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ARITHMETIC INDEX ON DERIVED k-JACOBSTHAL, DERIVED k-JACOBSTHAL LUCAS

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Abstract

In this paper we deliberate about sum of Derived k- Jacobsthal, Derived k- Jacobsthal Lucas with arithmetic index say $an + r$.We also investigate some properties, generating function ,alternated sum formula of Derived k-Jacobsthal and Derived k- Jacobsthal Lucas.

Keywords: Jacobsthal, Lucas, Derived k-Jacobsthal Lucas

Introduction

Jacobsthal numbers are generalized by different authors in different ways.[3],[4].In this paper we have two sections. In section 1 we discuss sum of Derived k-Jacobsthal number with arithmetic index,we also investigate generating function,[5,6] alternated sum formula of Derived k-Jacobsthal number. In section 2 we have sum of Derived k-Jacobsthal Lucas numbers with arithmetic index [7,8].we illustrate some properties, generating function alternated sum formula of Derived k-Jacobsthal Lucas numbers[9].

Definition

1.For n∈ N, any positive real number k, the ***Derived k- Jacobsthal Sequence***$\{D\hat{j}_{k,n}\}$ is defined by

$$\mathrm{D}\hat{\mathrm{j}}_{\mathrm{k,n+1}} = k\mathrm{D}\hat{\mathrm{j}}_{\mathrm{k},n} - 2\mathrm{D}\hat{\mathrm{j}}_{\mathrm{k},n-1}\ for\ n \geq 1\ \ _{with\ initial\ condition}\ \mathrm{D}\hat{\mathrm{j}}_{\mathrm{k},0} = 0, \mathrm{D}\hat{\mathrm{j}}_{\mathrm{k},1} = 1$$

2.For n∈ N, k>0 any real number then ***Derived k- Jacobsthal Lucas Sequence*** $\{\mathrm{D}\hat{\mathrm{j}}_{\mathrm{k,n}}\}$ is defined

by $D\hat{c}_{k,n+1} = kD\hat{c}_{k,n} - 2D\hat{c}_{k,n-1}$ $for\ n \geq 1\ with\ initial\ condition\ D\hat{c}_{k,0} = 2, D\hat{c}_{k,1} = k,$

3.BINET FORMULA: For $n \geq 0$ any integer, the Binet formula for n [th]Derived k- Jacobsthal, Derived k-Jacobsthal Lucas number are given by $\mathrm{D}\hat{\mathrm{j}}_{\mathrm{k,n}} = \left(\dfrac{r_1^n - r_2^n}{r_1 - r_2}\right)$;$\mathrm{D}\hat{c}_{k,n} = r_1^n + r_2^n$ where $r_1 = \frac{k+\sqrt{k^2-8}}{2}$;

$r_2 = \frac{k-\sqrt{k^2-8}}{2}$;(1)

SECTION:1 - On the Derived k – Jacobsthal number of kind$an + r$

In this session we discuss some properties, lemmas and formula for the sums of the Derived k-Jacobsthal number with index in an arithmetic sequence say $an + r$ for fixed integer a, r such that 0 ≤ r ≤ a-1.

Property 1.1: $\hat{Dj}_{k,m+n} = \hat{Dj}_{k,n+1}\,\hat{Dj}_{k,m} - 2\hat{Dj}_{k,n}\,\hat{Dj}_{k,m-1}$

Proof: Using (1) $\hat{Dj}_{k,n+1}\,\hat{Dj}_{k,m} - 2\hat{Dj}_{k,n}\,\hat{Dj}_{k,m-1}$

$$= \left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right)\left(\frac{r_1^{m} - r_2^{m}}{r_1 - r_2}\right) - 2\left(\frac{r_1^{n} - r_2^{n}}{r_1 - r_2}\right)\left(\frac{r_1^{m-1} - r_2^{m-1}}{r_1 - r_2}\right)$$

$$= \left(\frac{1}{(r_1 - r_2)^2}\right)\left\{\left(r_1^{n+1} - r_2^{n+1}\right)\left(r_1^{m} - r_2^{m}\right) - 2\left(r_1^{n} - r_2^{n}\right)\left(r_1^{m-1} - r_2^{m-1}\right)\right\}$$

$$= \frac{1}{(r_1 - r_2)^2}\left\{\left(r_1^{n+1} - r_2^{n+1}\right)\left(r_1^{m} - r_2^{m}\right) - 2\left(r_1^{n} - r_2^{n}\right)\left(r_1^{m-1} - r_2^{m-1}\right)\right\}$$

$$= \frac{1}{(r_1 - r_2)^2}\left(r_1^{m+n}(r_1 - r_2) - r_2^{m+n}(r_1 - r_2)\right)$$

$$= \frac{r_1^{m+n} - r_2^{m+n}}{r_1 - r_2} = \hat{Dj}_{k,m+n}$$

Property 1.2: $\hat{Dj}_{k,m}\,\hat{Dj}_{k,n+1} - \hat{Dj}_{k,m+1}\,\hat{Dj}_{k,n} = 2^n\,\hat{Dj}_{k,m-n}$

Proof: By mathematical induction the above result can be proved.

Lemma : 1.1 For all integer n ≥ 1, $r_1^n + r_2^n = \hat{Dj}_{k,n+1} - 2\hat{Dj}_{k,n-1}$ (2)

Proof: Using (1) $\hat{Dj}_{k,n+1} - 2\hat{Dj}_{k,n-1} = \frac{1}{r_1 - r_2}\left(r_1^{n+1} - r_2^{n+1} - 2r_1^{n+1} + 2r_2^{n-1}\right)$

After simplifications and using $r_1 r_2 = 2$ we obtain $\hat{Dj}_{k,n+1} - 2\hat{Dj}_{k,n-1} = r_1^n + r_2^n$

Lemma: 1.2 $\hat{Dj}_{k,\,a(n+2)+r} = \left(\hat{Dj}_{k,a+1} - 2\hat{Dj}_{k,\,a-1}\right)\hat{Dj}_{k,\,a(n+1)+r} - 2^a\,\hat{Dj}_{k,\,an+r}$

Proof: Using (1), (2) we can write

$$\left(\hat{Dj}_{k,a+1} - 2\hat{Dj}_{k,a-1}\right)\hat{Dj}_{k,a(n+1)+r} - 2^a\,\hat{Dj}_{k,an+r} = \left(r_1^a + r_2^a\right)\left(\frac{r_1^{a(n+1)+r} - r_2^{a(n+1)+r}}{r_1 - r_2}\right) - 2^a.\hat{Dj}_{k,an+r}$$

$$= \left(r_1^a + r_2^a\right)\left(\frac{r_1^{a(n+2)+r} - r_2^{a(n+2)+r}}{r_1 - r_2}\right) = \hat{Dj}_{k,a(n+2)+r}$$

Since $\hat{Dj}_{k,n} = \hat{Dj}_{k,n+1} - 2\hat{Dj}_{k,n-1}$. The above result can be simplified as

$$\hat{Dj}_{k,a(n+2)+r} = \hat{Dj}_{k,a}\,\hat{Dj}_{k,a(n+1)+r} - 2^a\,\hat{Dj}_{k,an+r}$$

Theorem :1`Generating function for the sequence $\left\{\hat{Dj}_{k,an+r}\right\}_{n=0}^{\infty}$

Let $\mathfrak{DJ}_{a,r}(k,x)$ be the generating function of the sequence $\left\{\hat{Dj}_{k,an+r}\right\}$ with $0 \le r \le a-1$. Then

$$\mathfrak{DJ}_{a,r}(k,x) = \hat{Dj}_{k,r} + \hat{Dj}_{k,a+r}x + \hat{Dj}_{k,2a+r}x^2 + \hat{Dj}_{k,3a+r}x^3 + \ldots$$

$$(1-\hat{Dj}_{k,a}x + 2^a x^2)\mathfrak{DJ}_{a,r}(k,x) = (1-\hat{Dj}_{k,a}x + 2^a x^2)(\hat{Dj}_{k,r} + \hat{Dj}_{k,a+r}x + \hat{Dj}_{k,2a+r}x^2 + \hat{Dj}_{k,3a+r}x^3 + \ldots$$

$$= \hat{Dj}_{k,r} + (\hat{Dj}_{k,a+r} - \hat{Dj}_{k,a}\hat{Dj}_{k,r})x + (\hat{Dj}_{k,2a+r} - \hat{Dj}_{k,a}\hat{Dj}_{k,a+r} + 2^a\hat{Dj}_{k,r})x^2 + \ldots$$

$$= \hat{Dj}_{k,r} + (\hat{Dj}_{k,a+r} - \hat{Dj}_{k,a}\hat{Dj}_{k,r})x + \sum_{n\ge 2}(\hat{Dj}_{k,a(n+2)r} - \hat{Dj}_{k,a}\hat{Dj}_{k,a(n+1)r} + 2^a\hat{Dj}_{k,an+r})x^n + \ldots$$

From lemma 1.2 the summation of right hand side of the above equation vanishes.

$$\therefore (1-\hat{Dj}_{k,a}x + 2^a x^2)\mathfrak{DJ}_{a,r}(k,x) = \hat{Dj}_{k,r} + (\hat{Dj}_{k,a+r} - \hat{Dj}_{k,a}\hat{Dj}_{k,r})x \qquad (3)$$

Using Property 1.1 we have $\hat{Dj}_{k,a+r} = \hat{Dj}_{k,r+1}\hat{Dj}_{k,a} - 2\hat{Dj}_{k,r}\hat{Dj}_{k,a-1}$

Subtract both sides by $\hat{Dj}_{k,r}\hat{Dj}_{k,a}$ we get $\hat{Dj}_{k,a+r} - \hat{Dj}_{k,r}\hat{Dj}_{k,a} = \hat{Dj}_{k,r+1}\hat{Dj}_{k,a} - (\hat{Dj}_{k,r}\hat{Dj}_{k,a} - 2\hat{Dj}_{k,r}\hat{Dj}_{k,a-1})$

$$= \hat{Dj}_{k,r+1}\hat{Dj}_{k,a} - \hat{Dj}_{k,r}\cdot\hat{Dj}_{k,a+1}$$

Using Property 1.2 the above equation can be simplified as $\hat{Dj}_{k,a+r} - \hat{Dj}_{k,r}\hat{Dj}_{k,a} = 2^r\hat{Dj}_{k,a-r}$

Hence $(1-\hat{Dj}_{k,a}x + 2^a x^2)\mathfrak{DJ}_{a,r}(k,x) = \hat{Dj}_{k,r} + 2^r\hat{Dj}_{k,a-r}x$

$$\therefore \mathfrak{DJ}_{a,r}(k,x) = \frac{\hat{Dj}_{k,r} + 2^r\hat{Dj}_{k,a-r}x}{1-\hat{Dj}_{k,a}x + 2^a x^2}$$

In Particular

1.a = 1, r = 0; $\mathfrak{DJ}_{1,0}(k,x) = \dfrac{x}{1-kx+2x^2}$; *is a Generting function of Derived k – Jacobsthal*

2. If a = 2 then at r = 0 $\mathfrak{DJ}_{2,0}(k,x) = \dfrac{kx}{1-(k^2-4)x+4x^2}$

3. If a = 2 then at r = 1 $\mathfrak{DJ}_{2,1}(k,x) = \dfrac{1-2x}{1-(k^2-4)x+4x^2}$

4. If a = 3 then at r = 0 $\mathfrak{DJ}_{3,0}(k,x) = \dfrac{(k^2-2)x}{1-(k^3-6k)x+8x^2}$

5.. If a = 3 then at r = 1 $\mathfrak{DJ}_{3,1}$(k,x) = $\dfrac{1-2kx}{1-(k^3-6k)x+8x^2}$

6. If a = 3 then at r = 2 $\mathfrak{DJ}_{3,2}$(k , x) = $\dfrac{k+4x}{1-(k^3-6k)x+8x^2}$

Theorem: 2 (Sum of Derived k-Jacobsthal number with arithmetic index $an+r$)

Sum of Derived k- Jacobsthal number of kind $an+r$ is

$$\sum_{i=0}^{n} \mathrm{D}\hat{\mathrm{j}}_{k,ai+r} = \frac{\mathrm{D}\hat{\mathrm{j}}_{k,a(n+1)+r} - 2^a \mathrm{D}\hat{\mathrm{j}}_{k,an+r} - \mathrm{D}\hat{\mathrm{j}}_{k,r} - 2^r \mathrm{D}\hat{\mathrm{j}}_{k,a-r}}{\mathrm{D}\hat{\mathrm{j}}_{k,a+1} - 2\mathrm{D}\hat{\mathrm{j}}_{k,a-1} - 2^a - 1} \tag{4}$$

Proof: Using (1) $\displaystyle\sum_{i=0}^{n} \mathrm{D}\hat{\mathrm{j}}_{k,ai+r} = \sum_{i=0}^{n}\left(\frac{r_1^{ai+r} - r_2^{ai+r}}{r_1 - r_2}\right)$

$$= \frac{1}{(r_1-r_2)}\left(\frac{r_1^{an+r+a} - r_1^r}{r_1^a - 1} - \frac{r_2^{an+r+a} - r_2^r}{r_2^a - 1}\right)$$

$$= \frac{1}{(r_1-r_2)\,((r_1r_2)^a - r_1^a - r_2^a + 1)}.\Big(r_1^{an+r}(r_1r_2)^a - r_1^r r_2^a - r_1^{a(n+1)+r} + r_1^r - r_2^{an+r}(r_1 r_2^a)$$

$$+ r_1^a r_2^r + r_2^{a(n+1)+r} - r_2^r\Big)$$

$$= \frac{1}{(2^a - (r_1^a + r_2^a) + 1)(r_1 - r_2)}\Big(2^a.\big(r_1^{an+r} - r_2^{an+r}\big) - \big(r_1^{a(n+1)+r} - r_2^{a(n+1)+r}\big) + \big(r_1^r - r_2^r\big) + \big(r_1 r_2\big)^r\big(r_1^{a-r} - r_2^{a-r}\big)\Big)$$

$$= \frac{-2^a \mathrm{D}\hat{\mathrm{j}}_{k,an+r} + \mathrm{D}\hat{\mathrm{j}}_{k,a(n+1)+r} - \mathrm{D}\hat{\mathrm{j}}_{k,r} + 2^r \mathrm{D}\hat{\mathrm{j}}_{k,a-r}}{\mathrm{D}\hat{\mathrm{j}}_{k,a+1} - 2\mathrm{D}\hat{\mathrm{j}}_{k,a-1} - 2^a - 1}$$

Corolory (i)(Sum of odd Derived k-Jacobsthal) If a = 2m+1 then, equation (4) becomes.

$$\sum_{i=0}^{n} \mathrm{D}\hat{\mathrm{j}}_{k,(2m+1)i+r} = \frac{\mathrm{D}\hat{\mathrm{j}}_{k,(2m+1)(n+1)+r} - 2^{(2m+1)}.\mathrm{D}\hat{\mathrm{j}}_{k,(2m+1)n+r} - \mathrm{D}\hat{\mathrm{j}}_{k,r} - 2^r \mathrm{D}\hat{\mathrm{j}}_{k,(2m+1-r)}}{\mathrm{D}\hat{\mathrm{j}}_{k,(2m+2)} - 2\mathrm{D}\hat{\mathrm{j}}_{k,2m} - 2^{(2m+1)} - 1}$$

Case 1: If m = 0, then a= 1, and r= 0 $\displaystyle\sum_{i=0}^{n} \mathrm{D}\hat{\mathrm{j}}_{k,i} = \frac{\mathrm{D}\hat{\mathrm{j}}_{k,(n+1)} - 2\mathrm{D}\hat{\mathrm{j}}_{k,n} - 1}{k-3}$

Case 2: If m = 1 then a = 3 $$\sum_{i=0}^{n} \hat{Dj}_{k,3i+r} = \frac{\hat{Dj}_{k,(3n+3+r)} - 8\hat{Dj}_{k,(3n+r)} - \hat{Dj}_{k,r} - 2^r \hat{Dj}_{k,(3-r)}}{\hat{Dj}_{k,4} - 2\hat{Dj}_{k,2} - 9}$$

(i). If r = 0 then $$\sum_{i=0}^{n} \hat{Dj}_{k,3i} = \frac{\hat{Dj}_{k,(3n+3)} - 8\hat{Dj}_{k,3n} - (k^2 - 2)}{k^3 - 6k - 9}$$

(ii). If r = 1 then $$\sum_{i=0}^{n} \hat{Dj}_{k,(3i+1)} = \frac{\hat{Dj}_{k,(3n+4)} - 8\hat{Dj}_{k,3n+1} - 1 - 2k}{k^3 - 6k - 9}$$

(iii). If r = 2 then $$\sum_{i=0}^{n} \hat{Dj}_{k,(3i+2)} = \frac{\hat{Dj}_{k,(3n+5)} - 8\hat{Dj}_{k,3n+2} - k - 4}{k^3 - 6k - 9}$$

Case 3: If m = 2 then a = 5 $$\sum_{i=0}^{n} \hat{Dj}_{k,5i+r} = \frac{\hat{Dj}_{k,(5n+5+r)} - 32\hat{Dj}_{k,(5n+r)} - \hat{Dj}_{k,r} - 2^r \hat{Dj}_{k,(5-r)}}{\hat{Dj}_{k,6} - 2\hat{Dj}_{k,4} - 33}$$

(i). If r = 0 then $$\sum_{i=0}^{n} \hat{Dj}_{k,5i} = \frac{\hat{Dj}_{k,(5n+5)} - 32\hat{Dj}_{k,5n} - k^4 + 6k^2 - 4}{k^5 - 10k^3 + 20k - 33}$$

(ii). If r = 1 then $$\sum_{i=0}^{n} \hat{Dj}_{k,5i+1} = \frac{\hat{Dj}_{k,(5n+6)} - 32\hat{Dj}_{k,5n+1} - 2k^3 + 8k - 1}{k^5 - 10k^3 + 20k - 33}$$

Corolary (ii):(Sum of even Derived k-Jacobsthal numbers.)

If a = 2m then equation (4) becomes $$\sum_{i=0}^{n} \hat{Dj}_{k,(2mi+r)} = \frac{\hat{Dj}_{k,(2m(n+1)+r)} - 2^{2m} \hat{Dj}_{k,(2mn+r)} - \hat{Dj}_{k,r} - 2^r \hat{Dj}_{k,(2m-r)}}{\hat{Dj}_{k,(2m+1)} - 2\hat{Dj}_{k,(2m-1)} - 2^{2m} - 1}$$

Case 1: If m = 1, then a = 2 we have $$\sum_{i=0}^{n} \hat{Dj}_{k,(2i+r)} = \frac{\hat{Dj}_{k,(2n+2+r)} - 4\hat{Dj}_{k,(2n+r)} - \hat{Dj}_{k,r} - 2^r \hat{Dj}_{k,(2-r)}}{\hat{Dj}_{k,3} - 2\hat{Dj}_{k,1} - 5}$$

(i). If r = 0 then $$\sum_{i=0}^{n} \hat{Dj}_{k,2i} = \frac{\hat{Dj}_{k,(2n+2)} - 4\hat{Dj}_{k,2n} - k}{k^2 - 9}$$

(ii). If r = 1 then $$\sum_{i=0}^{n} \hat{Dj}_{k,2i+1} = \frac{\hat{Dj}_{k,(2n+3)} - 4\hat{Dj}_{k,(2n+1)} - \hat{Dj}_{k,1} - 2\hat{Dj}_{k,1}}{k^2 - 9}$$

$$= \frac{\hat{Dj}_{k,(2n+3)} - 4\hat{Dj}_{k,(2n+1)} - 3}{k^2 - 9}$$

Case 2 : If m = 2 then a = 4 we have $$\sum_{i=0}^{n} \hat{Dj}_{k,4i+r} = \frac{\hat{Dj}_{k,(4n+4+r)} - 16\hat{Dj}_{k,(4n+r)} - \hat{Dj}_{k,r} - 2^r \hat{Dj}_{k,(4-r)}}{\hat{Dj}_{k,5} - 2\hat{Dj}_{k,3} - 17}$$

(i). If r = 0 then $\sum_{i=0}^{n} \hat{\mathrm{Dj}}_{k,4i} = \dfrac{\hat{\mathrm{Dj}}_{k,(4n+4)} - 16\hat{\mathrm{Dj}}_{k,4n} - k^3 + 4k}{k^4 - 8k^2 - 9}$

(ii). If r = 1 then $\sum_{i=0}^{n} \hat{\mathrm{Dj}}_{k,4i+1} = \dfrac{\hat{\mathrm{Dj}}_{k,(4n+5)} - 16\hat{\mathrm{Dj}}_{k,4n+1} - 2k^2 + 3}{k^4 - 8k^2 - 9}$

Theorem 3: (Alternating sum of the Derived k-Jacobsthal numbers with index $an + r$)

$$\sum_{i=0}^{n} (-1)^i \hat{\mathrm{Dj}}_{k,ai+r} = \frac{\hat{\mathrm{Dj}}_{k,r} + (-1)^n.\hat{\mathrm{Dj}}_{k,an+a+r} + (-1)^n 2^a \hat{\mathrm{Dj}}_{k,an+r} - 2^r \hat{\mathrm{Dj}}_{k,a-r}}{\hat{\mathrm{Dj}}_{k,a} + 2^a + 1}$$

Proof : Using (1) $\sum_{i=0}^{n} (-1)^i \hat{\mathrm{Dj}}_{k,ai+r} = \dfrac{1}{(r_1 - r_2)} \sum_{i=0}^{n} (-1)^i \left(r_1^{ai+r} - r_2^{ai+r}\right)$

$$= \frac{1}{(r_1 - r_2)}\left(r_1^r \sum_{i=1}^{n} ((-1)^i r_1^{ai} - r_2^r \sum_{i=1}^{n} (-1)^i r_2^{ai} \right)$$

$$= \frac{1}{(r_1 - r_2)}\left(\frac{r_1^r (1 + (-1)^n r_1^{a(n+1)})}{1 + r_1^a} - \frac{r_2^r \left(1 + (-1)^n r_2^{a(n+1)}\right)}{1 + r_2^a} \right)$$

$$= \frac{\dfrac{(r_1^r - r_2^r)}{(r_1 - r_2)} + (-1)^n \dfrac{(r_1^{an+a+r} - r_2^{an+a+r})}{(r_1 - r_2)} + \dfrac{(r_1^r r_2^a - r_2^a r_2^r)}{(r_1 - r_2)} + (-1)^n (r_1 r_2)^a \dfrac{(r_1^{an+r} - r_2^{an+r})}{(r_1 - r_2)}}{\hat{Dj}_{k,a} + 2^a + 1}$$

$$\sum_{i=0}^{n} (-1)^i \hat{\mathrm{Dj}}_{k,ai+r} = \frac{\hat{Dj}_{k,r} + (-1)^n \hat{Dj}_{k,(an+a+r)} + (-1)^n 2^a \hat{Dj}_{k,(an+r)} - 2^r \hat{Dj}_{k,(a-r)}}{\hat{Dj}_{k,a} + 2^a + 1}$$

Particular cases

Case 1: For a = 1, then r = 0

$$\sum_{i=0}^{n} (-1)^i \hat{\mathrm{Dj}}_{k,i} = \frac{\hat{Dj}_{k,0} + (-1)^n \hat{Dj}_{k,(n+1)} + (-1)^n 2^a \hat{Dj}_{k,n} - \hat{Dj}_{k,1}}{\hat{Dj}_{k,1} + 2 + 1}$$

$$= \frac{(-1)^n. \hat{Dj}_{k,(n+1)} + (-1)^n .2\hat{Dj}_{k,n} - 1}{k + 3}$$

Case 2: For a = 2 then $\sum_{i=0}^{n} (-1)^i \hat{\mathrm{Dj}}_{k,2i+r} = \dfrac{\hat{Dj}_{k,r} + (-1)^n \hat{Dj}_{k,(2n+2+r)} + (-1)^n 4.\hat{Dj}_{k,(2n+r)} - 2^r \hat{Dj}_{k,(2-r)}}{\hat{Dj}_{k,2} + 5}$

(i). If r = 0 then $\sum_{i=0}^{n} (-1)^i \hat{\mathrm{Dj}}_{k,2i} = \dfrac{(-1)^n \hat{Dj}_{k,(2n+2)} + 4(-1)^n \hat{Dj}_{k,2n} - k}{k^2 + 1}$

(ii). If r = 1 then $\sum_{i=0}^{n} (-1)^i \hat{Dj}_{k,2i+1} = \frac{(-1)^n D\hat{j}_{k,(2n+3)} + 4(-1)^n D\hat{j}_{k,(2n+1)} - 1}{k^2+1}$.

SECTION: 2-On Derived k-Jacobsthal Lucas of Arithmetic Index of $an + r$

Lemma 2.1 : For all integer n ≥ 1 $D\hat{c}_{k,n} = D\hat{c}_{k,n+1} - 2D\hat{c}_{k,n-1}$

Proof: Doing the same procedure as in Lemma 1.1 we can prove the result.

Lemma 2.2: Let a,r∈ N and 0 ≤ r ≤ a-1 then $D\hat{c}_{k,a(n+1)+r} = D\hat{c}_{k,a}\ D\hat{c}_{k,an+r} - 2^a D\hat{c}_{k,a(n-1)+r}$

Proof: Using the lemma 2.1 and following the same simplifications as in Lemma 1.2 we can prove it.

Particular cases

(i). For r = 0 then $D\hat{c}_{k,a(n+1)} = D\hat{c}_{k,a}.D\hat{c}_{k,an} - 2^a D\hat{c}_{k,a(n-1)}$ (5)

(a) Odd Derived k-Jacobsthal Lucas sequence. If a = 2p+1 then (5) becomes

$$D\hat{c}_{k,(2p+1)(n+1)} = D\hat{c}_{k,a}.D\hat{c}_{k,(2p+1)n} - 2^a D\hat{c}_{k,(2p+1)(n-1)}$$

(b) Even Derived k-Jacobsthal Lucas sequence If a = 2p then (5) becomes

$$D\hat{c}_{k,2p(n+1)} = D\hat{c}_{k,a}\ D\hat{c}_{k,2pn} - 2^n D\hat{c}_{k,2p(n-1)}$$

Theorem 4: Generating function for the sequence $\left\{D\hat{c}_{k,an+r}\right\}_{n=0}^{\infty}$

Let a,r∈ N and 0 ≤ r ≤ a-1 then $\mathfrak{DC}$ (k,x,a,r) = $\frac{D\hat{c}_{k,r} + (D\hat{c}_{k,a+r} - D\hat{c}_{k,a}.D\hat{c}_{k,r})x}{1 - D\hat{c}_{k,a}x + 2^a x^2}$

= $\frac{D\hat{c}_{k,r} + 2^r D\hat{c}_{k,a-r}x}{1 - D\hat{c}_{k,a}x + 2^a x^2}$

In Particular

(i). For a = 1, r = 0 $\mathfrak{DC}$ (k,x,1,0) = $\frac{2+kx}{1-kx+2x^2}$

(ii). For a = 2 then r = 0 $\mathfrak{DC}$ (k,x,2,0) = $\frac{2+(k^2-4)x}{1-(k^2-4)x+4x^2}$

(iii). For a= 2 then r = 1 $\mathfrak{DC}$ (k,x,2,1) = $\frac{k(1+2x)}{1-(k^2-4)x+4x^2}$

Theorem 5 (Sum of the Derived k-Jacobsthal Lucas numbers of arithmetic index.)

Let a,r∈N and 0 ≤ r ≤ a-1 then $\sum_{j=0}^{n} D\hat{c}_{k,aj+r} = \frac{D\hat{c}_{k,r} - 2^r D\hat{c}_{k,a-r} - D\hat{c}_{k,a(n+1)+r} + 2^a D\hat{c}_{k,an+r}}{1 - D\hat{c}_{k,a} + 2^a}$

Proof: Applying the same procedure as in Theorem 1.2 we can get the result.

Particular cases

(i). For r=0, a = 1 then $\sum_{j=0}^{n} D\hat{c}_{k,j} = \frac{D\hat{c}_{k,n+1} - 2D\hat{c}_{k,n} + k - 2}{k-3}$

(ii). For r = 0, a = 2p+1 then $\sum_{j=0}^{n} D\hat{c}_{k,(2p+1)j} = \frac{2 - D\hat{c}_{k,(2p+1)} - D\hat{c}_{k,(2p+1)(n+1)} + 2^{(2p+1)} D\hat{c}_{k,(2p+1)n}}{1 - D\hat{c}_{k,(2p+1)} + 2^{(2p+1)}}$

(iii). For r = 0, a = 2p then $\sum_{j=0}^{n} D\hat{c}_{k,(2p)j} = \frac{2 - D\hat{c}_{k,2p} - D\hat{c}_{k,2p(n+1)} + 2^{2p} D\hat{c}_{k,(2p)n}}{1 - D\hat{c}_{k,2p} + 2^{2p}}$

Theorem 6 (Alternating sum of the Derived k-Jacobsthal Lucas numbers with index $an + r$)

Let a,r∈N and 0 ≤ r ≤ a-1 then

$$\sum_{j=0}^{n} (-1)^{j} D\hat{c}_{k,aj+r} = \frac{D\hat{c}_{k,r} + (-1)^{n} D\hat{c}_{k,an+a+r} + (-1)^{n} 2^{a} D\hat{c}_{k,an+r} + 2^{r} D\hat{c}_{k,a-r}}{1 + D\hat{c}_{k,a} + 2^{a}}$$

Particular cases

(i). For a= 1, r= 0 then $\sum_{j=0}^{n} (-1)^{j} D\hat{c}_{k,j} = \frac{2 + (-1)^{n} D\hat{c}_{k,(n+1)} + 2(-1)^{n} D\hat{c}_{k,n} + k}{k+3}$

(ii). For a= 2, r = 0 then $\sum_{j=0}^{n} (-1)^{j} D\hat{c}_{k,2j} = \frac{2 + (-1)^{n} D\hat{c}_{k,(2n+2)} + 4(-1)^{n} D\hat{c}_{k,2n} + k^{2} - 4}{k^{2}+1}$

(iii). For a=2, r = 1 then $\sum_{j=0}^{n} (-1)^{j} D\hat{c}_{k,2j+1} = \frac{3k + (-1)^{n} D\hat{c}_{k,(2n+3)} + 4(-1)^{n} D\hat{c}_{k,(2n+1)}}{k^{2}+1}$.

Conclusion

We have studied arithmetic index of Derived k- Jacobsthal ,Derived k- JacobsthalLucas.We also seen Some properties, generating function and alternating sum of them.

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