Ordinary Differential Equations

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Critical Points

Consider an autonomous system

$$\begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y). \end{cases} \tag{1}$$

Let (x_0, y_0) be an isolated critical point of (2.1). If $C = [x(t), y(t)]$ is a path of (2.1), then we say that C approaches (x_0, y_0) as $t \to \infty$ if

$$\lim_{t\to\infty} x(t) = x_0 \text{ and } \lim_{t\to\infty} y(t) = y_0. \tag{2}$$

Geometrically, this means that if $P = (x, y)$ is a point that traces out C in accordance with the equations $x = x(t)$ and $y = y(t)$, then $P \to (x_0, y_0)$ as $t \to \infty$. nIf it is also true that

$$\lim_{t\to\infty} \frac{y(t) - y_0}{x(t) - x_0} \tag{3}$$

exists, or if the quotient in (2.3) becomes either positively or negatively infinite as $t \to \infty$, then we say that C enters the critical point (x_0, y_0) as $t \to \infty$.

Nodes:

A critical point like that in fig () is called a node. Such a point is approached and also entered by each path as $t \to \infty$ (or as $t \to -\infty$). For the node shown in fig (), there are four half- line paths, AO, BO, CO, and DO, which together with the origin make up the lines AB nd CD. all other paths resemble parts of parabolas, and as each of these paths approaches O its slope approaches that of the line AB.

Example

Consider the system

$$\begin{cases} \frac{dx}{dt} = x \\ \frac{dy}{dt} = -x + 2y. \end{cases} \tag{4}$$

Critical Points

It is clear that the origin is the only critical point, and the general solution can be found quite easily by the methods

$$\begin{cases} x = c_1 e^t \\ y = c_1 e^t + c_2 e^{2t}. \end{cases} \tag{5}$$

When $c_1 = 0$, we have $x = 0$ and $y = c_2 e^{2t}$. In this cases the path is the positive $y-$ axis when $c_2 > 0$, and the negative $y-$ axis when $c_2, 0$, and each path approaches and enters the origin as $t \to -\infty$.

When $c_2 = 0$, we have $x = c_1 e^t$ and $y = c_1 e^t$. This path is the half-line $y = x, x > 0$, when $c_1 .0$, and the half-line $y = x, x < 0$, when $c_1 < 0$, and again both paths approach and enter the origin as $t \to -\infty$. When both c_1 and c_2 are $\neq 0$, the paths lie on the parabolas $y = x + (c_2 | c_1^2) x^2$, which go through the origin with slope 1. It should be understood that each of these paths consists of only part of a parabola, the paths with $x .0$ if $c_1 > 0$, and the part with $x < 0$ if $c_1 < 0$. Each of these path also approaches and enters the origin as $t \to -\infty$; this can be seen at once from (2.5).

If we proceed directly from (2.4) to the differential equation

$$\frac{dy}{dx} = \frac{-x + 2y}{x} \tag{6}$$

giving the slope of the tangent to the path through (x, y) [provided $(x, y) \neq (0, 0)$] then on solving (2.6) as a homogeneous equation, we find that $y = x + cx^2$. This procedure yeilds the curves on which the paths lie(except those on the $y-$ axis), but gives no information about the manner in which the paths are traced out. It is clear from this discussion that the critical point $(0, 0)$ of the system (2.4) is a node.

Saddle points

: A critical point like that in fig () is called a saddle point. It is approached and entered by two half-line paths AO and B) as $t \to -\infty$, and these two paths lie on a line AB. It is also approached and entered by two half-line paths CO and DO as $t \to -\infty$, and these two paths lie on another line CD. Between the four half-line paths there are four region, and each contains a family of paths resembling hyperbolas. These paths do not approach O as $t \to \infty$ or as $t \to -\infty$, but instead are asymptotic to one or another of the half-line paths as $t \to \infty$ and as $t \to -\infty$.

Centers

A center (sometimes called a vortex) is a critical point that is surrounded by a family of closed paths. It is not approached by any path as $t \to \infty$ or $t \to -\infty$.

Example

The system

$$\begin{cases} \frac{dx}{dt} = -y \\ \frac{dy}{dt} = x \end{cases} \tag{7}$$

has the origin as its only critical point, and its general solution is

$$\begin{cases} x = -c_1 \sin t + c_2 \cos t \\ y = c_1 \cos t + c_2 \sin t. \end{cases} \tag{8}$$

The solution satisfying the conditions $x(0) = 1$ and $y(0) = 0$ is clearly

$$\begin{cases} x = \sin t = \cos(t - \frac{\pi}{2}) \\ y = -\cos t = \sin(t - \frac{\pi}{2}). \end{cases} \tag{9}$$

These two different solutions define the same path C (fig()), which is evidently the circle $x^2 + y^2 = 1$. Both (2.8) and (2.9) show that this path is traced out in the counterclockwise direction. If we eliminate t between the equations of the system, we get

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Whose general solution $x^2 + y^2 = c^2$ yeilds all the paths (but without their directions). It is obvious that the critical point $(0, 0)$ of the system(2.7) is a center.

Sprial

A critical point like that in fig() is called a sprial(or sometimes a focus). Such a point is approached in a sprial-like manner by a family of paths that wind around it an infinite number of times as $t \to \infty$ (or as $t \to -\infty$). Note particularly that while the paths approach O, they do not enter it. That is, a point P moving along such a path approaches O as $t \to \infty$ (or as $t \to -\infty$), but the line OP does not approach any definite direction.

Example

If a is an arbitrary constant, then the system

$$\begin{cases} \frac{dx}{dt} = ax - y \\ \frac{dy}{dt} = x + ay \end{cases} \tag{10}$$

has the origin as its only critical point (why?).

Spiral

The differential equation of the paths,

$$\frac{dy}{dx} = \frac{x + ay}{ax - y} \tag{11}$$

is most easily solved by introducing polar coordinates r and θ defined by $x = r\cos\theta$ and $y = r\sin\theta$. Since

$$r^2 = x^2 + y^2 \text{ and } \quad \theta = \tan^{-1}\frac{y}{x}$$

we see that

$$r\frac{dr}{dx} = x + y\frac{dy}{dx} \quad \text{and} \quad r^2\frac{d\theta}{dx} = x\frac{dy}{dx} - y.$$

Spiral

With the aid of these equations (2.11) can be written in the very simple form

$$\frac{dr}{d\theta} = ar.$$

So

$$r = ce^{a\theta} \tag{12}$$

is the polar equation of the paths. The two possible sprial configurations are shown in fig() and the direction in which these paths are traversed can be seen from the fact that $\frac{sx}{dt} = -y$ when $x = 0$. If $a = 0$, then (2.10) collapses to (2.7) and (2.12) becomes $r = c$, which is the polar equation of the family $x^2 + y^2 = c^2$ of all circles centered on the origin. This example therefore generalizes Example 2; and since the center shown in fig() stands on the borderline between the sprials of fig(), a critical point that ias a center is often called a borderline case.