Classical Dynamics

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Kinetic energy of a system in terms of its motion relative to a fixed point

Let $\vec{r}_i$ be the position vector of i^{th} particle relative to a point O fixed in a inertial frame and $\vec{r}_p$ be the position vector of arbitrary reference point P with respect to the inertial frame, the total kinetic energy of the system is the sum of the individual kinetic energy of the particles, namely,

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i^{\,2} \tag{1}$$

from the figure, we have

$$\vec{r}_i = \vec{r}_p + \vec{\rho}_i \tag{2}$$

Then, substituting from (1.2) into (1.1), we obtain

$$
\begin{aligned}
T &= \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\vec{r}}_p + \dot{\vec{\rho}}_i)(\dot{\vec{r}}_p + \dot{\vec{\rho}}_i) \\
&= \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_p^{\,2} + \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{\rho}}_i^{\,2} + \sum_{i=1}^{N} m_i \dot{\vec{r}}_p . \dot{\vec{\rho}}_i \\
&= \frac{1}{2} m \dot{\vec{r}}_p^{\,2} + \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{\rho}}_i^{\,2} + \dot{\vec{r}}_p . \sum_{i=1}^{N} m_i \dot{\vec{\rho}}_i
\end{aligned}
$$

The position vector ρ_c of the center of mass relative to P is

$$\rho_c = \frac{1}{m} \sum_{i=1}^{N} m_i \dot{\vec{\rho}}_i$$

Hence

$$T = \frac{1}{2} m \dot{\vec{r}}_p^{\,2} + \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{\rho}}_i^{\,2} + \dot{\vec{r}}_p . m \dot{\vec{\rho}}_c \tag{3}$$

Thus we find that the total Kinetic energy is the sum of three parts

- the kinetic energy due to a particle having a mass m and moving with the reference point P,
- the kinetic energy due to its motion relative to P,
- the scalar product of the velocity of the reference point and the linear momentum of the system relative to the reference point

If the reference point P is taken at the center of mass, then $\rho_c = 0$ and (1.3) reduce to

$$T = \frac{1}{2} m \dot{\vec{r}}_p^{\,2} + \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{\rho}}_i^{\,2}$$

Kinetic energy of a rigid body in terms of its motion relative to an arbitrary reference point P

The kinetic energy due to its motion relative to P is

$$T_{rot} = \frac{1}{2}m\dot{\vec{\rho_c}}^2 + \frac{1}{2}\sum_i \sum_j I_{ij} w_i w_j$$

where the moments and products of inertia are taken with respect to the mass center.
Then the kinteic energy is

$$T = \frac{1}{2}m\dot{\vec{r_p}}^2 + \frac{1}{2}m\dot{\vec{\rho_c}}^2 + \frac{1}{2}\sum_i \sum_j I_{ij} w_i w_j + \dot{\vec{r_p}}.m\dot{\vec{\rho_c}}$$

If P is fixed in the body, then the kinetic energy of reltive motion can be written in the form

$$T_{rot} = \frac{1}{2} \sum_i \sum_j I_{ij} w_i w_j$$

where the moments and products of inertia are now taken with respect to P.

Consider a system of particles whose positions are given as in the figure

The total angular momentum H with respect to a fixed point O is

$$H = \sum_{i=1}^{N} \vec{r}_i x m_i \vec{r}_i$$

(ie) it is the sum of the moments about O of the individual linear moment of the particles, assuming that each vector $m_i\dot{\vec{r}}_i$ has a line of action passing through the corresponding particle.

The position vector is

$$\vec{r}_i = \vec{r}_c + \vec{\rho}_i$$

substiting the expression for $\vec{r}_i$ into (1.1), we obtain

$$
\begin{aligned}
H &= \sum_{i=1}^{N} (\vec{r}_c + \vec{\rho}_i) \times m_i(\vec{r}_c \times \vec{\rho}_i) \\
&= \vec{r}_c \times m\dot{\vec{r}}_c + \vec{r}_c \times \sum_{i=1}^{N} m_i \dot{\vec{\rho}}_i + \sum_{i=1}^{N} m_i \vec{\rho}_i \times \dot{\vec{r}}_c + \sum_{i=1}^{N} (\vec{\rho}_i \times m_i \dot{\vec{\rho}}_i).
\end{aligned}
$$

But

$$
\begin{aligned}
\sum_{i=1}^{N} m_i \vec{\rho}_i &= \sum_{i=1}^{N} m_i(\vec{r}_i - \vec{r}_c) \\
&= \sum_{i=1}^{N} m_i \vec{r}_i - \sum_{i=1}^{N} m_i \vec{r}_c \\
&= \sum_{i=1}^{N} m_i \vec{r}_i - m_i \vec{r}_c = 0
\end{aligned}
$$

Hence

$$H = \vec{r}_c \times m\dot{\vec{r}}_c + \sum_{i=1}^{N} \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i$$

here the second term on the right is the angular momentum H_c about the center of mass, that is,

$$H_c = \sum_{i=1}^{N} (\vec{\rho}_i \times m_i \dot{\vec{\rho}}_i)$$

Summary: The angular momentum of the system of particles of total mass m about a fixed point O is equal to the angular momentum about O of a single particle of mass m which is moving with the center of mass plus the angular momentum of the system about the center of mass.

The angular momentum of a rigid body with respect to a fixed point O is

$$H = \vec{r}_c \times m\dot{\vec{r}}_c + H_c$$

where

$$H_c = \int_V \mu(\vec{\rho} \times (\vec{\omega} \times \vec{\rho}))dv$$

In terms of the moments and products of inertia about the center of mass, we can obtain the body axis components of H_c from the matrix equation

$$H_c = I\omega$$

The rotational kinetic energy can be written in the form

$$T_{rot} = \frac{1}{2}\omega . H_c$$