Classical Dynamics

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The angular momentum with respect to an arbitrary point P.

Let us obtain the angular momentum about the point P, as viewed by a nonrotating observer moving with that point(see the figure).
We can define H_p as

$$H_p = \sum_{i=1}^{N} \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i \tag{1}$$

where $\vec{\rho}_i$ is the position vector of the i^{th} particle with respect to the reference point P.
From the figure, we write $\vec{\rho}_i$ as

$$\vec{\rho}_i = \vec{r}_i - \vec{r}_c + \vec{\rho}_c \tag{2}$$

If we substitute (4.2) into (4.1)and note that the center of mass location is

$$r_c = \frac{1}{m} \sum_{i=1}^{N} m_i \vec{r}_i,$$

$$
\begin{aligned}
H_p &= (\vec{r}_i - \vec{r}_c + \vec{\rho}_i) \times m_i(\dot{\vec{r}}_i - \dot{\vec{r}}_c + \dot{\vec{\rho}}_c) \\
&= \sum_{i=1}^{N} \vec{r}_i \times m\dot{\vec{r}}_i - (\vec{r}_c \times m\dot{\vec{r}}_c) + \vec{\rho}_c \times m\dot{\vec{\rho}}_c \\
&= H - \vec{r}_c \times m\dot{\vec{r}}_c + \vec{\rho}_c \times m\dot{\vec{\rho}}_c
\end{aligned}
$$

Starting with H taken about O, the term $-\vec{r}_c \times m\dot{\vec{r}}_c$ shifts the reference point to the center of mass. Then the addition of $\vec{\rho}_c \times m\dot{\vec{\rho}}_c$ shifts the reference point away from the center of mass to the point P.

Problem

A simple pendulum consists of a particle of mass and a massless spring of length l that is attached at a point O which moves along the horizontal x-axis with a displacement $x_0 = A\sin\omega t$. Assume that the positive y-axis point upward and the coordinates (x, y) represent the location of the particle.

Now let us introduce a procedure which results in the ignorable coordinates begin eliminated from consideration as separate degrees of freedom in the Lagrangain formulation of the equations of motion. Suppose we consider a standard holonomic system whose configuration is given by n independent generlized coordinates, of which the first k are ignorable. In other words, the Lagrangian L is a function of $q_{k+1}, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t$. Now let us define a Routhian function $R(q_{k+1}, \cdots q_n, \quad \dot{q}_{k+1}, \cdots, \dot{q}_n, \beta_1, \cdots, \beta_k, t)$. as follows:

$$R = L - \sum_{i=1}^{k} \beta_i \dot{q}_i \tag{3}$$

where we eliminate the $\dot{q_1}'s$ corresponding to the ignorable coordinates by solving the k equations

$$\frac{\delta L}{\delta \dot{q_i}} = \beta_i \qquad (i = 1, 2, \cdots, k) \tag{4}$$

for $\dot{q}_1, \cdots, \dot{q}_k$ as functions of $q_{k+1}, \cdots q_n, \quad \dot{q_{k+1}}, \cdots, \dot{q}_n, \beta_1, \cdots, \beta_k, t$. These expressions for the $\dot{q_1}'s$ are linear in the $\beta's$. Next, let us make an arbitrary variation of all the variables in the Routhian function. We have

$$\delta R = \sum_{i=k+1}^{n} \frac{\delta R}{\delta q_i}\delta q_i + \sum_{i=k+1}^{n} \frac{\delta R}{\delta \dot{q_k}}\delta \dot{q_i} + \sum_{i=1}^{k} \frac{\delta R}{\delta \beta_i}\delta \beta_i + \frac{\delta R}{\delta t}\delta t \tag{5}$$

where we note that the $\beta' s$ are regarded as variables. A similar variation in the right-hand side of eq(4.3)

$$\delta L = \sum_{i=k+1}^{n} \frac{\delta L}{\delta q_i}\delta q_i + \sum_{i=1}^{k} \frac{\delta L}{\delta \dot{q}_i}\delta \dot{q}_i + \sum_{i=k+1}^{n} \frac{\delta L}{\delta \dot{q}_i}\delta \dot{q}_i + \frac{\delta L}{\delta t}\delta t \tag{6}$$

$$\delta \sum_{i=1}^{k} \beta_i \dot{q}_i = \sum_{i=1}^{k} \frac{\delta L}{\delta \dot{q}_i}\delta \dot{q}_i + \sum_{i=1}^{k} \dot{q}_i \delta \beta_i \tag{7}$$

$$\delta(L - \sum_{i=1}^{k} \beta_i \dot{q}_i) = \sum_{i=k+1}^{n} \frac{\delta L}{\delta q_i}\delta q_i + \sum_{i=1}^{k} \frac{\delta L}{\delta \dot{q}_i}\dot{q}_i - \sum_{i=1}^{k} \dot{q}_i \delta \beta_i + \frac{\delta L}{\delta t}\delta t \tag{8}$$

$$\frac{\delta L}{\delta q_i} = \frac{\delta R}{\delta q_i};$$
$$\frac{\delta L}{\delta \dot{q}_i} = \frac{\delta R}{\delta \dot{q}_i} \qquad (i = k+1, \cdots, n) \tag{9}$$

and

$$\dot{q}_i = -\frac{\delta R}{\delta \beta_i}; (i = 1, 2, \cdots, k)$$
$$\frac{\delta L}{\delta t} = \frac{\delta R}{\delta t} \tag{10}$$

$$\frac{d}{dt}\left(\frac{\delta R}{\delta \dot{q}_i}\right) - \frac{\delta R}{\delta q_i} = 0 \qquad (i = k+1, \cdots, n) \tag{11}$$

$$q_i = -\int \frac{\delta R}{\delta \beta_i} dt \qquad (i = 1, 2, \cdots, k) \tag{12}$$

$$\dot{\theta} = \frac{\beta}{r^2} \tag{13}$$

$$R = L - \beta\dot{\theta} = \frac{1}{2}\dot{r}^2 - \frac{\beta^2}{2r^2} + \frac{\mu}{r} \tag{14}$$

$$\frac{d}{dt}\left(\frac{\delta R}{\delta \dot{r}}\right) = r$$

$$\frac{\delta R}{\delta r} = \frac{\beta^2}{r^3} - \frac{\mu}{r^2}$$

$$r - \frac{\beta^2}{r^3} + \frac{\mu}{r^2} = 0 \tag{15}$$

$$R = T' - V' \tag{16}$$

$$
\begin{aligned}
T^{'} &= \frac{1}{2}\dot{r}^2 \\
V^{'} &= \frac{\beta^2}{2r^2} - \frac{\mu}{r}
\end{aligned} \tag{17}
$$

$$
Q_t = -\frac{\delta V}{\delta q_t} \tag{18}
$$

$$
W = \int_A^B Q.dq = \sum_{i=1}^{n} \int_{A_i}^{B_i} Q_i.dq_i \tag{19}
$$

$$
\sum_{i=1}^{n} a_{ji} dq_i = 0 \qquad (j = 1, 2, \cdots, m) \tag{20}
$$

$$
\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) - \frac{\delta L}{\delta q_i} = \sum_{j=1}^{m} \lambda_j a_{ji} \qquad (i = 1, 2, \cdots, n) \tag{21}
$$

$$\sum_{i=1}^{n} a_{ji}\dot{q}_i = 0 \qquad (j = 1, 2, \cdots, m) \tag{22}$$

$$\frac{dL}{dt} = \sum_{i=1}^{n} \frac{\delta L}{\delta \dot{q}_i}\dot{q}_i + \sum_{i=1}^{n} \frac{\delta L}{\delta q_i}\dot{q}_i \tag{23}$$

$$\frac{\delta L}{\delta q_i} = \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) - \sum_{j=1}^{m} \lambda_j a_{ji} \tag{24}$$

$$\frac{dL}{dt} = \sum_{i=1}^{n} \frac{\delta L}{\delta \dot{q}_i} + \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right)\dot{q}_i - \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_j a_j i \dot{q}_i \tag{25}$$

$$\frac{dL}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\delta L}{\delta \dot{q}_i}\dot{q}_i\right) \tag{26}$$

$$\sum_{i=1}^{n} \frac{\delta L}{\delta \dot{q}_i} \dot{q}_i - L = h \tag{27}$$