Measure Theory and Integration

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Problem

Example

If (I_n) be a finite set intervals covering the rationals in $[0, 1]$ such that $\sum l(I_n) \geq 1$.

Definition

E is lebesgue measureable if

$$l(I) = m^*(I \cap E) + m^*(I \cap E^c) \quad for \quad each \quad interval \quad I. \tag{1}$$

Theorem (Criteria)

E is measurable iff for any set $A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(a \cap E^c). \tag{2}$$

Proof.

(2) $\Rightarrow$ (1) is trivial . Assume that (1) holds. By the countable subadding m^*

$$m^*(A) = m^*((A \cap E) \cup (A \cap E^c))$$

$$\leq m^*(A \cap E) + m^*(A \cap E^c) \tag{3}$$

Let $\epsilon > 0$ be given, we choose a family $\{I_k\}_{k=1}^{\infty}$ of interval such that $A \subseteq \bigcup_{k=1}^{\infty} I_k$ and

$$\sum_{k=1}^{\infty} l(I_k) \leq m^*(A) + \epsilon \tag{4}$$

by (1)

$$l(I_k) = m^*(I_k \cap E) + m^*(I_k \cap E^c) \tag{5}$$

We have

$$\sum_{k=1}^{\infty} m(I_k \cap E) + m^*(I_k \cap E^c) \leq m^*(A) + \epsilon$$

$$m^*(\bigcup_{k=1}^{\infty}(I_k \cap E)) + m^*(\bigcup_{k=1}^{\infty}(I_k \cap E^c)) \leq m^*(A) + \epsilon$$

$$m^*((\bigcup_{k=1}^{\infty} I_k) \cap E) + m^*((\bigcup_{k=1}^{\infty} I_k) \cap E^c)) \leq m^*(A) + \epsilon$$

Since $A \cap E \subseteq (\cup I_k) \cap E$ and $A \cap E \subseteq (\cup I_k) \cap E^c$,

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A). \tag{6}$$

(2) follows from (6). Thus, we have that a set E in $\mathbb{R}$ is said to be lebesgue measurable if for all subset A of $\mathbb{R}$.

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Example

Example

ϕ is measurable and $\mathbb{R}$ is measurable.

Solution: $A \subseteq \mathbb{R}$.

$$\begin{aligned} m^*(A \cap \phi) + m^*(A \cap \phi^c) &= m^*(\phi) + m^*(A \cap \mathbb{R}) \\ &= m^*(\phi) + m^*(A). \\ &= m^*(A). \end{aligned}$$

Example

If E is measurable then E^c is measurable

$$\begin{aligned} m^*(A) &= m^*(A \cap E^c) + m^*(A \cap E). \\ &= m^*(A \cap E^c) + m^*(A \cap (E^c)^c). \end{aligned}$$

Therefore E^c is measureable.

ϕ is measurable $\Rightarrow \phi^c = \mathbb{R}$ is measurable.

Since m^* is subadditive, In order to to prove that the set E is measurable E, it is enough to prove that

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A).$$

Example

Intervals are measurable

$$\epsilon + m^*(A) \geq m^*(A \cap I) + m^*(A \cap I^c)$$

Let $I \subseteq \mathbb{R}$, let $\epsilon.0$.

Proof.

Take $A \subseteq \mathbb{R}$, there exists (J_k) such that $A \subseteq \cup J_k$ and

$$\sum_{k=1}^{\infty} l(I_k) \leq m^*(A) + \epsilon \tag{7}$$

$\bigcup_{k=1}^{\infty}(J_k \cap I) = (\bigcup_{k=1}^{\infty} J_k) \cap I.$

$$\supseteq A \cap I.$$

$\{J_k \cap I\}_{k=1}^{\infty}$ forms a cover of $A \cap I$.

$$m^*(A \cap I) \leq \sum_{k=1}^{\infty} l(J_k \cap I).$$

□

Proof.

Let $J_k' = J_k \cap I$ and $J_k'' = J_k \cap I^c = I_k' \cup I_k''$.

$$\bigcup_{k=1}^{\infty}(J_k \cap I^c) = (\bigcup_{k=1}^{\infty} J_k) \cap I^c.$$

$\Rightarrow m^*(A \cap I^c) \leq \sum_{k=1}^{\infty} m^*(J_k \cap I^c)$.

$$m^*(A \cap I) + m^*(A \cap I^c) \leq \sum_{k=1}^{\infty}\{l(J_k \cap I) + l(J_k \cap I^c)\}.$$

Since length ρ_n is countably additive.

Proof.

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} l\{(\cup(J_k \cap I) \cup (J_k \cap I^c)\} \\
&= \sum_{k=1}^{\infty} l(\cup(J_k \cap I \cup I^c)). \\
&= \sum_{k=1}^{\infty} l(J_k) \leq m^*(A) + \epsilon
\end{aligned}
$$

Theorem

If E and F are measurable set, $E \cup F$ is measurable.

Proof.

Let $A \subseteq \mathbb{R}$.

$$\begin{aligned} m^*(A) &\geq m^*(A(E \cup F)) + m^*(A \cap (E \cup F)^c). \\ &= m^*(A \cap (E \cup F)) + m^*(A \cap (E^c \cup F^c)) \end{aligned}$$

we know that E and F are measurable

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \quad \text{(Since E is measurable)} \tag{8}$$

$\square$

Proof.

$$\Rightarrow m^*(A \cap E^c) \geq m^*((A \cap E^C) \cap F) + m^*((A \cap E^c) \cap F^c)). \qquad (9)$$

We know that $m^*(A) \geq m^*(A \cap F) + m^*(A \cap F^c)$. (Since F is measurable).

$$\Rightarrow m^*(A) = m^*(A \cap E) + m^*((A \cap E^c) \cap F) + m^*((A \cap E^c) \cap F^c)$$

(Since F is measurable and A is replaced by $A \cap E^c$)

$$m^*(A) \geq m^*(A \cap E) + m^*((A \cap E^c) \cap F) + m^*(A \cap (E^c \cap F^c))$$

Since $A \cap (E \cup F) = (A \cap E) \cup (A \cap F) = (A \cap E) \cup ((A \cap E^c) \cap F)$

$$m^*(A) \geq m^*(A \cap (E \cup F)) + m^*(A \cap (E \cup F)^c).$$

Therefore $E \cup F$ is measurable.