Newton’s Equations of Motion

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The Mechanical System

The Newtonian mechanics is more intuitive than the more abstract Lagrangian theory, , even though in many cases the latter method is more useful and comfortable in setting up the equations of motion. Dynamics is the study of the motions of interacting bodies. Consider a mechanics system consisting of N particles, where a particle is an idealized material body having its mass concentratd at a point. The motion of the particle is therefore the motion of a point in space. Since a particle has no geometrical dimensions, we cannot specify the orientaion of a particle nor can we associate any particular rotational motion with it.

Newton's second law of motion

The differential equations of motion for a system od N particles can be obtained by applying Newtons' Laws of tion to the particles individually. For a single particle of mass m which is subject to a force F we obtain from Newton's second law the vector equation

$$F = ma \tag{1}$$

or

$$F = \frac{dp}{dt} = \dot{p} \tag{2}$$

where the linear momentum p is given by $p = mv$ and where the acceleration a (or v) is measured to an inertial frame of reference.

The existence of an inertial or Newtonian frame of reference is a fundamental postulated of Newtonian dynamics.

Example

Consider a frame with it origin at the sun and assume that it is nonrotating with respect to the so-called " fixed " stars. It can be shown that any other refernce frame that is not rotating, but that is translating with a uniform velocity relative to a given inertial frame, is itself an inertial frame.
Hence, the existence of a single inertial reference frame implies the existence of an infinity of other inertial frames which are equally valid (but not necessarily equally convenient) for the describtion of the motion of a particle, using the principles of Newtonian dynamics.

Assume that we have found a suitable inertial frame and that the vector r_i specifies the position of the ith particle relative to that frame. The equation of motion for the system of n particle can be written with the aid of Equation (1).

$$m_i\ddot{r}_i = F_i + R_i \quad (i = 1, 2, \cdots) \tag{3}$$

wher m_i is the mass of the ith particle and where we have broken the total force acting on the this particle into two vector components F_i and R_i. F_i is called the applied force and R_i is the constaint force.

The force acting on the body can be classified as follow:

- Applied force(External Force)
- Constraint force(Internal Force)

According to the mode application, the forces are classified into the following type:
Contact forces are transmitted to the body by a direct mechanical push or pull.
Body forces, on the hand, are associated with action at a distance and are represented by gravitational, electrical, other fields. It frequently occurs that body forces are applied throughout a body, but contact forces are applied only only at its boundary surface.

To study the motion of single particle,we use the cartesian coordinates (x, y, z) to represent the position of the particle and obtain

$$\begin{aligned} m\ddot{x} &= F_x + R_x \\ m\ddot{y} &= F_y + R_y \\ m\ddot{z} &= F_z + R_z \end{aligned} \tag{4}$$

where F_x and R_x are the x components of F and R, respectively and where F_y, R_y, F_z and R_z are defined similarly. The notations of equations (4) is somewhat unwieldly, however. In order to simplify the writing of the equations, let us denote the cartesian coordinates of the first particle by (x_1, x_2, x_3), of the second particle by (x_4, x_5, x_5) and so on. Then noting that the mass of the kth particle is $m_{3k-2} = m_{3k-1} = m_{3k}$ we can write the equations of motion in the form $m_i\ddot{x}_i = F_i + R_i, \qquad (i = 1, 2, 3, \cdots, 3N)$ where F_i and R_i are the x_i components of the applied forces and the constraint forces, respectively.

The equations of motion, whether they are written in the vector for of equation or in the scalar form, require that the variables be expressed in a consistent set of units. By consistent we mean that the quantities on both sides of each equation must be expressed in the same, or equaivalent, units. If we consider the dimensions of the units used in the equations of motion, we find that mass, length, time and force are present. Becuase the equations of motion must exhibit dimensional homogeneity, however, these four dimensions are not independent. In fact, any one dimension can be expressed n terms of the other three.

Definition

The number of degrees of freedom is equal to the number of coordinates minus the number of independent equations of constraints.

Example

Suppose that three particles are connected by rigid rods to form a triangular body with the particles at its corner. The configuration of the system is specified by giving the locations of the three particles, that is, by 9 cartesian coordinates. but each rigid rod is represented mathematically by an independent equation of constraint. So $3N - l = 9 - 3 = 6$, and the system has six degrees of freedom.

Definition (Coordinate transformation)

Consider two sets of coordinates which describes the same system. At any given time, the values of each set of coordinates are simply a group of numbers. The process of obtaining one set of numbers from the other is knows as coordinate transformation.

Definition (generalized coordinates)

any set of parameters which gives an unambiguous representation of the configuration of the system will serve as a system of coordinates in a more general sense. These parameter are known as generalized coordinates.

Theorem (Konig's theorem)

The total kinetic energy of the system is equal to the sum of (1) the kinetic energy due to a particle having a mass equal to the total mass of the system and moving with the velocity of the center of mass and (2) the kinetic energy due to the motion of the system relative to its center of mass.

Theorem

The kinetic energy of a system in terms of its motion relative to a fixed point is the sum of three parts:
(1) the kinetic energy due to a particle having a mass m and moving with the reference point P
(2) the kinetic energy of the system due to its motion relative to P, and
(3) the scalar product of the velocity of the reference point and the linear momentum of the system relative to the reference point.

Theorem

The angular momentum of a system of particles of total mass m about a fixed point O is equal to the angular momentum about O of a single particle of mass m which is moving with the center of mass plus the angular momentum of the system about the center of mass.

Definition

Consider a system whose configuration is described by n generalized coordinates. Let us define the Lagrangian function $L(q, \dot{q}, t)$ as follows:

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q, t).$$

The generalized momentum p_i associated with the generalized coordinates q_i is defined by the equation

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{5}$$