



Differential Geometry

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Level Sets

Definition

Given a function $f : U \rightarrow \mathbb{R}$, where $U \subset \mathbb{R}^{n+1}$, its **level sets** are the sets $f^{-1}(c)$ defined, for each real number c , by

$$f^{-1}(c) = \{(x_1, x_2, \dots, x_{n+1}) \in U : f(x_1, x_2, \dots, x_{n+1}) = c\}$$

Remark

Then the number c is called the height of the level set, and $f^{-1}(c)$ is called the level set at height c .

Remark

Since $f^{-1}(c)$ is the solution set of the equation $f(x_1, x_2, \dots, x_{n+1}) = c$, the level set $f^{-1}(c)$ is often described as “the set $f(x_1, x_2, \dots, x_{n+1}) = c$.”

Definition

The **graph** of a function $f : U \rightarrow \mathbb{R}$ is the subset of $\mathbb{R}^{n+2}$ defined by

$$\begin{aligned} \text{graph}(f) &= \{(x_1, x_2, \dots, x_{n+2}) \in \mathbb{R}^{n+2} : (x_1, x_2, \dots, x_{n+1}) \in U \\ \text{and } x_{n+2} &= f(x_1, x_2, \dots, x_{n+1})\} \end{aligned}$$

Remark

1. For $c \geq 0$, the level set of f at height c is just the set of all points in the domain of f over which the graph is at distance c .
2. For $c < 0$, the level set of f at height c is just the set of all points in the domain of f under which the graph is at distance $-c$.

Example

consider the function $f; \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ defined by $f(x_1, x_2, \dots, x_{n+1}) = x_1^2 + \dots + x_{n+1}^2$. The level sets $f^{-1}(c)$ are empty for $c < 0$, consist of a single point (the origin) if $c = 0$, and for $c > 0$ consist of two points if $n = 0$, circle centered at the origin with radius $\sqrt{c}$ if $n = 1$, spheres centered at the origin with radius $\sqrt{c}$ if $n = 2$, etc.

Example

Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $f(x, y, z) = x^2 + y^2 + z^2$. Then the level sets for the values of c are as follows:

Value of c	Level set
$c < 0$	empty set
$c = 0$	$\{(0,0)\}$
$c > 0$	sphere of radius $\sqrt{c}$ with center at origin

Sphere S^2

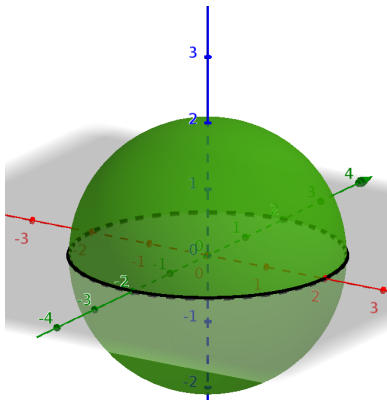


Figure: Level set of $(x, y, z) = x^2 + y^2 + z^2$ at $c = 4$

Cone in $\mathbb{R}^3$

Example

Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $f(x, y, z) = x^2 + y^2 - z^2$. Then the level sets for the values of c are as follows:

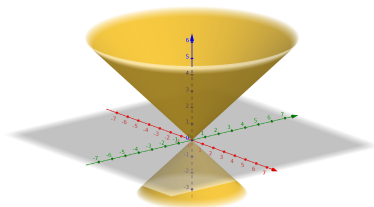


Figure: Level set of $f(x, y, z) = x^2 + y^2 - z^2$ at $c = 0$

Hyperboloid of one sheet in $\mathbb{R}^3$

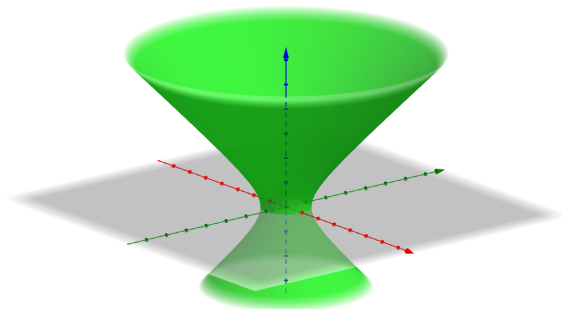


Figure: Level set of $f(x, y, z) = x^2 + y^2 - z^2$ at $c = 1$

Hyperboloid of two sheet in $\mathbb{R}^3$

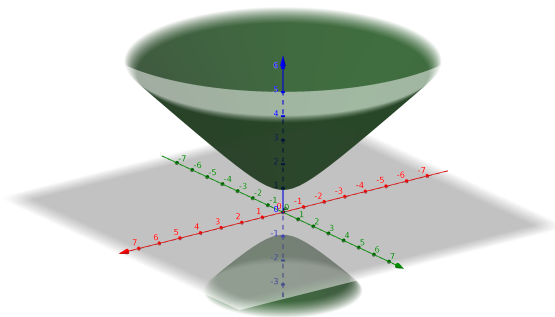


Figure: Level set of $f(x, y, z) = x^2 + y^2 + z^2$ at $c = -1$