Classical Mechanics

Central Force Motion: Kepler

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Integrable power law potentials

- Solving the problem – finding r and θ as functions of time with E, l, etc., as constants.
- We really seek the *equation of the orbit* – the dependence of r upon θ (eliminating the parameter t).
- The relation between a differential change dt and the corresponding change $d\theta$

$$l\,dt = mr^2 d\theta \tag{1}$$

- The relation between the derivaties

$$\frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta} \tag{2}$$

Integrable power law potentials

- Recall Lagrange equation for the r coordinate

$$m\ddot{r} - mr\dot{\theta}^2 = f(r) \tag{3}$$

- Substituting d/dt in the above equation

$$\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{1}{mr^2}\frac{dr}{d\theta}\right) - \frac{l}{mr^3} = f(r) \tag{4}$$

- Substituting $u = 1/r$ and expressing the results in terms on potential

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{d}{du}V\left(\frac{1}{u}\right) \tag{5}$$

Integrable power law potentials

- The resulting orbit is symmertric about two adjacent turning points.
- If the orbit is symmetric – possible to reflect it about the direction of the turning angle without producing any change.
- Let the coordinates are chosen such that the turning point occurs for $\theta = 0$ – reflection can be effected by subsituting $-\theta = \theta$.
- The differential equation for the orbit is obviously invarient under such a substitution.
- The orbit is invariant under reflection about the apsidal vectors.

Integrable power law potentials

- Any particular force law, the actual equation of the orbit can be obtained by eliminating t from the solution.

$$d\theta = \frac{l\,dr}{mr^2\sqrt{\frac{2}{m}\left[E - V(r) - \frac{l^2}{2mr^2}\right]}} \tag{6}$$

$$\theta = \int_{r_0}^{r} \frac{l\,dr'}{r'^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r'^2}}} + \theta_0 \tag{7}$$

Integrable power law potentials

- Changing the variable of the integration to $u = 1/r$

$$\theta = \theta_0 - \int_{r_0}^{r} \frac{l\,du'}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - u'^2}} \tag{8}$$

- The above integral can not be always expressed in terms of well-known functions.
- Only certain type of force laws – eg. power law functions

$$V = ar^{n+1} \tag{9}$$

- Trigonometric functions : $n = 1, -2, -3$
- Elliptic functions : $n = 5, 3, 0, -4, -5, -7$

The Kepler problem: Inverse square law of force

- The inverse square law is the most important of all the central force laws – The force and potential

$$f = -\frac{k}{r^2}, \quad V = -\frac{k}{r} \tag{10}$$

- Substituting V in Eq. (8)

$$\theta = \theta' - \int \frac{l\,du}{\sqrt{\frac{2mE}{l^2} - \frac{2mku}{l^2} - u^2}} \tag{11}$$

  where the integral is taken as idefinite.

- θ' can be determined from the initial conditions – not necessarily be the same as θ_0 at $t = 0$.

The Kepler problem: Inverse square law of force

- The above equation is of the standard form

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \cos^{-1}\left(-\frac{\beta + 2\gamma x}{\sqrt{q}}\right) \tag{12}$$

where $q = \beta^2 - 4\alpha\gamma$. We have

$$\alpha = \frac{2mE}{l^2}, \quad \beta = \frac{2mk}{l^2}, \quad \gamma = -1 \tag{13}$$

Therefore

$$q = \left(\frac{2mk}{l^2}\right)^2 \left(1 + \frac{2El^2}{mk^2}\right) \tag{14}$$

The Kepler problem: Inverse square law of force

- With these substitutions, eq. (11) becomes

$$\theta = \theta' - \cos^{-1}\left(\frac{\dfrac{l^2 u}{mk} - 1}{\sqrt{1 + \dfrac{2El^2}{mk^2}}}\right). \tag{15}$$

- Solving for $u = 1/r$, the equation of the orbit is found to be

$$\frac{1}{r} = \frac{mk}{l^2}\left[1 + \sqrt{1 + \frac{2El^2}{mk^2}}\cos\left(\theta - \theta'\right)\right] \tag{16}$$

- The constant of integration θ' can be identified as one of the turning angles of the orbit.

The Kepler problem: Inverse square law of force

- Integrating the conservation theorem for the angular momentum $mr^2 d\theta = l\,dt$, by means of eq. (16), one must additionally specify the initial angle θ_0.
- Now, the general equation of a conic with one of the focus at the origin is

$$\frac{1}{r} = C\left[1 + e\cos\left(\theta - \theta'\right)\right], \tag{17}$$

  where e is the eccentricity of the conic section.
- The orbit is always a conic section, with the eccentricity

$$e = \sqrt{1 + \frac{2El^2}{mk^2}}. \tag{18}$$

The Kepler problem: Inverse square law of force

- The nature of the orbit depends upon the magnitude of e according to the following:

$$
\begin{aligned}
&e > 1, \qquad && E > 0; \qquad && \text{hyperbola} \\
&e = 1, \qquad && E = 0; \qquad && \text{parabola} \\
&e < 1, \qquad && E < 0; \qquad && \text{ellipse} \\
&e = 0, \qquad && E = -\frac{mk^2}{2l^2}; \qquad && \text{circle}
\end{aligned}
$$

- The constant of integration θ' can be identified as one of the turning angles of the orbit.