Ordinary Differential Equations

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Theorem

Let x_n be an ordinary point of the differential equation

$$y^{''} + P(x)y^{'} + Q(x)y = 0. \tag{1}$$

and let a_0 and a_1 be arbitrary constants. Then there exists a unique function $y(x)$ that is analytic at x_n, is a solution of equation (1.1) in a certain neighborhood of this point, and satisfies the initial conditions $y(x_0) = a_0$ and $y(x_0) = a_1$. Furthermore, if the power series expansions of $P(x)$ and $Q(x)$ are valid on a interval $|x - x_0| < R, R > 0$, then the power series expansion of this solution is also valid on the same interval.

Proof

For the sake of convenience, we restrict our argument to the case in which x_0. This permits us to work with power series in x rather than $x - x_0$, and involves no real loss of generality. With this slight simplifaction, the hypothresis of the theorem is that $P(x)$ and $Q(x)$ are analytic at the origin and therefore have power series expansions

$$P(x) = \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1 x + p_2 x^2 + \cdots \tag{2}$$

and

$$Q(x) = \sum_{n=0}^{\infty} q_n x^n = q_0 + q_1 x + q_2 x^2 + \cdots \tag{3}$$

Proof

that converge on an interval $|x| < R$ for some $R > 0$. Keeping in mind the specified initial conditions, we try to find a solution for (1.1) in the form of a power series

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{4}$$

with radius of convergence at least R. Differentiao of (1.4) yeilds

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \tag{5}$$

Proof

and

$$y'' = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$$

$$= 2a_2 + 2.3a_3x + 3.4a_4x^2 + \cdots \tag{6}$$

It now follows from the rule for multiplying power series that

Proof

$$P(x)y' = (\sum_{n=0}^{\infty} p_n x^n)[\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n]$$

$$= \sum_{n=0}^{\infty}[\sum_{n=0}^{\infty} p_{n-k}(k+1)a_{k+1}]x^n \tag{7}$$

$$Q(x)y = (\sum_{n=0}^{\infty} q_n x^n)(\sum_{n=0}^{\infty} a_n x^n)$$

$$= \sum_{n=0}^{\infty}(\sum_{n=0}^{\infty} q_{n-k}a_k)x^n. \tag{8}$$

Proof

On substituing (1.6), (1.7) and (1.8) into (1.1) and adding the series term by term, we obtain

$$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} + \sum_{k=0}^{n} p_{n-k}(k+1)a_{k+1} + \sum_{k=0}^{n} q_{n-k}a_k]x^n = 0,$$

So we have the following recursion formula for the a_n:

$$(n+1)(n+2)a_{n+2} = -\sum_{k=1}^{n}[(k+1)p_{n-k}a_{k+1} + q_{n-k}a_k]. \qquad (9)$$

Proof

For $n = 0, 1, 2, \cdots$ this formula becomes

$$\begin{aligned} 2a_2 &= -(p_0 a_1 + q_0 a_0), \\ 2.3a_3 &= -(p_1 a_1 + 2p_0 a_2 + q_1 a_0 + q_0 a_1), \\ 3.4a_4 &= -(p_2 a_1 + 2p_1 a_2 + 3p_0 a_3 + q_2 a_0 + q_1 a_1 + q_0 a_2). \end{aligned}$$

These formulas determine $a_2, a_3, \cdots$ in terms of a_0 and a_1, so the resulting series (1.4), which formally satisfies (1.1) and the given initial conditions, is uniquely by these requirements.

Reference