Classical Mechanics

Central Force Motion

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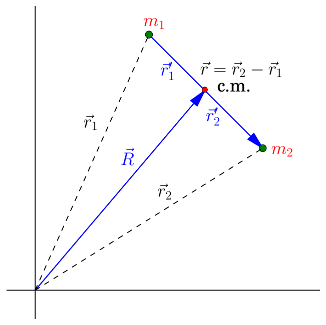
Reduction to one body problem

- Consider a monogenic system of two mass points m_1 and m_2 – the only forces are those due to an interaction potential U – assumed to be any function of $\vec{r}_2 - \vec{r}_1$ or of $\dot{\vec{r}}_2 - \dot{\vec{r}}_1$, or of any higher derivatives of $\vec{r}_2 - \vec{r}_1$.
- Six degrees of freedom – six independent generalized coordinates:



- 3 components of the radius vector to the center of mass, $\vec{R}$.
- 3 components of the difference vector $\vec{r} = \vec{r}_2 - \vec{r}_1$.
- Lagrangian

$$L = T(\dot{\vec{R}}, \dot{\vec{r}}) - U(\vec{r}, \dot{\vec{r}}, ..) \qquad (1)$$

Reduction to one body problem

- Kinetic energy, T – written as the sum of K.E. of the motion of the c.m., plus the K.E. of motion about the c.m.

$$T = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + T', \quad \text{where} \quad T' = \frac{1}{2}m_1\dot{\vec{r}}_1'^2 + \frac{1}{2}m_2\dot{\vec{r}}_2'^2. \tag{2}$$

- $\vec{r}_1'$ and $\vec{r}_2'$ – radii vectors of the two particles relative to the center of mass – are related to $\vec{r}$ by

$$\vec{r}_1' = -\frac{m_2}{m_1 + m_2}\vec{r}, \quad \text{and} \quad \vec{r}_2' = \frac{m_1}{m_1 + m_2}\vec{r}. \tag{3}$$

- Then T' takes the form

$$T' = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\vec{r}}^2. \tag{4}$$

Reduction to one body problem

- Total Lagrangian

$$L = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\vec{r}}^2 - U(\vec{r}, \dot{\vec{r}}, \ldots) \tag{5}$$

- Three coordinates $\vec{R}$ are cyclic – the center of mass will be either at rest or moving uniformly.
- None of the equations of motion for $\vec{r}$ will contain terms involving $\vec{R}$ or $\dot{\vec{R}}$ – simply drop the first term from the Lagrangian.
- A fixed center of force with single particle at a distance $\vec{r}$ from it, having a mass (reduced mass)

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{or} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{6}$$

Equation of motion

- Conservative central forces – the potential is $V(r)$, a function of $r = |\vec{r}|$ only.
- *Spherical symmetry* – any rotation, about any fixed axis can have no effect on the solution.
- The angle coordinate must be cyclic *(i.e., does not appear explicitly in the Lagrangian)*.
- The total angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is conserved.
- $\vec{r}$ is always perpendicular to the fixed direction of $\vec{L}$ – true only if $\vec{r}$ lies in a plane whose normal is parallel to $\vec{L}$.
- This reasoning breaks down if $\vec{L} = 0$, which requires $\vec{r}$ to be parallel to $\dot{\vec{r}}$ – satisfied only in straight line motion.
- *Central force motion is always motion in a plane*

Equation of motion

- Polar coordinates (r, θ, ψ) – radial distance, azimuth angle, and zenith angle or colatitude.
- Using polar axis to be in the direction of $\vec{L}$ – the motion always occur in the plane perpendicular to the polar axis.
- ψ is constant value $\pi/2 \implies$ two degrees of freedom.
- Lagrangian in plane polar coordinates

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r). \tag{7}$$

- θ is a cyclic coordinate – the corresponding canonical momentum is the angular momentum

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{8}$$

Equation of motion

- One of the two equations of motion

$$\dot{p}_\theta = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \implies mr^2\dot{\theta} = l \tag{9}$$

l is the constant magnitude of the angular momentum. The above equation can also be written as

$$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0, \quad \text{where } \frac{1}{2}r^2\dot{\theta} \text{ is the } \textit{areal velocity} \tag{10}$$

- The differential area swept out in time dt

$$dA = \frac{1}{2}r\left(rd\theta\right) \quad \text{and hence} \quad \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \tag{11}$$

Equation of motion

- Hence

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \tag{12}$$

- Thus the conservation of angular momentum is equivalent to saying the areal velocity is constant.
- This is the proof of the well-known Kepler's second law of planetary motion – *A line joining a planet and the Sun sweeps out equal areas in equal intervals of time.*
- The radial vector sweeps out equal areas in equal times.

Equation of motion

- The remaining Lagrange equation, for the coordinate r

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0. \tag{13}$$

$$m\ddot{r} - mr\dot{\theta}^2 = f(r), \tag{14}$$

where $f(r)$ is the value of the force.

- Eliminating $\dot{\theta}$ using the relation $mr^2\dot{\theta} = l$ [see (9)],

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r), \tag{15}$$

Equation of motion

- There is another first integral – the total energy E (since the forces are conservative)

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) \tag{16}$$

One can rewrite eq. (15) as

$$m\ddot{r} = \frac{d}{dr}\left(V + \frac{1}{2}\frac{l^2}{mr^2}\right) \tag{17}$$

Multiply by $\dot{r}$ on both sides of eq. (17). The right side becomes

$$m\ddot{r}\dot{r} = \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) \tag{18}$$

Equation of motion

- The right side of eq. (17) can be written as a total time derivative.
- Then eq. (17) is equivalent to

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{d}{dt}\left(V + \frac{1}{2}\frac{l^2}{mr^2}\right), \tag{19}$$

or

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + V + \frac{1}{2}\frac{l^2}{mr^2}\right) = 0, \tag{20}$$

and therefore

$$\frac{1}{2}m\dot{r}^2 + V + \frac{1}{2}\frac{l^2}{mr^2} = \text{ constant.} \tag{21}$$

Equation of motion

- The first two integrals give as the two quadratures necessary to complete the problem.
- Two variable r and θ – four integrations are needed.
- The first two integrations left the Lagrange equations as the following two first order equations

$$mr^2\dot{\theta} = l \tag{22}$$

$$\frac{1}{2}m\dot{r}^2 + V + \frac{1}{2}\frac{l^2}{mr^2} = E. \tag{23}$$

- Solving eq. (23) for $\dot{r}$

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}, \tag{24}$$

Equation of motion

- Equation (24) can be written as

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}, \tag{25}$$

- At time $t = 0$ let $r = r_0$

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}, \tag{26}$$

Equation of motion

- Once the solution for r is found, the solution θ can be written as

$$d\theta = \frac{l dt}{mr^2} \tag{27}$$

- Let θ_0 is the initial value of θ at $t = 0$

$$\theta = l \int_0^t \frac{dt}{mr^2(t)} + \theta_0 \tag{28}$$

Classification of Orbits

- A system of known energy and angular momentum – the magnitude and direction of the velocity can be determined in terms of the distance r.
- The magnitude v of the velocity from the conservation of energy

$$E = \frac{1}{2}mv^2 + V(r) \quad \Longrightarrow \quad v = \sqrt{\frac{2}{m}\left[E - V(r)\right]} \tag{29}$$

- The equation of motion expressed in r (with $\dot{\theta}$ expressed in terms of l) involves only r and its derivatives

Classification of Orbits

- The same equation can be obtained for a fictitious one-dimensional problem – a particle of mass m subject to a force

$$f' = f + \frac{l^2}{mr^3} \tag{30}$$

- The significance of the additional term is clear if it is written as $mr\dot{\theta}^2 = mv_\theta^2/r$ – centrifugal force.
- A one-dimensional problem with fictitious potential energy

$$V' = V + \frac{1}{2}\frac{l^2}{mr^2} \implies f' = -\frac{\partial V}{\partial r} = f(r) + \frac{l^2}{mr^3} \tag{31}$$

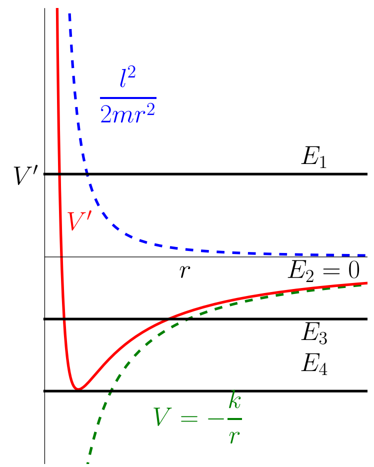
Classification of Orbits

- Consider the plot of V' against r for the specific case of an *attractive inverse-square law of force.*

$$f = -\frac{k}{r^2}, \quad V = -\frac{k}{r} \qquad (32)$$

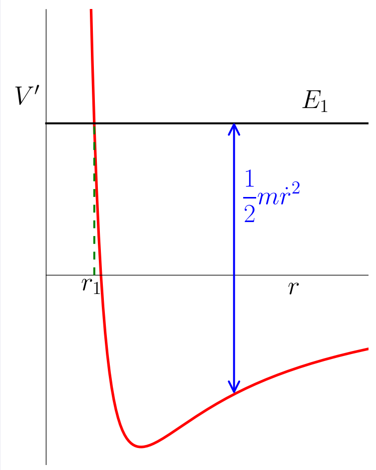
- $k > 0$ – the minus sign ensures that the force is toward the center of force.

$$V' = -\frac{k}{r} + \frac{l^2}{2mr^2} \qquad (33)$$



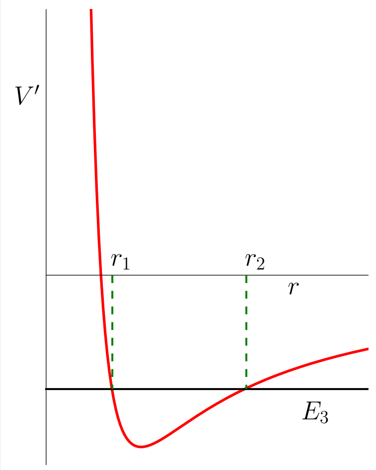
Classification of Orbits

- Consider the motion of a particle having energy E_1.
- This particle can never come closer than r_1.
- If $r < r_1$, V' exceeds E_1.
- K.E. negative $\implies$ imaginary velocity!! (physically not possible)
- There is no upper limit for r $\implies$ the orbit is not bounded



Classification of Orbits

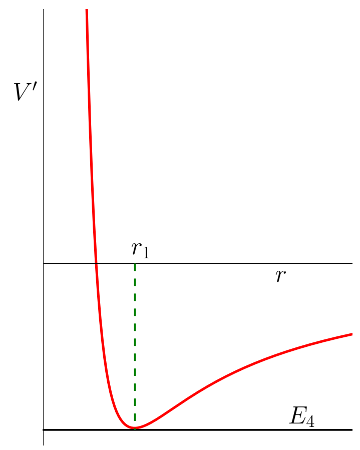
- Energy E_3: lower bound r_1 – also a maximum value of r_2 that cannot be exceeded by r with positive kinetic energy.
- The motion is bounded – two turning points, r_1 and r_2 – *apsidal distances*.
- This does not mean that the orbit is closed – the motion is contained between two circles of radius r_1 and r_2.



Classification of Orbits

- Energy is E_4 at the minimum of the fictitious potential – at the point where the two bounds coincide.
- Motion is possible at only one radius: $\dot{r} = 0$ – the orbit is a circle
- The effective "force" is negative of the slope of the V' curve – for circular orbit $f' = 0$

$$f(r) = -\frac{l^2}{mr^2} = -mr\dot{\theta}^2. \tag{34}$$



Classification of Orbits

- Condition for a circular orbit is that *the applied force be equal and opposite to the "reverse effective force" of the centripetal acceleration*.
- The above discussion of the orbits for various energies – at one value of l (angular momentum).
- Changing the value of l will change only the qualitative details – but it does not affect the results.

Summary

Attractive inverse square law of force

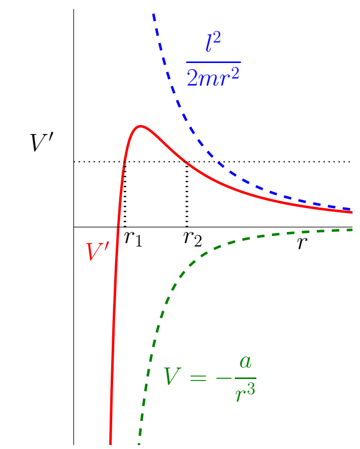
- Hyperbola for $E = E_1 > 0$
- Parabola for $E = E_2 = 0$
- Ellipse for $E_3 < 0$

Classification of Orbits

- Attractive potential

$$V(r) = -\frac{a}{r^3},$$
$$f(r) = -\frac{3a}{r^4}$$

- $r_0 < r_1$ – bounded motion – pass through center of force.
- $r_0 > r_2$ – motion is unbounded.
- $r_1 < r_0 < r_2$ – not physically possible



Classification of Orbits

Spherical pendulum

- Linear harmonic oscillator

$$V(r) = \frac{1}{2}kr^2,$$
$$f(r) = -kr$$

- $l = 0$ – motion along straight line ($V' = V$) – bounded for $E > 0$ (simple harmonic).
- $l \neq 0$ – always bounded – does not pass through center of force – elliptic orbit

$$f_x = -kx, \quad f_y = -ky \quad (35)$$