Measure Theory and Integration

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Countable sub-additivity

Theorem

Prove that m^* is countably sub-additive.

Proof.

Let $\{E_n\}$ be a countable collection of subsets of $\mathbb{R}$.
$m^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m^*(E_n)$. Let $E = \bigcup_{n=1}^{\infty} E_n$.
Then

$$m^*(E) \leq \sum_{n=1}^{\infty} m^*(E_n).$$

Let $\epsilon > 0$ be given, for each n, $\exists \quad (I_{nk})_{k=1}^{\infty}$ of the form $[a, b)$ such that $E_n \subseteq \bigcup_{k=1}^{\infty} I_{nk}$.

$\square$

Proof.

and

$$\sum_{k=1}^{\infty} l(I_{nk}) \leq m^*(E_n) + \epsilon|2^n.$$

Now,

$$\bigcup_{k=1}^{\infty}(\bigcup_{k=1}^{\infty} I_{nk}) \supseteq \bigcup_{k=1}^{\infty} E_n = E.$$

$$\Rightarrow E \subseteq \bigcup_{n,k=1}^{\infty} I_{nk}.$$

By the definition of outer measure m^*,

Proof.

$$
\begin{aligned}
m^*(E) &\leq \sum_{n,k=1}^{\infty} l(I_{nk}) \\
&= \sum_{n=1}^{\infty}(\sum_{k=1}^{\infty} l(I_{nk})) \\
&= \sum_{n=1}^{\infty}[m^*(E_n) + \epsilon|2^n] \\
&= \sum_{n=1}^{\infty} m^*(E_n) + \sum_{n=1}^{\infty} \epsilon|2^n \\
&= \sum_{n=1}^{\infty} m^*(E_n) + \epsilon
\end{aligned}
$$

Proof.

Therefore

$$m^*(E) \leq \sum_{n=1}^{\infty} m^*(E_n) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $m^*(E) \leq \sum_{n=1}^{\infty} m^*(E_n)$.

Theorem

Let $E \subseteq \mathbb{R}$. and ϵ be given. Then there exists an open set $U \in \mathbb{R}$ such that $E \subseteq U$ and $m^*(U) \leq m^*(E) + \epsilon$.

Proof.

By the definition of m^*, there exist $(I_k)_{k=1}^{\infty}$ such that $E \subseteq \bigcup_{k=1}^{\infty}$, and $\sum_{k=1}^{\infty} l(I_k) \leq m^*(E) + \epsilon$.
Let $I_k = [a_k, b_k)$. Then $I_k' = (a_k - \epsilon|2^{k+1}, b_k)$

$\square$

Proof.

Clearly $I_k \subseteq I_k^{'}$ and

$$l(I_K^{'}) = l(I_k) + \epsilon|2^{k+1}.$$

$$\begin{aligned} m^*(\bigcup I_k^{'}) &\leq \sum_{k=1}^{\infty} l(I_k^{'}) \\ &= \sum_{k=1}^{\infty} \{l(I_k) + \epsilon|2^{k+1}\} \\ &= \sum_{k=1}^{\infty} l(I_k) + \epsilon|2. \end{aligned}$$

Let $U = \bigcup_{k=1}^{\infty} I_k^{'}$. Then $E \subseteq U = \bigcup_{k=1}^{\infty} I_k^{'}$.

$$\Rightarrow m^*(U) \leq m^*(E) + \epsilon.$$

Various Outer Measures

If $k \subset I$ and K is compact $\Rightarrow m^*(k) = l(I) - m(I|k)$.
$E \subseteq \mathbb{R}, k \subseteq E$.

$$m_0(E) = sup\{m(K)|K \subseteq E\}.$$

$$I_k = [a, b)$$

$$m^*(E) = inf\{\sum_{k=1}^{\infty} l(I_k) \quad |E \subseteq \bigcup I_k\}.$$

$$m_c^*(E) = inf\{\sum l(I_k) \quad |E \subseteq \bigcup I_k\}, \quad I_k = [a, b].$$

$$m_{oc}^*(E) = inf\{\sum l(I_k) \quad |E \subseteq \bigcup I_k\}, \quad I_k = (a, b].$$

$$m_o^*(E) = inf\{\sum l(I_k) \quad |E \subseteq I_k\}, \quad I_k = (a, b).$$

$$m_m^*(E) \leq m^*(E).$$

$$m^* = m_m^* = m_o^* = m_o^* = m_c^*.$$

Proof.

Let $\epsilon > 0$,
Claim
$$m_o^*(E) \leq m_m^*(E).$$
We prove
$$m_o^*(E) \leq m_m^*(E) + \epsilon,$$
for all $\epsilon > 0$, $m_m^*(E) + \epsilon$ is not a l.b. there exist I_k such that $E \subseteq \bigcup I_k$ of any type with a_k, b_k as end points such that $E \subseteq \bigcup I_k$ and $\sum_{k=1}^{\infty} l(I_k) \leq m^*(E) + \epsilon$.
For each k, define an open interval I_k' such that $I_k' \supseteq I_k$.

Proof.

$$
\begin{aligned}
l(I_k') &= l(I_k) + \epsilon|2^k \\
I_k' &= (c_k, d_k)
\end{aligned}
$$

$$
E \subseteq \bigcup_{k=1}^{\infty} I_k'
$$

$$
\begin{aligned}
m_o^*(E) &\leq \sum_{k=1}^{\infty} l(I_k') = \sum_{k=1}^{\infty} \{l(I_k) + \epsilon|2^k\} \\
&= \sum_{k=1}^{\infty} l(I_k) + \epsilon \\
&\leq m_m^*(E) + 2\epsilon.
\end{aligned}
$$

Proof.

If $E \subseteq \mathbb{R}$ is a countable set $m^*(E) = 0$. $E = \{a_1, a_2, \cdots, \}$

$$\begin{aligned} 0 \leq m^*(E) &= m^*\left(\bigcup_{n=1}^{\infty} \{a_n\}\right) \\ &\leq \sum_{n=1}^{\infty} m^*(\{a_n\}) \\ &= \sum_{n=1}^{\infty} l(\{a_n\}) \\ &= 0. \end{aligned}$$

Therefore

$$m^*(E) = 0.$$