(6 pages)

S.No. 3139 **P 16 MA 22**

(For candidates admitted from 2016–2017 onwards)

M.Sc. DEGREE EXAMINATION, NOVEMBER 2022.

Mathematics

LINEAR ALGEBRA

Time : Three hours Maximum : 75 marks

PART A — (10 × 2 = 20)

Answer ALL questions.

1. Define Vector space.

2. What is Row reduce echelon matrix?

3. Define linear transformation.

4. Define a hyperspace.

5. Define an ideal.

6. Define algebraically closed.

7. State Cayley-Hamilton theorem.

8. Define minimal polynomial.

9. Define invariant under T.

10. When the linear operator is known to be nilpotent?

PART B — (5 × 5 = 25)

Answer ALL questions, choosing either (a) or (b).

11. (a) If A is an $n \times n$ matrix, then prove that the following are equivalent.

(i) A is invertible.

(ii) A is row-equivalent to the $n \times n$ identity matrix.

(iii) A is a product of elementary matrices.

Or

(b) Prove that every $m \times n$ matrix over the field F is row-equivalent to a row-reduced matrix.

12. (a) Let T be a linear transformation from V into W. Then prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

Or

(b) Let V be a finite-dimensional vector space over the field F, and let W be a subspace of V. Then prove that dim $W + W^0 = \dim V$.

13. (a) Let $p, f,$ and g be polynomials over the field F. Suppose that p is a prime polynomial and that p divides the product fg. Then prove that either p divides f or p divides g.

Or

(b) Let K be a commutative ring with identity, and let A and B be $n \times n$ matrices over K. Then show that $\det(AB) = (\det A)(\det B)$.

14. (a) Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then prove that T is diagonalizable if and only if the minimal polynomial for T has the form $p = (x - c_1)\ldots(x - c_k)$, where $c_1, \ldots, c_k$ are distinct elements of F.

Or

(b) Let T be a linear operator on an n – *dimensional vector* space V. Then prove that the characteristic and minimal polynomials for T have the same roots, except for multiplicities.

15. (a) Let P be an $m \times m$ matrix with entries in the polynomial algebra $F[x]$. Prove that the following are equivalent.

(i) P is invertible.

(ii) The determinant of P is a non-zero scalar polynomial.

(iii) P is row-equivalent to the $m \times m$ identity matrix.

(iv) P is a product of elementary matrices.

Or

(b) If $V = W_1 \oplus \ldots \oplus W_k$, then prove that there exist k linear operators $E_1, \ldots, E_k$ on V such that

(i) each E_i is a projection $(E_i^2 = E_i)$;

(ii) $E_i E_j = 0$, if $i \neq j$;

(iii) $I = E_1 + \ldots + E_k$;

(iv) The range of E_i is W_i.

PART C — (3 × 10 = 30)

Answer any THREE questions.

16. If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then prove that $W_1 + W_2$ is finite-dimensional and $\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$.

17. Let V and W be vector spaces over the field F. Let T and U be linear transformations from V into W. The function $(T+U)$ defined by

$(T+U)(\propto) = T \propto + U \propto$

Is a linear transformation sfrom V into W. If c is any element of F, the function (cT) defined by

$(cT)(\propto) = c(T \propto)$

Is a linear transformation from V into W. Then prove that the set of all linear transformations from V into W, together with the addition and scalar multiplication defined above, is a vector space over the field F.

18. State and prove Taylor's Formula.

19. State and prove Cayley-Hamilton theorem.

20. State and prove Primary Decomposition Theorem.