Hamilton's Equations of Motion

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Canonical form of equations of motion

Consider a holonomic system which can be described by the standard form of Lagrange's equations, namely,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, 2, \cdots, n) \qquad (1)$$

We remember that the generalized momentum conjugate to q_i is given by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad (i = 1, 2, \cdots, n) \qquad (2)$$

We can write (1) in the form

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \qquad (i = 1, 2, \cdots, n) \qquad (3)$$

Canonical form of equations of motion

Now let us define the Hamiltonian function $H(q, p, t)$ for the system as follows:

$$H(q, p, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q, q, t) \tag{4}$$

In general, the Hamiltonian function H is an explicit function of the q's, p's and t.

Since the right hand side of (4) contains $\dot{q}$'s, we must eliminate these quantities by expressing them in terms of the p's. This is accomplished by recalling from $p_i = \frac{\partial T}{\partial \dot{q}_i} = \sum_{i=1}^{n} m_{ij}\dot{q}_j + a_i$ that

$$p_i = \sum_{i=1}^{n} m_{ij}(q, t)\dot{q}_i + a_i(q, t) \tag{5}$$

Canonical form of equations of motion

Then we solve for the q's and obtain

$$\dot{q}_i = \sum_{j=1}^{n} b_{ij}(p_j - a_j) \tag{6}$$

where $b_{ij}(q, t)$ is an element of the matrix $b = m^{-1}$. The inertia matrix always be inverted since it is positive definite.
Now consider an arbitrary variation in the Hamiltonian function $H(q, p, t)$. We have

$$\delta H = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\delta p_i + \frac{\partial H}{\partial t}\delta t \tag{7}$$

In a similar manner, we obtain from (4) that

$$\delta H = \sum_{i=1}^{n} p_i \delta \dot{q}_i + \sum_{i=1}^{n} \dot{q}_i \delta p_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i - \frac{\partial L}{\partial t} \delta t. \tag{8}$$

Using the expression for p_i, given in (2) to cancel the first and fourth terms on the right-hand side, we find that

$$\delta H = \sum_{i=1}^{n} \dot{q}_i \delta p_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i - \frac{\partial L}{\partial t} \delta t. \tag{9}$$

The equations (7) and (9) are derived from the definitions of H and P_i. They do not contain any dynamical laws.

Canonical form of equations of motion

Now we use the Lagrange's equations of motion in the form given in (3). Substituting this expression into equation (9), we obtain

$$\delta H = \sum_{i=1}^{n} \dot{q}_i \delta p_i - \sum_{i=1}^{n} \dot{p}_i \delta q_i - \frac{\partial L}{\partial t}\delta t. \tag{10}$$

If we equate the equations (7) and (10), we have the following

$$\sum_{i=1}^{n}\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i - \sum_{i=1}^{n}\left(\dot{p}_i - \frac{\partial H}{\partial q_i}\right)\delta q_i - \left(\frac{\partial L}{\partial t} - \frac{\partial H}{\partial t}\right)\delta t = 0. \tag{11}$$

Canonical form of equations of motion

The variations $\delta q_i, \delta p_i$ and δt are mutually independent, so their coefficients must be zero. Hence

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$(i = 1, 2, \cdots, n) \tag{12}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

and

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \tag{13}$$

The $2n$ first-order equations given in Eq. (13) are known as Hamilton's canonical equations of motion.

Remarks:

The first n equations merely express the $\dot{q}$'s as linear functions of the p's and are equivalent to eq. (6).

The final n equations represent an application of the laws of motion to the system, and give the rates of change of the generalized momenta.

If we consider that the nq's and np's together to constitute a $2n$-vector x, then Hamilton's equations can be written as a first-order nonlinear vector equation of the form

$$\dot{x} = \nabla X(x, t) \tag{14}$$

where we note that all the $\dot{q}$'s and $\dot{p}$'s occur linearly on the left-hand side of the equation.

Modification of Hamiton's equations

Suppose that the generalized forces of a mechanical system are not all derivable from a potential function and that the system obeys the Lagrange's equations in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i' \qquad (i = 1, 2, \cdots, n), \tag{15}$$

where Q_i' is that potion of the generalized applied force which is not derivable from a potential function. Hamilton's equations for this system are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad (i = 1, 2, \cdots, n) \tag{16}$$

If eqch Q_i' can be expresses as a functions of the q's, p's, and t, then it is clear that eq. (16) is of the form $\dot{x} = \nabla X(x, t)$ given in eq (14). Suppose that we have a nonholonomic system whose configuration is described by the equations of motion are given in the form

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji} + Q_i' \qquad (i = 1, 2, \cdots, n) \quad (17)$$

has the following set of hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \sum_{j=1}^{m} \lambda_j aji + Q_j', \quad (i = 1, 2, \cdots, n) \qquad (18)$$

whre the m constraints equations are

$$\sum_{i=1}^{n} a_{ji}\dot{q}_i + a_{jt} = 0, \quad (j = 1, 2, \cdots, m). \qquad (19)$$

Here w have in eqs. (18) and (19) a set of $(2n + m)$ first order ordinary differential equations from which to solve the nq's, np's and $m\lambda$'s as