Ordinary Differential Equations

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Stability

We now introduce the concept of stability as it applies to the critical points of the system(2.1).
It was pointed out in the previous section that one of the most important questions in the study of a physical system is that of its steady states. However, a steady state has little physical significance unless it has a reasonable degree of permanence, i.e, unless it is stable. As a simple example, consider the pendulum of fig (). There are two steady states possible here: when the bob is at rest at the highest point, and when the bob is at rest at the lowest point. The first state is clearly unstable, and the second is stable. We now recall that the steady state of a simple physical system corresponds to an equilibrium point(or critical point) in the phase plane. These considerations suggest in a general way taht a small disturbance at an unstable equilibrium point leads to a larger and larger departure from this point, while the opposite is true at a stable equilibrium point.

We now formulate these intuitive ideas in a more precise way. Consider an isolated critical point of the system (2.1), and assume for the sake of convenience that this point is located at the origin $O = (0, 0)$ of the phase plane. This critical pont is said to be stable if for each positive number R there exists a positive number $r \leq R$ such that every path which is inside the circle $x^2 + y^2 = r^2$ for some $t = t_0$ remains inside the circle $x^2 + y^2 = R^2$ for all $t > t_0$ (fig()). Lossely speaking, a critical point is stable if all paths that get suffienciently close to the point stay close to the point. Further , our critical point is said to be asymptotically stable if it is stable and there exists a circle $x^2 + y^2 = r_0^2$ such that every path which is inside this circle for some $t = t_0$ approaches the origin as $t \to \infty$. Finally, if our critical point is not stable, then it is called unstable.

Critical points and Stability for Linear system

We consider the system

$$\begin{cases} \frac{dx}{dt} = a_1x + b_1y \\ \frac{dy}{dt} = a_2x + b_2y, \end{cases} \tag{1}$$

which has the origin $(0,0)$ as an obivous critical point . We assume throughtout this section that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \tag{2}$$

so that $(0,0)$ is the only critical point.

It was proved that (3.1) has anontrival solution of the form

$$\begin{cases} x = Ae^{mt} \\ y = Be^{mt} \end{cases}$$

whenever m is a root of the quadratic equation

$$m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0, \tag{3}$$

which is called the auxilliary equation of the system. Observe that condition (3.2) implies that zero cannot be a root of (3.3).

Let m_1 and m_2 be the root of (.3.). We shall prove that the nature of the critical point $(0,0)$ of the system(3.1) is determined by the nature of the number m_1 and m_2. It is reasonable to expect that three possiblities wil occur, according as m_1 and m_2 are real and distinct, real and equal, or conjugate complex. Unfortunately the situation is a little more complicated than this, and it is necessary to consider five cases, subdivided as follows.

Major cases:

(i) The roots m_1 and m_2 are real, distinct, and of the same sign (node).

(ii) The roots m_1 and m_2 are real, distinct, and of opposite sign (saddle point).

(iii) The roots m_1 and m_2 are conjugate complex but not pure imaginary (sprial).

(iv) The roots m_1 and m_2 are real and equal (node).

(v) The roots m_1 and m_2 are pure imaginary (center).

The reason for the distinction between the major cases and the borderline cases will become clear . For the present it suffices to remark that while the borderline cases are of mathematical interest they have little significance for applications, because the circumstances defining them are unlikely to arise in physical problems. We now turn to the proofs of the assertions in parentheses.

Case (i)

If the roots m_1 and m_2 are real, distinct, and of the same sign, then the critical point $(0,0)$ is a node.
We begin by assuming that m_1 and m_2 are both negative , and we choose the notation so that $m_1 < m_2 < 0$. By the general solution in this case is

$$\begin{cases} x = c_1 A_1 e^{m_1^t} + c_2 A_2 e^{m_2^t} \\ y = c_1 B_1 e^{m_1^t} + c_2 B_2 e^{m_2^t}, \end{cases} \tag{4}$$

where the $A's$ and $B's$ are definite constants such that $B_1 | A_1 \neq B_2 | A_2$, and where the $c's$ are arbitrary constants.

When $c_2 = 0$, we obtain the solutions

$$\begin{cases} x = c_1 A_1 e^{m_1^t} \\ y = c_1 B_1 e^{m_1^t}, .. \end{cases} \tag{5}$$

and when $c_1 = 0$, we obtain the solutions

$$\begin{cases} x = c_2 A_2 e^{m_2^t} \\ y = c_2 B_2 e^{m_2^t}. \end{cases} \tag{6}$$

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For any $c_1 > 0$, the solution (3.5) represent a path consisting of half of the line $A_1 y = B_1 x$ with slope $B_1|A_1$; and for any $c_1 < 0$, it represents a path consisting of the other half of this line. Since $m_1 < 0$, both of these half-line paths approach $(0, 0)$ as $t \to \infty$; and since $y|x = B_1|A_1$, both enter $(0, 0)$ with slope $B_1|A_1$ fig(). In exactly the same way, the solutions (3.6) represent two half-line paths lying on the line $A_2 y = B_2 x$ with slope $B_2|A_2$. These two paths also approach $(0, 0)$ as $t \to \infty$, and enter it with slope $B_2|A_2$.

If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (3.4) represents curved paths. Since $m_1 < 0$ and $m_2 < 0$, these paths also approach $(0,0)$ as $t \to \infty$. Futhermore, since $m_1 - m_2 < 0$ and

$$\frac{y}{x} = \frac{c_1 B_1 e^{m_1^t} + c_2 B_2 e^{m_2^t}}{c_1 A_1 e^{m_1^t} + c_2 A_2 e^{m_2^t}} = \frac{(c_1 B_1 | c_2) e^{(m_1 - m_2)t} + B_2}{(c_1 A_1 | c_2) e^{(m_1 - m_2)t} + A_2},$$

it is clear that $y|x \to B_2|A_2$ as $t \to \infty$, so all these paths enter $(0,0)$ with slope $B_2|A_2$. Figure () presents a qualitative picture of the situation. It is evident that our critical point is a node, and that it is asymptotically stable.

If m_1 and m_2 are both positive, and if we choose the notation so that $m_1 > m_2 > 0$, then the situtation is exactly the same except that all the paths now approach and enter $(0, 0)$ as $t \to -\infty$. The picture of the paths given in fig() is unchanged except that the arrows showing their directions are all reversed . We still have a node, but now it is unstable.