Classical Mechanics

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Scattering in a central force field

Solid angle

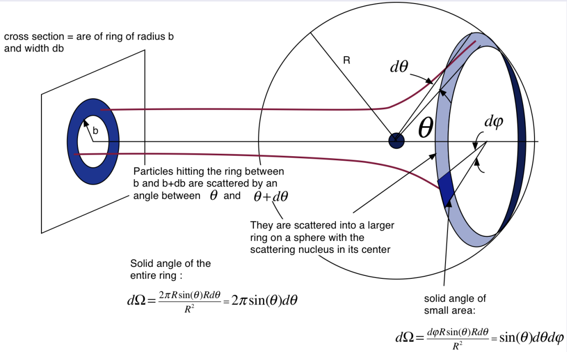

Figure: Geometry of the cross section and the solid angle

Scattering cross section

Differential scattering cross section

- Intensity (flux density) I – number of particles crossing per unit area normal to the incident beam in unit time
- The cross section for scattering in a give direction, $\sigma(\mathbf{\Omega})$

$$\sigma(\mathbf{\Omega})d\Omega = \frac{\text{number of particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident intensity}}$$

- $d\Omega$ is an element of solid angle in the direction $\mathbf{\Omega}$.

$$d\Omega = 2\pi \sin\theta d\theta$$

- "cross section" – $\sigma(\mathbf{\Omega})$ has the dimension of an area.

Scattering cross section

Impact parameter

- Angular momentum – in terms of the energy and the impact parameter, s.
- Impact parameter – perpendicular distance between the center of force and the incident velocity, v_0.

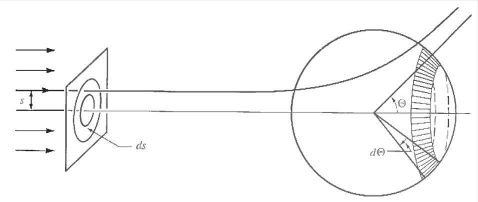

Figure: Scattering of an incident beam of particle by a center of force

$$l = mv_0 s = s\sqrt{2mE}$$

- Once E and s are fixed, the angle of scattering θ is then determined uniquely.

Scattering cross section

- number of particles scattered into a solid angle $d\Omega$ between θ and $\theta + d\theta$ must be equal to the number of incident particles with impact parameter lying between s and $s + ds$:

$$2\pi I s|ds| = 2\pi\sigma(\theta) I \sin\theta|d\theta|$$

- $s = s(\theta, E)$ – is a function of energy, then the differential cross section

$$\sigma(\theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right|$$

Scattering in a central force field

- $\theta = \pi - 2\Psi$, Ψ is the angle between the direction of the incoming asymptote and the periapsis (closest approach) direction.

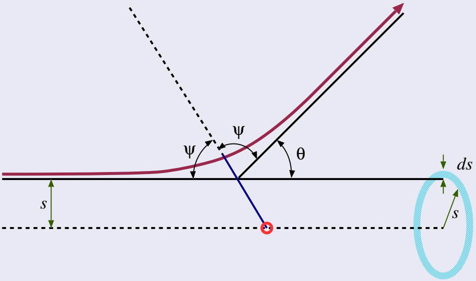

Figure: Orbit parameters and scattering angle

$$\Psi = \int_{r_m}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}}$$

Scattering in a central force field

Substituting $l = s\sqrt{2mE}$

$$\theta(s) = \pi - 2\int_{r_m}^{\infty} \frac{s\,dr}{r\sqrt{r^2\left(1 - \frac{V(r)}{E}\right) - s^2}}$$

or, changing r to $1/u$

$$\theta(s) = \pi - 2\int_{0}^{u_m} \frac{s\,du}{\sqrt{1 - \frac{V(1/u)}{E} - s^2u^2}}$$

$$f = \frac{ZZ'e^2}{r^2}$$

$$k = -ZZ'e^2$$

Repulsive scattering of charged particles by a Coulomb field

Scattering in a central force field

Energy E is greater than zero – orbit is a hyperbola with the ecentricity

$$e = \sqrt{1 + \frac{2El^2}{m(ZZ'e^2)^2}} = \sqrt{1 + \left(\frac{2Es}{ZZ'e^2}\right)^2}$$

If $\theta' = \pi$, the periapsis corresponds to $\theta = 0$ and the orbit equation becomes

$$\frac{1}{r} = \frac{mZZ'e}{l^2}\left(\epsilon \cos\theta - 1\right)$$

The direction of the incoming asymptote, Ψ – determined by the condition $r \to \infty$

$$\cos\Psi = \frac{1}{\epsilon} \quad \Longrightarrow \quad \sin\frac{\Theta}{2} = \frac{1}{\epsilon} \qquad (\Theta = \pi - 2\Psi)$$

Scattering in a central force field

$$\cot^2\frac{\Theta}{2} = \epsilon^2 - 1 \quad \Longrightarrow \quad \cot\frac{\Theta}{2} = \frac{2Es}{ZZ'e^2}$$

The function relationship between the impact parameter s and the scattering angle Θ

$$s = \frac{ZZ'e^2}{2E}$$

$$\sigma(\Theta) = \frac{1}{4}\left(\frac{ZZ'e^2}{2E}\right)^2 \csc^4\frac{\Theta}{2}$$

This is the famous **Rutherford** scattering cross section, orginally derived by Rutherford for the scattering of α particles by atomic nuclei.