $T_{\text{rotation}} = \frac{1}{2} I_{\alpha\beta} \omega_\alpha \omega_\beta$
- $I_{\alpha\beta} = m_i \left(\delta_{\alpha\beta} r_i^2 - r_{i\alpha} r_{i\beta}\right)$ is the moment of inertia tensor.
- For a rigid body with continuous distribution of density $\rho(\vec{r})$, the sum of the components of the moment of inertia tensor reduces to

$$I_{\alpha\beta} = \int_V \rho(\vec{r}) \left(\delta_{\alpha\beta} r^2 - r_\alpha r_\beta\right) dV.$$

Moment of inertia: example

- Consider a homogeneous cube of density ρ, mass M and side a.
- The origin is chosen to be at one corner and the three edges of adjacent to that corner lie on the $+x$, $+y$, and $+z$ axes.
- This means $\rho = M/a^3$, $r^2 = x^2 + y^2 + z^2$

$$I_{11} = \iiint \frac{M}{a^3}\left(y^2 + z^2\right) dxdydz$$
$$= \frac{M}{a^3}\frac{2a^5}{3} \equiv \frac{2}{3}Ma^2$$
$$I_{12} = \iiint \frac{M}{a^3}\left(-xy\right) dxdydz$$
$$= \frac{M}{a^3}\left(-\frac{a^5}{4}\right) \equiv -\frac{1}{4}Ma^2$$

$$\boldsymbol{I} = \begin{pmatrix} \frac{2}{3}b & -\frac{1}{4}b & -\frac{1}{4}b \\ -\frac{1}{4}b & \frac{2}{3}b & -\frac{1}{4}b \\ -\frac{1}{4}b & -\frac{1}{4}b & \frac{2}{3}b \end{pmatrix}, \quad b = Ma^2$$

Calculate the moment of inertia tensor for a solid cuboid of height h, width w, and depth d, and mass m by fixing the origin at one corner and the three edges of adjacent to that corner lie on the $+x$, $+y$, and $+z$ axes..

The Euler equations of motion

- The total kinetic energy and angular momentum split into one term relating to the translational motion of the center of mass and another involving motion about the centre of mass.

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

- A similar sort of division can be made for the potential energy also. Then the Lagrangian can be written as

$$L(q, \dot{q}) = L_c(q_c, \dot{q}_c) + L_b(q_b, \dot{q}_b).$$

  L_c is the part of the Lagrangian involving the generalized coordinates (and velocities $\dot{q}_c$) of the center of mass, and L_b the part relating to the orientation of the body about the center of mass (described by q_b and $\dot{q}_b$).

- It is convenient to work in terms of the principle axes system of the point of reference (kinetic energy takes simpler form).

The Euler equations of motion

- For rotational motion about a fixed point or the center of mass, the direct Newtonian approach leads to a set of equations known as **Euler's equations of motion**.
- Consider either an inertial frame whose origin is at the fixed point of the rigid body, or a system of space axes with origin at the center of mass. In these two situations

$$\left(\frac{d\vec{L}}{dt}\right)_s = \vec{N}.$$

The subscript s is used to denote the time derivative is with respect to axes that do not share the rotation of the body. However, the derivatives with respect to axes fixed in the body:

$$\left(\frac{d\vec{L}}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_b + \vec{\omega} \times \vec{L}$$

The Euler equations of motion

- Or simply dropping the 'body' subsscript

$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = \vec{N}$$

- The i-th component of the $\vec{L}$

$$\frac{dL_i}{dt} + \epsilon_{ijk}\omega_j L_k = N_i$$

- If the body axes are taken as the principle axes relative to the reference point, then the angular momentum components $L_i = I_i\omega_i$ (no summation over i).

$$I_i \frac{d\omega_i}{dt} + \epsilon_{ijk}\omega_j L_k = N_i$$

since the principal moments of inertia are time independent.

- In expanded form

$$\begin{aligned} I_1\dot{\omega}_1 - \omega_2\omega_3 (I_2 - I_3) &= N_1, \\ I_2\dot{\omega}_2 - \omega_3\omega_1 (I_3 - I_1) &= N_2, \\ I_3\dot{\omega}_3 - \omega_1\omega_2 (I_1 - I_2) &= N_3. \end{aligned}$$

- Euler's equation of the motion for a rigid body with one point fixed.
- The case $I_1 = I_2 \neq I_3$: A torque with components N_1 or N_2 will cause ω_1 and ω_2 to change without affecting ω_3.

Torque free motion of a rigid body

- Motion of a rigid body not subject to any net forces or torques. This means the centre of mass is either at rest or moving uniformly.
- The angular momentum arises only from rotation about the center of mass
- The Euler's equation are the equations of motion for the complete system.

$$
\begin{aligned}
I_1\dot{\omega}_1 &= \omega_2\omega_3\left(I_2 - I_3\right), \\
I_2\dot{\omega}_2 &= \omega_3\omega_1\left(I_3 - I_1\right), \\
I_3\dot{\omega}_3 &= \omega_1\omega_2\left(I_1 - I_2\right).
\end{aligned}
$$

- Integrals of motion: The kinetic energy and total angular momentum must be constant in time
- Possible to integrate completely in terms of Jacobian elliptic functions.
- An elegant geometrical description of the motion without knowing the complete solution — **Poinsot's construction**.

Poinsot's construction

- Consider a coordinate system oriented along the principal axes of the body.
- The axes measure the components of a vector $\vec{\rho}$ along the instantaneous axis of rotation.

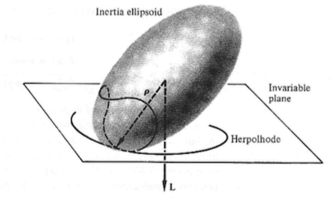
$$\vec{\rho} = \frac{\vec{\omega}}{\omega\sqrt{I}} = \frac{\vec{\omega}}{\sqrt{2T}},$$

- In this $\vec{\rho}$ space, define a function

$$F(\rho) = \vec{\rho} \cdot \mathbf{I} \cdot \vec{\rho} = \rho^2 I.$$

- Surfaces of constant F are ellipsoids
- $F = 1$ is the **inertia ellipsoid**

- Distance between the origin and the plane tangent at a point $\vec{\rho}$ must be constant.

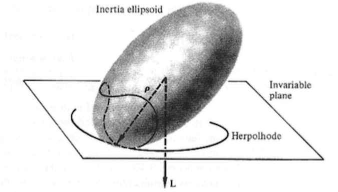


$$\frac{\vec{\rho} \cdot \vec{L}}{L} = \frac{\vec{\omega} \cdot \vec{L}}{L\sqrt{2T}} \equiv \frac{\sqrt{2T}}{L},$$

where $T = (\vec{\omega} \cdot \vec{L})/2$ is used.

Poinsot's construction

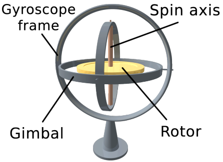
- Both T and $\vec{L}$ are constants of the motion – the tangent plane is always a fixed distance from the origin of the ellipsoid.
- The normal to the tangent plane, being along $\vec{L}$, also has a fixed direction and the plane is known as **invariable plane**.
- The force free motion of the rigid body can be visualized as being such that the inertia ellipsoid rolls (without slipping) on the invariable plane with the center of the ellipsoid a constant height about the plane.

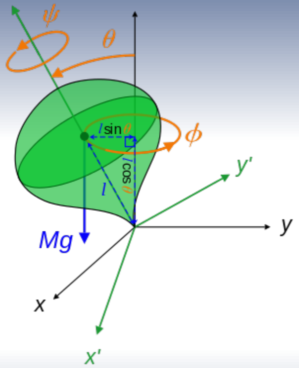


- The rolling occurs without slipping because the point of contact is defined by the position of $\vec{\rho}$ – the instantaneous axis of rotation – is the one direction in the body momentarily at rest.
- The curve traced out on the ellipsoid is the **polhode**, while the similar curve on the invariable plane is the **herpolhode**.

Heavy symmetrical top with one point fixed

- Let us consider a symmetrical body in a uniform gravitational field in which one point on the symmetry axis is fixed in space.
- Examples: Child's top, gyroscope, etc.

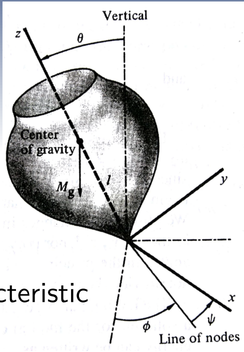




- The symmetry axis (one of the principal axes) is chosen the z axis of the coordinate system fixed on the rigid body.
- Since one point is stationary, the configuration of the top is completely specified by the three Euler angles.

Heavy symmetrical top with one point fixed

- θ gives the inclination of the z axis from the vertical.
- ϕ measures the azimuth of the top about the vertical.
- ψ is the rotation angle of the top about its own z axis.
- l is the distance of the center of gravity (located on the symmetry axis) from the fixed point.



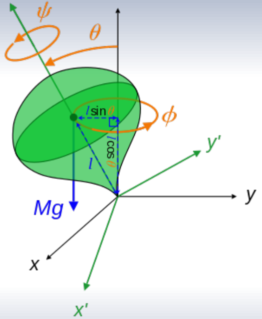
The rate of change of these three angles give the characteristic motions of the top

- $\dot{\psi} \implies$ rotation or spinning of the top about its own figure axis, z.
- $\dot{\phi} \implies$ precession or rotation of the figure axis z about the vertical z' axis.
- $\dot{\theta} \implies$ nutation or bobbling up and down of the z figure axis relative to the vertical space axis z'

Heavy symmetrical top with one point fixed

- For many cases of interest (eg. top, gyroscope etc.), we have $\dot{\psi} \gg \dot{\theta} \gg \dot{\phi}$
- The three principal moments of inertia: $I_1 = I_2 \neq I_3$.
- The Euler's equations become

$$
\begin{aligned}
I_1\dot{\omega}_1 + \omega_2\omega_3\left(I_3 - I_2\right) &= N_1, \\
I_2\dot{\omega}_2 + \omega_3\omega_1\left(I_1 - I_3\right) &= N_2, \\
I_3\dot{\omega}_3 &= N_3.
\end{aligned}
$$



- Consider the case where initially $N_2 = N_3 = 0$ and $N_1 \neq 0$, and $\omega_1 = \omega_2 = 0$ and $\omega_3 \neq 0$.
- The torque N_1 will cause ω_1 to change – the second equation requires that ω_2 begin to change.
- Euler's equation may not provide the most useful description of the motion.

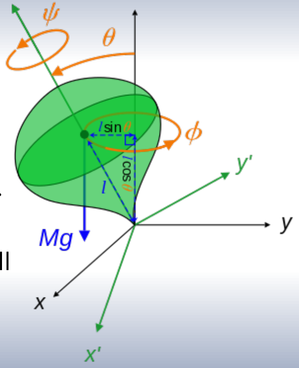
Heavy symmetrical top with one point fixed

- Lagrangian procedure will be used to obtain a solution
- The kinetic energy can be written as

$$T = \frac{1}{2} I_1 \left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2} I_3 \omega_3^2,$$

or in terms of Euler's angles

$$T = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2.$$



- The potential energy of the body is the sum over all the particles

$$V = -m_i \vec{r}_i \cdot \vec{g},$$

and is equivalent to

$$V = -M\vec{R} \cdot \vec{g} = Mgl\cos\theta$$

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- The Lagrangian

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} \cos \theta \right)^2 - Mgl \cos \theta,$$

- ϕ and ψ do not appear explicitly in the Lagrangian – they are **cyclic coordinates** – the corresponding generalized momenta are constant in time.
- The momentum conjugate to a rotation is the component of the total angular momentum along the axis of rotation — for ϕ is the vertical axis, and for ψ, the z axis.
- These components of angular momentum must be constant in time – there is no component of the torque along either the vertical or the body z-axis, as the torque due to gravity acts along the line of nodes (by definition both the axes are perpendicular to the line of nodes).

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- The two first integrals are

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 \left(\dot{\psi} + \dot{\phi} \cos\theta \right) = I_3 \omega_3 = I_1 a, \implies \omega_3 = \frac{I_1}{I_3} a$$

  and

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \left(I_1 \sin^2\theta + I_3 \cos^2\theta \right) \dot{\phi} + I_3 \dot{\psi} \cos\theta = I_1 b,$$

  where a and b are new constants.

- There is another first integral, the total energy as the system is conservative

$$E = T + V = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} \cos\theta \right)^2 + Mgl \cos\theta,$$

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- Only three additional quadratures are needed to solve the problem – they can be obtained from these three first integrals without directly using Lagrange equations. From the equation for p_ψ

$$I_3\dot{\psi} = I_1 a - I_3\dot{\phi}\cos\theta,$$

and substituting the above in p_ϕ equation

$$I_1\dot{\phi}\sin^2\theta + I_1 a\cos\theta = I_1 b \implies \dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}$$

- If θ is known as a function of time, the above $\dot{\phi}$ equation could be integrated to furnish the dependence of ϕ on time.

$$\dot{\psi} = \frac{I_1 a}{I_3} - \cos\theta\left(\frac{b - a\cos\theta}{\sin^2\theta}\right)$$

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- $E - I_3\omega_3^2/2 = E'$ is constant. Then

$$E' = \frac{I_1}{2}\dot{\theta}^2 + \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta,$$

- It has the form of an equivalent one-dimensional problem in the variable θ, with the effective potential V"

$$V' = Mgl\cos\theta + \frac{I_1}{2}\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2$$

- Thus, we have four constants — the two angular momenta p_ψ, p_ϕ, the energy term $E - I_3\omega_3^2/2$, and the potential energy term Mgl.
- Define four normalized constants

$$\alpha = \frac{2E - I_3\omega_3^2}{I_1}, \quad \beta = \frac{2Mgl}{I_1}, \quad a = \frac{p_\psi}{I_1}, \quad \text{and} \quad b = \frac{p_\phi}{I_1}$$

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- In terms of the constants

$$\alpha = \dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{\sin^2\theta} + \beta\cos\theta,$$

- The one-dimensional problem is similar to the description of the radial motion for the central force problem.
- In a convenient variable $u = \cos\theta$

$$\dot{u}^2 = (1 - u^2)(\alpha - \beta u) - (b - au)^2, \qquad \text{(A)}$$

  which can be reduced to a quadrature

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1 - u^2)(\alpha - \beta u) - (b - au)^2}}$$

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- ϕ and ψ can also be reduced to quadratures.
- The polynomial in the radical is a cubit and so one has to deal with elliptic integrals.
- The solution can also be generated numerically with the help of desktop computers.
- On the other hand, the general nature of the above problem can be discovered without actually performing the integrations.
- A simple analysis can be made by designating the right hand side of the equation (A) to a function $f(u)$

$$f(u) = (1 - u^2)(\alpha - \beta u) - (b - au)^2,$$

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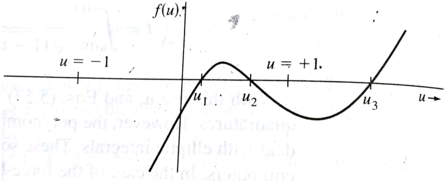
- For a gyroscope, $f(u)$ is only a quadratic equation since $\beta = 0$, while the full equation must be considered for the top.
- To understand the general motion of a spinning body, one should consider only cases where $\beta > 0$.
- The roots of the polynomial furnish the angles at which $\dot{\theta}$ changes sign, i.e., the **turning angles**.
- There is also a physical constraint that u must satisfy $-1 \leq u \leq +1$.



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