Ordinary Differential Equations

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The general first order ordinary differential equations is an equation of the form

$$F(t, x, x') = 0 \tag{1}$$

where F is a function of three variables and x' denotes the drivative $\frac{dx}{dt}$. Suppose that the point (t_0, x_0, x_0') in $\mathbb{R}^3$ satisfies equation (1) and suppose further that $D_3 F 9 t_0, x_0, x_0') \neq 0$. Then the implicite funtion theorem asserts that near the point (t_0, x_0, x_0'), equation (1) can be solved for x',

$$x' = f(t, x) \tag{2}$$

where f is a functin of two varibles defined in a neighborhoos of the point (t_0, x_0) in $\mathbb{R}^2$. We consider only ordinary differential equation of the form (2)

Caristi's p-cyclic map

A solution of the ordinary differential equation (2) on some interval I of the $t-$axis is a function $x(t)$ defined and continuously differentiable for $t \in I$ such that when $x(t)$ is substituted for x in (2), equation (2) becomes an identity in t for all $t \in I$. The general solution of (2) is the collection of all of its solutions.

If the function f in (2) depends only on t, the equation takes the elemetary form

$$\frac{dx}{dt} = f(t) \tag{3}$$

and , if f is continuous in some interval I, the general solution of (3) in the interval I is given by

$$x(t) = \int_{x_0}^{t} f(\tau)d\tau + c \tag{4}$$

where t_0 is any fixed point in I and c is an arbitrary constant.

Thus the general soltion of (3) is a one-parameter family of curves given by (4) where c is the parameter of the family. If in addition to equation (3) we require that the solution satisfy the condition

$$x = x_0 \text{ when } t = t_0, \tag{5}$$

then we must choose the parameter d in the general solution (4) to be equal to x_0.
Condition (5) is called an initial condition and the problem of finding the solution of equation eqref3 satisfying the initial condition (5) is called an initial value problem or a Cauchy problem. The solution of the initial value problem (3), (5) is given by

$$x = \int_{t_0}^{t} f(\tau) d\tau + x_0. \tag{6}$$

It is often useful to indicate the depence of the solution x given by (6) on the initial data t_0 and x_0. This is done by writing $x = x(t; t_0; x_0)$. It is clear from equation (6) that the solution of the initial value problem (3), (5) depends continuously on t_0 and x_0; that is $x(t : t); x_0)$ is a continuous function of t_0 and x_0. The solution is a continuously differentiable function of t in the interval I. Use some conditions on the function f, the general solution of equation (2) exists and also depends on one parameter, but it is usually difficult if not impossible to write a simple formula ofr it. It is easy to see, however, that if $x = x(t)$ must satisfy the integral equation

$$x(t) = \int_{t_0}^{t} f(\tau, x(\tau))d\tau + x_0. \tag{7}$$

Conversely, if $x(t)$ is a continuouly differnetiable function on I satisfying the integralequation (7). There is s fundamental theoem of ordinary differential equations which asserts the existence and uniqueness of solution of the initial value problem (2), (5). The proof of this theorem consists of showing that there is a unique solution of the initial value problem(2) (5). Before stating the theorem we must introduce some terminology and state the initial value problem in more precise language.
Let (t_0, x_0) be a point in $\mathbb{R}^2$ and let f be a real valued function of two variables t and x defined fr all points (t, x) of a rectangle Q centered at (t_0, x_0) and described by inequalities of the form

$$|t - t_0| < a. |x - x_o| < b \tag{8}$$

The initial value problem asks for a function $x(t)$ defined for t in some open interval I describibles by $|t - t_o| < h, 0 < h < a$ such that $x = x(t)$ satisfies the differential equation $\frac{dx(t)}{dt} = f(t, x(t))$ for $t \in I$ and the initial condition $x(t_0) = x_0$.

Caristi's p-cyclic map

A function f defined on Q is said to satisfy Lipschitz condition with respect to x if there is a constant $A > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leq A|x_1 - x_2| \tag{9}$$

for every pair of points (t, x_1) and (t, x_2) in Q.

Example

Let $f(t,x) = x^2$ and let Q be the rectangle given by $|t| < 2, |x - 1| < 1$ so that $(t_0, x_0) = (0, 1)$ and $a = 2, b = 1$. The function f is continuous in Q, and since $|D_2 f(t,x)| = |2x| < 4$ in Q, f satisfies the Lipschitz condition with $A = 4$. Now, it is easy to see that the unique solution to the initial value problem $\frac{dx}{dt} = x^2, x(0) = 1$. is given by $x = frac11 - t$, and this solution exists only in the subinterval I given by $|t| < 1$, of the original interval $|t| < 2$.

Theorem

Suppose in some rectangle Q defined by the inequalities of the form (8) *, the function f is continuous and satisfies a Lichitz conditionwith respect to the variable x. Then there exists a unique solution of* (2) *such that the function* $x(t)$ *belongs to* $C^1(I)$ *and x satisfies the equation* (2) *with the initial condition* (5).